

Covers Induced by Ext^1

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We prove a generalization of the flat cover conjecture by showing for any ring R that (1) each (right R -) module has a $\text{Ker Ext}(-, \mathcal{C})$ -cover, for any class of pure-injective modules \mathcal{C} , and that (2) each module has a $\text{Ker Tor}(-, \mathcal{B})$ -cover, for any class of left R -modules \mathcal{B} .

For Dedekind domains, we describe $\text{Ker Ext}(-, \mathcal{C})$ explicitly for any class of cotorsion modules \mathcal{C} ; in particular, we prove that (1) holds, and that $\text{Ker Ext}(-, \mathcal{C})$ is a cotilting torsion-free class. For right hereditary rings, we prove the consistency of the existence of special $\text{Ker Ext}(-, \mathcal{G})$ -precovers for any set of modules \mathcal{G} . © 2000

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1. INTRODUCTION

A classical result of Eckmann and Schopf says that if \mathcal{I} is the class of all injective (right R -) modules, then each module has an \mathcal{I} -envelope. Bass proved that if \mathcal{P} is the class of all projective modules, then each module

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has a \mathcal{P} -cover iff R is a right perfect ring. Bass's result is often interpreted as a lack of duality for modules over non-right-perfect rings.

Call a module M a *dual module* provided that there are a ring S , an R, S -bimodule N , and an injective cogenerator Q of $\text{Mod-}S$ such that $M \cong \text{Hom}_S(N, Q)$ (as right R -modules). There are two important instances of dual modules: if $S = \mathbb{Z}$ and $Q = \mathbb{Q}/\mathbb{Z}$, then the dual module is called the *character module* of the left R -module N ; if R is a k -algebra over a field k and $S = Q = k$, then any module M which is finite-dimensional over k is a dual module.

A well-known result (cf. [15]) says that the dual module M is injective iff N is a flat left R -module. So a natural candidate for dualizing the Eckmann-Schopf result (to arbitrary rings) is obtained by replacing \mathcal{P} with \mathcal{F} , the class of all flat modules. This led Enochs [11] to formulate the *flat cover conjecture* (FCC): "every module over every ring has an \mathcal{F} -cover." Only recently has the conjecture been proved, independently, by Enochs and El Bashir [4].

Enochs' proof proceeds by showing that the hypothesis of Corollary 11 of the authors' [10] is true for any ring R . The heart of his argument is a proof that there is a cardinal κ (depending only on R) such that every flat R -module A is the union of an increasing continuous sequence $(A_\alpha \mid \alpha \leq \sigma)$ of pure submodules (for some σ depending on A) such that for all $\alpha + 1 \leq \sigma$, $\text{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ and $A_{\alpha+1}/A_\alpha$ is flat. The hypothesis of [10, Corollary 11] then follows as in the proof of Corollary 10 below.

This motivates the following definition.

DEFINITION 1. For any right R -module A and any cardinal κ , a κ -*refinement* of A (of length σ) is an increasing sequence $(A_\alpha \mid \alpha \leq \sigma)$ of pure submodules of A such that $A_0 = 0$, $A_\sigma = A$, $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $\text{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ for all $\alpha + 1 \leq \sigma$.

In homological terms, the FCC says that every module has an \mathcal{A} -cover, where \mathcal{A} is the kernel of the contravariant Ext functor $\text{Ext}(-, \mathcal{C})$ and \mathcal{C} is the class of *all* dual modules (or, respectively, \mathcal{A} is the kernel of the covariant Tor functor $\text{Tor}(-, \mathcal{B})$ and \mathcal{B} is the class of *all* left R -modules). (Precise definitions are given in the next section.)

Using κ -refinements, we will generalize the FCC by replacing \mathcal{C} with *any* class of *pure-injective* modules (resp., replacing \mathcal{B} by *any* class of left R -modules). (See Corollaries 10 and 11.)

In Theorem 16, we prove that \mathcal{C} can be *any* class of *cotorsion* modules when R is a Dedekind domain; in that case, we also give a full description of the kernel. Assuming Gödel's Axiom of Constructibility ($V = L$), we prove the existence of special $\text{Ker Ext}(-, \mathcal{G})$ -precovers for *any set* of modules \mathcal{G} provided that R is a right hereditary ring (Theorem 14).

2. PRELIMINARIES

For a ring R , denote by $\text{Mod-}R$ the category of all right R -modules. We will use “module” to mean right R -module. Also, Hom , Ext , and Tor will stand for Hom_R , Ext_R^1 , and Tor_1^R , respectively.

DEFINITION 2. Let $\mathcal{C} \subseteq \text{Mod-}R$ and let \mathcal{B} be a class of left R -modules. We define

$${}^\perp\mathcal{C} = \text{Ker Ext}(-, \mathcal{C}) = \{D \mid \text{Ext}(D, C) = 0 \text{ for all } C \in \mathcal{C}\},$$

and similarly

$$\mathcal{C}^\perp = \text{Ker Ext}(\mathcal{C}, -) = \{D \mid \text{Ext}(C, D) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\text{Ker Tor}(-, \mathcal{B}) = \{A \mid \text{Tor}(A, B) = 0 \text{ for all } B \in \mathcal{B}\}.$$

For a module C , we will write ${}^\perp C$ instead of ${}^\perp\{C\}$.

We start by recalling a lemma relating κ -refinements to the vanishing of Ext :

LEMMA 3. Let C be a module. Suppose that $A = A_\mu$ is the union of a continuous chain of submodules, $A = \cup_{\alpha < \mu} A_\alpha$, such that $A_0 \in {}^\perp C$ and for all $\alpha + 1 < \mu$, $A_{\alpha+1}/A_\alpha \in {}^\perp C$. Then $A \in {}^\perp C$.

Proof. The proof is well known (see [7, Theorem 1.2; 12, Lemma IV.2.1; or 10, Lemma 1]). ■

We will often use the following notions and facts concerning precovers and covers:

DEFINITION 4. Let $\mathcal{A} \subseteq \text{Mod-}R$ and $M \in \text{Mod-}R$.

A homomorphism $\phi \in \text{Hom}(A, M)$ with $A \in \mathcal{A}$ is called an \mathcal{A} -precover of M if the induced map $\text{Hom}(A', A) \rightarrow \text{Hom}(A', M)$ is surjective for all $A' \in \mathcal{A}$. An \mathcal{A} -precover $\phi \in \text{Hom}(A, M)$ is an \mathcal{A} -cover provided that each $\psi \in \text{Hom}(A, A)$ satisfying $\phi = \phi\psi$ is an automorphism of A .²

A precover ϕ is called *special* provided that $\text{Ker}(\phi) \in \mathcal{A}^\perp$ and ϕ is surjective.

A (special) \mathcal{A} -preenvelope and an \mathcal{A} -envelope are defined dually; see [22, Sect. 1.2].

²Here, we follow the terminology of Enochs and Xu [22]. The corresponding terminology of Auslander, Reiten, and Smalø (e.g. in [2]) is that of a right approximation and a minimal right approximation.

If $\phi: A \rightarrow M$ is surjective, $A \in \mathcal{A}$, and $\text{Ker}(\phi) \in \mathcal{A}^\perp$, then ϕ is a special \mathcal{A} -precover of M (see [22, 2.1.3]). Furthermore, by [22, 2.1.1 and 2.2.12], we have

THEOREM 5. *Let $\mathcal{A} \subseteq \text{Mod-}R$ be a class containing all projective modules and closed under direct limits and extensions. Assume that a module M has an \mathcal{A} -precover. Then M has an \mathcal{A} -cover, the \mathcal{A} -cover is special, and it is uniquely determined up to isomorphism.*

A submodule A of a module B is a *pure* submodule ($A \subseteq_* B$, for short) if for each finitely presented module F , the functor $\text{Hom}(F, -)$ preserves the exactness of the sequence $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$. (See, for example, [17, pp. 53ff] or [14, pp. 94ff].) We will need the following properties of pure submodules:

LEMMA 6. *Let R be a ring and let $\kappa \geq \text{card}(R) + \aleph_0$.*

- (i) *Let M be a module and X be a subset of M with $\text{card}(X) \leq \kappa$. Then there is a pure submodule $N \subseteq_* M$ such that $X \subseteq N$ and $\text{card}(N) \leq \kappa$.*
- (ii) *Assume $C \subseteq_* A$ and $B/C \subseteq_* A/C$. Then $B \subseteq_* A$.*
- (iii) *Assume $A_0 \subseteq \cdots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \cdots$ is a chain of pure submodules of M . Then $\cup_\alpha A_\alpha$ is a pure submodule of M .*

Proof. The proof is well known (see [14, Theorem 6.4]). ■

For convenience we state a consequence of [10, Theorem 10] in the terminology of this paper.

THEOREM 7. *If \mathcal{C} is a class of modules such that $({}^\perp\mathcal{C})^\perp = Q^\perp$ for some module Q , then every module has a special ${}^\perp\mathcal{C}$ -precover.*

Proof. This follows from [10, Theorem 10] because $({}^\perp\mathcal{C}, ({}^\perp\mathcal{C})^\perp)$ is a cotorsion theory and to say that it has enough projectives is to say that every module has a special ${}^\perp\mathcal{C}$ -precover. ■

3. COVERS INDUCED BY EXT AND TOR

Modules that are injective with respect to pure embeddings are called *pure-injective* [14, Sect. 7]. For example, any dual module is pure injective.

THEOREM 8. *Let R be a ring and \mathcal{C} be a class of pure-injective modules. Let $\kappa = \text{card}(R) + \aleph_0$. Then the following conditions are equivalent for any module A :*

- (i) $A \in {}^\perp\mathcal{C}$;

(ii) *there is a cardinal λ such that A has a κ -refinement $(A_\alpha \mid \alpha \leq \lambda)$ with $A_{\alpha+1}/A_\alpha \in {}^\perp\mathcal{C}$ for all $\alpha < \lambda$.*

Proof. (i) implies (ii). Let $\kappa = \text{card}(R) + \aleph_0$. If $\text{card}(A) \leq \kappa$, we can let $\lambda = 1$, $A_0 = 0$, and $A_1 = A$. So we can assume that $\text{card}(A) > \kappa$. Let $\lambda = \text{card}(A)$. Then $A \cong F/K$, where $F = R^{(A)}$ is a free module. We enumerate the elements of F in a λ -sequence: $F = \{x_\alpha \mid \alpha < \lambda\}$. By induction on α , we will define a sequence $(A_\alpha \mid \alpha \leq \lambda)$ such that for all $\alpha \leq \lambda$, A_α is pure in A and belongs to ${}^\perp\mathcal{C}$. Since each $C \in \mathcal{C}$ is pure-injective, it will follow from the long exact sequence induced by

$$0 \rightarrow A_\alpha \rightarrow A_{\alpha+1} \rightarrow A_{\alpha+1}/A_\alpha \rightarrow 0$$

that $A_{\alpha+1}/A_\alpha \in {}^\perp\mathcal{C}$ for all $\alpha < \lambda$.

A_α will be constructed so that it equals $(R^{(I_\alpha)} + K)/K$ for some $I_\alpha \subseteq \lambda$ such that $R^{(I_\alpha)} \cap K$ is pure in K . Let $A_0 = 0$. Assume A_β has been defined for all $\beta < \sigma$. Suppose first that $\sigma = \alpha + 1$. By induction on $n < \omega$ we will define an increasing chain $F_0 \subseteq F_1 \subseteq \dots$ and then put $A_{\alpha+1} = \bigcup_{n < \omega} (F_n + K)/K$. We require that $\text{card}(F_{n+1}/F_n) \leq \kappa$ for all $n < \omega$, and furthermore, for n odd, that $(F_n + K)/K$ be pure in F/K ; for n even, that $F_n = R^{(J_n)}$ for some $J_n \supseteq J_{n-2} \supseteq \dots \supseteq J_0$ and $F_n \supseteq K'_n$, where $F_{n-1} \cap K \subseteq K'_n \subseteq_* K$.

First, put $F_0 = R^{(I_\alpha)}$ and let $J_0 = I_\alpha$ and $K'_0 = R^{(I_\alpha)} \cap K$. Assume F_{n-1} has been constructed and n is odd. By part (i) of Lemma 6 there is a pure submodule $(F_n + K)/(F_{n-2} + K) \subseteq_* F/(F_{n-2} + K)$ of cardinality $\leq \kappa$ containing $(x_\alpha R + F_{n-1} + K)/(F_{n-2} + K)$. Moreover, we can choose F_n so that $\text{card}(F_n/F_{n-1}) \leq \kappa$. By part (ii) of Lemma 6, $(F_n + K)/K$ is pure in F/K .

Assume $n > 0$ is even. We first define K'_n : by part (i) of Lemma 6, we find a pure submodule $K'_n/K'_{n-2} \subseteq_* K/K'_{n-2}$ of cardinality $\leq \kappa$ containing $(F_{n-1} \cap K)/K'_{n-2}$. This is possible since $K'_{n-2} \supseteq F_{n-3} \cap K$ and $(F_{n-1} \cap K)/(F_{n-3} \cap K)$ embeds in F_{n-1}/F_{n-3} , so it has cardinality $\leq \kappa$. By part (ii) of Lemma 6, we have $K'_n \subseteq_* K$.

We can choose $J_n \subseteq \lambda$ so that $\text{card}(J_n - J_{n-2}) \leq \kappa$ and $F_{n-1} + K'_n \subseteq R^{(J_n)} = F_n$. This is possible since $\text{card}((F_{n-1} + K'_n)/F_{n-2}) \leq \kappa$; indeed, we have the exact sequence

$$0 \rightarrow F_{n-1}/F_{n-2} \rightarrow (F_{n-1} + K'_n)/F_{n-2} \rightarrow (F_{n-1} + K'_n)/F_{n-1} \rightarrow 0,$$

and $(F_{n-1} + K'_n)/F_{n-1} \cong K'_n/(F_{n-1} \cap K)$ has cardinality $\leq \kappa$ because it is a homomorphic image of K'_n/K'_{n-2} .

Now, define $A_{\alpha+1} = \bigcup_{n < \omega} (F_n + K)/K$ and $I_{\alpha+1} = \bigcup_{n < \omega} J_{2n}$. By part (iii) of Lemma 6, $A_{\alpha+1} \subseteq_* A$. Clearly, $\text{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$.

We have $A_{\alpha+1} \cong F'/K'$, where $F' = \bigcup_{n < \omega} F_{2n}$ and $K' = F' \cap K$. Also, $F' = R^{(I_{\alpha+1})}$ is free, and $K' = \bigcup_{n < \omega} K'_{2n}$ is pure in K by construction and part (iii) of Lemma 6.

Let $C \in \mathcal{C}$. In order to prove that $\text{Ext}(A_{\alpha+1}, C) = 0$, we have to extend any $f \in \text{Hom}(K', C)$ to an element of $\text{Hom}(F', C)$. First, f extends to K , since $K' \subseteq_* K$ and C is pure-injective. By the assumption (i), we can extend further to F , and then restrict to F' .

Finally, if $\sigma \leq \lambda$ is a limit ordinal, let $A_\sigma = \bigcup_{\beta < \sigma} A_\beta$; that A_σ has the desired properties follows from Lemma 3 and part (iii) of Lemma 6.

(ii) implies (i). This is clear by Lemma 3. ■

LEMMA 9. *If $\mathcal{A} \subseteq \text{Mod-}R$ is equal to ${}^\perp\mathcal{C}$ for a class \mathcal{C} of pure-injective modules, then every module M which has an \mathcal{A} -precover has an \mathcal{A} -cover.*

Proof. This follows from Theorem 5 and the following observation of Angeleri, Mantese, Tonolo, and Trlifaj: Assume P is a pure-injective module. Then ${}^\perp P$ is closed under homomorphic images of pure epimorphisms. The canonical map of a direct sum onto a direct limit is well-known to be a pure epimorphism (cf. [21, 33.9(2)]). So ${}^\perp P$ is closed under direct limits. ■

COROLLARY 10. *Let R be a ring and \mathcal{C} be a class of pure-injective modules. Then every module has a ${}^\perp\mathcal{C}$ -cover.*

Proof. Let $\kappa = \text{card}(R) + \aleph_0$. Denote by H the direct sum of a representative set of the class $\{A \mid \text{card}(A) \leq \kappa \text{ \& } \text{Ext}(A, \mathcal{C}) = 0\}$. Clearly, $({}^\perp\mathcal{C})^\perp \subseteq H^\perp$. Conversely, take $D \in H^\perp$. Let $A \in {}^\perp\mathcal{C}$; by Theorem 8, A has a κ -refinement $(A_\alpha \mid \alpha \leq \lambda)$. By choice of H , $\text{Ext}(A_{\alpha+1}/A_\alpha, D) = 0$ for all $\alpha < \lambda$ and hence, by Lemma 3, $\text{Ext}(A, D) = 0$. So $D \in ({}^\perp\mathcal{C})^\perp$. This proves that $({}^\perp\mathcal{C})^\perp = H^\perp$. By Theorem 7, every module has a special ${}^\perp\mathcal{C}$ -precover. An application of Lemma 9 finishes the proof. ■

If \mathcal{C} is the class of *all* pure-injective modules then ${}^\perp\mathcal{C}$ is the class of all flat modules, so Corollary 10 implies the FCC. However, in general, ${}^\perp\mathcal{C}$ will be larger than the class of flat modules.

Theorem 8 and Corollary 10 remain true for any notion of “pure” that satisfies properties (i)–(iii) in Lemma 6. For example, this happens for the RD-purity [12, II. Sect. 3]; hence we get analogous results for the particular case where \mathcal{C} is a class of RD-injective modules.

There is an analogue of Theorem 8 for the bifunctor Tor :

COROLLARY 11. *Let R be a ring and \mathcal{B} be any class of left R -modules. Let $\kappa = \text{card}(R) + \aleph_0$. The following conditions are equivalent for any module A :*

(i) $A \in \text{Ker Tor}(-, \mathcal{B})$,

(ii) *there is a cardinal λ such that A has a κ -refinement $(A_\alpha \mid \alpha \leq \lambda)$ such that $A_{\alpha+1}/A_\alpha \in \text{Ker Tor}(-, \mathcal{B})$ for all $\alpha < \lambda$.*

Proof. For each $B \in \mathcal{B}$, let $C(B) = \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ be the character module of B . Put $\mathcal{C} = \{C(B) \mid B \in \mathcal{N}\}$. Then \mathcal{C} is a class of pure-injective modules. By [6, Proposition VI.5.1], ${}^{\perp}\mathcal{C} = \text{Ker Tor}(-, \mathcal{B})$, so the assertion follows from Theorem 8. ■

THEOREM 12. *Let R be a ring.*

(i) *Let \mathcal{B} be a class of left R -modules. Then every module has a $\text{Ker Tor}(-, \mathcal{B})$ -cover.*

(ii) *Let \mathcal{D} be a class of dual modules. Then every module has a ${}^{\perp}\mathcal{D}$ -cover.*

Proof. (i) As above, we have $\mathcal{A} = \text{Ker Tor}(-, \mathcal{B}) = {}^{\perp}\mathcal{C}$, where \mathcal{C} is a class of pure-injective modules. Then every module has an \mathcal{A} -cover by Corollary 10.

(ii) Since any dual module is pure-injective, every module has a ${}^{\perp}\mathcal{D}$ -cover by Corollary 10. ■

Taking \mathcal{B} to be the class of all left R -modules, we obtain the FCC again, this time as a consequence of Theorem 12(i).

COROLLARY 13. (i) *Let k be a field and R be a k -algebra. Let \mathcal{M} be a class of k -finite dimensional modules. Then every module has a ${}^{\perp}\mathcal{M}$ -cover.*

(ii) *Assume that R is a right pure-semisimple ring. Let \mathcal{M} be any class of modules. Then every module has a ${}^{\perp}\mathcal{M}$ -cover.*

Proof. (i) Since any finite-dimensional module is dual (in the k -vector space duality), the assertion follows from Theorem 12(ii).

(ii) Since every R -module is pure-injective (see [14, Theorem 8.4]) this follows from Corollary 10. ■

4. HEREDITARY RINGS

By [10, Theorem 10], every module has a special M^{\perp} -preenvelope, for any module M . When M is pure-injective, Theorem 8 (for the class $\mathcal{C} = \{M\}$) yields the dual assertion that every module has a special ${}^{\perp}M$ -precover. It is an open problem (even for $R = \mathbb{Z}$) whether for every M (or even for $M = \mathbb{Z}$) every module has a special ${}^{\perp}M$ -precover. However, we can prove a consistency result in the case where R is a right hereditary ring:

THEOREM 14. *Assume $V = L$. Let R be a right hereditary ring and let \mathcal{G} be a set of modules. Let $\kappa = \prod_{G \in \mathcal{G}} \text{card}(G) + \text{card}(R) + \aleph_0$.*

- (i) Let A be a module of cardinality $\rho > \kappa$ such that $A \in {}^\perp\mathcal{G}$. Then there is a κ -refinement $(A_\alpha \mid \alpha \leq \rho)$ such that $A_{\alpha+1}/A_\alpha \in {}^\perp\mathcal{G}$ for all $\alpha < \rho$.
- (ii) Every module has a special ${}^\perp\mathcal{G}$ -precover.

Proof. Replacing \mathcal{G} by $\prod_{G \in \mathcal{G}} G$, we can, without loss of generality, assume that $\mathcal{G} = \{G\}$ for a single module G . Part (i) is then a consequence of Theorem 5.5(2) on page 50 of [8], which is proved there for the ring \mathbb{Z} , but which has the same proof for any hereditary ring R . The sequence $(A_\nu \mid \nu \leq \text{cf}(\rho))$ given there has quotients $A_{\nu+1}/A_\nu$ which are of cardinality $< \rho$, but by induction on $\rho \geq \kappa^+$, we can refine this sequence by inserting between A_ν and $A_{\nu+1}$, whenever $\text{card}(A_{\nu+1}/A_\nu) > \kappa$, a sequence $(A_{\nu,\tau} \mid \tau \leq \rho_\nu)$ such that $\rho_\nu = \text{card}(A_{\nu+1}/A_\nu)$, $A_{\nu,0} = A_\nu$, $A_{\nu,\rho_\nu} = A_{\nu+1}$, and for all $\tau < \rho_\nu$, $\text{card}(A_{\nu,\tau+1}/A_{\nu,\tau}) \leq \kappa$. Moreover, one can check that each member of the refined sequence $(A_{\nu,\tau} \mid \nu \leq \text{cf}(\rho), \tau \leq \rho_\nu)$ has fewer than ρ predecessors, and hence the whole sequence has length ρ . (See also [3, Theorem 3.1], where the result is proved for $\mathcal{G} = \{R\}$.)

Part (ii) follows as in Corollary 10. ■

Remark 15. (1) The proof does not extend to proper classes of modules because of the dependence of κ on the cardinality of \mathcal{G} . This contrasts with Theorem 8, where $\kappa = \text{card}(R) + \aleph_0$ does not depend on \mathcal{C} . Also, Theorem 14(ii) cannot be improved to claim the existence of ${}^\perp\mathcal{G}$ -covers. Indeed, if R is right hereditary, but not right perfect, and $\mathcal{G} = \{F\}$ where F is the free module of rank $2^{\text{card}(R)}$, then (under $V = L$) ${}^\perp\mathcal{G}$ is the class of all projective modules; cf. [20, Theorem 3.13(ii)]. Since R is not right perfect, there exist modules without ${}^\perp\mathcal{G}$ -covers, by the classical result of Bass.

(2) In order to be able to conclude that every module has a special ${}^\perp\mathcal{G}$ -precover, it is not necessary that the length of the refined sequence be a cardinal, $\rho = \text{card}(A)$, rather than just an ordinal. We do not know if it is provable in ordinary set theory, ZFC (say for $R = \mathbb{Z}$), that for every G there is a κ such that every A satisfying $\text{Ext}(A, G) = 0$ has a κ -refinement (of some length σ) whose factors (i.e. $A_{\alpha+1}/A_\alpha$) are in ${}^\perp G$.

(3) For the case of $G = \mathbb{Z} = R$, in any model of ZFC in which there are non-free Whitehead groups, there exists $A \in {}^\perp\mathbb{Z}$ such that there is no \aleph_0 -refinement of A whose factors are in ${}^\perp\mathbb{Z}$: take A to be a non-free Whitehead group and use the fact that countable Whitehead groups are free. Furthermore, for any explicitly given cardinal κ (e.g., κ is \aleph_{586} or $\aleph_{\omega_1+\omega 3+29}$), there is no theorem of ZFC which says that every $A \in {}^\perp\mathbb{Z}$ has a κ -refinement whose factors are in ${}^\perp\mathbb{Z}$; this is because there is a model of ZFC in which there are non-free Whitehead groups, but every Whitehead group of size $\leq \kappa$ is free (see [9, 2.8]).

(4) There is a model of ZFC + GCH such that for any non-cotorsion \mathbb{Z} -module G and for any κ , there is an A such that $\text{Ext}(A, G) = 0$ but there is no κ -refinement of A of length $= \text{card}(A)$ whose factors are in ${}^\perp G$. We use a model of the uniformization principle designated UP in [20, p. 1526]. As there, or as in [19], given κ , we can construct a \mathbb{Z} -module A of some cardinality $\lambda > \kappa$ such that $\text{Ext}(A, G) = 0$ and A has a λ -filtration $\bigcup_{\nu < \lambda} A'_\nu$ such that for a stationary set of ν , $A'_{\nu+1}/A'_\nu \cong \mathbb{Q}$. A standard argument then shows that for any κ -refinement $(A_\alpha \mid \alpha \leq \lambda)$ there is an $\alpha < \beta < \lambda$ such that \mathbb{Q} is a submodule of A_β/A_α , and hence $\text{Ext}(A_\beta/A_\alpha, G) \neq 0$, since G is not cotorsion.

In contrast to Remark 15(4) we have the following theorem for cotorsion modules over Dedekind domains.

Recall that a module C is *cotorsion* if $\text{Ext}(F, C) = 0$ for every flat module F (cf. [22, p. 52]). For example, any pure-injective module is cotorsion. If R is a Dedekind domain, then C is cotorsion iff $\text{Ext}(Q(R), C) = 0$ where $Q(R)$ is the quotient field of R .

For a module M , denote by $\text{Cog}(M)$ the class of all modules cogenerated by M (i.e., the class of all submodules of products of copies of M). A module M is *cotilting* if ${}^\perp M = \text{Cog}(M)$. $\text{Cog}(M)$ is then a torsion-free class, called the *cotilting torsion-free class*; cf. [5, Sect. 1].

THEOREM 16. *Let R be a Dedekind domain and let $\text{Spec}(R)$ be the spectrum of R . Let \mathcal{C} be a class of cotorsion modules.*

- (i) *There is a set $S_{\mathcal{C}} \subseteq \text{Spec}(R)$ such that*

$${}^\perp \mathcal{C} = \{A \in \text{Mod-}R \mid \forall P \in S_{\mathcal{C}} : R/P \not\subseteq A\}.$$

In fact, $S_{\mathcal{C}} = \{P \in \text{Spec}(R) \mid \exists C \in \mathcal{C} : R/P \notin {}^\perp C\}$.

(ii) *There is a class \mathcal{P} of pure-injective modules such that ${}^\perp \mathcal{C} = {}^\perp \mathcal{P}$. This is a consequence of any one of the following facts for an arbitrary cotorsion module C :*

- (a) ${}^\perp C = {}^\perp \prod \{\hat{R}_P \mid P \in S_C\}$, where $\hat{R}_P = \text{Hom}(E(R/P), E(R/P))$;
 - (b) ${}^\perp C = {}^\perp PE(C)$, where $PE(C)$ is the pure-injective envelope of C ;
 - (c) ${}^\perp C = {}^\perp F$, where F is the flat cover of C ; moreover, F is pure-injective.
- (iii) ${}^\perp \mathcal{C}$ is a cotilting torsion-free class and every module has a ${}^\perp \mathcal{C}$ -cover.

Proof. (i) Let A be a module. Denote by $T(A)$ the torsion part of A . Since every element of \mathcal{C} is cotorsion and $A/T(A)$ is flat, we have $A \in {}^\perp \mathcal{C}$ iff $T(A) \in {}^\perp \mathcal{C}$. We also have $\text{Soc}(E(T(A))) \trianglelefteq T(A) \trianglelefteq E(T(A))$ and $\text{Soc}(E(T(A))) = \text{Soc}(T(A)) \cong \bigoplus_{0 \neq P \in \text{Spec}(R)} (R/P)^{(\alpha_P)}$ for some cardinals α_P . By Matlis' theory [16] (see also [17, Theorem 18.4]) we have $E(R/P) =$

$\cup_{n < \omega} P^{-n}(R/P)$, so $E(R/P)$ has an (infinite) composition series with factors isomorphic to R/P , for every $0 \neq P \in \text{Spec}(R)$. By Lemma 3 we get that $T(A) \in {}^\perp \mathcal{C}$ iff $\text{Soc}(T(A)) \in {}^\perp \mathcal{C}$ iff $R/P \in {}^\perp \mathcal{C}$ for all $0 \neq P \in \text{Spec}(R)$ such that R/P is a submodule of A . Note that $R/P \in {}^\perp \mathcal{C}$ iff $P \notin S_{\mathcal{C}}$. It follows that $A \in {}^\perp \mathcal{C}$ iff R/P is not a submodule of A for all $P \in S_{\mathcal{C}}$.

(ii) (a) Let $0 \neq P \in \text{Spec}(R)$. By part (i), ${}^\perp \hat{R}_P = \{A \mid \forall Q \in S_{\hat{R}_P} : R/Q \not\subseteq A\}$, where $S_{\hat{R}_P} = \{Q \in \text{Spec}(R) \mid R/Q \notin {}^\perp \hat{R}_P\}$.

By Matlis' theory, if $q \in R \setminus P$, then $q \cdot$ is an automorphism of $E(R/P)$, and hence of \hat{R}_P . Since $\hat{R}_P = \text{Hom}(E(R/P), \bigoplus_{Q \in \text{Spec}(R)} E(R/Q))$, \hat{R}_P is pure-injective and flat, but not injective. Since $q \cdot$ is a monomorphism of $E(\hat{R}_P)$ we infer that the torsion module $M_P = E(\hat{R}_P)/\hat{R}_P$ is q -torsion-free. We also have $\text{Ext}(R/Q, \hat{R}_P) = \text{Hom}(R/Q, M_P)$ for all $0 \neq Q \in \text{Spec}(R)$. It follows that $\text{Ext}(R/Q, \hat{R}_P) = 0$ for all $Q \neq P$. Since $\text{Soc}(M_P) \neq 0$ and $\text{Soc}(M_P)$ is a direct sum of copies of R/P , we get $\text{Ext}(R/P, \hat{R}_P) \neq 0$.

This proves that $S_{\hat{R}_P} = \{P\}$. If $J = \prod \{\hat{R}_P \mid P \in S_C\}$, then J is pure-injective and $S_J = S_C$, so ${}^\perp C = {}^\perp J$ by part (i).

(b) By part (i) it suffices to show that for all P in $\text{Spec}(R)$, $R/P \in {}^\perp C$ if and only if $R/P \in {}^\perp \text{PE}(C)$. But C is elementarily equivalent to $\text{PE}(C)$ ([18]; see also [14, Theorem 7.51]). Once we show that there is a first-order sentence θ_P in the language of R -modules such that, for any module M , $\text{Ext}(R/P, M) = 0$ if and only if $M \models \theta_P$, we are done. Now P is generated by two elements, say p_1, p_2 , and is finitely presented; say the relations are generated by $\{\sum_{i=1}^2 r_{ij} p_i = 0 \mid j = 1, \dots, m\}$. Also, $\text{Ext}(R/P, M) = 0$ if and only if every homomorphism from P to M extends to a homomorphism from R to M . Therefore $\text{Ext}(R/P, M) = 0$ if and only if

$$M \models \forall x_1 \forall x_2 [(\bigwedge_{j=1}^m \sum_{i=1}^2 r_{ij} x_i = 0) \Rightarrow (\exists y (\bigwedge_{i=1}^2 p_i y = x_i))].$$

(c) Since C is cotorsion, F is flat and cotorsion and hence pure-injective [22, Lemma 3.2.3]. For each $P \in \text{Spec}(R)$, denote by R_P the localization of R at P , by P_P the (unique) maximal ideal of R_P , and by $k(P)$ the residue field R_P/P_P . By [22, Theorem 4.1.15], $F \cong \prod_{P \in \text{Spec}(R)} T_P$, where T_P is the completion of a free R_P -module of rank π_P in the P_P -adic topology. The cardinals π_P ($P \in \text{Spec}(R)$) are uniquely determined by C and are called the *0th dual Bass numbers* of C [22, Sect. 5.2].

Xu's formula for computing dual Bass numbers [22, Theorem 5.2.2] gives $\pi_P = \dim_{k(P)} k(P) \otimes_{R_P} \text{Hom}(R_P, C)$. In particular, $\pi_P = 0$ iff $k(P) \otimes_{R_P} \text{Hom}(R_P, C) = 0$ iff $\text{Im}(\nu_P \otimes_{R_P} 1) = \text{Hom}(R_P, C)$, where ν_P is the embedding of P_P into R_P . The latter is equivalent to $P_P \cdot \text{Hom}(R_P, C) = \text{Hom}(R_P, C)$.

Since R_P is a noetherian valuation domain, the ideal P_P is principal, and $P_P = s \cdot R_P$ for some $s \in P_P$. So $\pi_P = 0$ iff $s \cdot \text{Hom}(R_P, C) = \text{Hom}(R_P, C)$.

On the other hand, if $0 \neq P \in \text{Spec}(R)$, then $R/P \cong k(P)$ as R -modules, so $R/P \in {}^\perp C$ iff $\text{Hom}(\nu_P, C)$ is surjective. The latter is equivalent to $s.\text{Hom}(R_P, C) = \text{Hom}(R_P, C)$, and hence to $\pi_P = 0$. It follows that $S_C = \{P \in \text{Spec}(R) \mid P \neq 0 \text{ \& } \pi_P \neq 0\}$.

By [22, Lemma 4.1.5], $T_P \cong \text{Hom}(E(R/P), E(R/P)^{(\pi_P)})$, so $q.$ is an automorphism of T_P for each $q \in R \setminus P$, and as in part (a) we get $S_{T_P} = \{P\}$ whenever $P \neq 0$ and $\pi_P \neq 0$. Since $S_{T_0} = \emptyset$, we infer that $S_F = S_C$, so ${}^\perp C = {}^\perp F$ by part (i).

(iii) By part (ii) and Corollary 10, every module has a special ${}^\perp \mathcal{C}$ -cover. Since ${}^\perp \mathcal{C}$ is closed under submodules and products, [1, Theorem 2.5] gives that ${}^\perp \mathcal{C}$ is a cotilting torsion-free class. ■

In [13, Sect. 2], cotilting torsion-free classes of abelian groups were characterized. We have the following for modules over Dedekind domains:

COROLLARY 17. *Let R be a Dedekind domain and \mathcal{T} be a class of modules. Then the following conditions are equivalent:*

- (i) \mathcal{T} is a cotilting torsion-free class such that \mathcal{T} is closed under direct limits.
- (ii) There is a set of non-zero prime ideals, \mathcal{P} , such that

$$\mathcal{T} = \{A \in \text{Mod-}R \mid \forall P \in \mathcal{P} : R/P \not\subseteq A\}.$$

Proof. (i) implies (ii). We have $\mathcal{T} = {}^\perp C$ for a cotilting module C . Since \mathcal{T} is closed under direct limits and contains all projective modules, C is cotorsion. By part (i) of Theorem 16, we can take $\mathcal{P} = S_C$.

(ii) implies (i). By the proof of part (ii) of Theorem 16, we have $S_{\hat{R}_P} = \{P\}$ for each non-zero prime ideal P . So $\mathcal{T} = {}^\perp \prod \{\hat{R}_P \mid P \in \mathcal{P}\}$, and \mathcal{T} is a cotilting torsion-free class closed under direct limits by (the proof of) part (iii). ■

5. OPEN PROBLEMS

(1) Characterize the rings R such that for each $M \in \text{Mod-}R$, every module has a special ${}^\perp M$ -precover. By Theorem 14, this is the case for any right hereditary ring R assuming Gödel's Axiom of Constructibility ($V = L$). Also, this is true in ZFC in the case when R is right pure-semisimple, by Corollary 13(ii).

(2) Denote by \mathcal{W} the class of all Whitehead groups [9]. Does every abelian group have a special \mathcal{W} -precover (in ZFC)? This is a particular case of (1) for $R = M = \mathbb{Z}$. Under $V = L$, every Whitehead group is free, so the answer is positive.

(3) *Can Theorem 16 be extended to wider classes of rings (such as Prüfer domains or commutative Noetherian rings of finite Krull dimension)?* In particular, for which rings is it the case that for every class \mathcal{C} of cotorsion modules, every module has a special ${}^{\perp}\mathcal{C}$ -precover?

Note added in proof. The first author and S. Shelah have shown that it is consistent with ZFC+GCH that there is no Q such that ${}^{\perp}\mathcal{W}^{\perp} = Q^{\perp}$.

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