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Artinian level modules and cancellable sequences

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Abstract

In order to use dualization to study Hilbert functions of artinian level algebras we extend the notion of level sequences and cancellable sequences, introduced by Geramita and Lorenzini, to include Hilbert functions of certain artinian modules. As in the case of algebras a level sequence is cancellable, but now by dualization its reverse is also cancellable which gives a new condition on level sequences. We also give a characterization of the cancellable sequences involving Macaulay representations.

Keywords: Level algebra; Level module; Hilbert function; Betti numbers; Graded algebra; Graded module; Lexicographic ideal; Lexicographic submodule; Level sequence; Cancellable sequence; Graded dual

1. Introduction

There have been a lot of interest in artinian level algebras, that is, finitely generated graded algebras over a field with socle concentrated in one degree, and especially in their Hilbert functions. In [1] Boij introduced the concept of a level module as a generalization of a level algebra. When we are interested in Hilbert functions of artinian level algebras this generalization is motivated by the following. The class of artinian level modules is closed under dualization and truncation and this means that if a sequence $(h_0, h_1, \dots, h_s)$

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is the Hilbert function of an artinian level module then its reverse $(h_s, h_{s-1}, \dots, h_0)$ and all its truncations, that is, $(h_i, h_{i+1}, \dots, h_j)$ for all $0 \leqslant i \leqslant j \leqslant s$, are too.

For example, let $(1, h_1, h_2, \dots, h_s)$ be the Hilbert function of an artinian level algebra. Then the reverse $(h_s, h_{s-1}, \dots, h_1, 1)$ is the Hilbert function of an artinian level module. This module is generated in degree 0 and knowing this we can use Macaulay's theorem for modules (see Hulett [2]) to get an upper bound for the growth of its Hilbert function in each step. This yields for example that $(1, 3, \dots, \geqslant 14, 7, 3)$ is not the Hilbert function of a level algebra since no graded module generated in degree 0 over a polynomial ring with three variables can grow like $(3, 7, 14, \dots)$.

Geramita and Lorenzini [3] introduced the notion of a cancellable sequence. In Section 2 we extend this notion to include not only Hilbert functions of certain artinian algebras but also of certain artinian modules. The point of this is that the Hilbert function of an artinian level module is cancellable and thus by dualization its reverse is also cancellable, whereas in general the reverse of a cancellable sequence is not cancellable. Thus the condition that the reverse of a level Hilbert function is cancellable gives something new.

In Section 3 we study the cancellable sequences through a result of Eliahou and Kervaire [4] and end up with Theorem 18. In Section 4 we recall the necessary facts about dualization and truncation from Boij [1] and see that we actually gain something from the generalization to modules.

2. Cancellation in resolutions

In this section we will explain the notion of artinian level modules, level sequences, cancellable sequences and cancellation in resolutions. Cancellation in resolutions were first considered for level algebras by Geramita and Lorenzini in [3]. Given the Hilbert function of a graded algebra there is a special graded algebra having maximal Betti numbers among all graded algebras with this Hilbert function. Peeva shows in [5] that the Betti numbers of any graded algebra can be obtained from these maximal Betti numbers by a sequence of operations called consecutive cancellations. The point is that by looking at the maximal Betti numbers we can say that certain Betti numbers cannot appear for the given Hilbert function. We will explain these results and see that they hold for graded modules as well.

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k. Consider R as a graded ring by giving each x_i degree one and let $\mathfrak{m} = \bigoplus_{i \geqslant 1} R_i$ be the unique graded maximal ideal. If M is a finitely generated graded R-module with a minimal free resolution given by

$$0 \to \bigoplus_j R(-j)^{\beta_{n,j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}} \to M \to 0$$

then $\beta_{i,j}(M) = \beta_{i,j}$ are the graded Betti numbers of M. The graded Betti numbers are independent of the resolution since $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_j^R(M, k)_i$. Let F be a graded free R-module with basis $e_1, \dots, e_m$ such that e_i has degree d_i and $d_1 \leqslant \cdots \leqslant d_m$.

Definition 1. A *monomial* of F is an element on the form ue_i for some monomial u in R. The *lexicographic order* on monomials of R is the order in which $x_1^{\alpha_1}\dots x_n^{\alpha_n} > x_1^{\beta_1}\dots x_n^{\beta_n}$ if $\alpha_i > \beta_i$ for the largest index i such that $\alpha_i \neq \beta_i$. The lexicographic order on monomials of F is the order in which $ue_i > u'e_j$ if $i < j$ or if $i = j$ and $u > u'$ where u and u' are monomials in R. A graded submodule L of F is called a *lexicographic submodule* if each graded component L_i of L is spanned as a vector space over k by the $\dim_k L_i$ largest monomials in lexicographic order. A *lexicographic ideal* is a lexicographic submodule of R.

Remark 2. Note that if L is a lexicographic submodule of F then $L = I_1e_1 + \cdots + I_me_m$ for some lexicographic ideals I_j. It follows that $F/L \cong \bigoplus_{j=1}^m (R/I_j)(-d_j)$ and that

$$\beta_{s,t}(F/L) = \sum_{j=1}^m \beta_{s,t+d_j}(R/I_j).$$

If M is a graded submodule of F then there is a lexicographic submodule L such that $\dim_k L_i = \dim_k M_i$ for every i. This was proved by Macaulay [6] when $F = R$ and in the general case by Hulett [7]. Since in this case L_i must be the vector space spanned by the $\dim_k M_i$ largest monomials of F in lexicographic order we see that there is only one choice for L. Furthermore, Bigatti, Hulett and Pardue has shown that the module F/L has the largest Betti numbers among all modules with the same Hilbert function as F/M.

Theorem 3 (Bigatti–Hulett–Pardue). *Let M be a graded submodule of the graded free module F and let L be a lexicographic submodule of F such that* $\dim_k L_i = \dim_k M_i$ *for every i. Then $\beta_{i,j}(F/M) \leqslant \beta_{i,j}(F/L)$ for every i and j.*

Proof. See Bigatti [8], Hulett [7] and Pardue [9]. □

Let M be a graded submodule of the graded free module F and let L be the lexicographic submodule of F with the same Hilbert function as M. Then

$$\begin{array}{ccc} (1-t)^n \displaystyle\sum_{j=0}^{\infty} \dim_k(F/M)t^j & = & \displaystyle\sum_{j=0}^{\infty}\sum_{i=0}^{n}(-1)^i\beta_{i,j}(F/M)t^j \\ \| & & \\ (1-t)^n \displaystyle\sum_{j=0}^{\infty} \dim_k(F/L)t^j & = & \displaystyle\sum_{j=0}^{\infty}\sum_{i=0}^{n}(-1)^i\beta_{i,j}(F/L)t^j \end{array}$$

and we see that $\sum_{i=0}^n(-1)^i\beta_{i,j}(F/M) = \sum_{i=0}^n(-1)^i\beta_{i,j}(F/L)$ for all j. Since we also know that $\beta_{i,j}(F/M) \leqslant \beta_{i,j}(F/L)$ for all i and j it follows that the numbers $\beta_{i,j}(F/M)$ can be obtained from $\beta_{i,j}(F/L)$ by a sequence of *cancellations* defined as follows. Choose i and i' such that one is odd and one is even and replace $\beta_{i,j}(F/L)$ with $\beta_{i,j}(F/L) - 1$ and $\beta_{i',j}(F/L)$ with $\beta_{i',j}(F/L) - 1$. A cancellation is called a *consecutive cancellation* if $i' = i + 1$. Peeva shows in [5] that we actually only need consecutive cancellations.

Theorem 4 (Peeva). *Let M be a graded submodule of F and let L be the lexicographic submodule with the same Hilbert function. Then the graded Betti numbers $\beta_{i,j}(F/M)$ can be obtained from $\beta_{i,j}(F/L)$ by a sequence of consecutive cancellations.*

Proof. See Peeva [5, Theorem 1.1]. This theorem is stated for graded algebras but the proof holds for graded modules as well. □

We will now recall the definition of an artinian level module from Boij [1].

Definition 5. Let M be a graded R-module. Then

$$\operatorname{Soc} M = \{x \in M\colon\ \mathfrak{m}x = 0\}$$

is called *the socle of M*.

Definition 6. Let $M = M_0 \oplus \cdots \oplus M_s$ be a graded artinian R-module. Then M is a *level module* if it is generated by M_0 and $\operatorname{Soc} M = M_s$.

If M is an artinian module then there is an integer s such that $M_i = 0$ for every $i > s$. Thus we can write the Hilbert function of M as a sequence of finite length $(h_0, h_1, \dots, h_s)$. Such a sequence is called a *level sequence* if it is the Hilbert function of an artinian level module. We will now see that if $h = (h_0, h_1, \dots, h_s)$ is a level sequence and $\{\beta_{i,j}\}$ is the set of maximal Betti numbers associated with h then $\beta_{n-1,n+i} \geqslant \beta_{n,n+i}$ for every $i \neq s$.

Definition 7. A sequence of integers $h = (h_0, h_1, \dots, h_s)$ is called *cancellable* if there is a free graded R-module F, generated in degree 0, and a lexicographic submodule L of F such that h is the Hilbert function of F/L and the Betti numbers of F/L satisfy

$$\beta_{n-1,n+i}(F/L) \geqslant \beta_{n,n+i}(F/L)$$

for every $i \neq s$.

Proposition 8. *A level sequence is cancellable.*

Proof. Let M be a graded R-module. By calculating $\operatorname{Tor}_n^R(M,k)$ from the Koszul resolution of k one can show that $\operatorname{Tor}_n^R(M,k) \cong (\operatorname{Soc} M)(-n)$ and this means that $\dim_k(\operatorname{Soc} M)_i = \beta_{n,n+i}(M)$. Thus if M is artinian level with socle in degree s we have that $\beta_{n,n+i}(M) = 0$ for all $i \neq s$.

Now let $h = (h_0, h_1, \dots, h_s)$ be a level sequence. Then there is a free graded R-module F generated in degree 0 and a graded submodule N such that F/N is artinian level with Hilbert function given by h. Let L be the lexicographic submodule of F with the same Hilbert function as N and choose an integer $i \neq s$. Then $\beta_{n,n+i}(F/N) = 0$ and by Theorem 4 there is a sequence of consecutive cancellations on

$$\big(\beta_{1,n+i}(F/L), \beta_{2,n+i}(F/L), \dots, \beta_{n,n+i}(F/L)\big)$$

such that $\beta_{n,n+i}(F/L)$ becomes zero. Thus $\beta_{n-1,n+i}(F/L) \geqslant \beta_{n,n+i}(F/L)$ and the proposition follows. $\square$

3. Calculation of maximal Betti numbers

Eliahou and Kervaire gave in [4] an explicit minimal resolution for a family of ideals called stable ideals. From their resolution it is possible to get an expression for the Betti numbers of R/I in terms of the minimal generators of I when I is, for example, a lexicographic ideal. We will use this to calculate the difference $\beta_{n-1,n+i}(F/L) - \beta_{n,n+i}(F/L)$ directly from the Hilbert function of F/L when L is a lexicographic submodule of F and F is generated in degree 0. This leads to a characterization of the cancellable sequences.

Definition 9. For every monomial u in R we define

$$m(u) = \min\{i : x_i \text{ divides } u\}.$$

An ideal I of R is *stable* if it is generated by monomials $u_1, \dots, u_r$ such that if $m = m(u_j)$ then $x_i u_j / x_m \in I$ for every $i > m$. If u is any monomial and $i > m(u)$ then $x_i u / x_{m(u)} > u$ and this means that a lexicographic ideal is stable.

Proposition 10 (Eliahou–Kervaire). *Let I be a stable ideal of R and let G_d be the set of minimal monomial generators of I of degree d. Then*

$$\beta_{s,t}(R/I) = \sum_{u \in G_{t-s+1}} \binom{n - m(u)}{s-1}.$$

Proof. See Eliahou and Kervaire [4] $\square$

Remark 11. Compared to Eliahou and Kervaire [4] we have renumbered the variables of R by applying $x_i \mapsto x_{n-i+1}$ and, according to this, changed the definition of the function m and the definition of a stable ideal. This also explains why the expression for the Betti numbers looks slightly different.

Definition 12. Let d be a positive integer. Then any positive integer a can be written uniquely in the form

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1},$$

where $k_d > k_{d-1} > \cdots > k_1$ (see Bruns and Herzog [10]). This sum is called *the dth Macaulay representation* of a and $k_d, \dots, k_1$ *the dth Macaulay coefficients* of a. Furthermore if $i, j \in \mathbb{Z}$ then we define

$$[a_{(d)}]^i_j = \binom{k_d + i}{d + j} + \binom{k_{d-1} + i}{d - 1 + j} + \cdots + \binom{k_s + i}{s + j},$$

where s is the smallest number such that $k_s \neq 0$. (We believe that this notation was first introduced by Robbiano in [11].)

Next follows a description of some special sets of monomials that we need for the proof of Lemma 14. The description is borrowed from Bruns and Herzog [10]. Let $u = x_{j(1)}x_{j(2)}\ldots x_{j(d)}$ with $1 \leqslant j(1) \leqslant \cdots \leqslant j(d) \leqslant n$ be a monomial in R_d and denote by $\mathcal{L}_u$ all monomials in R_d smaller than u, that is $\mathcal{L}_u = \{v \in R_d\colon\ v < u\}$. Denote by $[x_1, \ldots, x_i]_t$ the set of monomials of degree t in the variables $x_1, \ldots, x_i$. Then we can write $\mathcal{L}_u$ as a disjoint union

$$\mathcal{L}_u = \bigcup_{i=1}^{d} [x_1, \ldots, x_{j(i)-1}]_i x_{j(i+1)} \ldots x_{j(d)}$$

called the *natural decomposition* of $\mathcal{L}_u$. It follows that

$$|\mathcal{L}_u| = \sum_{i=1}^{d} \binom{k_i}{i}$$

where $k_i = j(i) + i - 2$ and that $k_d, \ldots, k_1$ are the Macaulay coefficients of $|\mathcal{L}_u|$.

Remark 13. For the following lemma we need to define the binomial coefficient $\binom{f}{g}$ when f, g or both are negative in a certain way. Define $\binom{f}{g} = 0$ when $f < g$ and then recursively for all f and g through the usual rule for binomial coefficients

$$\binom{f}{g} = \binom{f+1}{g} - \binom{f}{g-1}.$$

Lemma 14. *Let $u_1 < u_2 < \cdots$ be the monomials of degree d in R written in lexicographic order. Then, for any integer $0 < a \leqslant \dim_k R_d$, the number of monomials v in $\{u_1, u_2, \ldots, u_a\}$ such that $m(v) = r$ is given by $[a_{(d)}]_{-1}^{-r}$.*

Proof. If $u = u_{a+1}$ then $\{u_1, u_2, \ldots, u_a\} = \mathcal{L}_u$ and we can describe this set by its natural decomposition. Write u as $u = x_{j(1)}x_{j(2)}\ldots x_{j(d)}$ for some integers $1 \leqslant j(1) \leqslant \cdots \leqslant j(d) \leqslant n$ and let $k_i = j(i) + i - 2$. Then

$$\{u_1, u_2, \ldots, u_a\} = \bigcup_{i=1}^{d} [x_1, \ldots, x_{j(i)-1}]_i\, x_{j(i+1)} \ldots x_{j(d)}$$

and $k_d, \ldots, k_1$ are the dth Macaulay coefficients of a. We start by looking at the parts of this decomposition. The subset of monomials v in $[x_1, \ldots, x_{j(i)-1}]_i$ such that $m(v) = r$ is given by $[x_r, \ldots, x_{j(i)-1}]_{i-1}x_r$ where $[x_r, \ldots, x_{j(i)-1}]_{i-1} = \emptyset$ if $r > j(i) - 1$ and $[x_r, \ldots, x_{j(i)-1}]_{i-1} = \{1\}$ if $i - 1 = 0$ and $r \leqslant j(i) - 1$. The number of elements in this set equals the number of monomials of degree $i-1$ in $j(i)-r$ variables, that is, the number of elements in $[x_r,\dots,x_{j(i)-1}]_{i-1}x_r$ is

$$\binom{j(i)-r+i-2}{i-1}=\binom{k_i-r}{i-1}.$$

Note that with the definition of the binomial coefficient given in Remark 13 this holds even if $r>j(i)-1$ or $i-1=0$. It follows that the monomials v in $\{u_1,u_2,\dots,u_a\}$ such that $m(v)=r$ are

$$\bigcup_{i=1}^{d}[x_r,\dots,x_{j(i)-1}]_{i-1}\,x_r x_{j(i+1)}\dots x_{j(d)}$$

and that the number of elements in this set is

$$\sum_{i=1}^{d}\binom{k_i-r}{i-1}=[a_{(d)}]_{-1}^{-r}. \qquad \square$$

Proposition 15. *Let I be a lexicographic ideal and let $H(i)=H(R/I,i)$ be the Hilbert function of R/I. Then*

$$\beta_{s,t}(R/I)=\sum_{i=1}^{n}\binom{n-i}{s-1}\left(\left[H(d-1)_{(d-1)}\right]_0^{-i+1}-\left[H(d)_{(d)}\right]_{-1}^{-i}\right)$$

where $d=t-s+1$.

Proof. We will use Proposition 10 and for that we need to calculate the minimal monomial generators of degree d of I. Let $u_1<u_2<\cdots$ be the monomials in R_d written in lexicographic order. Then the k-vector space $(R/I)_d$ is spanned by $u_1,\dots,u_{H(d)}$ since I is a lexicographic ideal and $\dim_k(R/I)_d=H(d)$. By Macaulay's theorem for lexicographic ideals we know that $R_1(R/I)_{d-1}$ is spanned by $u_1,\dots,u_p$ where $p=[H(d-1)_{(d-1)}]_1^1$. Thus the minimal generators of I of degree d are $G_d=\{u_{H(d)-1},\dots,u_p\}$. Now we use Proposition 10 and get

$$\beta_{s,t}(R/I)=\sum_{u\in G_d}\binom{n-m(u)}{s-1}=\sum_{i=1}^{p}\binom{n-m(u_i)}{s-1}-\sum_{i=1}^{H(d)}\binom{n-m(u_i)}{s-1}.$$

By applying Lemma 14 to $\{u_1,\dots,u_p\}$ and $\{u_1,\dots,u_{H(d)}\}$ we can count the number of terms in the two sums above on the right that equal $\binom{n-i}{s-1}$ for $1\leqslant i\leqslant n$ and we get

$$\beta_{s,t}(R/I)=\sum_{i=1}^{n}\binom{n-i}{s-1}\left([p_{(d)}]_{-1}^{-i}-\left[H(d)_{(d)}\right]_{-1}^{-i}\right).$$

Now $[p_{(d)}]_{-1}^{-i} = [[H(d-1)_{(d-1)}]_{1(d)}^{1}]_{-1}^{-i} = [H(d-1)_{(d-1)}]_{0}^{-i+1}$ and this ends the proof. □

Lemma 16. *Let I be a lexicographic ideal and let $H(i) = H(R/I, i)$ be the Hilbert function of R/I. Then*

$$\begin{aligned}&\beta_{n-1,n+i}(R/I) - \beta_{n,n+i}(R/I)\\ &\quad = n\big(H(i+1) - \big[H(i+2)_{(i+2)}\big]_{-1}^{-1}\big) - H(i) + \big[H(i+2)_{(i+2)}\big]_{-2}^{-2}.\end{aligned}$$

Proof. By Proposition 15 we have

$$\begin{aligned}\beta_{n,n+i} &= \sum_{j=1}^{n} \binom{n-j}{n-1} \big(\big[H(i)_{(i)}\big]_{0}^{-j+1} - \big[H(i+1)_{(i+1)}\big]_{-1}^{-j}\big)\\ &= H(i) - \big[H(i+1)_{(i+1)}\big]_{-1}^{-1}\end{aligned}$$

since $\binom{n-j}{n-1} = 0$ when $j > 1$. Furthermore

$$\begin{aligned}\beta_{n-1,n+i} &= \sum_{j=1}^{n} \binom{n-j}{n-2} \big(\big[H(i+1)_{(i+1)}\big]_{0}^{-j+1} - \big[H(i+2)_{(i+2)}\big]_{-1}^{-j}\big)\\ &= (n-1)\big(H(i+1) - \big[H(i+2)_{(i+2)}\big]_{-1}^{-1}\big)\\ &\quad + \big[H(i+1)_{(i+1)}\big]_{0}^{-1} - \big[H(i+2)_{(i+2)}\big]_{-1}^{-2}.\end{aligned}$$

Thus

$$\begin{aligned}\beta_{n-1,n+i} - \beta_{n,n+i} &= n\big(H(i+1) - \big[H(i+2)_{(i+2)}\big]_{-1}^{-1}\big)\\ &\quad + \big[H(i+1)_{(i+1)}\big]_{0}^{-1} + \big[H(i+1)_{(i+1)}\big]_{-1}^{-1} - H(i+1)\\ &\quad - H(i) + \big[H(i+2)_{(i+2)}\big]_{-1}^{-1} - \big[H(i+2)_{(i+2)}\big]_{-1}^{-2}.\end{aligned}$$

Now since $\binom{f-1}{g} + \binom{f-1}{g-1} = \binom{f}{g}$ holds for all f and g we see that

$$\big[H(i+1)_{(i+1)}\big]_{0}^{-1} + \big[H(i+1)_{(i+1)}\big]_{-1}^{-1} = H(i+1).$$

In the same way we get

$$\big[H(i+2)_{(i+2)}\big]_{-1}^{-1} - \big[H(i+2)_{(i+2)}\big]_{-1}^{-2} = \big[H(i+2)_{(i+2)}\big]_{-2}^{-2}$$

and the lemma follows. □

Proposition 17. *Let F be a free graded module generated in degree* 0, *L a lexicographic submodule of F and $H(i) = H(F/L, i)$ the Hilbert function of F/L. Fix an integer i and let q be the quotient and r the remainder when $H(i+2)$ is divided by* $\dim_k R_{i+2}$. *Then*

$$\begin{aligned}&\beta_{n-1,n+i}(F/L) - \beta_{n,n+i}(F/L)\\&\quad = n\big(H(i+1) - q\dim_k R_{i+1} - [r_{(i+2)}]_{-1}^{-1}\big) - H(i) + q\dim_k R_i + [r_{(i+2)}]_{-2}^{-2}.\end{aligned}$$

Proof. Assume that F is generated by $e_1, \dots, e_m$ all of degree 0. As noted in Remark 2 we may write $L = I_1e_1 + \cdots + I_me_m$ for some lexicographic ideals I_j and then $\beta_{s,t}(F/L) = \sum_j \beta_{s,t}(R/I_j)$. To simplify the expressions let $a_j = H(R/I_j, i)$, $b_j = H(R/I_j, i+1)$ and $c_j = H(R/I_j, i+2)$ and note that $\sum_j a_j = H(i)$, $\sum_j b_j = H(i+1)$ and $\sum_j c_j = H(i+2)$. Now by Lemma 16

$$\begin{aligned}\beta_{n-1,n+i}(F/L) - \beta_{n,n+i}(F/L) &= \sum_j \big(\beta_{n-1,n+i}(R/I_j) - \beta_{n,n+i}(R/I_j)\big)\\ &= \sum_j \big(n\big(b_j - [c_{j(i)}]_{-1}^{-1}\big) - a_j + [c_{j(i)}]_{-2}^{-2}\big)\\ &= n\Big(H(i+1) - \sum_j [c_{j(i)}]_{-1}^{-1}\Big) - H(i) + \sum_j [c_{j(i)}]_{-2}^{-2}.\end{aligned}$$

We need to compute the numbers $c_j = H(R/I_j, i+2)$ for every j. It follows from the definition of a lexicographic submodule that L_{i+2} is given by

$$L_{i+2} = Ve_{q+1} + R_{i+2}e_{q+2} + \cdots + R_{i+2}e_m$$

where V is the subspace of R_{i+2} spanned by the $\dim_k R_{i+2} - r$ largest monomials in lexicographic order. We see that

$$c_j = \begin{cases} \dim_k R_{i+2} & \text{if } j \leqslant q,\\ r & \text{if } j = q+1,\\ 0 & \text{if } j > q+1,\end{cases}$$

and thus

$$\sum_j [c_{j(i+2)}]_{-1}^{-1} = q[\dim_k R_{i+2(i)}]_{-1}^{-1} + [r_{(i)}]_{-1}^{-1}$$

and

$$\sum_j [c_{j(i+2)}]_{-2}^{-2} = q[\dim_k R_{i+2(i)}]_{-2}^{-2} + [r_{(i+2)}]_{-2}^{-2}.$$

Now $[\dim_k R_{i+2(i+2)}]_{-1}^{-1} = \dim_k R_{i+1}$ and $[\dim_k R_{i+2(i+2)}]_{-2}^{-2} = \dim_k R_i$ and the proposition follows. □

Theorem 18. *Let $h = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of a graded R-module generated in degree 0. Then h is cancellable if and only if for each $0 \leqslant i < s$ we have*

$$n\big(h_{i+1} - q \dim_k R_{i+1} - [r_{(i+2)}]_{-1}^{-1}\big) - h_i + q \dim_k R_i + [r_{(i+2)}]_{-2}^{-2} \geqslant 0,$$

where q is the quotient and r the remainder when h_{i+2} is divided by $\dim_k R_{i+2}$.

Proof. This follows immediately from Proposition 17 and the definition of a cancellable sequence. □

Corollary 19. *Let $(h_0, h_1, \ldots, h_s)$ be a cancellable sequence and let i be an integer such that $0 \leqslant i \leqslant s$ and assume that $h_{i+2} \leqslant \dim_k R_{i+2}$. Then*

$$n\big(h_{i+1} - [h_{i+2(i+2)}]_{-1}^{-1}\big) - h_i + [h_{i+2(i+2)}]_{-2}^{-2} \geqslant 0.$$

Proof. If $h_{i+2} \leqslant \dim_k R_{i+2}$ then $q = 0$ and $r = h_{i+2}$ in Theorem 18. □

Corollary 20. *Let $(h_0, h_1, \ldots, h_s)$ be a cancellable sequence and let i be an integer such that $0 \leqslant i \leqslant s$ and assume that $h_{i+2} \leqslant i + 2$. Then*

$$n\big(h_{i+1} - h_{i+2}\big) - h_i + h_{i+2} \geqslant 0.$$

Proof. For any positive integer a we have that if $j \geqslant a$ then the jth Macaulay representation of a is

$$a = \underbrace{\binom{j}{j} + \binom{j-1}{j-1} + \cdots + \binom{j-a+1}{j-a+1}}_{a \text{ number of terms}}.$$

Thus $[a_{(j)}]_r^r = a$ for any integer r and from this the corollary follows. □

4. Dualization and truncation

In Sections 2 and 3 we have considered artinian level modules which are a generalization of artinian level algebras. With our definition, an artinian level algebra is nothing but a cyclic artinian level module, at least when it comes to its module structure. In this section we will motivate this generalization. We will see that the class of artinian level modules is closed under dualization and truncation. This means that if a sequence $(h_0, h_1, \ldots, h_s)$ is level then its reverse $(h_s, h_{s-1}, \ldots, h_0)$ and all its truncations, that is, $(h_i, h_{i+1}, \ldots, h_j)$ for all $0 \leqslant i \leqslant j \leqslant s$, are too. This is, of course, not true if we only consider Hilbert functions of artinian level algebras since they always have $h_0 = 1$.

We have seen that a level sequence is cancellable and thus, by dualization, its reverse is too. Furthermore, the reverse of a cancellable sequence is not in general cancellable so if we remove from the set of all cancellable sequences the sequences whose reverse is not cancellable we get something smaller and the set of all level sequences will be a subset of this set.

In some cases where the level sequences are known we have used the criterion given in Theorem 18 to see how many of the cancellable sequences have a cancellable reverse and how many of them are level.

First we recall what we need about dualization and truncation from Boij [1].

Definition 21. Let M be a finitely generated graded R-module. Then the *graded dual* of M is defined to be $M^\vee = {}^*\mathrm{Hom}_k(M,k) = \bigoplus_i \mathrm{Hom}_k(M_i,k)$. By regarding $M^\vee$ as a subset of $\mathrm{Hom}_k(M,k)$ we let the module structure on $M^\vee$ be defined by $x\phi(y) = \phi(xy)$ for all $x \in R$, $\phi \in M^\vee$ and $y \in M$. The grading is given by $M_i^\vee = \mathrm{Hom}_k(M_{-i},k)$.

Remark 22. Since $\dim_k \mathrm{Hom}_k(M_{-i},k) = \dim_k M_{-i}$ we get that $H(M^\vee,i) = H(M,-i)$. Thus if the Hilbert function of M is given by $(h_0,h_1,\dots,h_s)$. Then its reverse $(h_s,h_{s-1},\dots,h_0)$ is the Hilbert function of $M^\vee(-s)$.

Proposition 23. *If M is an artinian level R-module with socle in degree s, then $M^\vee(-s)$ is artinian level with socle in degree s.*

Proof. See Boij [1, Proposition 2.3]. □

Proposition 24. *Let $M = \bigoplus_i M_i$ be an artinian R-module with socle in degree s and let i, j be integers such that $0 \leqslant i \leqslant j \leqslant s$. Then the ith twist of $M_i \oplus \cdots \oplus M_j$ is an artinian level module.*

Proof. See Boij [1, Proposition 2.4]. □

Example 25. Let M be an artinian level R-module with Hilbert function given by $(\dots,c,n,2)$. By truncating the module M and taking the dual we get a level module with Hilbert function $(2,n,c)$. By Proposition 8 a level sequence is cancellable so it follows from Corollary 19 that

$$n\bigl(n - [c_{(2)}]_{-1}^{-1}\bigr) - 2 + [c_{(2)}]_{-2}^{-2} \geqslant 0.$$

It is easy to see that this implies

$$c \leqslant \binom{n}{2} + 1.$$

In fact, as noted by Fabrizio Zanello (private communication), this upper bound on c is sharp since the R-module $R/(x_n + \mathfrak{m}^3) \oplus R/(x_1^3, x_2, x_3, \dots, x_n)$ is artinian level and its Hilbert function is $(2, n, \binom{n}{2} + 1)$.

If we want to use Theorem 18 to decide if a sequence $(h_0,h_1,\dots,h_s)$ is cancellable we must first see that it is the Hilbert function of a graded R-module generated in degree 0 and this can be checked by Macaulay's theorem for modules.

Proposition 26. *A sequence $(h_i)_{i\geqslant 0}$ is the Hilbert function of a graded R-module generated in degree 0 if and only if for each i*

$$h_{i+1} \leqslant q \dim_k R_{i+1} + [r_{(i)}]_1^1,$$

where q is the quotient and r the remainder when h_i is divided by $\dim_k R_i$.

Proof. This is a special case of Hulett [2, Corollary 6]. □

Since the reverse of a level sequence, by dualization, is level it is interesting to look at cancellable sequences whose reverse also is cancellable. We will see that not all such sequences are level, which is not surprising as indicated by the following argument. It is well known that all cyclic artinian level R-modules of type one, that is, all artinian Gorenstein algebras, have symmetric Hilbert functions. The condition in Theorem 18 depends only on three adjacent positions in the sequence at a time and it is unlikely that this would force a sequence to be symmetric. Nevertheless it is interesting to see how many of the cancellable sequences that are level and what we gain by looking at the reverse of the sequences. We will do this in some special cases next.

Let s and t be integers and denote by $M_{s,t}$ the set of all sequences of positive integers $h = (1, n, h_2, h_3, \dots, h_s)$ such that $h_s = t$ and both h and its reverse are bounded by Macaulay's theorem for modules, that is, satisfy the condition in Proposition 26. Let $F_{s,t}$ be the subset of $M_{s,t}$ of all cancellable sequences and let $B_{s,t}$ be the subset of $M_{s,t}$ of all sequences whose reverse is cancellable. Denote by $L_{s,t}$ the set of all level sequences on the form $(1, n, h_2, h_3, \dots, h_s)$ where $h_s = t$. We have seen that $L_{s,t} \subseteq F_{s,t} \cap B_{s,t}$.

Using Proposition 26 and Theorem 18 it is easy to generate the sets $M_{s,t}$, $F_{s,t}$ and $B_{s,t}$ with a computer. In general we do not know very much about the set $L_{s,t}$ but for some values of s, t and n we do. In [12, Theorem 4.2] Stanley describes precisely the set of all Gorenstein sequences when $n \leqslant 3$, that is, all cyclic artinian level R-modules of type one. Using this result we can generate the set $L_{s,t}$ for all s when $t = 1$ and $n \leqslant 3$. With $n = 3$, Geramita et al. in [13] use several different techniques to record the sets $L_{s,t}$ for all $t \geqslant 2$ when $s \leqslant 5$ and for $t = 2$ when $s = 6$. Thus for $n = 3$ we have complete knowledge of the sets $L_{s,t}$ for all t when $s \leqslant 5$ and for $t \leqslant 2$ when $s = 6$.

The number of elements in the sets $M_{s,t}$, $F_{s,t}$, $B_{s,t}$, $F_{s,t} \cap B_{s,t}$ and $L_{s,t}$ for these values of s and t are displayed in Tables 1–4. Note that it is when $|F_{s,t} \cap B_{s,t}| < |F_{s,t}|$ that we actually gain something by looking at the reverse of the sequences. We see that this happens at several places in the tables and we have marked these places by writing the corresponding numbers with bold face.

Table 1
Socle degree 6

t	$M_{6,t}$	$F_{6,t}$	$B_{6,t}$	$F_{6,t} \cap B_{6,t}$	$L_{6,t}$
1	34	**23**	23	**22**	11
2	148	**85**	81	**71**	58

Table 2
Socle degree 5

t	$M_{5,t}$	$F_{5,t}$	$B_{5,t}$	$F_{5,t} \cap B_{5,t}$	$L_{5,t}$
1	12	10	10	10	4
2	44	**31**	29	**27**	23
3	59	**39**	41	**37**	34
4	56	**38**	45	**36**	34
5	49	**34**	42	**33**	32
6	49	30	39	30	26
7	37	24	32	24	22
8	27	19	26	19	18
9	20	16	20	16	15
10	15	12	15	12	12
11	15	11	15	11	10
12	10	8	10	8	8
13	7	6	7	6	6
14	5	5	5	5	5
15	5	4	5	4	4
16	3	3	3	3	3
17	2	2	2	2	2
18	2	2	2	2	2
19	1	1	1	1	1
20	1	1	1	1	1
21	1	1	1	1	1

Table 3
Socle degree 4

t	$M_{4,t}$	$F_{4,t}$	$B_{4,t}$	$F_{4,t} \cap B_{4,t}$	$L_{4,t}$
1	5	5	5	5	4
2	14	**11**	10	**10**	10
3	17	13	13	13	12
4	14	11	13	11	11
5	14	10	11	10	9
6	10	8	10	8	8
7	7	6	7	6	6
8	5	5	5	5	5
9	5	4	5	4	4
10	3	3	3	3	3
11	2	2	2	2	2
12	2	2	2	2	2
13	1	1	1	1	1
14	1	1	1	1	1
15	1	1	1	1	1

Table 4
Socle degree 3

t	$M_{3,t}$	$F_{3,t}$	$B_{3,t}$	$F_{3,t} \cap B_{3,t}$	$L_{3,t}$
1	2	2	2	2	1
2	5	4	4	4	4
3	4	4	4	4	4
4	4	3	4	3	3
5	3	3	3	3	3
6	2	2	2	2	2
7	2	2	2	2	2
8	1	1	1	1	1
9	1	1	1	1	1
10	1	1	1	1	1

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