Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the TD–D canonical form and the LB–UB canonical form

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Abstract

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider an ordered pair of linear transformations $A : V \to V$ and $B : V \to V$ which satisfy both (i), (ii) below.

 (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal.
 (ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal.

We call such a pair a *Leonard pair on V*. We introduce two canonical forms for Leonard pairs. We call these the *TD–D canonical form* and the *LB–UB canonical form*. In the TD–D canonical form the Leonard pair is represented by an irreducible tridiagonal matrix and a diagonal matrix, subject to a certain normalization. In the LB–UB canonical form the Leonard pair is represented by a lower bidiagonal matrix and an upper bidiagonal matrix, subject to a certain normalization. We describe the two canonical forms in detail. As an application we obtain the following results. Given square matrices A, B over $\mathbb{K}$, with A tridiagonal and B diagonal, we display a necessary and sufficient condition for A, B to represent a Leonard pair. Given square matrices A, B over $\mathbb{K}$, with A lower bidiagonal and B upper bidiagonal, we display a necessary and sufficient condition for

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A, B to represent a Leonard pair. We briefly discuss how Leonard pairs correspond to the q-Racah polynomials and some related polynomials in the Askey scheme. We present some open problems concerning Leonard pairs.

1. Introduction

We begin by recalling the notion of a *Leonard pair* [16,27–29,35,36]. We will use the following terms. Throughout this paper, when we refer to a matrix, we mean a square matrix. A matrix is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 [35]. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair on V* we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy both (i), (ii) below.

 (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

Note 1.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We emphasize we are not using this convention. In a Leonard pair A, A^*, the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

We give some background on Leonard pairs. There is a correspondence between Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and some related polynomials in the Askey scheme. This correspondence is discussed in [28,29], [35, Appendix A] and in Section 27 below. Reference [17] contains detailed information about the Askey scheme.

Leonard pairs play a role in representation theory. For instance Leonard pairs arise naturally in the representation theory of the Lie algebra sl_2 [16], the quantum algebra $U_q(sl_2)$ ([18–22], [25, Chapter 4], [27,36]), the Askey–Wilson algebra [7–9,37] and the tridiagonal algebra [16,35,36].

Leonard pairs play a role in combinatorics. For instance Leonard pairs can be constructed from certain partially ordered sets [28]. Also, there exists a combinatorial object called a P- and Q-polynomial association scheme [1,2,24,30,34]. Leonard pairs have been used to describe certain irreducible modules for the subconstituent algebra of these association schemes [31–33]. See [3–6,16] for more information on Leonard pairs and association schemes.

Leonard pairs are closely related to the work of Grünbaum and Haine on the "bispectral problem" [11,12]. See [10,13–15] for related work.

The rest of this introduction contains a detailed summary of the present paper.

In this paper we introduce two canonical forms for Leonard pairs. The first of these is called the *TD–D canonical form*. In this form the Leonard pair is represented by an irreducible tridiagonal matrix and a diagonal matrix, subject to a certain normalization. To describe the second form we make a definition. A matrix is said to be *lower bidiagonal* whenever each nonzero entry lies on either the diagonal or the subdiagonal. A matrix is said to be *upper bidiagonal* whenever its transpose is lower bidiagonal. We call our second form the *LB–UB canonical form*. In this form the Leonard pair is represented by a lower bidiagonal matrix and an upper bidiagonal matrix, subject to a certain normalization.

We fix some notation. Let d denote a nonnegative integer. We let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d + 1$ by $d + 1$ matrices which have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Any $\mathbb{K}$-algebra which is isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$ will be called a *matrix algebra over $\mathbb{K}$ of diameter d*.

Before proceeding we sharpen our concept of a Leonard pair. Let $\mathcal{A}$ denote a matrix algebra over $\mathbb{K}$ and let V denote an irreducible left $\mathcal{A}$-module. By a *Leonard pair in $\mathcal{A}$* we mean an ordered pair of elements taken from $\mathcal{A}$ which act on V as a Leonard pair in the sense of Definition 1.1. Let A, A^* denote a Leonard pair in $\mathcal{A}$. Then A and A^* together generate $\mathcal{A}$ [29, Corollary 3.2]. By a *Leonard pair over $\mathbb{K}$* we mean a sequence $\mathcal{A}$, A, A^* where $\mathcal{A}$ is a matrix algebra over $\mathbb{K}$ and A, A^* is a Leonard pair in $\mathcal{A}$. We call $\mathcal{A}$ the *ambient algebra* and suppress it in the notation, referring only to A, A^*. Let A, A^* denote a Leonard pair over $\mathbb{K}$. By the *diameter* of this pair we mean the diameter of its ambient algebra. By the *underlying module* for this pair we mean an irreducible left module for its ambient algebra. For the rest of this section, when we refer to a scalar we mean an element of $\mathbb{K}$. When we refer to a Leonard pair it is assumed to be over $\mathbb{K}$.

We recall the notion of an *eigenvalue sequence* for a Leonard pair. Let A, A^* denote a Leonard pair. By definition there exists a basis for the underlying module with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal. In the matrix representing A the diagonal entries are the eigenvalues of A and it turns out these are mutually distinct [35, Lemma 1.3]. Therefore the sequence of diagonal entries gives an ordering of the eigenvalues of A. We call this sequence an *eigenvalue sequence* for A, A^*. Given an eigenvalue sequence for A, A^*, if we invert the order of the sequence we get another eigenvalue sequence for A, A^*. Moreover A, A^* has no further eigenvalue sequence. To clarify this let d denote the diameter of A, A^*. Then A, A^* has exactly two eigenvalue sequences if $d \geqslant 1$ and a single eigenvalue sequence if $d = 0$. By a *dual eigenvalue sequence* for A, A^* we mean an eigenvalue sequence for the Leonard pair A^*, A.

A *Leonard system* is essentially a Leonard pair, together with an eigenvalue sequence and a dual eigenvalue sequence for that pair. For the duration of this section we take this as the definition of a Leonard system. (In the main part of our paper we will define a Leonard system in a slightly different manner in which the eigenvalues are replaced by their corresponding primitive idempotents.)

We mentioned each Leonard system involves a Leonard pair; we call this pair the *associated* Leonard pair. The set of Leonard systems associated with a given Leonard pair

will be called the *associate class* for that pair. In order to describe the associate classes we use the following notation. Let Φ denote a Leonard system. If we invert the ordering on the eigenvalue sequence of Φ we get a Leonard system which we denote by $\Phi^{\Downarrow}$. If we instead invert the ordering on the dual eigenvalue sequence of Φ we get a Leonard system which we denote by $\Phi^{\downarrow}$. We view $\downarrow, \Downarrow$ as permutations on the set of all Leonard systems. These permutations are commuting involutions and therefore induce an action of the Klein 4-group on the set of all Leonard systems. The orbits of this action coincide with the associate classes.

We discuss the notion of *isomorphism* for Leonard pairs and Leonard systems. Let A, A^* and B, B^* denote Leonard pairs. By an *isomorphism of Leonard pairs from A, A* to B, B** we mean an isomorphism of $\mathbb{K}$-algebras from the ambient algebra of A, A^* to the ambient algebra of B, B^* which sends A to B and A^* to B^*. We say A, A^* and B, B^* are *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to B, B^*. We say two given Leonard systems are *isomorphic* whenever

  (i) their associated Leonard pairs are isomorphic;
 (ii) their eigenvalue sequences coincide; and
(iii) their dual eigenvalue sequences coincide.

The set of Leonard systems is partitioned into both isomorphism classes and associate classes. These partitions are related as follows. Let A, A^* denote a Leonard pair and let d denote the diameter. If $d \geqslant 1$ then the corresponding associate class contains four Leonard systems and these are mutually nonisomorphic. If $d = 0$ then the corresponding associate class contains a single Leonard system.

Before proceeding with Leonard systems we introduce the notion of a *parameter array*. A parameter array is a finite sequence of scalars which satisfy a certain list of equations and inequalities. We care about parameter arrays because it turns out they are in bijection with the isomorphism classes of Leonard systems. A parameter array is defined as follows. Let d denote a nonnegative integer. By a *parameter array of diameter d* we mean a sequence of scalars $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ which satisfy (i)–(v) below.

  (i) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leqslant i \leqslant d)$.
 (ii) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leqslant i, j \leqslant d)$.
(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$ $(1 \leqslant i \leqslant d)$.
(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$ $(1 \leqslant i \leqslant d)$.
 (v) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$.

We give a bijection from the set of isomorphism classes of Leonard systems to the set of parameter arrays. Let Φ denote a Leonard system. To Φ we attach the following four sequences of scalars. The first two sequences are the eigenvalue sequence of Φ and the dual eigenvalue sequence of Φ. Let us denote these by $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, respectively. By a slightly technical construction which we omit for now, we obtain a third sequence of scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$. We call this the *first split sequence* of Φ. We let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence for $\Phi^{\Downarrow}$ and call this the *second split sequence* of Φ. By [35, Theorem 1.9] a sequence of scalars $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is a parameter array if and only if there exists a Leonard system Φ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$. If Φ exists then Φ is unique up to isomorphism. In this case we call p the *parameter array of Φ*. The map which sends a Leonard system to its parameter array induces the desired bijection from the set of isomorphism classes of Leonard systems to the set of parameter arrays.

Earlier we described an action of the Klein 4-group on the set of Leonard systems. The above bijection induces an action of the same group on the set of parameter arrays. We now describe this action. Let Φ denote a Leonard system and let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array of Φ. The parameter array of $\Phi^{\downarrow}$ is $p^{\downarrow}$ where $p^{\downarrow} := (\theta_i, \theta_{d-i}^*, i = 0, \ldots, d; \phi_{d-j+1}, \varphi_{d-j+1}, j = 1, \ldots, d)$. The parameter array of $\Phi^{\Downarrow}$ is $p^{\Downarrow}$ where $p^{\Downarrow} := (\theta_{d-i}, \theta_i^*, i = 0, \ldots, d; \phi_j, \varphi_j, j = 1, \ldots, d)$ [35, Theorem 1.11].

Let A, A^* denote a Leonard pair. By a *parameter array of A, A^** we mean the parameter array of an associated Leonard system. We observe that if p is a parameter array of A, A^* then so are $p^{\downarrow}$, $p^{\Downarrow}$, $p^{\downarrow\Downarrow}$ and A, A^* has no further parameter arrays. We comment on the distinctness of these arrays. Let d denote the diameter of A, A^*. Then p, $p^{\downarrow}$, $p^{\Downarrow}$, $p^{\downarrow\Downarrow}$ are mutually distinct if $d \geqslant 1$ and identical if $d = 0$. Therefore A, A^* has exactly four parameter arrays if $d \geqslant 1$ and just one parameter array if $d = 0$.

We now describe the TD–D canonical form.

We define what it means for a given Leonard system to be in TD–D canonical form. Let Φ denote a Leonard system with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Let A, A^* denote the associated Leonard pair. Then Φ is in *TD–D canonical form* whenever

(i) the ambient algebra of A, A^* is $\mathrm{Mat}_{d+1}(\mathbb{K})$;
(ii) A is tridiagonal and A^* is diagonal;
(iii) A has constant row sum θ_0 and $A_{00}^* = \theta_0^*$.

We describe the Leonard systems which are in TD–D canonical form. In order to do this we consider the set of parameter arrays. We define two functions on this set. We call these functions T and D. Let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array. The image p^T is the tridiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has the following entries. The diagonal entries are

$$p_{ii}^T = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}$$

for $0 \leqslant i \leqslant d$, where $\varphi_0 = 0$, $\varphi_{d+1} = 0$ and where $\theta_{-1}^*, \theta_{d+1}^*$ denote indeterminates. The superdiagonal and subdiagonal entries are

$$
p_{i-1,i}^T = \varphi_i \frac{\prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*)}{\prod_{h=0}^{i-1}(\theta_i^* - \theta_h^*)}, \qquad p_{i,i-1}^T = \phi_i \frac{\prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*)}{\prod_{h=i}^{d}(\theta_{i-1}^* - \theta_h^*)}
$$

for $1 \leqslant i \leqslant d$. The image p^D is $\mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$. The significance of T and D is the following. Given a Leonard system in TD–D canonical form the associated Leonard pair is p^T, p^D where p denotes the corresponding parameter array.

Let Φ denote a Leonard system. By a *TD–D canonical form for Φ*, we mean a Leonard system which is isomorphic to Φ and which is in TD–D canonical form. We show there exists a unique TD–D canonical form for Φ.

Let A, A^* denote a Leonard pair and consider its set of associated Leonard systems. From the construction this set contains at most one Leonard system which is in TD–D canonical form. The case in which this Leonard system exists is of interest; to describe this case we define a TD–D canonical form for Leonard pairs. We do this as follows.

We define what it means for a Leonard pair to be in TD–D canonical form. Let A, A^* denote a Leonard pair and let $\theta_0, \theta_1, \ldots, \theta_d$ denote an eigenvalue sequence for this pair. Then A, A^* is in *TD–D canonical form* whenever

 (i) the ambient algebra of A, A^* is $\mathrm{Mat}_{d+1}(\mathbb{K})$;
 (ii) A is tridiagonal and A^* is diagonal;
(iii) A has constant row sum and this sum is θ_0 or θ_d.

We just defined the TD–D canonical form for Leonard pairs and earlier we defined this form for Leonard systems. These two versions are related as follows. A given Leonard pair is in TD–D canonical form if and only if there exists an associated Leonard system which is in TD–D canonical form.

Let A, A^* denote a Leonard pair. By a *TD–D canonical form for A, A^** we mean a Leonard pair which is isomorphic to A, A^* and which is in TD–D canonical form. We describe the TD–D canonical forms for A, A^*. To do this we give a bijection from the set of parameter arrays for A, A^* to the set of TD–D canonical forms for A, A^*. This bijection sends each parameter array p to the pair p^T, p^D. To clarify this let d denote the diameter of A, A^*. If $d \geqslant 1$ then there exists exactly four TD–D canonical forms for A, A^*. If $d = 0$ then there exists a unique TD–D canonical form for A, A^*.

We give several applications of our theory. For the first application let d denote a non-negative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. We give a necessary and sufficient condition for A, A^* to be a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form. Indeed we show the following are equivalent:

 (i) the pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form;
 (ii) there exists a parameter array p of diameter d such that $A = p^T$ and $A^* = p^D$.

Our second application is similar to the first but more general. Again let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let us assume A is tridiagonal and A^* is diagonal. We give a necessary and sufficient condition for A, A^* to be a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. This condition is given in Theorem 25.1.

This completes our description of the TD–D canonical form. Our description of the LB–UB canonical form runs along similar lines; we save the details for the main body of the paper. We comment that in the main body of the paper it will be convenient to treat the LB–UB canonical form before the TD–D canonical form.

As we proceed through the paper we illustrate our results using two running examples which involve specific parameter arrays.

Near the end of the paper we discuss how Leonard pairs correspond to the q-Racah polynomials and some related polynomials in the Askey scheme. The general idea is the following. Given a Leonard pair A, A^* the corresponding polynomials give the entries in a transition matrix which takes a basis satisfying Definition 1.1(i) to a basis satisfying Definition 1.1(ii). We compute these polynomials explicitly for our two examples. For these examples the polynomials turn out to be Krawtchouk polynomials and q-Racah polynomials.

At the end of the paper we present some open problems concerning Leonard pairs.

2. Leonard systems

We now begin our formal argument. Our first goal is to recall our working definition of a Leonard system. We begin with some notation.

Let d denote a nonnegative integer. We let $\mathbb{K}^{d+1}$ denote the vector space over $\mathbb{K}$ consisting of all $d+1$ by 1 matrices which have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$ under matrix multiplication. We observe this module is irreducible. We let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. From now on when we refer to an $\mathcal{A}$-module we mean a left $\mathcal{A}$-module. Let V denote an irreducible $\mathcal{A}$-module. We remark that V is unique up to isomorphism of $\mathcal{A}$-modules, and that V has dimension $d+1$. Let $v_0, v_1, \ldots, v_d$ denote a basis for V. For $X \in \mathcal{A}$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents* X *with respect to* $v_0, v_1, \ldots, v_d$ whenever $X v_j = \sum_{i=0}^{d} Y_{ij} v_i$ for $0 \leqslant j \leqslant d$. For $A \in \mathcal{A}$, we say A is *multiplicity-free* whenever it has $d+1$ distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leqslant i \leqslant d$ put

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{1}$$

where I denotes the identity of $\mathcal{A}$. We observe

(i) $A E_i = \theta_i E_i$ $(0 \leqslant i \leqslant d)$;
(ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant d)$;
(iii) $\sum_{i=0}^{d} E_i = I$;
(iv) $A = \sum_{i=0}^{d} \theta_i E_i$.

Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. Using (i)–(iv) we find the sequence $E_0, E_1, \ldots, E_d$ is a basis for the $\mathbb{K}$-vector space $\mathcal{D}$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Let V denote an irreducible $\mathcal{A}$-module. Then

$$V = E_0 V + E_1 V + \cdots + E_d V \quad \text{(direct sum)}. \tag{2}$$

For $0 \leqslant i \leqslant d$, $E_i V$ is the (one-dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

Definition 2.1. Let d denote a nonnegative integer and let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let A, A^* denote an ordered pair consisting of multiplicity-free elements in $\mathcal{A}$. By an *idempotent sequence* for A, A^* we mean an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of A such that

$$E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1, \\ \neq 0, & \text{if } |i - j| = 1, \end{cases} \quad (0 \leqslant i, j \leqslant d).$$

We observe that if $E_0, E_1, \ldots, E_d$ is an idempotent sequence for A, A^* then so is $E_d, E_{d-1}, \ldots, E_0$ and A, A^* has no further idempotent sequence. By a *dual idempotent sequence* for A, A^* we mean an idempotent sequence for A^*, A.

Definition 2.2 [35]. Let d denote a nonnegative integer and let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. By a *Leonard system in* $\mathcal{A}$ we mean a sequence

$$\Phi = \left(A, A^*; E_i, E_i^*, i = 0, \ldots, d \right) \tag{3}$$

which satisfies (i)–(iii) below.

  (i) Each of A, A^* is a multiplicity-free element of $\mathcal{A}$.
 (ii) $E_0, E_1, \ldots, E_d$ is an idempotent sequence for A, A^*.
(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is a dual idempotent sequence for A, A^*.

We call d the *diameter* of Φ and say Φ is *over* $\mathbb{K}$. We call $\mathcal{A}$ the *ambient algebra* of Φ.

3. The relatives of a Leonard system

A given Leonard system can be modified in several ways to get a new Leonard system. For instance, let Φ denote the Leonard system from (3), and let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{K}$ such that $\alpha \neq 0$, $\alpha^* \neq 0$. Then

$$\left(\alpha A + \beta I, \alpha^* A^* + \beta^* I; E_i, E_i^*, i = 0, \ldots, d \right) \tag{4}$$

is a Leonard system in $\mathcal{A}$. Also, each of the following three sequences is a Leonard system in $\mathcal{A}$.

$$\Phi^* := \left(A^*, A; E_i^*, E_i, i = 0, \dots, d\right),$$

$$\Phi^\downarrow := \left(A, A^*; E_i, E_{d-i}^*, i = 0, \dots, d\right),$$

$$\Phi^\Downarrow := \left(A, A^*; E_{d-i}, E_i^*, i = 0, \dots, d\right).$$

We refer to Φ^* (respectively $\Phi^\downarrow$; respectively $\Phi^\Downarrow$) as the *dual* (respectively *first inversion*; respectively *second inversion*) of Φ. Viewing $*$, $\downarrow$, $\Downarrow$ as permutations on the set of all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{5}$$

$$\Downarrow * = *\downarrow, \qquad \downarrow * = *\Downarrow, \qquad \downarrow \Downarrow = \Downarrow \downarrow. \tag{6}$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the relations (5), (6) is the dihedral group D_4. We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action. The relatives of Φ are as follows:

Name	Relative
Φ	$(A, A^*; E_i, E_i^*, i = 0, \dots, d)$
$\Phi^\downarrow$	$(A, A^*; E_i, E_{d-i}^*, i = 0, \dots, d)$
$\Phi^\Downarrow$	$(A, A^*; E_{d-i}, E_i^*, i = 0, \dots, d)$
$\Phi^{\downarrow\Downarrow}$	$(A, A^*; E_{d-i}, E_{d-i}^*, i = 0, \dots, d)$
Φ^*	$(A^*, A; E_i^*, E_i, i = 0, \dots, d)$
$\Phi^{\downarrow*}$	$(A^*, A; E_{d-i}^*, E_i, i = 0, \dots, d)$
$\Phi^{\Downarrow*}$	$(A^*, A; E_i^*, E_{d-i}, i = 0, \dots, d)$
$\Phi^{\downarrow\Downarrow*}$	$(A^*, A; E_{d-i}^*, E_{d-i}, i = 0, \dots, d)$

4. Leonard pairs and Leonard systems

In view of our comments in the previous section, when we discuss a Leonard system we are often not interested in the orderings of the primitive idempotents, we just care how the elements A, A^* interact. This brings us back to the notion of a Leonard pair.

Definition 4.1. Let d denote a nonnegative integer and let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. By a *Leonard pair in* $\mathcal{A}$ we mean an ordered pair A, A^* which satisfies (i)–(iii) below.

(i) Each of A, A^* is a multiplicity-free element of $\mathcal{A}$.
(ii) There exists an idempotent sequence for A, A^*.
(iii) There exists a dual idempotent sequence for A, A^*.

By [35, Lemma 1.7] the preceding definition of a Leonard pair is equivalent to the definition given in the Introduction.

Let Φ denote the Leonard system from (3). Then the pair A, A^* from that line forms a Leonard pair in $\mathcal{A}$. We say this pair is *associated* with Φ.

Each Leonard system is associated with a unique Leonard pair. Let A, A^* denote a Leonard pair. By the *associate class* for A, A^* we mean the set of Leonard systems which are associated with A, A^*. By Definition 4.1 this associate class contains at least one Leonard system Φ. Apparently this associate class contains Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$ and no other Leonard systems.

Let A, A^* denote the Leonard pair from Definition 4.1. Then the pair A^*, A is a Leonard pair in $\mathcal{A}$. We call this pair the *dual* of A, A^*. We observe two Leonard systems are relatives if and only if their associated Leonard pairs are equal or dual.

5. Isomorphisms of Leonard pairs and Leonard systems

We recall the notion of *isomorphism* for Leonard pairs and Leonard systems. We begin with a comment.

Lemma 5.1 [29, Corollary 3.2]. *Let A, A^* denote the Leonard pair from Definition* 4.1. *Then A and A^* together generate $\mathcal{A}$.*

Let Φ denote the Leonard system from (3) and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathbb{K}$-algebras. We write $\Phi^\sigma := (A^\sigma, A^{*\sigma}; E_i^\sigma, E_i^{*\sigma}, i = 0, \ldots, d)$ and observe Φ^σ is a Leonard system in $\mathcal{A}'$.

Definition 5.2. Let Φ and Φ' denote Leonard systems over $\mathbb{K}$. By an *isomorphism of Leonard systems from Φ to Φ'* we mean an isomorphism σ of $\mathbb{K}$-algebras from the ambient algebra of Φ to the ambient algebra of Φ' such that $\Phi^\sigma = \Phi'$. By Lemma 5.1 there exists at most one isomorphism of Leonard systems from Φ to Φ'. We say Φ and Φ' are *isomorphic* whenever this isomorphism exists.

We now consider the notion of isomorphism for Leonard pairs.

Let A, A^* denote the Leonard pair from Definition 4.1 and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of $\mathbb{K}$-algebras. We observe the pair A^σ, $A^{*\sigma}$ is a Leonard pair in $\mathcal{A}'$.

Definition 5.3. Let A, A^* and B, B^* denote Leonard pairs over $\mathbb{K}$. By an *isomorphism of Leonard pairs from A, A^* to B, B^** we mean an isomorphism σ of $\mathbb{K}$-algebras from the ambient algebra of A, A^* to the ambient algebra of B, B^* such that $A^\sigma = B$ and $A^{*\sigma} = B^*$. By Lemma 5.1 there exists at most one isomorphism of Leonard pairs from A, A^* to B, B^*. We say A, A^* and B, B^* are *isomorphic* whenever this isomorphism exists.

We have a comment.

Lemma 5.4. *Let A, A^* denote a Leonard pair and let d denote the diameter. If $d \geqslant 1$ then the corresponding associate class contains four Leonard systems and these are mutually*

nonisomorphic. If $d = 0$ then the corresponding associate class contains a single Leonard system.

Proof. Let Φ denote a Leonard system associated with A, A^*. Then the associate class of Φ contains Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$ and no other Leonard systems. Suppose $d \geqslant 1$. Then Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$ are mutually nonisomorphic; if not the isomorphism involved would stabilize each of A, A^* and is therefore the identity map by Lemma 5.1. Suppose $d = 0$. Then Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$ are identical by the construction. $\square$

We finish this section with a remark. Let $\mathcal{A}$ denote a matrix algebra over $\mathbb{K}$. Let $\sigma : \mathcal{A} \to \mathcal{A}$ denote any map. By the Skolem–Noether theorem [26, Corollary 9.122], σ is an isomorphism of $\mathbb{K}$-algebras if and only if there exists an invertible $S \in \mathcal{A}$ such that $X^\sigma = SXS^{-1}$ for all $X \in \mathcal{A}$.

6. The adjacency relations

Definition 6.1. Let A, A^* denote the Leonard pair from Definition 4.1. Consider the set consisting of the primitive idempotents of A. We define a symmetric binary relation $\sim$ on this set. Let $E_0, E_1, \ldots, E_d$ denote an idempotent sequence for A, A^*. For $0 \leqslant i, j \leqslant d$ we define $E_i \sim E_j$ whenever $|i - j| = 1$. We call $\sim$ the *first adjacency relation* for A, A^*. We let $\approx$ denote the first adjacency relation for the Leonard pair A^*, A and call $\approx$ the *second adjacency relation* for A, A^*.

We make several observations.

Lemma 6.2. *Let A, A^* denote the Leonard pair from Definition 4.1. Let $E_0, E_1, \ldots, E_d$ (respectively $E_0^*, E_1^*, \ldots, E_d^*$) denote an ordering of the primitive idempotents of A (respectively A^*). Then $E_0, E_1, \ldots, E_d$ is an idempotent sequence for A, A^* if and only if $E_0 \sim E_1 \sim \cdots \sim E_d$. Moreover $E_0^*, E_1^*, \ldots, E_d^*$ is a dual idempotent sequence for A, A^* if and only if $E_0^* \approx E_1^* \approx \cdots \approx E_d^*$.*

Lemma 6.3. *Let A, A^* denote the Leonard pair from Definition 4.1. Let E and F denote primitive idempotents of A. Then the following are equivalent*:

  (i) $E \sim F$;
 (ii) $E \neq F$ and $EA^*F \neq 0$;
(iii) $E \neq F$ and $FA^*E \neq 0$.

Let E^* and F^* denote primitive idempotents of A^*. Then the following are equivalent:

  (i) $E^* \approx F^*$;
 (ii) $E^* \neq F^*$ and $E^*AF^* \neq 0$;
(iii) $E^* \neq F^*$ and $F^*AE^* \neq 0$.

7. The eigenvalue sequences

Definition 7.1. Let Φ denote the Leonard system from (3). For $0 \leqslant i \leqslant d$ we let θ_i (respectively θ_i^*) denote the eigenvalue of A (respectively A^*) associated with E_i (respectively E_i^*). We call $\theta_0, \theta_1, \ldots, \theta_d$ the *eigenvalue sequence* of Φ. We call $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the *dual eigenvalue sequence* of Φ. We observe $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct and contained in $\mathbb{K}$. Similarly $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct and contained in $\mathbb{K}$.

Definition 7.2. Let A, A^* denote a Leonard pair. By an *eigenvalue sequence* for this pair, we mean the eigenvalue sequence for an associated Leonard system. We remark that if $\theta_0, \theta_1, \ldots, \theta_d$ is an eigenvalue sequence for A, A^* then so is $\theta_d, \theta_{d-1}, \ldots, \theta_0$ and A, A^* has no further eigenvalue sequence. By a *dual eigenvalue sequence* for A, A^* we mean an eigenvalue sequence for the Leonard pair A^*, A.

We will use the following results.

Lemma 7.3. *Let d denote a nonnegative integer and let A, A^* denote a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume*

 (i) *A is lower triangular*; *and*
(ii) *$A_{ij}^* = 0$ if $j - i > 1$ $(0 \leqslant i, j \leqslant d)$.*

Then the sequence of diagonal entries $A_{00}, A_{11}, \ldots, A_{dd}$ is an eigenvalue sequence for A, A^*. Moreover $A_{j-1,j}^* \neq 0$ for $1 \leqslant j \leqslant d$.

Proof. We assume the pair A, A^* is a Leonard pair so A is multiplicity-free. We assume A is lower triangular so the sequence of diagonal entries $A_{00}, A_{11}, \ldots, A_{dd}$ gives an ordering of the eigenvalues of A. We show this sequence is an eigenvalue sequence for A, A^*. For $0 \leqslant i \leqslant d$ let E_i denote the primitive idempotent of A associated with the eigenvalue A_{ii}. We show $E_{j-1} \sim E_j$ for $1 \leqslant j \leqslant d$. This will follow once we show

$$E_i \nsim E_j \quad \text{if } j - i > 1 \ (0 \leqslant i, j \leqslant d). \tag{7}$$

We abbreviate $V = \mathbb{K}^{d+1}$. For $0 \leqslant r \leqslant d$ let V_r denote the subspace of V consisting of those vectors which have 0 in coordinates $0, 1, \ldots, r - 1$. The matrix A is lower triangular so $AV_r \subseteq V_r$. The restriction of A to V_r has eigenvalues $A_{rr}, \ldots, A_{dd}$ so $V_r = E_r V + \cdots + E_d V$. Apparently $E_r V \subseteq V_r$ and moreover each of $E_0, \ldots, E_{r-1}$ vanishes on V_r. From our assumption about A^* we find $A^* V_r \subseteq V_{r-1}$ for $1 \leqslant r \leqslant d$. Let i, j denote integers $(0 \leqslant i, j \leqslant d)$ and assume $j - i > 1$. From our above comments we find

$$E_i A^* E_j V \subseteq E_i A^* V_j \subseteq E_i V_{j-1} = 0.$$

Apparently $E_i A^* E_j V = 0$ so $E_i A^* E_j = 0$. Now $E_i \nsim E_j$ by Lemma 6.3. We now have (7) and it follows $E_{j-1} \sim E_j$ for $1 \leqslant j \leqslant d$. Applying Lemma 6.2 we find $E_0, E_1, \ldots, E_d$ is an idempotent sequence for A, A^*. Now $A_{00}, A_{11}, \ldots, A_{dd}$ is an eigenvalue sequence

for A, A^* by Definition 7.2. To finish the proof we show $A^*_{j-1,j} \neq 0$ for $1 \leqslant j \leqslant d$. Let j be given and suppose $A^*_{j-1,j} = 0$. Then $A^* V_j \subseteq V_j$. We mentioned earlier that $A V_j \subseteq V_j$. The matrices A and A^* together generate $\mathrm{Mat}_{d+1}(\mathbb{K})$ by Lemma 5.1 so $X V_j \subseteq V_j$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. The space V is irreducible as a module for $\mathrm{Mat}_{d+1}(\mathbb{K})$, so $V_j = 0$ or $V_j = V$. From the definition of V_j and since $1 \leqslant j \leqslant d$ we find $V_j \neq 0$ and $V_j \neq V$. This is a contradiction and we conclude $A^*_{j-1,j} \neq 0$. $\square$

Lemma 7.4. *Let d denote a nonnegative integer and let A, A^* denote a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume*

 (i) *A is upper triangular*; *and*
(ii) *$A^*_{ij} = 0$ if $i - j > 1$ ($0 \leqslant i, j \leqslant d$).*

Then the sequence of diagonal entries $A_{00}, A_{11}, \ldots, A_{dd}$ is an eigenvalue sequence for A, A^*. Moreover $A^*_{i,i-1} \neq 0$ for $1 \leqslant i \leqslant d$.

Proof. Using Definition 4.1 we find A^t, A^{*t} is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$, where t denotes transpose. To obtain the result apply Lemma 7.3 to this pair. $\square$

We give a corollary to Lemmas 7.3 and 7.4. In order to state it we make a definition.

Definition 7.5. Let d denote a nonnegative integer and let A denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$. We say A is *lower bidiagonal* whenever each nonzero entry lies on either the diagonal or the subdiagonal. We say A is *upper bidiagonal* whenever the transpose of A is lower bidiagonal.

Corollary 7.6. *Let d denote a nonnegative integer and let A, A^* denote a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume A is lower bidiagonal and A^* is upper bidiagonal. Then* (i)–(iv) *hold below.*

 (i) *The sequence $A_{00}, A_{11}, \ldots, A_{dd}$ is an eigenvalue sequence for A, A^*.*
 (ii) *$A_{i,i-1} \neq 0$ for $1 \leqslant i \leqslant d$.*
(iii) *The sequence $A^*_{00}, A^*_{11}, \ldots, A^*_{dd}$ is a dual eigenvalue sequence for A, A^*.*
(iv) *$A^*_{i-1,i} \neq 0$ for $1 \leqslant i \leqslant d$.*

Proof. (i) and (iv). Apply Lemma 7.3 to A, A^*.
   (ii) and (iii). Apply Lemma 7.4 to the Leonard pair A^*, A. $\square$

The following fact may seem intuitively clear from Definition 4.1, but strictly speaking it requires proof.

Corollary 7.7. *Let d denote a nonnegative integer and let A, A^* denote a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume A is tridiagonal and A^* is diagonal. Then* (i), (ii) *hold below.*

(i) *A is irreducible.*

(ii) *The sequence $A^*_{00}, A^*_{11}, \ldots, A^*_{dd}$ is a dual eigenvalue sequence for A, A^*.*

Proof. (i) Applying Lemma 7.4 to the Leonard pair A^*, A we find $A_{i,i-1} \neq 0$ for $1 \leqslant i \leqslant d$. Applying Lemma 7.3 to A^*, A we find $A_{i-1,i} \neq 0$ for $1 \leqslant i \leqslant d$.

(ii) Apply Lemma 7.3 to the Leonard pair A^*, A. □

8. The split sequences

In Definition 7.1 we defined the eigenvalue sequence and the dual eigenvalue sequence of a Leonard system. There are two more parameter sequences of interest to us. In order to define these, we review some results from [16,29,35]. Let Φ denote the Leonard system in (3) and let V denote an irreducible $\mathcal{A}$-module. For $0 \leqslant i \leqslant d$ we define

$$U_i = \left(E^*_0 V + E^*_1 V + \cdots + E^*_i V \right) \cap (E_i V + E_{i+1} V + \cdots + E_d V). \tag{8}$$

We showed in [35, Lemma 3.8] that each of $U_0, U_1, \ldots, U_d$ has dimension 1, and that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)}. \tag{9}$$

The elements A and A^* act on the U_i as follows. By [35, Lemma 3.9], both

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_d I)U_d = 0, \tag{10}$$

$$\left(A^* - \theta^*_i I\right)U_i = U_{i-1} \quad (1 \leqslant i \leqslant d), \qquad \left(A^* - \theta^*_0 I\right)U_0 = 0, \tag{11}$$

where the θ_i, θ^*_i are from Definition 7.1. Pick an integer i ($1 \leqslant i \leqslant d$). By (11) we find $(A^* - \theta^*_i I)U_i = U_{i-1}$ and by (10) we find $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently U_i is an eigenspace for $(A - \theta_{i-1} I)(A^* - \theta^*_i I)$ and the corresponding eigenvalue is a nonzero element of $\mathbb{K}$. We denote this eigenvalue by φ_i. We display a basis for V which illuminates the significance of φ_i. Setting $i = 0$ in (8) we find $U_0 = E^*_0 V$. Combining this with (10) we find

$$U_i = (A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I) E^*_0 V \quad (0 \leqslant i \leqslant d). \tag{12}$$

Let η^*_0 denote a nonzero vector in $E^*_0 V$. From (12) we find that for $0 \leqslant i \leqslant d$ the vector $(A - \theta_{i-1} I) \cdots (A - \theta_0 I)\eta^*_0$ is a basis for U_i. From this and (9) we find the sequence

$$(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I)\eta^*_0 \quad (0 \leqslant i \leqslant d) \tag{13}$$

is a basis for V. With respect to this basis the matrices representing A and A^* are

$$\begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}, \qquad (14)$$

respectively. We call the sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ the *first split sequence* of Φ. We let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence for $\Phi^{\Downarrow}$ and call this the *second split sequence* of Φ. For notational convenience we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$, $\phi_0 = 0$, $\phi_{d+1} = 0$.

9. A classification of Leonard systems

We recall our classification of Leonard systems.

Theorem 9.1 [35, Theorem 1.9]. *Let d denote a nonnegative integer and let*

$$\theta_0, \theta_1, \ldots, \theta_d; \quad \theta_0^*, \theta_1^*, \ldots, \theta_d^*; \quad \varphi_1, \varphi_2, \ldots, \varphi_d; \quad \phi_1, \phi_2, \ldots, \phi_d$$

denote scalars in $\mathbb{K}$. Then there exists a Leonard system Φ over $\mathbb{K}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$ if and only if (i)–(v) *hold below.*

(i) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leqslant i \leqslant d)$.

(ii) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leqslant i, j \leqslant d)$.

(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d)$ $(1 \leqslant i \leqslant d)$.

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0)$ $(1 \leqslant i \leqslant d)$.

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$.

Moreover, if (i)–(v) *hold above then Φ is unique up to isomorphism of Leonard systems.*

We view Theorem 9.1 as a linear algebraic version of a theorem of D.A. Leonard ([23], [1, p. 260]). This is discussed in [35].

10. The notion of a parameter array

In view of Theorem 9.1 we make the following definition.

Definition 10.1. Let d denote a nonnegative integer. By a *parameter array over* $\mathbb{K}$ *with diameter* d, we mean a sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ of scalars taken from $\mathbb{K}$ which satisfy conditions (i)–(v) in Theorem 9.1.

We give several examples of a parameter array.

Example 10.2. Let d denote a nonnegative integer and consider the following scalars in $\mathbb{K}$.

$$\theta_i = d - 2i, \quad \theta_i^* = d - 2i \quad (0 \leqslant i \leqslant d),$$

$$\varphi_i = -2i(d - i + 1), \quad \phi_i = 2i(d - i + 1) \quad (1 \leqslant i \leqslant d).$$

To avoid degenerate situations, we assume the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d. Then the sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is a parameter array over $\mathbb{K}$.

Proof. The sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ satisfies Theorem 9.1(i)–(v) so this sequence is a parameter array over $\mathbb{K}$. □

Example 10.3. Let d denote a nonnegative integer. Let q, s, s^*, r_1, r_2 denote nonzero scalars in $\mathbb{K}$ such that $r_1 r_2 = ss^* q^{d+1}$. Assume none of q^i, $r_1 q^i$, $r_2 q^i$, $s^* q^i / r_1$, $s^* q^i / r_2$ is equal to 1 for $1 \leqslant i \leqslant d$ and that neither of sq^i, $s^* q^i$ is equal to 1 for $2 \leqslant i \leqslant 2d$. Define

$$\theta_i = q^{-i} + sq^{i+1}, \qquad \theta_i^* = q^{-i} + s^* q^{i+1}$$

for $0 \leqslant i \leqslant d$ and

$$\varphi_i = q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1 q^i)(1 - r_2 q^i),$$

$$\phi_i = q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^*$$

for $1 \leqslant i \leqslant d$. Then the sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is a parameter array over $\mathbb{K}$.

Proof. The sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ satisfies Theorem 9.1(i)–(v) so this sequence is a parameter array over $\mathbb{K}$. □

11. Parameter arrays and Leonard systems

In this section we discuss the relationship between parameter arrays and Leonard systems.

Definition 11.1. Let Φ denote a Leonard system over $\mathbb{K}$, with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$. By Theorem 9.1 the sequence $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is a parameter array over $\mathbb{K}$. We call this array the *parameter array of Φ*.

We remark that by Theorem 9.1 the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over $\mathbb{K}$ to the set of parameter arrays over $\mathbb{K}$.

Earlier we discussed several ways to modify a given Leonard system to get a new Leonard system. We now consider how these modifications affect the corresponding parameter array.

Lemma 11.2. *Let Φ denote the Leonard system from* (3) *and let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the corresponding parameter array. Let α, α^*, β, β^* denote scalars in $\mathbb{K}$ such that $\alpha \neq 0$, $\alpha^* \neq 0$. Then the Leonard system* (4) *has parameter array*

$$\left(\alpha\theta_i + \beta, \, \alpha^*\theta_i^* + \beta^*, \, i = 0, \ldots, d; \, \alpha\alpha^*\varphi_j, \alpha\alpha^*\phi_j, \, j = 1, \ldots, d\right).$$

Proof. Routine. $\square$

Lemma 11.3 [35, Theorem 1.11]. *Let Φ denote a Leonard system and let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the corresponding parameter array. Then* (i)–(iii) *hold below.*

(i) *The parameter array of Φ^* is p^* where $p^* := (\theta_i^*, \theta_i, i = 0, \ldots, d; \varphi_j, \phi_{d-j+1}, j = 1, \ldots, d)$.*

(ii) *The parameter array of $\Phi^\downarrow$ is $p^\downarrow$ where $p^\downarrow := (\theta_i, \theta_{d-i}^*, i = 0, \ldots, d; \phi_{d-j+1}, \varphi_{d-j+1}, j = 1, \ldots, d)$.*

(iii) *The parameter array of $\Phi^\Downarrow$ is $p^\Downarrow$ where $p^\Downarrow := (\theta_{d-i}, \theta_i^*, i = 0, \ldots, d; \phi_j, \varphi_j, j = 1, \ldots, d)$.*

The following equations will be useful.

Corollary 11.4. *Let d denote a positive integer and let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array over $\mathbb{K}$. Then* (i)–(iii) *hold below.*

(i) $\dfrac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} = \dfrac{\theta_i^* - \theta_{d-i}^*}{\theta_0^* - \theta_d^*}$ $(0 \leqslant i \leqslant d)$.

(ii) $\varphi_i = \phi_d \sum_{h=0}^{i-1} \dfrac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i - \theta_0)(\theta_{i-1}^* - \theta_d^*)$ $(1 \leqslant i \leqslant d)$.

(iii) $\phi_i = \varphi_d \sum_{h=0}^{i-1} \dfrac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_{d-i} - \theta_d)(\theta_{i-1}^* - \theta_d^*)$ $(1 \leqslant i \leqslant d)$.

Proof. Each of (i)–(iii) is an algebraic consequence of the conditions in Theorem 9.1. Below we give a more intuitive proof using Lemma 11.3. Let Φ denote a Leonard system over $\mathbb{K}$ which has the given parameter array.

(i) Applying Theorem 9.1(iv) to Φ^* and using Lemma 11.3(i) we obtain

$$\phi_{d-i+1} = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h^* - \theta_{d-h}^*}{\theta_0^* - \theta_d^*} + (\theta_i - \theta_0)\left(\theta_{d-i+1}^* - \theta_0^*\right) \tag{15}$$

for $1 \leqslant i \leqslant d$. To finish the proof, in (15) replace i by $d - i + 1$ and compare the result with Theorem 9.1(iv).

(ii) Apply Theorem 9.1(iii) to Φ^* and simplify the result using (i) above and Lemma 11.3(i).

(iii) Apply (ii) above to $\Phi^{\Downarrow}$ and use Lemma 11.3(iii). $\quad\square$

12. The parameter arrays of a Leonard pair

In this section we define the notion of a parameter array for a Leonard pair.

Definition 12.1. Let A, A^* denote a Leonard pair. By a *parameter array of A, A^** we mean the parameter array of an associated Leonard system.

The parameter arrays of a Leonard pair are related as follows.

Lemma 12.2. *Let A, A^* denote the Leonard pair from Definition 4.1. Let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array of A, A^*. Then the following are parameter arrays of A, A^*.*

$$p = \left(\theta_i, \theta_i^*,\ i = 0, \ldots, d;\ \varphi_j, \phi_j,\ j = 1, \ldots, d\right),$$
$$p^{\downarrow} = \left(\theta_i, \theta_{d-i}^*,\ i = 0, \ldots, d;\ \phi_{d-j+1}, \varphi_{d-j+1},\ j = 1, \ldots, d\right),$$
$$p^{\Downarrow} = \left(\theta_{d-i}, \theta_i^*,\ i = 0, \ldots, d;\ \phi_j, \varphi_j,\ j = 1, \ldots, d\right),$$
$$p^{\downarrow\Downarrow} = \left(\theta_{d-i}, \theta_{d-i}^*,\ i = 0, \ldots, d;\ \varphi_{d-j+1}, \phi_{d-j+1},\ j = 1, \ldots, d\right).$$

The Leonard pair A, A^* has no further parameter arrays.

Proof. By Definition 12.1 there exists a Leonard system Φ which is associated with A, A^* and which has parameter array p. The above sequences are the parameter arrays for Φ, $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$ and these are the Leonard systems associated with A, A^*. $\quad\square$

Corollary 12.3. *Let A, A^* denote the Leonard pair from Definition 4.1. Then A, A^* has exactly four parameter arrays if $d \geqslant 1$ and a unique parameter array if $d = 0$.*

Proof. Referring to Lemma 12.2, the parameter arrays p, $p^{\downarrow}$, $p^{\Downarrow}$, $p^{\downarrow\Downarrow}$ are mutually distinct if $d \geqslant 1$ and identical if $d = 0$. $\quad\square$

We have a comment.

Lemma 12.4. *Let A, A^* denote a Leonard pair over $\mathbb{K}$ and let B, B^* denote a Leonard pair over $\mathbb{K}$. These pairs are isomorphic if and only if they share a parameter array. In this case the set of parameter arrays for A, A^* coincides with the set of parameter arrays for B, B^*.*

Proof. Suppose A, A^* and B, B^* share a parameter array p. By Definition 12.1 there exists a Leonard system Φ which is associated with A, A^* and which has parameter array p. Similarly there exists a Leonard system Φ' which is associated with B, B^* and which has parameter array p. Observe Φ, Φ' are isomorphic since they have the same parameter array. Observe the isomorphism involved is an isomorphism of Leonard pairs from A, A^* to B, B^*. Apparently A, A^* and B, B^* are isomorphic. The remaining claims of the lemma are clear. □

13. The LB–UB canonical form; preliminaries

We now turn our attention to the LB–UB canonical form. We begin with some comments.

Definition 13.1. Let Φ denote the Leonard system from (3) and let V denote an irreducible $\mathcal{A}$-module. By a *Φ-LB–UB basis* for V we mean a sequence of the form (13), where $\theta_0, \theta_1, \ldots, \theta_d$ denotes the eigenvalue sequence for Φ and η_0^* denotes a nonzero vector in E_0^*V.

Lemma 13.2. *Let Φ denote the Leonard system from (3). Let $\theta_0, \theta_1, \ldots, \theta_d$ denote the eigenvalue sequence for Φ. Let V denote an irreducible $\mathcal{A}$-module and let $v_0, v_1, \ldots, v_d$ denote a sequence of vectors in V, not all zero. Then this sequence is a Φ-LB–UB basis for V if and only if both*

(i) $v_0 \in E_0^*V$; *and*
(ii) $Av_i = \theta_i v_i + v_{i+1}$ *for* $0 \leqslant i \leqslant d - 1$.

Proof. Routine. □

Definition 13.3. Let Φ denote the Leonard system from (3). We define a map $\natural : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ as follows. Let V denote an irreducible $\mathcal{A}$-module. For all $X \in \mathcal{A}$ we let $X^\natural$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which represents X with respect to a Φ-LB–UB basis for V. We observe $\natural : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras. We call $\natural$ the *LB–UB canonical map* for Φ.

Before proceeding we introduce some notation.

Definition 13.4. Consider the set of all parameter arrays over $\mathbb{K}$. We define two functions on this set. We call these functions L and U. Let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j =$

$1, \ldots, d$) denote a parameter array over $\mathbb{K}$. The images p^L and p^U are the following matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

$$
p^L = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad p^U = \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}.
$$

Lemma 13.5. *Let Φ denote the Leonard system from* (3). *Let $\natural$ denote the LB–UB canonical map for Φ, from Definition* 13.3. *Then $A^\natural = p^L$ and $A^{*\natural} = p^U$, where p denotes the parameter array for Φ.*

Proof. Write $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$. Each of $A^\natural$, p^L is equal to the matrix on the left in (14) so $A^\natural = p^L$. Each of $A^{*\natural}$, p^U is equal to the matrix on the right in (14) so $A^{*\natural} = p^U$. $\square$

14. The LB–UB canonical form for Leonard systems

In this section we introduce the LB–UB canonical form for Leonard systems. We define what it means for a given Leonard system to be in LB–UB canonical form. We describe the Leonard systems which are in LB–UB canonical form. We show every Leonard system is isomorphic to a unique Leonard system which is in LB–UB canonical form.

Definition 14.1. Let Φ denote the Leonard system from (3). Let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (respectively dual eigenvalue sequence) of Φ. We say Φ is in *LB–UB canonical form* whenever (i)–(iv) hold below.

  (i) $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$.
 (ii) A is lower bidiagonal and A^* is upper bidiagonal.
(iii) $A_{i,i-1} = 1$ for $1 \leqslant i \leqslant d$.
 (iv) $A_{00} = \theta_0$ and $A_{00}^* = \theta_0^*$.

Lemma 14.2. *Let Φ denote the Leonard system from* (3). *Assume Φ is in LB–UB canonical form, so that $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$ by Definition* 14.1(i). *For $0 \leqslant i \leqslant d$ let v_i denote the vector in $\mathbb{K}^{d+1}$ which has ith coordinate* 1 *and all other coordinates* 0. *Then the sequence $v_0, v_1, \ldots, v_d$ is a Φ-LB–UB basis for $\mathbb{K}^{d+1}$. Let $\natural$ denote the LB–UB canonical map for Φ, from Definition* 13.3. *Then $\natural$ is the identity map.*

Proof. Let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (respectively dual eigenvalue sequence) for Φ. By Definition 14.1, A is lower bidiagonal with $A_{i,i-1} = 1$ for $1 \leqslant i \leqslant d$. By Corollary 7.6(i) and since $A_{00} = \theta_0$ we find $A_{ii} = \theta_i$ for $0 \leqslant i \leqslant d$. Apparently $Av_i = \theta_i v_i + v_{i+1}$ for $0 \leqslant i \leqslant d - 1$. By Definition 14.1, A^* is upper bidiagonal with $A_{00}^* = \theta_0^*$. Apparently v_0 is an eigenvector for A^* with eigenvalue θ_0^*.

Therefore $v_0 \in E_0^* V$. Applying Lemma 13.2 (with $V = \mathbb{K}^{d+1}$) we find $v_0, v_1, \ldots, v_d$ is a Φ-LB–UB basis for $\mathbb{K}^{d+1}$. From the construction each element in $\mathrm{Mat}_{d+1}(\mathbb{K})$ represents itself with respect to $v_0, v_1, \ldots, v_d$. Therefore $\natural$ is the identity map in view of Definition 13.3. $\quad\square$

Theorem 14.3. *Let Φ denote the Leonard system from* (3) *and assume Φ is in LB–UB canonical form. Then $A = p^L$ and $A^* = p^U$, where L, U are from Definition* 13.4 *and p is the parameter array of Φ.*

Proof. Let $\natural$ denote the LB–UB canonical map for Φ, from Definition 13.3. We assume Φ is in LB–UB canonical form, so $\natural$ is the identity map by Lemma 14.2. Applying Lemma 13.5 we find $A = p^L$ and $A^* = p^U$. $\quad\square$

Corollary 14.4. *Let Φ and Φ' denote Leonard systems over $\mathbb{K}$ which are in LB–UB canonical form. Then the following are equivalent*:

 (i) Φ *and* Φ' *are isomorphic*;
(ii) $\Phi = \Phi'$.

Proof. (i) $\Rightarrow$ (ii). The Leonard systems Φ, Φ' have a common parameter array which we denote by p. By Theorem 14.3 the Leonard pair associated with each of Φ, Φ' is equal to p^L, p^U. Apparently Φ and Φ' are in the same associate class. By this and since Φ, Φ' are isomorphic we find $\Phi = \Phi'$ in view of Lemma 5.4.

(ii) $\Rightarrow$ (i). Clear. $\quad\square$

Definition 14.5. Let Φ denote the Leonard system from (3). By an *LB–UB canonical form for Φ* we mean a Leonard system over $\mathbb{K}$ which is isomorphic to Φ and which is in LB–UB canonical form.

Theorem 14.6. *Let Φ denote the Leonard system from* (3). *Then there exists a unique LB–UB canonical form for Φ. This form is $\Phi^\natural$, where $\natural$ denotes the LB–UB canonical map for Φ from Definition* 13.3.

Proof. We first show $\Phi^\natural$ is an LB–UB canonical form for Φ. Since Φ is a Leonard system in $\mathcal{A}$ and since $\natural : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras, we find $\Phi^\natural$ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is isomorphic to Φ. We show $\Phi^\natural$ is in LB–UB canonical form. To do this we show $\Phi^\natural$ satisfies conditions (i)–(iv) of Definition 14.1. Observe $\Phi^\natural$ satisfies Definition 14.1(i) since $\mathrm{Mat}_{d+1}(\mathbb{K})$ is the ambient algebra of $\Phi^\natural$. Observe $\Phi^\natural$ satisfies Definition 14.1(ii)–(iv) by Definition 13.4 and Lemma 13.5. We have now shown $\Phi^\natural$ satisfies Definition 14.1(i)–(iv) so $\Phi^\natural$ is in LB–UB canonical form. Apparently $\Phi^\natural$ is a Leonard system over $\mathbb{K}$ which is isomorphic to Φ and which is in LB–UB canonical form. Therefore $\Phi^\natural$ is an LB–UB canonical form for Φ by Definition 14.5. To finish the proof we let Φ' denote an LB–UB canonical form for Φ and show $\Phi' = \Phi^\natural$. Observe Φ', $\Phi^\natural$ are isomorphic since they are both isomorphic to Φ. The Leonard systems Φ', $\Phi^\natural$ are isomorphic and in LB–UB canonical form so $\Phi' = \Phi^\natural$ by Corollary 14.4. $\quad\square$

Corollary 14.7. *Consider the set of Leonard systems over $\mathbb{K}$ which are in LB–UB canonical form. We give a bijection from this set to the set of parameter arrays over $\mathbb{K}$. The bijection sends each Leonard system to its own parameter array.*

Proof. By the remark following Definition 11.1, the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over $\mathbb{K}$ to the set of parameter arrays over $\mathbb{K}$. By Theorem 14.6 each of these isomorphism classes contains a unique element which is in LB–UB canonical form. The result follows. $\square$

15. The LB–UB canonical form for Leonard pairs

In this section we define and discuss the LB–UB canonical form for Leonard pairs. We begin with a comment.

Lemma 15.1. *Let A, A^* denote the Leonard pair from Definition 4.1. Then there exists at most one Leonard system which is associated with A, A^* and which is in LB–UB canonical form.*

Proof. Let Φ and Φ' denote Leonard systems which are associated with A, A^* and which are in LB–UB canonical form. We show $\Phi = \Phi'$. Since Φ, Φ' are in the same associate class, this will follow once we show Φ, Φ' have the same eigenvalue sequence and the same dual eigenvalue sequence. Observe by Theorem 14.3 that the sequence of diagonal entries for A is the common eigenvalue sequence for Φ, Φ'. Similarly the sequence of diagonal entries for A^* is the common dual eigenvalue sequence for Φ, Φ'. Apparently $\Phi = \Phi'$. $\square$

Referring to the above lemma, we now consider those Leonard pairs for which there exists an associated Leonard system which is in LB–UB canonical form. In order to describe these we introduce the LB–UB canonical form for Leonard pairs.

Definition 15.2. Let A, A^* denote the Leonard pair from Definition 4.1. We say this pair is in *LB–UB canonical form* whenever (i)–(iii) hold below.

  (i) $\mathcal{A} = \text{Mat}_{d+1}(\mathbb{K})$.
 (ii) A is lower bidiagonal and A^* is upper bidiagonal.
(iii) $A_{i,i-1} = 1$ for $1 \leqslant i \leqslant d$.

We just defined the LB–UB canonical form for Leonard pairs, and in Definition 14.1 we defined this form for Leonard systems. We now compare these two versions. We will use the following definition.

Definition 15.3. Let d denote a nonnegative integer and let A, A^* denote a Leonard pair in $\text{Mat}_{d+1}(\mathbb{K})$. We assume A is lower bidiagonal and A^* is upper bidiagonal. We make some comments and definitions.

(i) By Corollary 7.6(i) the sequence $A_{00}, A_{11}, \ldots, A_{dd}$ is an eigenvalue sequence for A, A^*. We call this sequence the *designated eigenvalue sequence* for A, A^*.

(ii) By Corollary 7.6(iii) the sequence $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$ is a dual eigenvalue sequence for A, A^*. We call this sequence the *designated dual eigenvalue sequence* for A, A^*.

(iii) By the *designated Leonard system* for A, A^* we mean the Leonard system which is associated with A, A^* and which has eigenvalue sequence $A_{00}, A_{11}, \ldots, A_{dd}$ and dual eigenvalue sequence $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$.

(iv) By the *designated parameter array* for A, A^* we mean the parameter array of the designated Leonard system for A, A^*.

Lemma 15.4. *Let A, A^* denote the Leonard pair from Definition 4.1. Then the following are equivalent*:

(i) *A, A^* is in LB–UB canonical form*;

(ii) *there exists a Leonard system Φ which is associated with A, A^* and which is in LB–UB canonical form.*

Suppose (i), (ii) *hold. Then Φ is the designated Leonard system of A, A^*.*

Proof. (i) $\Rightarrow$ (ii). Let Φ denote the designated Leonard system for A, A^*, from Definition 15.3(iii). From the construction Φ is associated with A, A^* and in LB–UB canonical form.

(ii) $\Rightarrow$ (i). Compare Definitions 14.1 and 15.2.

Now suppose (i), (ii) hold. Then Φ is the designated Leonard system for A, A^* by Lemma 15.1 and the proof of (i) $\Rightarrow$ (ii) above. $\quad\square$

Corollary 15.5. *We give a bijection from the set of Leonard systems over $\mathbb{K}$ which are in LB–UB canonical form, to the set of Leonard pairs over $\mathbb{K}$ which are in LB–UB canonical form. The bijection sends each Leonard system to its associated Leonard pair. The inverse bijection sends each Leonard pair to its designated Leonard system.*

Proof. This is a reformulation of Lemma 15.4. $\quad\square$

Theorem 15.6. *We give a bijection from the set of parameter arrays over $\mathbb{K}$ to the set of Leonard pairs over $\mathbb{K}$ which are in LB–UB canonical form. The bijection sends each parameter array p to the Leonard pair p^L, p^U. The inverse bijection sends each Leonard pair to its designated parameter array.*

Proof. Composing the inverse of the bijection from Corollary 14.7, with the bijection from Corollary 15.5, we obtain a bijection from the set of parameter arrays over $\mathbb{K}$ to the set of Leonard pairs over $\mathbb{K}$ which are in LB–UB canonical form. Let p denote a

parameter array over $\mathbb{K}$ and let A, A^* denote the image of p under this bijection. We show $A = p^L$ and $A^* = p^U$. By Corollary 14.7 there exists a unique Leonard system over $\mathbb{K}$ which is in LB–UB canonical form and which has parameter array p. Let us denote this system by Φ. By the construction A, A^* is associated with Φ. Applying Theorem 14.3 to Φ we find $A = p^L$ and $A^* = p^U$. To finish the proof we show p is the designated parameter array for A, A^*. We mentioned A, A^* is associated with Φ and Φ is in LB–UB canonical form so Φ is the designated Leonard system for A, A^* by Corollary 15.5. We mentioned p is the parameter array for Φ so p is the designated parameter array for A, A^* by Definition 15.3(iv). $\square$

Definition 15.7. Let A, A^* denote the Leonard pair from Definition 4.1. By an *LB–UB canonical form for A, A^** we mean a Leonard pair over $\mathbb{K}$ which is isomorphic to A, A^* and which is in LB–UB canonical form.

Theorem 15.8. *Let A, A^* denote the Leonard pair from Definition* 4.1. *We give a bijection from the set of parameter arrays for A, A^* to the set of LB–UB canonical forms for A, A^*. This bijection sends each parameter array p to the pair p^L, p^U. (The parameter arrays for A, A^* are given in Lemma* 12.2.) *The inverse bijection sends each LB–UB canonical form for A, A^* to its designated parameter array.*

Proof. Let B, B^* denote a Leonard pair over $\mathbb{K}$ which is in LB–UB canonical form. Let p denote the designated parameter array for B, B^*. In view of Theorem 15.6 it suffices to show the following are equivalent: (i) A, A^* and B, B^* are isomorphic; (ii) p is a parameter array for A, A^*. These statements are equivalent by Lemma 12.4. $\square$

Corollary 15.9. *Let A, A^* denote the Leonard pair from Definition* 4.1. *If $d \geqslant 1$ then there exist exactly four LB–UB canonical forms for A, A^*. If $d = 0$ there exists a unique LB–UB canonical form for A, A^*.*

Proof. Immediate from Theorem 15.8 and Corollary 12.3. $\square$

16. How to recognize a Leonard pair in LB–UB canonical form

Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let us assume A is lower bidiagonal and A^* is upper bidiagonal. We give a necessary and sufficient condition for A, A^* to be a Leonard pair which is in LB–UB canonical form.

Theorem 16.1. *Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Assume A is lower bidiagonal and A^* is upper bidiagonal. Then the following* (i), (ii) *are equivalent.*

 (i) *The pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in LB–UB canonical form.*
 (ii) *There exists a parameter array $(\theta_i, \theta_i^*, i = 0, \dots, d; \varphi_j, \phi_j, j = 1, \dots, d)$ over $\mathbb{K}$ such that*

$$A_{ii} = \theta_i, \quad A_{ii}^* = \theta_i^* \quad (0 \leqslant i \leqslant d), \tag{16}$$

$$A_{i,i-1} = 1, \quad A_{i-1,i}^* = \varphi_i \quad (1 \leqslant i \leqslant d). \tag{17}$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *above is uniquely determined by A, A*. This parameter array is the designated parameter array for A, A* in the sense of Definition* 15.3.

Proof. This is a reformulation of Theorem 15.6. $\quad\square$

17. Leonard pairs *A*, *A** with *A* lower bidiagonal and *A** upper bidiagonal

Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let us assume A is lower bidiagonal and A^* is upper bidiagonal. We give a necessary and sufficient condition for A, A^* to be a Leonard pair.

Theorem 17.1. *Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Assume A lower bidiagonal and A^* is upper bidiagonal. Then the following* (i), (ii) *are equivalent.*

 (i) *The pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*
(ii) *There exists a parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ over $\mathbb{K}$ such that*

$$A_{ii} = \theta_i, \quad A_{ii}^* = \theta_i^* \quad (0 \leqslant i \leqslant d), \tag{18}$$

$$A_{i,i-1} A_{i-1,i}^* = \varphi_i \quad (1 \leqslant i \leqslant d). \tag{19}$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *above is uniquely determined by A, A*. This parameter array is the designated parameter array for A, A* in the sense of Definition* 15.3.

Proof. (i) $\Rightarrow$ (ii). By Corollary 7.6(ii) we have $A_{i,i-1} \neq 0$ for $1 \leqslant i \leqslant d$. Let S denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has diagonal entries $S_{ii} = A_{10} A_{21} \cdots A_{i,i-1}$ for $0 \leqslant i \leqslant d$. Each of $S_{00}, S_{11}, \ldots, S_{dd}$ is nonzero so S^{-1} exists. Let $\sigma : \mathrm{Mat}_{d+1}(\mathbb{K}) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the isomorphism of $\mathbb{K}$-algebras which satisfies $X^\sigma = S^{-1} X S$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. From the construction A^σ (respectively $A^{*\sigma}$) is lower bidiagonal (respectively upper bidiagonal) with entries

$$A_{ii}^\sigma = A_{ii}, \quad A_{ii}^{*\sigma} = A_{ii}^* \quad (0 \leqslant i \leqslant d), \tag{20}$$

$$A_{i,i-1}^\sigma = 1, \quad A_{i-1,i}^{*\sigma} = A_{i,i-1} A_{i-1,i}^* \quad (1 \leqslant i \leqslant d). \tag{21}$$

Apparently $A^\sigma, A^{*\sigma}$ is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in LB–UB canonical form. Applying Theorem 16.1 to this pair we find there exists a parameter array $(\theta_i, \theta_i^*, i =$

$0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d$) over $\mathbb{K}$ such that both $A_{ii}^\sigma = \theta_i$, $A_{ii}^{*\sigma} = \theta_i^*$ for $0 \leqslant i \leqslant d$ and $A_{i-1,i}^{*\sigma} = \varphi_i$ for $1 \leqslant i \leqslant d$. Combining these facts with (20), (21) we find this parameter array satisfies (18), (19).

(ii) $\Rightarrow$ (i). For $1 \leqslant i \leqslant d$ we have $A_{i,i-1} \neq 0$ by (19) and since $\varphi_i \neq 0$. Let $\sigma : \mathrm{Mat}_{d+1}(\mathbb{K}) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the isomorphism of $\mathbb{K}$-algebras from the proof of (i) $\Rightarrow$ (ii) above. We routinely find both $A_{ii}^\sigma = \theta_i$, $A_{ii}^{*\sigma} = \theta_i^*$ for $0 \leqslant i \leqslant d$ and both $A_{i,i-1}^\sigma = 1$, $A_{i-1,i}^{*\sigma} = \varphi_i$ for $1 \leqslant i \leqslant d$. Apparently A^σ, $A^{*\sigma}$ satisfies Theorem 16.1(ii). Applying that theorem to this pair we find A^σ, $A^{*\sigma}$ is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in LB–UB canonical form. In particular A^σ, $A^{*\sigma}$ is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. By this and since σ is an isomorphism we find A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Suppose (i), (ii) hold above. Let p denote a parameter array which satisfies (ii) above. We show p is the designated parameter array for A, A^*. We first show p is a parameter array for A, A^*. Observe p is a parameter array for A^σ, $A^{*\sigma}$ by Theorem 16.1 and the proof of (ii) $\Rightarrow$ (i) above. Also A, A^* is isomorphic to A^σ, $A^{*\sigma}$ so p is a parameter array for A, A^*. Observe p is the designated parameter array for A, A^* by Definition 15.3. $\quad\square$

18. Examples of Leonard pairs A, A^* with A lower bidiagonal and A^* upper bidiagonal

Example 18.1. Let d denote a nonnegative integer. Let A and A^* denote the following matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

$$A = \begin{pmatrix} d & & & & \mathbf{0} \\ -1 & d-2 & & & \\ & -2 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & -d & -d \end{pmatrix}, \qquad A^* = \begin{pmatrix} d & 2d & & & & \mathbf{0} \\ & d-2 & 2d-2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 2 \\ \mathbf{0} & & & & & -d \end{pmatrix}.$$

To avoid degenerate situations, we assume the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d. Then the pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The corresponding designated parameter array from Definition 15.3 is the parameter array given in Example 10.2.

Proof. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array from Example 10.2. We routinely find this parameter array satisfies Theorem 17.1(ii); applying that theorem we find A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is the designated parameter array of A, A^* by the last line of Theorem 17.1. $\quad\square$

Example 18.2. Let d, q, s, s^*, r_1, r_2 be as in Example 10.3. Let A and A^* denote the following matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The matrix A is lower bidiagonal with entries

$$A_{ii} = q^{-i} + sq^{i+1} \quad (0 \leqslant i \leqslant d),$$

$$A_{i,i-1} = \left(1 - q^{-i}\right)\left(1 - r_1 q^i\right) \quad (1 \leqslant i \leqslant d).$$

The matrix A^* is upper bidiagonal with entries

$$A_{ii}^* = q^{-i} + s^* q^{i+1} \quad (0 \leqslant i \leqslant d),$$
$$A_{i-1,i}^* = \left(q^{-d} - q^{1-i}\right)\left(1 - r_2 q^i\right) \quad (1 \leqslant i \leqslant d).$$

Then the pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The corresponding designated parameter array from Definition 15.3 is the parameter array given in Example 10.3.

Proof. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array from Example 10.3. We routinely find this array satisfies Theorem 17.1(ii); applying that theorem we find A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is the designated parameter array for A, A^* by the last line of Theorem 17.1. □

19. The TD–D canonical form; preliminaries

We now turn our attention to the TD–D canonical form. We begin with some comments.

Lemma 19.1 [29, Lemma 5.1]. *Let Φ denote the Leonard system from* (3) *and let V denote an irreducible $\mathcal{A}$-module. Let η_0 denote a nonzero vector in $E_0 V$. Then the sequence*

$$E_0^* \eta_0, E_1^* \eta_0, \ldots, E_d^* \eta_0 \tag{22}$$

is a basis for V.

Definition 19.2. Let Φ denote the Leonard system from (3) and let V denote an irreducible $\mathcal{A}$-module. By a *Φ-TD–D basis* for V we mean a sequence of the form (22), where η_0 denotes a nonzero vector in $E_0 V$.

The concept of a Φ-TD–D basis will play an important role in what follows. Therefore we examine it carefully. In each of the next two lemmas we give a characterization of this type of basis.

Lemma 19.3. *Let Φ denote the Leonard system from* (3) *and let V denote an irreducible $\mathcal{A}$-module. Let $v_0, v_1, \ldots, v_d$ denote a sequence of vectors in V, not all 0. Then this sequence is a Φ-TD–D basis for V if and only if both* (i), (ii) *hold below.*

 (i) $v_i \in E_i^* V$ for $0 \leqslant i \leqslant d$.
(ii) $\sum_{i=0}^{d} v_i \in E_0 V$.

Proof. To prove the lemma in one direction, assume $v_0, v_1, \ldots, v_d$ is a Φ-TD–D basis for V. By Definition 19.2 there exists a nonzero $\eta_0 \in E_0 V$ such that $v_i = E_i^* \eta_0$ for $0 \leqslant i \leqslant d$. Apparently $v_i \in E_i^* V$ for $0 \leqslant i \leqslant d$ so (i) holds. Let I denote the identity element of $\mathcal{A}$ and observe $I = \sum_{i=0}^{d} E_i^*$. Applying this to η_0 we find $\eta_0 = \sum_{i=0}^{d} v_i$ and (ii) follows. We have now proved the lemma in one direction. To prove the lemma in the other direction, assume $v_0, v_1, \ldots, v_d$ satisfy (i), (ii) above. We define $\eta_0 = \sum_{i=0}^{d} v_i$ and observe $\eta_0 \in E_0 V$. Using (i) we find $E_i^* v_j = \delta_{ij} v_j$ for $0 \leqslant i, j \leqslant d$; it follows $v_i = E_i^* \eta_0$ for $0 \leqslant i \leqslant d$. Observe $\eta_0 \neq 0$ since at least one of $v_0, v_1, \ldots, v_d$ is nonzero. Now $v_0, v_1, \ldots, v_d$ is a Φ-TD–D basis for V by Definition 19.2. $\quad\square$

We recall some notation. Let d denote a nonnegative integer and let B denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let α denote a scalar in $\mathbb{K}$. Then B is said to have *constant row sum* α whenever $B_{i0} + B_{i1} + \cdots + B_{id} = \alpha$ for $0 \leqslant i \leqslant d$.

Lemma 19.4. *Let Φ denote the Leonard system from* (3). *Let $\theta_0, \theta_1, \ldots, \theta_d$* (*respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$*) *denote the eigenvalue sequence* (*respectively dual eigenvalue sequence*) *of Φ. Let V denote an irreducible $\mathcal{A}$-module and let $v_0, v_1, \ldots, v_d$ denote a basis for V. Let B* (*respectively B^**) *denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which represents A* (*respectively A^**) *with respect this basis. Then $v_0, v_1, \ldots, v_d$ is a Φ-TD–D basis for V if and only if* (i), (ii) *hold below.*

 (i) *B has constant row sum θ_0.*
 (ii) *$B^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$.*

Proof. Observe

$$A \sum_{j=0}^{d} v_j = \sum_{i=0}^{d} v_i (B_{i0} + B_{i1} + \cdots + B_{id}).$$

Recall $E_0 V$ is the eigenspace for A and eigenvalue θ_0. Apparently B has constant row sum θ_0 if and only if $\sum_{i=0}^{d} v_i \in E_0 V$. Recall that for $0 \leqslant i \leqslant d$, $E_i^* V$ is the eigenspace for A^* and eigenvalue θ_i^*. Apparently $B^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$ if and only if $v_i \in E_i^* V$ for $0 \leqslant i \leqslant d$. The result follows in view of Lemma 19.3. $\quad\square$

20. The TD–D canonical map

Let Φ denote the Leonard system from (3). In this section we use Φ to define a certain isomorphism $\flat : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$. We call $\flat$ the *TD–D canonical map* for Φ. We describe the entries of $A^\flat$ and $A^{*\flat}$.

Definition 20.1. Let Φ denote the Leonard system from (3). We define a map $\flat : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ as follows. Let V denote an irreducible $\mathcal{A}$-module. For all $X \in \mathcal{A}$ we let $X^\flat$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which represents X with respect to a Φ-TD–D basis for V.

We observe $\flat: \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras. We call $\flat$ the *TD–D canonical map* for Φ.

Referring to Definition 20.1, we now describe $A^\flat$ and $A^{*\flat}$. We begin with a comment.

Lemma 20.2. *Let Φ denote the Leonard system from* (3). *Let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (respectively dual eigenvalue sequence) of Φ. Let $\flat$ denote the TD–D canonical map for Φ, from Definition* 20.1. *Then* (i), (ii) *hold below.*

(i) $A^\flat$ *has constant row sum θ_0.*
(ii) $A^{*\flat} = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$.

Proof. Combine Lemma 19.4 and Definition 20.1. $\quad\square$

Referring to Definition 20.1, we now describe $A^\flat$ and $A^{*\flat}$ from another point of view. We use the following notation.

Definition 20.3. Consider the set of all parameter arrays over $\mathbb{K}$. We define two functions on this set. We call these functions T and D. Let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array over $\mathbb{K}$. The image p^T is the tridiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has the following entries. The diagonal entries are

$$p_{ii}^T = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}$$

for $0 \leqslant i \leqslant d$, where we recall $\varphi_0 = 0$, $\varphi_{d+1} = 0$ and where θ_{-1}^*, θ_{d+1}^* denote indeterminates. The superdiagonal and subdiagonal entries are

$$p_{i-1,i}^T = \varphi_i \frac{\prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*)}{\prod_{h=0}^{i-1}(\theta_i^* - \theta_h^*)}, \qquad p_{i,i-1}^T = \phi_i \frac{\prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*)}{\prod_{h=i}^{d}(\theta_{i-1}^* - \theta_h^*)}$$

for $1 \leqslant i \leqslant d$. The image p^D is the following matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$:

$$p^D = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*).$$

Theorem 20.4. *Let Φ denote the Leonard system from* (3). *Let $\flat$ denote the TD–D canonical map for Φ, from Definition* 20.1. *Then $A^\flat = p^T$ and $A^{*\flat} = p^D$, where p denotes the parameter array for Φ.*

Proof. Observe $A^{*\flat} = p^D$ by Lemma 20.2(ii). We have $A^\flat = p^T$ by [29, Theorem 11.2]. $\quad\square$

We finish this section with an observation.

Corollary 20.5. *Let $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array over $\mathbb{K}$. Then the matrix p^T has constant row sum θ_0.*

Proof. By the remark after Definition 11.1 there exists a Leonard system Φ over $\mathbb{K}$ which has parameter array p. For notational convenience let us assume Φ is the Leonard system (3). Let $\flat$ denote the TD–D canonical map for Φ, from Definition 20.1. Then $A^\flat$ has constant row sum θ_0 by Lemma 20.2 and $A^\flat = p^T$ by Theorem 20.4 so p^T has constant row sum θ_0. □

21. The TD–D canonical form for Leonard systems

In this section we introduce the TD–D canonical form for Leonard systems. We define what it means for a given Leonard system to be in TD–D canonical form. We describe the Leonard systems which are in TD–D canonical form. We show every Leonard system is isomorphic to a unique Leonard system which is in TD–D canonical form.

Definition 21.1. Let Φ denote the Leonard system from (3). Let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote the eigenvalue sequence (respectively dual eigenvalue sequence) of Φ. We say Φ is in *TD–D canonical form* whenever (i)–(iii) hold below.

  (i) $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$.
 (ii) A is tridiagonal and A^* is diagonal.
(iii) A has constant row sum θ_0 and $A_{00}^* = \theta_0^*$.

Lemma 21.2. *Let Φ denote the Leonard system from (3). Assume Φ is in TD–D canonical form, so that $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$ by Definition 21.1(i). For $0 \leqslant i \leqslant d$ let v_i denote the vector in $\mathbb{K}^{d+1}$ which has ith coordinate 1 and all other coordinates 0. Then the sequence $v_0, v_1, \ldots, v_d$ is a Φ-TD–D basis for $\mathbb{K}^{d+1}$. Let $\flat$ denote the TD–D canonical map for Φ, from Definition 20.1. Then $\flat$ is the identity map.*

Proof. Observe $v_0, v_1, \ldots, v_d$ is a basis for $\mathbb{K}^{d+1}$, and that with respect to this basis each element of $\mathrm{Mat}_{d+1}(\mathbb{K})$ represents itself. Let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the dual eigenvalue sequence for Φ. By Corollary 7.7(ii) and since $A_{00}^* = \theta_0^*$ we find $A^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$. Applying Lemma 19.4 (with $V = \mathbb{K}^{d+1}$), we find $v_0, v_1, \ldots, v_d$ is a Φ-TD–D basis for $\mathbb{K}^{d+1}$. We mentioned each element in $\mathrm{Mat}_{d+1}(\mathbb{K})$ represents itself with respect to $v_0, v_1, \ldots, v_d$, so $\flat$ is the identity map in view of Definition 20.1. □

Theorem 21.3. *Let Φ denote the Leonard system from (3), and assume Φ is in TD–D canonical form. Then $A = p^T$ and $A^* = p^D$, where T, D are from Definition 20.3 and p is the parameter array for Φ.*

Proof. Let $\flat$ denote the TD–D canonical map for Φ, from Definition 20.1. We assume Φ is in TD–D canonical form, so $\flat$ is the identity map by Lemma 21.2. Applying Theorem 20.4 we find $A = p^T$ and $A^* = p^D$. □

Corollary 21.4. *Let Φ and Φ' denote Leonard systems over $\mathbb{K}$ which are in TD–D canonical form. Then the following are equivalent*:

(i) *Φ and Φ' are isomorphic*;
(ii) *$\Phi = \Phi'$.*

Proof. (i) $\Rightarrow$ (ii). The Leonard systems Φ, Φ' have a common parameter array which we denote by p. By Theorem 21.3 the Leonard pair associated with each of Φ, Φ' is equal to p^T, p^D. Apparently Φ and Φ' are in the same associate class. By this and since Φ, Φ' are isomorphic we find $\Phi = \Phi'$ in view of Lemma 5.4.

(ii) $\Rightarrow$ (i). Clear. $\quad\square$

Definition 21.5. Let Φ denote the Leonard system from (3). By a *TD–D canonical form for Φ* we mean a Leonard system over $\mathbb{K}$ which is isomorphic to Φ and which is in TD–D canonical form.

Theorem 21.6. *Let Φ denote the Leonard system from* (3). *Then there exists a unique TD–D canonical form for Φ. This is $\Phi^\flat$, where $\flat$ denotes the TD–D canonical map for Φ from Definition* 20.1.

Proof. We first show $\Phi^\flat$ is a TD–D canonical form for Φ. Since Φ is a Leonard system in $\mathcal{A}$ and since $\flat: \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras, we find $\Phi^\flat$ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is isomorphic to Φ. We show $\Phi^\flat$ is in TD–D canonical form. To do this we show $\Phi^\flat$ satisfies conditions (i)–(iii) of Definition 21.1. Observe $\Phi^\flat$ satisfies Definition 21.1(i) since $\mathrm{Mat}_{d+1}(\mathbb{K})$ is the ambient algebra of $\Phi^\flat$. Observe $\Phi^\flat$ satisfies Definition 21.1(ii), (iii) by Lemma 20.2 and Theorem 20.4. We have now shown $\Phi^\flat$ satisfies Definition 21.1(i)–(iii) so $\Phi^\flat$ is in TD–D canonical form. Apparently $\Phi^\flat$ is a Leonard system over $\mathbb{K}$ which is isomorphic to Φ and which is in TD–D canonical form. Therefore $\Phi^\flat$ is a TD–D canonical form for Φ by Definition 21.5. To finish the proof we let Φ' denote a TD–D canonical form for Φ and show $\Phi' = \Phi^\flat$. Observe Φ', $\Phi^\flat$ are isomorphic since they are both isomorphic to Φ. The Leonard systems Φ', $\Phi^\flat$ are isomorphic and in TD–D canonical form so $\Phi' = \Phi^\flat$ by Corollary 21.4. $\quad\square$

Corollary 21.7. *Consider the set of Leonard systems over $\mathbb{K}$ which are in TD–D canonical form. We give a bijection from this set to the set of parameter arrays over $\mathbb{K}$. The bijection sends each Leonard system to its own parameter array.*

Proof. By the remark following Definition 11.1, the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over $\mathbb{K}$ to the set of parameter arrays over $\mathbb{K}$. By Theorem 21.6 each of these isomorphism classes contains a unique element which is in TD–D canonical form. The result follows. $\quad\square$

22. The TD–D canonical form for Leonard pairs

In this section we define and discuss the TD–D canonical form for Leonard pairs. We begin with a comment.

Lemma 22.1. *Let A, A^* denote the Leonard pair from Definition* 4.1. *Then there exists at most one Leonard system which is associated with A, A^* and which is in TD–D canonical form.*

Proof. Let Φ and Φ' denote Leonard systems which are associated with A, A^* and which are in TD–D canonical form. We show $\Phi = \Phi'$. Let $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta_0', \theta_1', \ldots, \theta_d'$) denote the eigenvalue sequence for Φ (respectively Φ'). Let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ (respectively $\theta_0^{*\prime}, \theta_1^{*\prime}, \ldots, \theta_d^{*\prime}$) denote the dual eigenvalue sequence for Φ (respectively Φ'). Observe Φ, Φ' are in the same associate class so Φ' is one of Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$. Therefore $\theta_i' = \theta_i$ for $0 \leqslant i \leqslant d$ or $\theta_i' = \theta_{d-i}$ for $0 \leqslant i \leqslant d$. Also $\theta_i^{*\prime} = \theta_i^*$ for $0 \leqslant i \leqslant d$ or $\theta_i^{*\prime} = \theta_{d-i}^*$ for $0 \leqslant i \leqslant d$. To show $\Phi = \Phi'$ it suffices to show $\theta_i = \theta_i'$ and $\theta_i^* = \theta_i^{*\prime}$ for $0 \leqslant i \leqslant d$. Each of θ_0, θ_0' is equal to the common row sums of A so $\theta_0 = \theta_0'$. Apparently $\theta_i = \theta_i'$ for $0 \leqslant i \leqslant d$. Each of $\theta_0^*, \theta_0^{*\prime}$ is equal to A_{00}^* so $\theta_0^* = \theta_0^{*\prime}$. Apparently $\theta_i^* = \theta_i^{*\prime}$ for $0 \leqslant i \leqslant d$. We conclude $\Phi = \Phi'$. $\square$

Referring to the above lemma, we now consider those Leonard pairs for which there exists an associated Leonard system which is in TD–D canonical form. In order to describe these we introduce the TD–D canonical form for Leonard pairs.

Definition 22.2. Let A, A^* denote the Leonard pair from Definition 4.1 and let $\theta_0, \theta_1, \ldots, \theta_d$ denote an eigenvalue sequence for this pair. We say A, A^* is in *TD–D canonical form* whenever (i)–(iii) hold below.

(i) $\mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$.
(ii) A is tridiagonal and A^* is diagonal.
(iii) A has constant row sum and this sum is θ_0 or θ_d.

We just defined the TD–D canonical form for Leonard pairs, and in Definition 21.1 we defined this form for Leonard systems. We now compare these two versions. We will use the following definition.

Definition 22.3. Let A, A^* denote the Leonard pair from Definition 4.1, and assume this pair is in TD–D canonical form. We make several comments and definitions.

(i) By Definitions 22.2(iii) and 7.2, there exists a unique eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ for A, A^* such that A has constant row sum θ_0. We call this the *designated eigenvalue sequence* for A, A^*.
(ii) By Corollary 7.7(ii) the sequence $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$ is a dual eigenvalue sequence for A, A^*. We call this the *designated dual eigenvalue sequence* for A, A^*.

(iii) By the *designated Leonard system* for A, A^* we mean the Leonard system which is associated with A, A^* and which has eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$ and dual eigenvalue sequence $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$.

(iv) By the *designated parameter array* for A, A^* we mean the parameter array of the designated Leonard system for A, A^*.

Lemma 22.4. *Let A, A^* denote the Leonard pair from Definition* 4.1. *Then the following are equivalent*:

 (i) *A, A^* is in TD–D canonical form*;

(ii) *there exists a Leonard system Φ which is associated with A, A^* and which is in TD–D canonical form.*

Suppose (i), (ii) *hold. Then Φ is the designated Leonard system of A, A^*.*

Proof. (i) $\Rightarrow$ (ii). Let Φ denote the designated Leonard system for A, A^*, from Definition 22.3(iii). From the construction Φ is associated with A, A^* and in TD–D canonical form.

(ii) $\Rightarrow$ (i). Compare Definitions 21.1 and 22.2.

Now suppose (i), (ii) hold. Then Φ is the designated Leonard system for A, A^* by Lemma 22.1 and the proof of (i) $\Rightarrow$ (ii) above. $\quad\square$

Corollary 22.5. *We give a bijection from the set of Leonard systems over $\mathbb{K}$ which are in TD–D canonical form, to the set of Leonard pairs over $\mathbb{K}$ which are in TD–D canonical form. The bijection sends each Leonard system to its associated Leonard pair. The inverse bijection sends each Leonard pair to its designated Leonard system.*

Proof. This is a reformulation of Lemma 22.4. $\quad\square$

Theorem 22.6. *We give a bijection from the set of parameter arrays over $\mathbb{K}$ to the set of Leonard pairs over $\mathbb{K}$ which are in TD–D canonical form. The bijection sends each parameter array p to the Leonard pair p^T, p^D. The inverse bijection sends each Leonard pair to its designated parameter array.*

Proof. Composing the inverse of the bijection from Corollary 21.7, with the bijection from Corollary 22.5, we obtain a bijection from the set of parameter arrays over $\mathbb{K}$ to the set of Leonard pairs over $\mathbb{K}$ which are in TD–D canonical form. Let p denote a parameter array over $\mathbb{K}$ and let A, A^* denote the image of p under this bijection. We show $A = p^T$ and $A^* = p^D$. By Corollary 21.7 there exists a unique Leonard system over $\mathbb{K}$ which is in TD–D canonical form and which has parameter array p. Let us denote this system by Φ. By the construction A, A^* is associated with Φ. Applying Theorem 21.3 to Φ we find $A = p^T$ and $A^* = p^D$. To finish the proof we show p is the designated parameter array for A, A^*. We mentioned A, A^* is associated with Φ and Φ is in TD–D canonical form so Φ is the designated Leonard system for A, A^* by Corollary 22.5. We mentioned p is the parameter array for Φ so p is the designated parameter array for A, A^* by Definition 22.3(iv). $\quad\square$

Definition 22.7. Let A, A^* denote the Leonard pair from Definition 4.1. By a *TD–D canonical form for A, A** we mean a Leonard pair over $\mathbb{K}$ which is isomorphic to A, A^* and which is in TD–D canonical form.

Theorem 22.8. *Let A, A^* denote the Leonard pair from Definition 4.1. We give a bijection from the set of parameter arrays for A, A^* to the set of TD–D canonical forms for A, A^*. This bijection sends each parameter array p to the pair p^T, p^D.* (*The parameter arrays for A, A^* are given in Lemma 12.2.*) *The inverse bijection sends each TD–D canonical form for A, A^* to its designated parameter array.*

Proof. Let B, B^* denote a Leonard pair over $\mathbb{K}$ which is in TD–D canonical form. Let p denote the designated parameter array for B, B^*. In view of Theorem 22.6 it suffices to show the following are equivalent: (i) A, A^* and B, B^* are isomorphic; (ii) p is a parameter array for A, A^*. These statements are equivalent by Lemma 12.4. $\quad\square$

Corollary 22.9. *Let A, A^* denote the Leonard pair from Definition 4.1. If $d \geqslant 1$ then there exist exactly four TD–D canonical forms for A, A^*. If $d = 0$ then there exists a unique TD–D canonical form for A, A^*.*

Proof. Immediate from Theorem 22.8 and Corollary 12.3. $\quad\square$

23. How to recognize a Leonard pair in TD–D canonical form

Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let us assume A is tridiagonal and A^* is diagonal. We give a necessary and sufficient condition for A, A^* to be a Leonard pair which is in TD–D canonical form. We present two versions of our result.

Theorem 23.1. *Let d denote a nonnegative integer and let A, A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Assume A is tridiagonal and A^* is diagonal. Then the following* (i), (ii) *are equivalent.*

(i) *The pair A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form.*

(ii) *There exists a parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ over $\mathbb{K}$ such that*

$$A_{ii} = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (0 \leqslant i \leqslant d),$$

$$A_{i-1,i} = \varphi_i \frac{\prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*)}{\prod_{h=0}^{i-1}(\theta_i^* - \theta_h^*)} \quad (1 \leqslant i \leqslant d),$$

$$A_{i,i-1} = \phi_i \frac{\prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*)}{\prod_{h=i}^{d}(\theta_{i-1}^* - \theta_h^*)} \quad (1 \leqslant i \leqslant d),$$

$$A_{ii}^* = \theta_i^* \quad (0 \leqslant i \leqslant d).$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *above is uniquely determined by A, A*. This parameter array is the designated parameter array for A, A* in the sense of Definition 22.3.*

Proof. This is a reformulation of Theorem 22.6. □

Theorem 23.2. *Let d denote a nonnegative integer and let A, A* denote matrices in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume A is tridiagonal and A* is diagonal. Then the following* (i), (ii) *are equivalent.*

(i) *The pair A, A* is a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$ *which is in TD–D canonical form.*
(ii) *There exists a parameter array* $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ *over* $\mathbb{K}$ *such that A has constant row sum* θ_0 *and*

$$A_{i-1,i} = \varphi_i \frac{\prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*)}{\prod_{h=0}^{i-1}(\theta_i^* - \theta_h^*)} \quad (1 \leqslant i \leqslant d),$$

$$A_{i,i-1} = \phi_i \frac{\prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*)}{\prod_{h=i}^{d}(\theta_{i-1}^* - \theta_h^*)} \quad (1 \leqslant i \leqslant d),$$

$$A_{ii}^* = \theta_i^* \quad (0 \leqslant i \leqslant d).$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *above is uniquely determined by A, A*. This parameter array is the designated parameter array for A, A* in the sense of Definition 22.3.*

Proof. Combine Theorem 23.1 and Corollary 20.5. □

24. Examples of Leonard pairs in TD–D canonical form

In this section we give a few examples of Leonard pairs which are in TD–D canonical form.

Example 24.1. Let d denote a nonnegative integer. Let A and A^* denote the following matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

$$A = \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad A^* = \mathrm{diag}(d, d-2, d-4, \ldots, -d).$$

To avoid degenerate situations, we assume the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d. Then the pair A, A^* is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form. The corresponding designated parameter array from Definition 22.3 is the parameter array given in Example 10.2.

Proof. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array from Example 10.2. We routinely verify this parameter array satisfies Theorem 23.2(ii); applying that theorem we find A, A^* is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form. The parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is the designated parameter array for A, A^* by the last line of Theorem 23.2. $\square$

Example 24.2. Let d, q, s, s^*, r_1, r_2 be as in Example 10.3. Let A and A^* denote the following matrices in $\text{Mat}_{d+1}(\mathbb{K})$. The matrix A is tridiagonal with entries

$$A_{01} = \frac{(1 - q^{-d})(1 - r_1 q)(1 - r_2 q)}{1 - s^* q^2},$$

$$A_{i-1,i} = \frac{(1 - q^{i-d-1})(1 - s^* q^i)(1 - r_1 q^i)(1 - r_2 q^i)}{(1 - s^* q^{2i-1})(1 - s^* q^{2i})} \quad (2 \leqslant i \leqslant d),$$

$$A_{i,i-1} = \frac{(1 - q^i)(1 - s^* q^{i+d+1})(r_1 - s^* q^i)(r_2 - s^* q^i)}{s^* q^d (1 - s^* q^{2i})(1 - s^* q^{2i+1})} \quad (1 \leqslant i \leqslant d-1),$$

$$A_{d,d-1} = \frac{(1 - q^d)(r_1 - s^* q^d)(r_2 - s^* q^d)}{s^* q^d (1 - s^* q^{2d})}$$

and constant row sum $1 + sq$. The matrix A^* is diagonal with entries

$$A_{ii}^* = q^{-i} + s^* q^{i+1} \quad (0 \leqslant i \leqslant d).$$

Then the pair A, A^* is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form. The corresponding designated parameter array from Definition 22.3 is the parameter array given in Example 10.3.

Proof. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array from Example 10.3. We routinely verify this parameter array satisfies Theorem 23.2(ii); applying that theorem we find A, A^* is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{K})$ which is in TD–D canonical form. The parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is the designated parameter array for A, A^* by the last line of Theorem 23.2. $\square$

25. Leonard pairs A, A^* with A tridiagonal and A^* diagonal

Let d denote a nonnegative integer and let A, A^* denote matrices in $\text{Mat}_{d+1}(\mathbb{K})$. Let us assume A is tridiagonal and A^* is diagonal. We give a necessary and sufficient condition for A, A^* to be a Leonard pair.

Theorem 25.1. *Let d denote a nonnegative integer and let A, A^* denote matrices in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Assume A is tridiagonal and A^* is diagonal. Then the following* (i), (ii) *are equivalent.*

(i) *The pair A, A^* is a Leonard pair in* $\mathrm{Mat}_{d+1}(\mathbb{K})$.
(ii) *There exists a parameter array* $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ *over $\mathbb{K}$ such that*

$$A_{ii} = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (0 \leqslant i \leqslant d), \tag{23}$$

$$A_{i,i-1} A_{i-1,i} = \varphi_i \phi_i \frac{\prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*) \prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*)}{\prod_{h=0}^{i-1}(\theta_i^* - \theta_h^*) \prod_{h=i}^{d}(\theta_{i-1}^* - \theta_h^*)} \quad (1 \leqslant i \leqslant d), \tag{24}$$

$$A_{ii}^* = \theta_i^* \quad (0 \leqslant i \leqslant d). \tag{25}$$

Suppose (i), (ii) *hold and let R denote the set of parameter arrays which satisfy* (ii) *above. Then R consists of the parameter arrays* $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ *for A, A^* which satisfy $\theta_i^* = A_{ii}^*$ for $0 \leqslant i \leqslant d$. If $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is in R then so is $(\theta_{d-i}, \theta_i^*, i = 0, \ldots, d; \phi_j, \varphi_j, j = 1, \ldots, d)$ and R contains no further elements.*

Proof. (i) $\Rightarrow$ (ii). We assume A is tridiagonal and A^* is diagonal so $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$ is a dual eigenvalue sequence for A, A^* by Corollary 7.7(ii). For notational convenience we set $\theta_i^* = A_{ii}^*$ for $0 \leqslant i \leqslant d$. By Definition 7.2 there exists a Leonard system Φ which is associated with A, A^* and which has dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote the eigenvalue sequence for Φ. Let $\varphi_1, \varphi_2, \ldots, \varphi_d$ (respectively $\phi_1, \phi_2, \ldots, \phi_d$) denote the first (respectively second) split sequence for Φ. We abbreviate $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ and observe p is the parameter array for Φ. We show p satisfies the conditions of (ii) above. Observe p is over $\mathbb{K}$ since the Leonard pair A, A^* is over $\mathbb{K}$. We show p satisfies (23)–(25). Let $\flat$ denote the TD–D canonical map for Φ. We recall $A^\flat = p^T$ and $A^{*\flat} = p^D$ by Theorem 20.4. Since $\mathrm{Mat}_{d+1}(\mathbb{K})$ is the ambient algebra of Φ the domain of $\flat$ is equal to $\mathrm{Mat}_{d+1}(\mathbb{K})$. Since the range of $\flat$ is equal to $\mathrm{Mat}_{d+1}(\mathbb{K})$ as well, there exists an invertible matrix $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $X^\flat = SXS^{-1}$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Observe $A^{*\flat} = A^*$ so $SA^* = A^*S$. The matrix A^* is diagonal with diagonal entries mutually distinct so S is diagonal. From this and since $A^\flat = SAS^{-1}$ we find $A_{ii}^\flat = A_{ii}$ for $0 \leqslant i \leqslant d$ and $A_{i,i-1}^\flat A_{i-1,i}^\flat = A_{i,i-1} A_{i-1,i}$ for $1 \leqslant i \leqslant d$. By these comments the parameter array p satisfies (23) and (24). From the construction p satisfies (25).

(ii) $\Rightarrow$ (i). Let $p := (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array over $\mathbb{K}$ which satisfies (23)–(25). Let Φ denote a Leonard system over $\mathbb{K}$ which has parameter array p. Recall Φ is only determined up to isomorphism; replacing Φ with an isomorphic Leonard system if necessary we may assume Φ is in TD–D canonical form by Theorem 21.6. Let B, B^* denote the Leonard pair associated with Φ. Then $B = p^T$ and $B^* = p^D$ by Theorem 21.3. Apparently $B^* = A^*$; moreover $B_{ii} = A_{ii}$ for $0 \leqslant i \leqslant d$ and

$B_{i,i-1}B_{i-1,i} = A_{i,i-1}A_{i-1,i}$ for $1 \leqslant i \leqslant d$. Let S denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has diagonal entries $S_{ii} = \prod_{h=1}^{i} A_{i,i-1}/B_{i,i-1}$ for $0 \leqslant i \leqslant d$. We observe $S_{ii} \neq 0$ for $0 \leqslant i \leqslant d$ so S^{-1} exists. Let $\sigma : \mathrm{Mat}_{d+1}(\mathbb{K}) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the isomorphism of $\mathbb{K}$-algebras which satisfies $X^{\sigma} = SXS^{-1}$ for all $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. From our above comments we find $B^{\sigma} = A$ and $B^{*\sigma} = A^*$. By this and since B, B^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$ we find A, A^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Suppose (i), (ii) hold. Let R' denote the set of parameter arrays for A, A^* which have dual eigenvalue sequence $A_{00}^*, A_{11}^*, \ldots, A_{dd}^*$. From Lemma 12.2 we find that if $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is in R' then so is $(\theta_{d-i}, \theta_i^*, i = 0, \ldots, d; \phi_j, \varphi_j, j = 1, \ldots, d)$ and R' contains no further elements. We now show $R = R'$. From the proof of (i) $\Rightarrow$ (ii) above we find $R' \subseteq R$. We show $R \subseteq R'$. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array in R. By the proof of (ii) $\Rightarrow$ (i) above we find this array is for A, A^* in the sense of Definition 12.1. By (25) we find $\theta_i^* = A_{ii}^*$ for $0 \leqslant i \leqslant d$. Apparently $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ is contained in R' and it follows $R \subseteq R'$. We have now shown $R = R'$ and the proof is complete. $\quad\square$

26. How to compute the parameter arrays which satisfy Theorem 25.1(ii)

Let d denote a positive integer and let A, A^* denote a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let us assume A is tridiagonal and A^* is diagonal. Suppose we wish to verify that A, A^* is a Leonard pair. In order to do this it suffices to display a parameter array $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ which satisfies Theorem 25.1(ii). We give a method for obtaining this array from the entries of A and A^*. Our method is summarized as follows. From (25) we find $\theta_i^* = A_{ii}^*$ for $0 \leqslant i \leqslant d$. To obtain the rest of the array we proceed in two steps: (i) we obtain θ_0, θ_d as the roots of a certain quadratic polynomial whose coefficients are rational expressions involving $A_{00}, A_{11}, A_{dd}, A_{10}A_{01}$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$; (ii) we obtain θ_i $(1 \leqslant i \leqslant d-1)$ and φ_i, ϕ_i $(1 \leqslant i \leqslant d)$ as rational expressions involving $\theta_0, \theta_d, A_{00}, A_{dd}$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. For convenience we discuss step (ii) before step (i). To prepare for step (ii) we give a lemma.

Lemma 26.1. *Let d denote a positive integer and let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array over $\mathbb{K}$. For notational convenience we define*

$$\vartheta_i := \sum_{h=0}^{i-1} \frac{\theta_h^* - \theta_{d-h}^*}{\theta_0^* - \theta_d^*} \quad (0 \leqslant i \leqslant d). \tag{26}$$

Then (i)–(iii) *hold below.*

(i) $\theta_i = \theta_0 + \frac{\varphi_i - \phi_d \vartheta_i}{\theta_{i-1}^* - \theta_d^*}$ $(1 \leqslant i \leqslant d)$.

(ii) $\theta_i = \theta_d + \frac{\varphi_{i+1} - \phi_1 \vartheta_{i+1}}{\theta_{i+1}^* - \theta_0^*}$ $(0 \leqslant i \leqslant d-1)$.

(iii) $\frac{\varphi_{i+1} - \phi_1 \vartheta_{i+1}}{\theta_{i+1}^* - \theta_0^*} = \frac{\varphi_i - \phi_d \vartheta_i}{\theta_{i-1}^* - \theta_d^*} + \theta_0 - \theta_d$ $(1 \leqslant i \leqslant d-1)$.

Proof. (i) Let the integer i be given. Evaluating Corollary 11.4(ii) using Corollary 11.4(i) we find $\varphi_i = \phi_d \vartheta_i + (\theta_i - \theta_0)(\theta_{i-1}^* - \theta_d^*)$. Solving this equation for θ_i we get the result.

(ii) Similar to the proof of (i) above, except use Theorem 9.1(iii) instead of Corollary 11.4(ii).

(iii) Combine (i), (ii) above. $\square$

Theorem 26.2. *Let d denote a positive integer and let A, A^* denote a Leonard pair in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Assume A is tridiagonal and A^* is diagonal. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote a parameter array which satisfies Theorem 25.1(ii). Then $\theta_i \ (1 \leqslant i \leqslant d-1)$ and $\varphi_i, \phi_i \ (1 \leqslant i \leqslant d)$ are obtained from $\theta_0, \theta_d, A_{00}, A_{dd}$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ as follows.*

(i) *To obtain $\varphi_1, \varphi_d, \phi_1, \phi_d$ use*

$$\varphi_1 = (A_{00} - \theta_0)(\theta_0^* - \theta_1^*), \qquad \varphi_d = (A_{dd} - \theta_d)(\theta_d^* - \theta_{d-1}^*), \tag{27}$$

$$\phi_1 = (A_{00} - \theta_d)(\theta_0^* - \theta_1^*), \qquad \phi_d = (A_{dd} - \theta_0)(\theta_d^* - \theta_{d-1}^*). \tag{28}$$

(ii) *To obtain $\varphi_2, \varphi_3, \ldots, \varphi_{d-1}$ recursively apply Lemma 26.1(iii).*
(iii) *To obtain $\theta_1, \theta_2, \ldots, \theta_{d-1}$ use Lemma 26.1(i) or (ii).*
(iv) *To obtain $\phi_2, \phi_3, \ldots, \phi_{d-1}$ use Theorem 9.1(iv).*

Proof. (i) To obtain the equation on the left (respectively right) in (27) set $i = 0$ (respectively $i = d$) in (23) and rearrange terms. Equation (28) is just (27) with the original parameter array replaced by the parameter array $(\theta_{d-i}, \theta_i^*, i = 0, \ldots, d; \phi_j, \varphi_j, j = 1, \ldots, d)$.

(ii)–(iv). Clear. $\square$

Theorem 26.3. *With reference to Theorem 26.2, the scalars θ_0, θ_d are the roots of the quadratic polynomial*

$$(\lambda - A_{00})(\lambda - \alpha/\varepsilon) - A_{10}A_{01}/\varepsilon, \tag{29}$$

where ε, α are defined as follows. If $d = 1$ then $\varepsilon = 1$ and $\alpha = A_{11}$. If $d \geqslant 2$ then

$$\varepsilon = \frac{(\theta_1^* - \theta_d^*)(\theta_1^* - \theta_{d-1}^*) \cdots (\theta_1^* - \theta_2^*)}{(\theta_0^* - \theta_d^*)(\theta_0^* - \theta_{d-1}^*) \cdots (\theta_0^* - \theta_2^*)} \tag{30}$$

and

$$\alpha = A_{11} \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_2^*} - A_{00} \frac{\theta_1^* - \theta_d^*}{\theta_0^* - \theta_2^*} \frac{\theta_0^* - \theta_1^*}{\theta_0^* - \theta_d^*} + A_{dd} \frac{\theta_{d-1}^* - \theta_d^*}{\theta_0^* - \theta_2^*} \frac{\theta_0^* - \theta_1^*}{\theta_0^* - \theta_d^*}. \tag{31}$$

Proof. First suppose $d = 1$. Then θ_0, θ_d are the roots of the characteristic polynomial of A and this polynomial is $(\lambda - A_{00})(\lambda - A_{11}) - A_{10}A_{01}$. Next suppose $d \geqslant 2$. We claim the scalar ε from (30) satisfies

$$\varepsilon = 1 - \frac{\theta_0^* - \theta_1^*}{\theta_0^* - \theta_2^*} \frac{\theta_0^* + \theta_1^* - \theta_{d-1}^* - \theta_d^*}{\theta_0^* - \theta_d^*}. \tag{32}$$

To obtain (32) we recall by Corollary 20.5 that p^T has constant row sum θ_0, where $p = (\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$. Considering row 1 of p^T we find $p_{10}^T + p_{11}^T + p_{12}^T = \theta_0$. We evaluate the left-hand side of this equation using Definition 20.3. In the resulting equation we eliminate φ_1, φ_2 using Theorem 9.1(iii) and we simplify the result using Corollary 11.4(i). Equation (32) follows and our claim is proved. To show θ_0, θ_d are the roots of (29) we show both

$$\theta_0 + \theta_d = A_{00} + \alpha/\varepsilon, \tag{33}$$

$$\theta_0 \theta_d = A_{00}\alpha/\varepsilon - A_{10}A_{01}/\varepsilon. \tag{34}$$

To verify (33) we consider the expression α given in (31). We simplify this expression by evaluating A_{11} in terms of $\theta_0, \theta_d, A_{00}, A_{dd}$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ using (23) and Theorem 26.2. Simplifying the result further using (32) we find $\alpha = \varepsilon(\theta_0 + \theta_d - A_{00})$ and (33) follows. To verify (34) we evaluate the product $A_{10}A_{01}$ in terms of $\theta_0, \theta_d, A_{00}, A_{dd}$ and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ using (24) and Theorem 26.2. Simplifying the result using (30) we obtain $A_{10}A_{01} = -\varepsilon(A_{00} - \theta_0)(A_{00} - \theta_d)$. Combining this with (33) we routinely obtain (34). $\square$

27. Transition matrices and polynomials

Let Φ denote a Leonard system over $\mathbb{K}$ and let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the corresponding parameter array. Let $\mathcal{A}$ denote the ambient algebra of Φ. Let $\flat : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the TD–D canonical map for Φ, from Definition 20.1. Let $\sharp : \mathcal{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the TD–D canonical map for Φ^*. We describe how $\flat$ and $\sharp$ are related. To do this we cite some facts from [29, Section 16]. For $0 \leqslant i, j \leqslant d$ we define the scalar

$$\mathcal{P}_{ij} = \sum_{n=0}^{d} \frac{(\theta_i - \theta_0)(\theta_i - \theta_1) \cdots (\theta_i - \theta_{n-1})(\theta_j^* - \theta_0^*)(\theta_j^* - \theta_1^*) \cdots (\theta_j^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}. \tag{35}$$

Let P denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has entries

$$P_{ij} = k_j \mathcal{P}_{ij} \quad (0 \leqslant i, j \leqslant d),$$

where $\mathcal{P}_{ij}$ is from (35) and where k_j equals

$$\frac{\varphi_1 \varphi_2 \cdots \varphi_j}{\phi_1 \phi_2 \cdots \phi_j}$$

times

$$\frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_d^*)}{(\theta_j^* - \theta_0^*)\cdots(\theta_j^* - \theta_{j-1}^*)(\theta_j^* - \theta_{j+1}^*)\cdots(\theta_j^* - \theta_d^*)}$$

for $0 \leqslant j \leqslant d$. Then $P_{i0} = 1$ for $0 \leqslant i \leqslant d$ and $X^\sharp P = P X^\flat$ for all $X \in \mathcal{A}$. Let P^* denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which has entries

$$P_{ij}^* = k_j^* \mathcal{P}_{ji} \quad (0 \leqslant i, j \leqslant d),$$

where $\mathcal{P}_{ji}$ is from (35) and k_j^* equals

$$\frac{\varphi_1 \varphi_2 \cdots \varphi_j}{\phi_d \phi_{d-1} \cdots \phi_{d-j+1}}$$

times

$$\frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)}{(\theta_j - \theta_0)\cdots(\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1})\cdots(\theta_j - \theta_d)}$$

for $0 \leqslant j \leqslant d$. Then $P_{i0}^* = 1$ for $0 \leqslant i \leqslant d$ and $X^\flat P^* = P^* X^\sharp$ for all $X \in \mathcal{A}$. Moreover $P P^* = \nu I$ where

$$\nu = \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_d^*)}{\phi_1 \phi_2 \cdots \phi_d}.$$

We comment on (35). For $0 \leqslant i, j \leqslant d$, $\mathcal{P}_{ij}$ is a polynomial of degree j in θ_i and a polynomial of degree i in θ_j^*. The class of polynomials which can be obtained from a parameter array in this fashion coincides with the class of polynomials which are contained in the Askey scheme [17] and which are orthogonal with respect to a measure which has finitely many nonzero values. This class consists of the Krawtchouk, Hahn, dual Hahn, Racah, the q-analogs of these, and some polynomials obtained from the q-Racah by letting $q = -1$. See [35, Appendix A] and [1, p. 260] for more details. To illustrate this we obtain some Krawtchouk and q-Racah polynomials from the parameter arrays given in Examples 10.2 and 10.3, respectively.

Example 27.1 [29, Section 16]. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array in Example 10.2. Referring to the discussion in the first part of this section, for $0 \leqslant i, j \leqslant d$ we have

$$\mathcal{P}_{ij} = \sum_{n=0}^{d} \frac{(-i)_n(-j)_n 2^n}{(-d)_n n!} \tag{36}$$

where

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad n = 0, 1, 2, \ldots.$$

Moreover

$$k_j = \binom{d}{j}, \quad k_j^* = \binom{d}{j} \quad (0 \leqslant j \leqslant d)$$

and $v = 2^d$. We have $P = P^*$ and $P^2 = 2^d I$. For $0 \leqslant i, j \leqslant d$ the expression on the right in (36) is equal to the hypergeometric series

$$_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \,\middle|\, 2 \right). \tag{37}$$

From this we find $\mathcal{P}_{ij}$ is a Krawtchouk polynomial of degree j in θ_i and a Krawtchouk polynomial of degree i in θ_j^*.

Example 27.2 [29, Section 16]. Let $(\theta_i, \theta_i^*, i = 0, \ldots, d; \varphi_j, \phi_j, j = 1, \ldots, d)$ denote the parameter array in Example 10.3. Referring to the discussion in the first part of this section, for $0 \leqslant i, j \leqslant d$ we have

$$\mathcal{P}_{ij} = \sum_{n=0}^{d} \frac{(q^{-i}; q)_n (sq^{i+1}; q)_n (q^{-j}; q)_n (s^*q^{j+1}; q)_n q^n}{(r_1 q; q)_n (r_2 q; q)_n (q^{-d}; q)_n (q; q)_n} \tag{38}$$

where

$$(a; q)_n := (1 - a)(1 - aq)\left(1 - aq^2\right) \cdots \left(1 - aq^{n-1}\right), \quad n = 0, 1, 2 \ldots.$$

Moreover

$$k_j = \frac{(r_1 q; q)_j (r_2 q; q)_j (q^{-d}; q)_j (s^*q; q)_j (1 - s^*q^{2j+1})}{s^j q^j (q; q)_j (s^*q/r_1; q)_j (s^*q/r_2; q)_j (s^*q^{d+2}; q)_j (1 - s^*q)},$$

$$k_j^* = \frac{(r_1 q; q)_j (r_2 q; q)_j (q^{-d}; q)_j (sq; q)_j (1 - sq^{2j+1})}{s^{*j} q^j (q; q)_j (sq/r_1; q)_j (sq/r_2; q)_j (sq^{d+2}; q)_j (1 - sq)}$$

for $0 \leqslant j \leqslant d$ and

$$v = \frac{(sq^2; q)_d (s^*q^2; q)_d}{r_1^d q^d (sq/r_1; q)_d (s^*q/r_1; q)_d}.$$

For $0 \leqslant i, j \leqslant d$ the expression on the right in (38) is equal to the basic hypergeometric series

$$_4\phi_3\left(\begin{matrix} q^{-i}, \ sq^{i+1}, \ q^{-j}, \ s^*q^{j+1} \\ r_1 q, \ r_2 q, \ q^{-d} \end{matrix} \,\middle|\, q, q \right).$$

By this and since $r_1 r_2 = ss^*q^{d+1}$ we find $\mathcal{P}_{ij}$ is a q-Racah polynomial of degree j in θ_i and a q-Racah polynomial of degree i in θ_j^*.

28. Directions for further research

In this section we give some suggestions for further research.

Problem 28.1. Let Φ denote the Leonard system from (3). Let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{K}$ such that $\alpha \neq 0$ and $\alpha^* \neq 0$. Recall the sequence $(\alpha A + \beta I, \alpha^* A^* + \beta^* I; E_i, E_i^*, i = 0, \dots, d)$ is a Leonard system in $\mathcal{A}$. In some cases this system is isomorphic to a relative of Φ; describe all the cases where this occurs.

Problem 28.2. Let d denote a nonnegative integer. Find all Leonard pairs A, A^* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which satisfy the following two conditions:

 (i) A is irreducible tridiagonal;
(ii) A^* is lower bidiagonal with $A_{i,i-1} = 1$ for $1 \leqslant i \leqslant d$.

Problem 28.3. Let d denote a nonnegative integer. Find all Leonard pairs A, A^* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that each of A, A^* is irreducible tridiagonal.

Problem 28.4. Let d denote a nonnegative integer. Find all Leonard pairs A, A^* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which satisfy the following two conditions:

 (i) each of A, A^* is irreducible tridiagonal;
(ii) there exists a diagonal matrix H in $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $A = H A^* H^{-1}$.

Problem 28.5. Let A, A^* denote the Leonard pair from Definition 4.1. Determine when does there exist invertible elements U, U^* in $\mathcal{A}$ which satisfy (i)–(iii) below:

  (i) $UA = AU$;
 (ii) $U^* A^* = A^* U^*$;
(iii) $U A^* U^{-1} = U^{*-1} A U^*$.

This problem arises naturally in the context of a spin model contained in a Bose–Mesner algebra of P- and Q-polynomial type [5].

Problem 28.6. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Leonard triple on V, we mean a three-tuple of linear transformations $A : V \to V$, $A^* : V \to V$, $A^\varepsilon : V \to V$ which satisfy conditions (i)–(iii) below.

  (i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ε are each irreducible tridiagonal.
 (ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A and A^ε are each irreducible tridiagonal.
(iii) There exists a basis for V with respect to which the matrix representing A^ε is diagonal and the matrices representing A and A^* are each irreducible tridiagonal.

Find all the Leonard triples.

Remark 28.7. Referring to Problem 28.5, let A^ε denote the common value of UA^*U^{-1}, $U^{*-1}AU^*$. Then A, A^*, A^ε is a Leonard triple.

Conjecture 28.8. *Let Φ denote the Leonard system from* (3) *and let I denote the identity element of $\mathcal{A}$. Then for all $X \in \mathcal{A}$ the following are equivalent*:

(i) *both*

$$E_i X E_j = 0 \quad \textit{if } |i - j| > 1 \ (0 \leqslant i, j \leqslant d),$$

$$E_i^* X E_j^* = 0 \quad \textit{if } |i - j| > 1 \ (0 \leqslant i, j \leqslant d);$$

(ii) *X is a $\mathbb{K}$-linear combination of I, A, A^*, AA^*, A^*A.*

Conjecture 28.9. *Let Φ denote the Leonard system from* (3). *Then for $0 \leqslant r \leqslant d$ the elements*

$$E_0^*, E_1^*, \ldots, E_r^*, E_r, E_{r+1}, \ldots, E_d$$

together generate $\mathcal{A}$.

Acknowledgments

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