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Composition factors of functors

Ergün Yaraneri

Department of Mathematics, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey

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ABSTRACT

Let $\mathcal{X}$ be a family of finite groups satisfying certain conditions and $\mathbb{K}$ be a field. We study composition factors, radicals, and socles of biset and related functors defined on $\mathcal{X}$ over $\mathbb{K}$. For such a functor M and for a group H in $\mathcal{X}$, we construct bijections between some classes of maximal (respectively, simple) subfunctors of M and some classes of maximal (respectively, simple) $\mathbb{K}\,\mathrm{Out}(H)$-submodules of $M(H)$. We use these bijections to relate the multiplicity of a simple functor $S_{H,V}$ in M to the multiplicity of V in a certain $\mathbb{K}\,\mathrm{Out}(H)$-module related to $M(H)$. We then use these general results to study the structure of one of the important biset and related functors, namely the Burnside functor $\mathrm{B}_{\mathbb{K}}$ which assigns to each group G in $\mathcal{X}$ its Burnside algebra $\mathrm{B}_{\mathbb{K}}(G) = \mathbb{K} \otimes_{\mathbb{Z}} B(G)$ where $B(G)$ is the Burnside ring of G. We find the radical and the socle of $\mathrm{B}_{\mathbb{K}}$ in most cases of $\mathcal{X}$ and $\mathbb{K}$. For example, if $\mathbb{K}$ is of characteristic $p > 0$ and $\mathcal{X}$ is a family of finite abelian p-groups, we find the radical and the socle series of $\mathrm{B}_{\mathbb{K}}$ considered as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. We finally study restrictions of functors to nonfull subcategories. For example, we find some conditions forcing a simple deflation functor to remain simple as a Mackey functor. For an inflation functor M defined on abelian groups over a field of characteristic zero, we also obtain a criterion for M to be semisimple, in terms of the images of inflation and induction maps on M.

1. Introduction

The main purpose of the paper is to develop some methods that can be used in order to find composition factors of biset and related functors. We especially obtain some results allowing us to find the radicals and the socles of arbitrary biset and related functors. We use these results to study the

E-mail address: eyaraneri@yahoo.com.

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structure of the Burnside functor, aiming to find its radical and socle series. We also study restrictions of functors to nonfull subcategories, for instance we study structures of inflation and deflation functors considered as (global) Mackey functors.

The notion of biset functors was introduced and developed by Bouc [2]. One of the most important examples of biset functors is the Burnside functor $\mathrm{B}_{\mathbb{K}}$ on $\mathcal{X}$ over $\mathbb{K}$ which assigns to each group G in the family $\mathcal{X}$ its Burnside algebra $\mathbb{K} \otimes_{\mathbb{Z}} B(G)$ where $\mathbb{K}$ is a field and $\mathcal{X}$ is a family of finite groups satisfying some certain conditions and $B(G)$ is the Burnside ring of G.

For an arbitrary functor M and a group H in $\mathcal{X}$, we first construct a bijective correspondence between maximal subfunctors of M whose simple quotients have H as minimal groups and some maximal $\mathbb{K}\mathrm{Out}(H)$-submodules of the Brauer quotient of M at H. These bijections allow us to find maximal subfunctors of M in terms of the maximal $\mathbb{K}\mathrm{Out}(H)$-submodules of the Brauer quotients of M at groups H in $\mathcal{X}$, because determining maximal submodules of the Brauer quotients of M are usually easier than determining maximal subfunctors of M. We already used a similar approach in [17] to study the structure of Mackey functors for a fixed group. Therefore, we here (in Sections 4 and 6) extend some parts of [17] to functors defined globally.

The subfunctors of the Burnside functor $\mathrm{B}_{\mathbb{K}}$, considered as a biset functor on the family of all finite p-groups (p is a prime), are found explicitly in Bouc and Thévenaz [3] when the characteristic of the field $\mathbb{K}$ is different from p. In this case each algebra $\mathrm{B}_{\mathbb{K}}(G)$ admits a $\mathbb{K}$-basis consisting of primitive idempotents, and these primitive idempotents are used in [3] in a crucial way. In a similar case, when $\mathbb{K}$ is of characteristic $p > 0$ and $\mathcal{X}$ is a family of nilpotent p'-groups, the composition factors of the Burnside functor considered as a biset functor on $\mathcal{X}$ over $\mathbb{K}$ is studied by Bourizk [8]. We mainly study the structure of $\mathrm{B}_{\mathbb{K}}$ in the remaining cases, and we also consider $\mathrm{B}_{\mathbb{K}}$ as an inflation, a (global) Mackey, or a deflation functor and study its structure in each case. In most of the cases we study the structure of $\mathrm{B}_{\mathbb{K}}$, the Burnside algebra $\mathrm{B}_{\mathbb{K}}(G)$ is local and so it has no $\mathbb{K}$-basis consisting of idempotents. Biset functors together with these related functors are called globally defined Mackey functors or group functors by some authors. We here do not use this terminology.

An important quotient functor of $\mathrm{B}_{\mathbb{K}}$ defined on p-groups is the rational representation functor whose subfunctors are found in Bouc [2] (when the characteristic of $\mathbb{K}$ is 0) and in Bouc [4] (when the characteristic of $\mathbb{K}$ is $p > 0$). The rational representation functor and its subfunctors are studied also by Bourizk [7,9].

In Section 5 we study the radical of $\mathrm{B}_{\mathbb{K}}$. We obtain some conditions that must be satisfied by $\mathrm{B}_{\mathbb{K}}$ in order to have a maximal subfunctor. For instance, we show that the Burnside functor, considered as a deflation or a Mackey functor on the family of all finite p-groups over a field of characteristic $p > 0$, has no maximal subfunctors. In Theorem 5.4 we obtain that the Burnside functor $\mathrm{B}_{\mathbb{K}}$, considered as a Mackey functor on a family $\mathcal{X}$ over $\mathbb{K}$, has a maximal subfunctor J satisfying the property that H is a minimal group of the simple functor $\mathrm{B}_{\mathbb{K}}/J$ if and only if for any $K \in \mathcal{X}$ having a subgroup isomorphic to H and for any subgroup A of K isomorphic to H the index $|N_K(A) : A|$ is not divisible by the characteristic of the field $\mathbb{K}$. One of the consequence of this result is that the Burnside functor considered as a Mackey functor on a family of p'-groups over a field of characteristic $p > 0$ is semisimple. A related result that can be found in Webb [15] states that over any field of characteristic 0 the category of (global) Mackey functors is semisimple.

We also obtain similar results related to simple subfunctors. For example, using a result of Bourizk [6] we find in Proposition 7.4 that the socle of the Burnside functor, considered as a biset functor on the family of all p-groups of order less than or equal to p^m (m is a natural number with $m \geqslant 2$) over a field of characteristic $p > 0$, is isomorphic to

$$\bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^m and $S_{H,\mathbb{K}}$ is the simple functor parameterized by the pair $(H, \mathbb{K})$ (see Section 2).

We devote Section 8 to the study of the Burnside functor considered as a deflation or a Mackey functor. We discover some results about the simple functors appearing in the radical quotients of the Burnside functor. For example, if we consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on the family of all p-groups of order less than or equal to p^m (m is a natural number) over a field of characteristic $p > 0$, we show that the simple functor $S_{1,\mathbb{K}}$ (whose multiplicity in the Burnside functor is 1) appears in the radical quotient J_m/J_{m+1} where $J_k = \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}})$ denotes the kth radical of the Burnside functor.

In Section 9, we consider the Burnside functor as a biset functor on a family of finite abelian p-groups, and find its radical and socle series. For instance, letting $\mathbb{K}$ be of characteristic $p > 0$ and $\mathcal{X}$ be the family of all finite abelian p-groups, and considering the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$, and for any natural number $k \geqslant 1$ putting $J_k = \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}})$, in Theorem 9.4 we find the radical series of $\mathrm{B}_{\mathbb{K}}$ as follows:

$$B_{\mathbb{K}}/J_1 \cong S_{1,\mathbb{K}} \quad \text{and} \quad J_k/J_{k+1} \cong \bigoplus_H S_{H,\mathbb{K}},$$

for any natural number $k \geqslant 1$, where H ranges over a complete set of isomorphism classes of all groups of order p^{k+1} in $\mathcal{X}$. We also observe that the radical and the socle series of $\mathrm{B}_{\mathbb{K}}$ coincides with each other, except that the socle series reaches to $\mathrm{B}_{\mathbb{K}}$ only if one places a bound on the orders of the groups in $\mathcal{X}$.

We finally study restrictions of functors to nonfull subcategories. For instance, we observe that if a simple Mackey functor $S_{H,V}$ parameterized by the pair (H, V) is projective then the simple deflation functor $S_{H,V}$ parameterized by the pair (H, V) remains to be simple when considered as a Mackey functor. For another example, we obtain a semisimplicity criterion for inflation functors. That is, an inflation functor M defined on a family $\mathcal{X}$ of abelian groups over a field of characteristic 0 is semisimple if and only if

$$\left(\sum_{N \leqslant H:\ N \neq 1} \mathrm{Inf}_{H/N}^{H} M(H/N)\right) \subseteq \left(\sum_{P<H} \mathrm{Ind}_P^H M(P)\right)$$

for any $H \in \mathcal{X}$.

Most of our notations are standard and tend to follow [2]. Let $H \leqslant G \geqslant K$ be finite groups. By the notation $HgK \subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H, K) in G. The notations $S \leqslant_G G$ and $S \leqslant_* G$ appearing in an index set both mean that S ranges over all non-G-conjugate subgroups of G. The notation $S <_G G$ means that $S \leqslant_G G$ and $S \neq G$. A quotient group of a subgroup of G is called a section of G. Thus a section of G is of the form A/B where $B \trianglelefteq A \leqslant G$. By a proper section of G we mean a section of G whose order is less than the order of G. For any set $\mathcal{S}$ we denote by $|\mathcal{S}|$ the number of elements in $\mathcal{S}$. For any prime number p by a p'-group we mean a group whose order is not divisible p. If p is the characteristic of a field and $p = 0$, by a p'-group we mean any finite group.

For a functor M we denote by $\mathrm{Jac}(M)$ the Jacobson radical of M, the intersection of all maximal subfunctors. It may happen that M has no maximal subfunctors, in which case we have $\mathrm{Jac}(M) = M$. In a dual way we denote by $\mathrm{Soc}(M)$ the socle of M, the sum of all simple subfunctors of M. If M has no simple subfunctors then $\mathrm{Soc}(M) = 0$. We also define the higher radicals and socles as: $\mathrm{Jac}^i(M) = \mathrm{Jac}(\mathrm{Jac}^{i-1}(M))$ and $\mathrm{Soc}^i(M)/\mathrm{Soc}^{i-1}(M) = \mathrm{Soc}(M/\mathrm{Soc}^{i-1}(M))$ for any natural number $i \geqslant 1$ where $\mathrm{Jac}^0(M) = M$ and $\mathrm{Soc}^0(M) = 0$. One then has the radical and the socle series

$$M = \mathrm{Jac}^0(M) \supseteq \mathrm{Jac}^1(M) \supseteq \mathrm{Jac}^2(M) \supseteq \cdots,$$
$$0 = \mathrm{Soc}^0(M) \subseteq \mathrm{Soc}^1(M) \subseteq \mathrm{Soc}^2(M) \subseteq \cdots.$$

The successive quotients of each series are either zero or semisimple, because a functor (whose evaluations at each group in $\mathcal{X}$ is a finite dimensional $\mathbb{K}$-module), with zero radical is semisimple, see the explanation given after 3.3. If there are only finitely many groups, up to isomorphism, in $\mathcal{X}$ and if $M(G)$ is a finite dimensional $\mathbb{K}$-module for each $G \in \mathcal{X}$, it follows from the explanation given in the next paragraph that the radical and the socle series reach to 0 and M, respectively, and they have equal finite lengths called the Loewy length of M.

Let M be a functor on $\mathcal{X}$ and S be a simple functor on $\mathcal{X}$ parameterized by the pair (H, V). We sometimes write $M^{\mathcal{X}}$ to indicate that we are considering M as a functor on $\mathcal{X}$. We say that S is a composition factor of M if there are subfunctors $K \subseteq L$ of M such that $L/K \cong S$. Let $\mathcal{Y}$ be a subfamily of $\mathcal{X}$ (satisfying certain conditions) such that $S^{\mathcal{Y}} \neq 0$. It follows from the explanation given after 3.8 that $S^{\mathcal{X}}$ is a composition factor of $M^{\mathcal{X}}$ if and only if $S^{\mathcal{Y}}$ is a composition factor of $M^{\mathcal{Y}}$. By the multiplicity of S in M we mean the multiplicity of $S^{\mathcal{Z}}$ as a composition factor of $M^{\mathcal{Z}}$ where $\mathcal{Z}$ is the subfamily of $\mathcal{X}$ such that any group in $\mathcal{Z}$ is isomorphic to a section of H. Although $M^{\mathcal{X}}$ may not have a composition series, $M^{\mathcal{Z}}$ must have a composition series. The reason for this is that $M^{\mathcal{Z}}$ may be identified with a module of the category algebra of the skeletal category of its domain category (i.e., any of $\mathfrak{b}$, $\mathfrak{i}$, $\mathfrak{d}$, or $\mathfrak{m}$ defined in Section 2), and it is a finite dimensional $\mathbb{K}$-algebra because there are only finitely many groups, up to isomorphism, in $\mathcal{Z}$. See, for instance, Barker [1] and Webb [15] for more details about the category algebras. Furthermore, it follows from above that if $M^{\mathcal{X}}$ has a composition series then the multiplicity of $S^{\mathcal{X}}$ in $M^{\mathcal{X}}$ is equal to the multiplicity of $S^{\mathcal{Z}}$ in $M^{\mathcal{Z}}$. More to the point, we observe in 4.10 that the multiplicity of S in M is equal to the multiplicity of V as a composition factor of $\mathrm{End}(H)$-module $M(H)$.

Throughout the paper, R is a commutative unital ring, $\mathbb{K}$ is a field, and $\mathcal{X}$ is a family of finite groups which is closed under taking subgroups, quotients, and isomorphisms.

2. Preliminaries

In this section, we simply collect some crucial results on bisets and functors in Bouc [2]. Let G, H, and K be finite groups. A (G, H)-biset is a finite set U having a left G-action and a right H-action such that the two actions commute. Given a (G, H)-biset U and an (H, K)-biset V, the cartesian product $U \times V$ becomes a right H-set with the action $(u, v)h = (uh, h^{-1}v)$. If we let $u \otimes v$ denote the H-orbit of $U \times V$ containing (u, v), then the set $U \times_H V$ of the H-orbits of $U \times V$ becomes a (G, K)-biset with the actions $g(u \otimes v)k = gu \otimes vk$. Any (G, H)-biset U is a left $G \times H$-set by the action $(g, h)u = guh^{-1}$, and conversely. Terminology for (G, H)-bisets is inherited from terminology for $G \times H$-sets. Thus transitive (G, H)-bisets are isomorphic to bisets of the form $(G \times H)/L$ where L is a subgroup $G \times H$. We write $[U]$ for the isomorphism class of a biset U.

Let L be a subgroup of $G \times H$. We define

$$p_1(L) = \{g \in G\colon \exists h \in H,\ (g, h) \in L\} \quad \text{and} \quad k_1(L) = \{g \in G\colon (g, 1) \in L\},$$
$$p_2(L) = \{h \in H\colon \exists g \in G,\ (g, h) \in L\} \quad \text{and} \quad k_2(L) = \{h \in H\colon (1, h) \in L\}.$$

Then $k_i(L)$ is a normal subgroup $p_i(L)$, and $k_1(L) \times k_2(L)$ is a normal subgroup of L, and the three quotient groups which we denote by $q(L)$ are isomorphic. If $L \leqslant G \times H$ and $M \leqslant H \times K$ we write

$$L * M = \{(g, k) \in G \times K\colon \exists h \in H,\ (g, h) \in L,\ (h, k) \in M\}.$$

Proposition 2.1. *(See [2].) Let $L \leqslant G \times H$ and $M \leqslant H \times K$. Then*

$$\big((G \times H)/L\big) \times_H \big((H \times K)/M\big) \cong \sum_{p_2(L)hp_1(M) \subseteq H} (G \times K)/\big(L * {}^{(h,1)}M\big).$$

There are five types of basic bisets so that any transitive biset is isomorphic to a product of them. For $H \leqslant G \trianglerighteq N$ and isomorphism of groups $\psi : G \to G'$, they are

$$\mathrm{Ind}_H^G = (G \times H)/\big\{(h,h)\colon h \in H\big\},$$
$$\mathrm{Res}_H^G = (H \times G)/\big\{(h,h)\colon h \in H\big\},$$
$$\mathrm{Inf}_{G/N}^G = (G \times G/N)/\big\{(g,gN)\colon g \in G\big\},$$
$$\mathrm{Def}_{G/N}^G = (G/N \times G)/\big\{(gN,g)\colon g \in G\big\},$$
$$\mathrm{Iso}_G^{G'}(\psi) = \big(G' \times G\big)/\big\{\big(\psi(g),g\big)\colon g \in G\big\}.$$

Proposition 2.2. *(See [2].) For any $L \leqslant G \times H$ we have*

$$(G \times H)/L \cong \mathrm{Ind}_{p_1(L)}^G \mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \mathrm{Iso}_{p_2(L)/k_2(L)}^{p_1(L)/k_1(L)}(\psi)\, \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \mathrm{Res}_{p_2(L)}^H$$

where $\psi(hk_2(L)) = gk_1(L)$ if and only if $(g,h) \in L$.

Let $\mathcal{X}$ be a family of finite groups closed under taking subgroups, taking isomorphisms and taking quotients. We define the biset category $\mathfrak{b}$ (on $\mathcal{X}$ over R), which is R-linear, as follows:

- The objects are the groups in $\mathcal{X}$.
- If H and G are in $\mathcal{X}$ then $\mathrm{Hom}_{\mathfrak{b}}(H,G) = RB(G \times H)$ is the Burnside group of (G,H)-bisets, with coefficients in R.
- Composition of morphisms is obtained by R-linearity from the product $(U,V) \mapsto U \times_H V$.

Any R-linear (covariant) functor from the category $\mathfrak{b}$ to the category of left R-modules is called a biset functor (on $\mathcal{X}$ over R). We denote by $\mathfrak{F}_{\mathfrak{b}}$ the category of biset functors, which is an abelian category.

We also want to consider some nonfull subcategories of $\mathfrak{b}$ and R-linear functors from these subcategories to the category of left R-modules. Let $\mathfrak{i}$ be the subcategory of $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{i}}(H,G) = \bigoplus_{L \leqslant_* G \times H\colon\, k_2(L)=1} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{i}$ to the category of left R-modules is called an inflation functor (on $\mathcal{X}$ over R). We denote by $\mathfrak{F}_{\mathfrak{i}}$ the category of inflation functors.

Let $\mathfrak{d}$ be the subcategory of $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{d}}(H,G) = \bigoplus_{L \leqslant_* G \times H\colon\, k_1(L)=1} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{d}$ to the category of left R-modules is called a deflation functor (on $\mathcal{X}$ over R). We denote by $\mathfrak{F}_{\mathfrak{d}}$ the category of deflation functors.

Let $\mathfrak{m}$ be the subcategory of $\mathfrak{b}$ with the same objects and with the morphisms

$$\mathrm{Hom}_{\mathfrak{m}}(H,G) = \bigoplus_{L \leqslant_* G \times H\colon\, k_1(L)=1=k_2(L)} R\big[(G \times H)/L\big].$$

An R-linear functor from $\mathfrak{m}$ to the category of left R-modules is called a (global) Mackey functor (on $\mathcal{X}$ over R). We denote by $\mathfrak{F}_{\mathfrak{m}}$ the category of Mackey functors. Mackey functors can also be defined on a family $\mathcal{X}$ of finite groups closed under taking subgroups and taking isomorphism.

These four functor categories have similar theories. For example their simple objects are parameterized in the same manner. From now on in this section, a functor means any of biset, inflation, deflation or Mackey.

For any groups X and Y in $\mathcal{X}$ the composition of morphism gives an $(\mathrm{End}(Y), \mathrm{End}(X))$-bimodule structure on $\mathrm{Hom}(X, Y)$, and for a functor M we have an $\mathrm{End}(X)$-module structure on $M(X)$ given by $fm_X = M(f)(m_X)$.

For a group X in $\mathcal{X}$ and an $\mathrm{End}(X)$-module V we define a functor $L_{X,V}$ and an its subfunctor $J_{X,V}$ as follows:

$$L_{X,V}(Y) = \mathrm{Hom}(X, Y) \otimes_{\mathrm{End}(X)} V,$$
$$L_{X,V}(f) : L_{X,V}(Y) \to L_{X,V}(Z), \quad \theta \otimes v \mapsto f\theta \otimes v,$$
$$J_{X,V}(Y) = \bigcap_{f \in \mathrm{Hom}(Y,X)} \mathrm{Ker}\big(L_{X,V}(f)\big),$$

where $\mathrm{Ker}(L_{X,V}(f))$ denotes the kernel of the map $L_{X,V}(f)$.

Having defined the functors $L_{X,V}$ we define two important functors between the functor category $\mathfrak{F}$ (i.e., any of $\mathfrak{F}_{\mathfrak{b}}$, $\mathfrak{F}_{\mathfrak{i}}$, $\mathfrak{F}_{\mathfrak{d}}$ or $\mathfrak{F}_{\mathfrak{m}}$) and $\mathrm{End}(X)$-module category.

$$L_{X,-} : \mathrm{End}(X)\text{-Mod} \to \mathfrak{F}, \quad V \mapsto L_{X,V},$$

and if $\varphi : V \to W$ is an $\mathrm{End}(X)$-module homomorphism then $L_{X,-}(\varphi) : L_{X,V} \to L_{X,W}$ is the natural transformation whose $Y \in \mathcal{X}$ component is the map $L_{X,V}(Y) \to L_{X,W}(Y)$, given by $f \otimes v \mapsto f \otimes \varphi(v)$.

$$e_X : \mathfrak{F} \to \mathrm{End}(X)\text{-Mod}, \quad M \mapsto M(X),$$

and if $\pi : M \to N$ is a morphism of functors (i.e., a natural transformation) then $e_X(\pi)$ is the X-component $\pi_X : M(X) \to N(X)$ of π.

Proposition 2.3. *(See [2].) Let X be a group in $\mathcal{X}$. Then*:

(1) *e_X is an exact functor and $L_{X,-}$ is a right exact functor.*
(2) *$(L_{X,-}, e_X)$ is an adjoint pair.*
(3) *If V is a projective* $\mathrm{End}(X)$*-module then $L_{X,V}$ is a projective functor.*
(4) *If V is an indecomposable* $\mathrm{End}(X)$*-module then $L_{X,V}$ is an indecomposable functor.*

Let M be a functor. A group H in $\mathcal{X}$ is called a minimal group of M if $M(H) \neq 0$ and $M(K) = 0$ for all $K \in \mathcal{X}$ with $|K| < |H|$.

Proposition 2.4. *(See [2].) Let X be a group in $\mathcal{X}$ and let V be a simple* $\mathrm{End}(X)$*-module. Then, $J_{X,V}$ is the unique maximal subfunctor of $L_{X,V}$ and $L_{X,V}/J_{X,V}$ is a simple functor whose evaluation at X is V. However, X may not be a minimal subgroup of this simple functor.*

Proposition 2.5. *(See [2].) For a group G in $\mathcal{X}$, there is a direct sum decomposition*

$$\mathrm{End}(G) = Ext(G) \oplus I_G$$

where I_G is a two sided ideal of $\mathrm{End}(G)$ *with an R-basis consisting of the elements $[(G \times G)/L]$ of* $\mathrm{End}(G)$ *with $|q(L)| < |G|$, and $Ext(G)$ is a unital subalgebra of* $\mathrm{End}(G)$ *isomorphic to the group algebra $R\,\mathrm{Out}(G)$ of the group of outer automorphisms of G.*

A simple functor S with a minimal group H is denoted by $S_{H,V}$ if $S(H) = V$.

Theorem 2.6. *(See [2].) In the following an $R\,\mathrm{Out}(H)$-module is considered as an $\mathrm{End}(H)$-module via the natural projection map $\mathrm{End}(H) \to Ext(H) \cong R\,\mathrm{Out}(H)$ given in 2.5.*

(1) *Let H be a group in $\mathcal{X}$ and let V be a simple $R\,\mathrm{Out}(H)$-module. Then H is a minimal subgroup of the simple functor $L_{H,V}/J_{H,V}$. So $L_{H,V}/J_{H,V} = S_{H,V}$.*
(2) *Let S be a simple functor and let H be a minimal subgroup S. Then I_H annihilates $S(H)$, and $S(H)$ is a simple $R\,\mathrm{Out}(H)$-module, and $S \cong S_{H,V}$ where $S(H) = V$.*
(3) *$S_{H,V} \cong S_{K,W}$ if and only if there is a group isomorphism $H \to K$ transporting V to W.*
(4) *If $S_{H,V}(G) \neq 0$ for some group G, then H is isomorphic to a section of G (to a subgroup of G in the case of Mackey functors).*

3. Linear functors in general

Throughout this section, $\mathfrak{A}$ is an (small) R-linear category, and $\mathfrak{F}$ is the category of R-linear (covariant) functors from $\mathfrak{A}$ to the category of (left) R-modules.

Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$. Composition of morphisms of $\mathfrak{A}$ induces an (left) $\mathrm{End}_{\mathfrak{A}}(X)$-module structure on the R-module $M(X)$ defined by $fm = M(f)(m)$ for any $f \in \mathrm{End}_{\mathfrak{A}}(X)$ and any $m \in M(X)$. The main purpose of this section is to find some relations between the maximal (respectively, simple) subfunctors of M and the maximal (respectively, simple) $\mathrm{End}_{\mathfrak{A}}(X)$-submodules of $M(X)$.

For a functor $M \in \mathfrak{F}$, an object X of $\mathfrak{A}$, and an $\mathrm{End}_{\mathfrak{A}}(X)$-submodule V of $M(X)$, we define two subfunctors $\mathrm{Im}^M_{X,V}$ and $\mathrm{Ker}^M_{X,V}$ of M whose evaluations at any object Y of $\mathfrak{A}$ are given as

$$\mathrm{Im}^M_{X,V}(Y) = \sum_{f \in \mathrm{Hom}_{\mathfrak{A}}(X,Y)} M(f)(V),$$

$$\mathrm{Ker}^M_{X,V}(Y) = \bigcap_{f \in \mathrm{Hom}_{\mathfrak{A}}(Y,X)} M(f)^{-1}(V),$$

where for an $f \in \mathrm{Hom}_{\mathfrak{A}}(Y,X)$ we denote by $M(f)^{-1}(V)$ the set of all elements $y \in M(Y)$ such that $M(f)(y) \in V$. It is obvious from the definitions that they are subfunctors of M and that the evaluations of subfunctors $\mathrm{Im}^M_{X,V}$ and $\mathrm{Ker}^M_{X,V}$ at X are both equal to V. Moreover, $\mathrm{Im}^M_{X,V}$ is the smallest subfunctor of M in the sense that it is contained in any subfunctor I of M satisfying $V \subseteq I(X)$, and $\mathrm{Ker}^M_{X,V}$ is the largest subfunctor of M in the sense that it contains any subfunctor J of M satisfying $J(X) \subseteq V$. We note that the subfunctor $J_{X,V}$ of $L_{X,V}$ described in Section 2 is the $\mathrm{Ker}_{X,0}$ subfunctor of $L_{X,V}$. Some elementary properties and applications of these subfunctors can be found in [16].

Lemma 3.1. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$. Then*:

(1) *The maps $J \to J(X)$ and $\mathrm{Ker}^M_{X,V} \leftarrow V$ define a bijective correspondence between the largest elements J of the set of all subfunctors I of M satisfying the property $\mathrm{Im}^M_{X,M(X)} \nsubseteq I$, and the maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$.*
(2) *The maps $J \to J(X)$ and $\mathrm{Im}^M_{X,V} \leftarrow V$ define a bijective correspondence between the smallest elements J of the set of all subfunctors I of M satisfying the property $I \nsubseteq \mathrm{Ker}^M_{X,0}$, and the simple $\mathrm{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$.*

We skip the proof of the above result, which follows easily from the definitions of Im and Ker subfunctors. Note that the largest (respectively smallest) subfunctors J considered in the above result may not be the maximal (respectively simple) subfunctors of M unless $\mathrm{Im}^M_{X,M(X)} = M$ (respectively $\mathrm{Ker}^M_{X,0} = 0$). If we assume further that $\mathrm{Im}^M_{X,M(X)} = M$ (respectively $\mathrm{Ker}^M_{X,0} = 0$) then the above result implies Propositions 3.3 and 3.5 in [16]. We also note that the conditions $\mathrm{Im}^M_{X,M(X)} \nsubseteq I$ and $I \nsubseteq \mathrm{Ker}^M_{X,0}$ are equivalent to the conditions $I(X) \neq M(X)$ and $I(X) \neq 0$, respectively.

The following is an immediate consequence of 3.1.

Proposition 3.2. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$. Then*:

(1) *The maps $J \to J(X)$ and $\mathrm{Ker}^M_{X,V} \leftarrow V$ define a bijective correspondence between the maximal subfunctors J of M satisfying the property $\mathrm{Im}^M_{X,M(X)} \nsubseteq J$, and the maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$ satisfying the property*

$$\mathrm{Im}^M_{X,M(X)} + \mathrm{Ker}^M_{X,V} = M.$$

(2) *The maps $J \to J(X)$ and $\mathrm{Im}^M_{X,V} \leftarrow V$ define a bijective correspondence between the simple subfunctors J of M satisfying the property $J \nsubseteq \mathrm{Ker}^M_{X,0}$, and the simple $\mathrm{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$ satisfying the property*

$$\mathrm{Im}^M_{X,V} \cap \mathrm{Ker}^M_{X,0} = 0.$$

The following characterization of simple functors (see, for instance, Corollary 3.6 of [16]) is an easy consequence of 3.2.

Remark 3.3. Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X) \neq 0$. Then, M is simple if and only if $M(X)$ is a simple $\mathrm{End}_{\mathfrak{A}}(X)$-module, $\mathrm{Im}^M_{X,M(X)} = M$, and $\mathrm{Ker}^M_{X,0} = 0$.

Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X) \neq 0$. It follows from 3.2 that any maximal subfunctor of M which does not contain $\mathrm{Im}^M_{X,M(X)}$ must be of the form $\mathrm{Ker}^M_{X,V}$ for some maximal $\mathrm{End}_{\mathfrak{A}}(X)$-submodule V of $M(X)$, and so it contains $\mathrm{Ker}^M_{X,0}$. Consequently, we must have that

$$\mathrm{Ker}^M_{X,0} \cap \mathrm{Im}^M_{X,M(X)} \subseteq \mathrm{Jac}(M) \quad \text{and} \quad \mathrm{Jac}\big(M(X)\big) \subseteq \mathrm{Jac}(M)(X),$$

where $\mathrm{Jac}(M)$ denotes the radical of the functor M and $\mathrm{Jac}(M(X))$ denotes the radical of the $\mathrm{End}_{\mathfrak{A}}(X)$-module $M(X)$. Moreover, it is clear from the definitions that $\mathrm{Ker}^M_{X,0} \cap \mathrm{Im}^M_{X,M(X)}$ is equal to $\mathrm{Ker}^{I_X}_{X,0}$ where $I_X = \mathrm{Im}^M_{X,M(X)}$. Now we assume further that $\mathrm{Jac}(M) = 0$ and that $M(Y)$ is an artinian $\mathrm{End}_{\mathfrak{A}}(Y)$-module for each object Y of $\mathfrak{A}$. For any object Y of $\mathfrak{A}$, it follows from what we observed above that $M(Y)$ is a semisimple $\mathrm{End}_{\mathfrak{A}}(Y)$-module and that $\mathrm{Ker}^{I_Y}_{Y,0} = 0$. Then, it follows from 3.3 that each I_Y is a semisimple functor. Consequently, M must be a semisimple functor because M is equal to the sum of the semisimple functors I_Y where Y is ranging in the set of all objects Y of $\mathfrak{A}$ with $M(Y) \neq 0$.

We apply 3.3 to derive the following result.

Remark 3.4. Let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$ and $\mathfrak{F}_{\mathfrak{B}}$ be the category of R-linear functors from $\mathfrak{B}$ to the category of R-modules. Any functor $M \in \mathfrak{F}$ defines a functor $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} M = M \circ I \in \mathfrak{F}_{\mathfrak{B}}$, called the restriction of M to $\mathfrak{B}$, where $I : \mathfrak{B} \to \mathfrak{A}$ is the inclusion functor. If $\mathfrak{B}$ is a full subcategory of $\mathfrak{A}$ and $S \in \mathfrak{F}$ is a simple functor, then the restriction of S to $\mathfrak{B}$ is either zero or a simple functor in $\mathfrak{F}_{\mathfrak{B}}$.

Lemma 3.5. *Let $M \in \mathfrak{F}$ be a semisimple functor and X be an object of $\mathfrak{A}$. For any $\mathrm{End}_{\mathfrak{A}}(X)$-submodule V of $M(X)$,*

$$\mathrm{Im}^M_{X,M(X)} + \mathrm{Ker}^M_{X,V} = M \quad \textit{and} \quad \mathrm{Im}^M_{X,V} \cap \mathrm{Ker}^M_{X,0} = 0.$$

Proof. Letting $T = M/\operatorname{Ker}^M_{X,V}$ we may see that $\operatorname{Ker}^T_{X,0} = 0$. Therefore, $S(X) \neq 0$ for any simple subfunctor S of T. As T is semisimple, by using 3.3 and the definition of Im subfunctors we obtain that $\operatorname{Im}^T_{X,T(X)} = T$, which implies that the sum of $\operatorname{Im}^M_{X,M(X)}$ and $\operatorname{Ker}^M_{X,V}$ is M. What remains can be justified similarly. □

The following refinement of 3.2 is an easy consequence of 3.5 and 3.2.

Proposition 3.6. *Let $M \in \mathfrak{F}$ be a semisimple functor and X be an object of $\mathfrak{A}$. Then:*

(1) *The maps $J \to J(X)$ and $\operatorname{Ker}^M_{X,V} \leftarrow V$ define a bijective correspondence between the maximal subfunctors J of M satisfying the property $J(X) \neq M(X)$, and the maximal $\operatorname{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$.*
(2) *The maps $J \to J(X)$ and $\operatorname{Im}^M_{X,V} \leftarrow V$ define a bijective correspondence between the simple subfunctors J of M satisfying the property $J(X) \neq 0$, and the simple $\operatorname{End}_{\mathfrak{A}}(X)$-submodules V of $M(X)$.*

We have also the following relations between socles and Im subfunctors, and radicals and Ker subfunctors.

Remark 3.7. Let $M, N \in \mathfrak{F}$ be functors, $S \in \mathfrak{F}$ be a simple functor, and X be an object of $\mathfrak{A}$ such that $S(X) \neq 0$. Put $I = \operatorname{Im}^M_{X,M(X)}$ and $K = \operatorname{Ker}^M_{X,0}$. Then:

(1) If $\operatorname{Im}^N_{X,N(X)} = N$ then $\operatorname{Hom}_{\mathfrak{F}}(N, M) \cong \operatorname{Hom}_{\mathfrak{F}}(N, I)$ as R-modules. In particular, the multiplicities of S in the socles of M and I are equal.
(2) If $\operatorname{Ker}^N_{X,0} = 0$ then $\operatorname{Hom}_{\mathfrak{F}}(M, N) \cong \operatorname{Hom}_{\mathfrak{F}}(M/K, N)$ as R-modules. In particular, the multiplicities of S in the heads of M and M/K are equal.

Proof. For any natural transformation $\pi : N \to M$ it follows that

$$\pi(N) = \pi\big(\operatorname{Im}^N_{X,N(X)}\big) \subseteq \operatorname{Im}^M_{X,M(X)} = I,$$

and it follows from 3.3 that $\operatorname{Im}^S_{X,S(X)} = S$, proving the first part. The second part can be proved similarly. □

Proposition 3.8. *Let $M, N \in \mathfrak{F}$ be functors and let X be an object of $\mathfrak{A}$. Let*

$$\phi : \operatorname{Hom}_{\mathfrak{F}}(M, N) \to \operatorname{Hom}_{\operatorname{End}_{\mathfrak{A}}(X)}\big(M(X), N(X)\big), \quad \pi \mapsto \pi_X,$$

be the R-module (R-algebra if $M = N$) homomorphism sending a natural transformation π to its X-component π_X. Then:

(1) *If $\operatorname{Ker}^N_{X,0} = 0$ then ϕ is a monomorphism.*
(2) *If $\operatorname{Ker}^N_{X,0} = 0$ and $\operatorname{Im}^M_{X,M(X)} = M$ then ϕ is an isomorphism.*

Proof. (1) Let $\pi : M \to N$ be a natural transformation with $\pi_X = 0$. Then,

$$0 = \pi_X\big(M(X)\big) = \pi(M)(X),$$

implying that

$$\pi(M) \subseteq \operatorname{Ker}^N_{X,0} = 0.$$

Thus, $\pi = 0$ if $\pi_X = 0$.

(2) Let $f : M(X) \to N(X)$ be an $\mathrm{End}_{\mathfrak{A}}(X)$-module homomorphism. We will construct a natural transformation $\pi : M \to N$ with $\pi_X = f$. Let Y be an object of $\mathfrak{A}$ and $u \in M(Y)$. As $\mathrm{Im}^M_{X,M(X)} = M$, there are elements $v_1, \dots, v_n$ in $M(X)$ and morphisms $f_1, \dots, f_n$ in $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$ for some natural number n, such that

$$u = M(f_1)(v_1) + \cdots + M(f_n)(v_n).$$

We define π_Y as

$$\pi_Y(u) = N(f_1)\big(f(v_1)\big) + \cdots + N(f_n)\big(f(v_n)\big).$$

One may see that π with this definition is a natural transformation with $\pi_X = f$. We here justify only that π_Y defined above is a well-defined map. For this end, let $w_1, \dots, w_m$ be elements of $M(X)$ and $g_1, \dots, g_m$ be morphisms in $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$ such that

$$M(g_1)(w_1) + \cdots + M(g_m)(w_m) = 0.$$

We need to show that $a = 0$ where

$$a = N(g_1)\big(f(w_1)\big) + \cdots + N(g_m)\big(f(w_m)\big).$$

Indeed, let g be any morphism in $\mathrm{Hom}_{\mathfrak{A}}(Y, X)$. Then, each $g \circ g_i$ is in $\mathrm{End}_{\mathfrak{A}}(X)$, and as f is an $\mathrm{End}_{\mathfrak{A}}(X)$-module homomorphism we must have that $N(g \circ g_i)(f(w_i)) = f(M(g \circ g_i)(w_i))$. Hence,

$$\begin{aligned}
N(g)(a) &= N(g)\big(N(g_1)\big(f(w_1)\big) + \cdots + N(g_m)\big(f(w_m)\big)\big) \\
&= N(g \circ g_1)\big(f(w_1)\big) + \cdots + N(g \circ g_m)\big(f(w_m)\big) \\
&= f\big(M(g \circ g_1)(w_1)\big) + \cdots + f\big(M(g \circ g_m)(w_m)\big) \\
&= f\big(M(g)\big(M(g_1)(w_1) + \cdots + M(g_m)(w_m)\big)\big) \\
&= f\big(M(g)(0)\big) = 0,
\end{aligned}$$

showing that $a \in \mathrm{Ker}^N_{X,0}$. Therefore, $a = 0$. $\square$

Let $M \in \mathfrak{F}$ be a functor and $S \in \mathfrak{F}$ be a simple functor. Let $\mathfrak{B}$ be a full subcategory of $\mathfrak{A}$ such that $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} S \neq 0$. Suppose that there are subfunctors $K \subseteq L$ of $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} M$ such that L/K is isomorphic to $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} S$. Take any object X of $\mathfrak{B}$ such that $S(X) \neq 0$. It follows that $\mathrm{Hom}_{\mathrm{End}_{\mathfrak{B}}(X)}(L(X), S(X))$ is nonzero. As $\mathrm{End}_{\mathfrak{B}}(X)$ is equal to $\mathrm{End}_{\mathfrak{A}}(X)$, we see by using 3.8 and 3.3 that $\mathrm{Hom}_{\mathfrak{F}}(\mathrm{Im}^M_{X,L(X)}, S)$ is nonzero. Therefore, S appears in the head of $\mathrm{Im}^M_{X,L(X)}$, which is a subfunctor of M. Consequently, we observed that if $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} S$ is a composition factor of $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} M$ then S is a composition factor of M. The converse of this observation is also true and it follows from 3.4. A consequence of this observation is that, for any simple functors S_1 and S_2 in $\mathfrak{F}$ and for any full subcategory $\mathfrak{B}$ of $\mathfrak{A}$, if $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} S_1$ and $\downarrow^{\mathfrak{A}}_{\mathfrak{B}} S_2$ are nonzero isomorphic functors then S_1 and S_2 are isomorphic functors in $\mathfrak{F}$.

We have the following obvious consequence of 3.8.

Corollary 3.9. *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $\mathrm{Im}^M_{X,M(X)} = M$ and $\mathrm{Ker}^M_{X,0} = 0$. Suppose*

$$M = M_1 \oplus \cdots \oplus M_n$$

is a decomposition of M into nonzero functors in $\mathfrak{F}$. Then,

$$M(X) = M_1(X) \oplus \cdots \oplus M_n(X)$$

is a decomposition of $M(X)$ into nonzero $\mathrm{End}_{\mathfrak{A}}(X)$-modules such that the functors M_i and M_j are isomorphic if and only if the $\mathrm{End}_{\mathfrak{A}}(X)$-modules $M_i(X)$ and $M_j(X)$ are isomorphic. Moreover, M_i is an indecomposable functor if and only if $M_i(X)$ is an indecomposable $\mathrm{End}_{\mathfrak{A}}(X)$-module.

4. Maximal subfunctors and Brauer quotients

Throughout this section, by a functor we mean any of biset functor, inflation functor, (global) Mackey functor, or deflation functor, defined on $\mathcal{X}$ over $\mathbb{K}$. Whenever we consider Mackey functors, we do not need to assume that the family $\mathcal{X}$ is closed under taking quotients, and the words "section" may be replaced with the words "subgroup".

We begin with recalling the notion of the Brauer quotient of a functor, see [15]. Let M be a functor and H be a group in $\mathcal{X}$, we put

$$b_H(M) = \sum_{f,K} M(f)\big(M(K)\big)$$

where K ranges over all groups in $\mathcal{X}$ having no sections isomorphic to H and f ranges in $\mathrm{Hom}(K,H)$. It is clear that $b_H(M)$ is a $\mathbb{K}\,\mathrm{Out}(H)$-submodule of $M(H)$. The quotient module $M(H)/b_H(M)$ is called the Brauer quotient of M at H, and denoted by $\overline{M}(H)$.

For a functor M, and groups H and K in $\mathcal{X}$, and $f \in \mathrm{Hom}(K,H)$, we sometimes use the notation f to denote the $\mathbb{K}$-module homomorphism $M(f): M(K) \to M(H)$. For instance, by the expression $I_H M$ in the below we mean the sum of all $\mathbb{K}$-modules $M(f)(M(H))$ where f ranges in the ideal I_H of $\mathrm{End}(H)$ described in 2.5.

Remark 4.1. Let M be a functor and H be a group in $\mathcal{X}$. Then, $I_H M \subseteq b_H(M)$ so that $b_H(M)$ is an $\mathrm{End}(H)$-submodule of $M(H)$ where I_H is the ideal of $\mathrm{End}(H)$ described in 2.5. In particular, any $\mathbb{K}\,\mathrm{Out}(H)$-submodule of $M(H)$ containing $b_H(M)$ is an $\mathrm{End}(H)$-submodule of $M(H)$.

Proof. As the ideal I_H of $\mathrm{End}(H)$ is spanned by the transitive (H,H)-bisets

$$\big[(H \times H)/L\big]$$

with $|q(L)| < |H|$, by using 2.2 we may factorize

$$\big[(H \times H)/L\big]$$

as fg for some $K \in \mathcal{X}$ with $|K| < |H|$ and $f \in \mathrm{Hom}(K,H)$ and $g \in \mathrm{Hom}(H,K)$. In particular, K has no sections isomorphic to H. As M is a functor,

$$M(fg)\big(M(H)\big) \subseteq M(f)\big(M(K)\big) \subseteq b_H(M). \qquad \square$$

The above result shows that the notation $\mathrm{Ker}^M_{H,V}$ makes sense for $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $M(H)$ containing $b_H(M)$, where $\mathrm{Ker}^M_{X,W}$ subfunctors of a functor M are defined in the previous section for $\mathrm{End}(X)$-submodules W of $M(X)$.

Theorem 4.2. *Let M be a functor and H be a group in $\mathcal{X}$. Then, the maps $J \to J(H)$ and $\operatorname{Ker}^M_{H,V} \leftarrow V$ define a bijective correspondence between the largest elements J of the set of all subfunctors I of M satisfying the property that H is a minimal group of M/I, and the maximal $\mathbb{K}\operatorname{Out}(H)$-submodules $V/b_H(M)$ of $\overline{M}(H)$. Moreover, $\overline{M}(H) = 0$ if and only if M has no quotient functor having H as a minimal group.*

Proof. Let I be a subfunctor of M satisfying the property that H is a minimal group of M/I. We will observe that $b_H(M) \subseteq I(H)$. In particular, $\overline{M}(H) \neq 0$: Indeed, for any group K in $\mathcal{X}$ having no sections isomorphic to H, and any $L \leqslant H \times K$, we must have that $|q(L)| < |H|$, and then 2.2 implies that

$$\big[(H \times K)/L\big]$$

can be factorized as fg for some $A \in \mathcal{X}$ with $|A| < |H|$ and $f \in \operatorname{Hom}(A, H)$ and $g \in \operatorname{Hom}(K, A)$. Moreover, $M(A) = I(A)$ as H is a minimal group of M/I and as $|A| < |H|$. Now

$$M(fg)\big(M(K)\big) \subseteq M(f)\big(M(A)\big) = M(f)\big(I(A)\big) \subseteq I(H).$$

Hence $b_H(M) \subseteq I(H)$.

Let $\mathcal{A}$ be the set of all subfunctors I of M satisfying the property that H is a minimal group of M/I, and let $\mathcal{B}$ be the set of all subfunctors I of M satisfying the property that

$$\operatorname{Im}^M_{H,M(H)} \nsubseteq I,$$

so that we have $\mathcal{A} \subseteq \mathcal{B}$. We will show that any largest element J of the set $\mathcal{A}$ remains to be a largest element in the set $\mathcal{B}$: Indeed, let J be a largest element of $\mathcal{A}$. If there is an element I of $\mathcal{B}$ such that $J \subseteq I$, then

$$M(K) = J(K) \subseteq I(K)$$

for any group K in $\mathcal{X}$ with $|K| < |H|$ because H is a minimal group of M/J. This shows that H is also a minimal group of M/I, and so $I \in \mathcal{A}$ proving that $J = I$.

Let V be a proper $\mathbb{K}\operatorname{Out}(H)$-submodule of $M(H)$ containing $b_H(M)$. We will show that H is a minimal group of the functor M/I where $I = \operatorname{Ker}^M_{H,V}$. In particular, I is an element of the set $\mathcal{A}$ defined above: Indeed, as $I(H) = V$ the functor M/I is nonzero at H. Let K be a group in $\mathcal{X}$ such that $|K| < |H|$. Then K has no sections isomorphic to H, implying for any subgroup L of $H \times K$ that

$$\big[(H \times K)/L\big]M(K) \subseteq b_H(M) \subseteq V.$$

Hence $I(K) = M(K)$ which shows that H is a minimal group of M/I.

Finally, the theorem follows from part (1) of 3.1. $\square$

The subfunctors J mentioned in the previous result may not be maximal subfunctors of M. For maximal subfunctors we have the following result as an immediate consequence of 3.2 and 4.2.

Corollary 4.3. *Let M be a functor and H be a group in $\mathcal{X}$. Then, the maps $J \to J(H)$ and $\operatorname{Ker}^M_{H,V} \leftarrow V$ define a bijective correspondence between the maximal subfunctors J of M satisfying the property that H is a minimal group of M/J, and the maximal $\mathbb{K}\operatorname{Out}(H)$-submodules $V/b_H(M)$ of $\overline{M}(H)$ satisfying the property that*

$$\operatorname{Im}^M_{H,M(H)} + \operatorname{Ker}^M_{H,V} = M.$$

In terms of multiplicities in heads, 4.3 may be stated as follows.

Corollary 4.4. *Let M be a functor and H be a group in $\mathcal{X}$. For any simple $\mathbb{K}\operatorname{Out}(H)$-module V, let n be the multiplicity of V in the $\mathbb{K}\operatorname{Out}(H)$-module $\overline{M}(H)/\operatorname{Jac}(\overline{M}(H))$ and let m be the multiplicity of the simple functor $S_{H,V}$ in $M/\operatorname{Jac}(M)$. Then, $m \leqslant n$. In particular, if $m \neq 0$ and $n = 1$ then $m = 1$.*

Proof. Firstly, it follows from 4.3 that the multiplicity of $S_{H,V}$ in $M/\operatorname{Jac}(M)$ is finite. There are m maximal subfunctors $J_1, \ldots, J_m$ of M such that each quotient M/J_i is isomorphic to $S_{H,V}$ and such that the product of natural epimorphisms

$$\psi : M \to \prod_{i=1}^{m} M/J_i$$

is surjective. From 2.3, the evaluation functor e_H is exact so that the H-component

$$\psi_H : M(H) \to \prod_{i=1}^{m} M(H)/J_i(H)$$

of (the natural transformation) ψ is a surjective $\operatorname{End}(H)$-module homomorphism. We know from 4.3 that each $J_i(H)$ contains $b_H(M)$ and $J_i(H)/b_H(M)$ is a maximal $\mathbb{K}\operatorname{Out}(H)$-submodule of $\overline{M}(H)$ and its quotient is isomorphic to V. Thus ψ_H induces a $\mathbb{K}\operatorname{Out}(H)$-module homomorphism

$$\overline{M}(H) \to mV$$

which is surjective. Hence, $n \geqslant m$. $\square$

The previous two results will be the main tool we use to find the maximal subfunctors of a given functor M and multiplicities of simple functors in the head of M. For this end, we first need to find the maximal $\mathbb{K}\operatorname{Out}(H)$-submodules $V/b_H(M)$ of the Brauer quotients $\overline{M}(H)$ so that maximal subfunctors are of the form $\operatorname{Ker}^M_{H,V}$, but for $\operatorname{Ker}^M_{H,V}$ to be a maximal subfunctor, V must satisfy the given condition in 4.3. The next result illustrate some groups H for which this condition satisfied automatically for any maximal $\mathbb{K}\operatorname{Out}(H)$-submodules $V/b_H(M)$ of $\overline{M}(H)$.

Proposition 4.5. *Let M be a functor and H be a group in $\mathcal{X}$. Suppose that $\overline{M}(H) \neq 0$ and that $\overline{M}(K) = 0$ for any group K in $\mathcal{X}$ having a proper section isomorphic to H. Then*:

(1) *The maps $J \to J(H)$ and $\operatorname{Ker}^M_{H,V} \leftarrow V$ define a bijective correspondence between the maximal subfunctors J of M satisfying the property that H is a minimal group of M/J, and the maximal $\mathbb{K}\operatorname{Out}(H)$-submodules $V/b_H(M)$ of $\overline{M}(H)$.*
(2) *For any simple $\mathbb{K}\operatorname{Out}(H)$-module V, the multiplicity of V in $\overline{M}(H)/\operatorname{Jac}(\overline{M}(H))$ is equal to the multiplicity of $S_{H,V}$ in $M/\operatorname{Jac}(M)$.*

Proof. (1) Let $V/b_H(M)$ be a maximal $\mathbb{K}\operatorname{Out}(H)$-submodule of $\overline{M}(H)$. From 4.3 it is enough to show that $I = M$ where I is the functor defined as

$$I = \operatorname{Im}^M_{H,M(H)} + \operatorname{Ker}^M_{H,V}.$$

Assume that $I \neq M$. Then M/I is nonzero, and so it has a minimal group K. It follows from 4.2 that $\overline{M}(K) \neq 0$. The condition on H implies that K has no proper sections isomorphic to H. Moreover, K is not isomorphic to H because $I(H) = M(H)$ and K is a minimal group of M/I. We will show that

$$M(K) = \operatorname{Ker}^M_{H,V}(K) \subseteq I(K),$$

which contradicts the fact that K is a minimal group of M/I: The group K has no sections isomorphic to H so that for any $f \in \mathrm{Hom}(K, H)$ we have

$$M(f)\big(M(K)\big) \subseteq b_H(M) \subseteq V.$$

Therefore $M(K) \subseteq \mathrm{Ker}^M_{H,V}(K)$, as desired.

(2) Let n be the multiplicity of V and m be the multiplicity of $S_{H,V}$. The inequality $m \leqslant n$ is known from 4.4. There are n maximal $\mathbb{K}\mathrm{Out}(H)$-submodules $V_1, \ldots, V_n$ of $M(H)$ containing $b_H(M)$ such that each quotient $M(H)/V_i$ is isomorphic to V and such that the product of natural homomorphisms

$$\phi : M(H) \to \prod_{i=1}^{n} M(H)/V_i$$

is surjective. From the first part, we know that each $J_i = \mathrm{Ker}^M_{H,V_i}$ is a maximal subfunctor of M and M/J_i is isomorphic to $S_{H,V}$. We will show that the product of natural homomorphisms

$$\psi : M \to \prod_{i=1}^{n} M/J_i$$

is surjective, which gives the inequality $n \leqslant m$. For this end, we first put

$$\widetilde{J}_i = \bigcap_{j=1:\ j \neq i}^{n} J_j$$

for any i. Surjectivity of ψ will follow if we show that $J_i + \widetilde{J}_i = M$ for any i. Indeed, if the sum $J_i + \widetilde{J}_i$ is not M, then $\widetilde{J}_i \subseteq J_i$ (as J_i is a maximal subfunctor of M), implying that $\widetilde{J}_i(H) \subseteq J_i(H)$, equivalently

$$\bigcap_{j=1:\ j \neq i}^{n} V_j \subseteq V_i.$$

But then, for any v in $M(H)$ which is not in V_i, the element of

$$\prod_{i=1}^{n} M(H)/V_i$$

whose j-components are all equal to 0 for $j \neq i$ and whose i-component is $v + V_i$ has no preimage under the map ϕ, contradicting to the surjectivity of ϕ. $\square$

Proposition 4.6. *Let M be a semisimple functor and H be a group in $\mathcal{X}$. Then:*

1. *The maps $J \to J(H)$ and $\mathrm{Ker}^M_{H,V} \leftarrow V$ define a bijective correspondence between the maximal subfunctors J of M satisfying the property that H is a minimal group of M/J, and the maximal $\mathbb{K}\mathrm{Out}(H)$-submodules $V/b_H(M)$ of $\overline{M}(H)$.*
2. *$\overline{M}(H)$ is a semisimple $\mathbb{K}\mathrm{Out}(H)$-module.*
3. *For any simple $\mathbb{K}\mathrm{Out}(H)$-module V, the multiplicity of V in $\overline{M}(H)$ is equal to the multiplicity of $S_{H,V}$ in M.*

Proof. (1) follows from 3.5 and 4.3.

(2) As M is semisimple, it follows from 3.3 that $M(H)$ is a semisimple $\mathrm{End}(H)$-module, and so $M(H)/I_H M$ is a semisimple $\mathbb{K}\,\mathrm{Out}(H)$-module where I_H is the ideal of $\mathrm{End}(H)$ described in 2.5. We now obtain the result by using 4.1 stating that $\overline{M}(H)$ is a quotient of $M(H)/I_H M$.

(3) follows from the previous parts, 4.4, and the proof of the second part of 4.5. □

For an arbitrary functor M, which is not necessarily semisimple, let

$$b_H(M)/b_H(M) = V_0/b_H(M) \subset V_1/b_H(M) \subset \cdots \subset V_n/b_H(M) = \overline{M}(H)$$

be a composition series of the $\mathbb{K}\,\mathrm{Out}(H)$-module $\overline{M}(H)$. Letting $M_i = \mathrm{Ker}^M_{H,V_i}$ for each i, we obtain a series

$$\mathrm{Ker}^M_{H,b_H(M)} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of the functor M. We see for each i that $\mathrm{Ker}_{H,0}$ subfunctor of the quotient M_i/M_{i-1} is 0, and then we deduce by using 3.2 that M_i/M_{i-1} has a unique simple subfunctor, namely its $\mathrm{Im}_{H,(M_i/M_{i-1})(H)}$ subfunctor which is isomorphic to $S_{H,V_i/V_{i-1}}$. We then conclude that the multiplicity of $S_{H,V_i/V_{i-1}}$ in M_i/M_{i-1} is 1 because $(M_i/M_{i-1})(H) \cong V_i/V_{i-1}$. Thus, we justified the following.

Proposition 4.7. *Let M be a functor, H be a group in $\mathcal{X}$, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. Then, the multiplicity of V in $\overline{M}(H)$ is equal to the multiplicity of $S_{H,V}$ in $M/\mathrm{Ker}^M_{H,b_H(M)}$. In particular, the multiplicity of V in $\overline{M}(H)$ is less than or equal to the multiplicity of $S_{H,V}$ in M.*

Corollary 4.8. *Let M be a functor and $H \in \mathcal{X}$ be a group with $b_H(M) = 0$. For any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $M(H)$ is equal to the multiplicity of $S_{H,V}$ in M.*

Proof. Put $T = \mathrm{Ker}^M_{H,b_H(M)}$. As $T(H) = b_H(M) = 0$, we see that T has no composition factor having H as a minimal group. Therefore, the multiplicities of $S_{H,V}$ in M and M/T are equal. The result follows from 4.7. □

The following result (in which $b_H(M) = 0$) is an immediate consequence of 4.8.

Corollary 4.9. *Let M be a functor and $H \in \mathcal{X}$ be a minimal group of M. For any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $M(H)$ is equal to the multiplicity of $S_{H,V}$ in M.*

Remark 4.10. Let M be a functor, H be a group in $\mathcal{X}$, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. Then:

(1) The multiplicity of $S_{H,V}$ in M is equal to the multiplicity of the simple $\mathrm{End}(H)$-module V in the $\mathrm{End}(H)$-module $M(H)$.
(2) The multiplicity of $S_{H,V}$ in M is less than or equal to the multiplicity of the simple $\mathbb{K}\,\mathrm{Out}(H)$-module V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $M(H)$.

Proof. (1) For any simple functor $S_{H,V}$ on $\mathcal{X}$ over any field $\mathbb{K}$ (which is not assumed to be algebraically closed), it follows from 3.8 that the endomorphism algebras $\mathrm{End}_{\mathfrak{F}}(S_{H,V})$ and $\mathrm{End}_{\mathrm{End}(H)}(V)$ are isomorphic. Moreover, let $P(V)$ be the projective cover of the simple $\mathrm{End}(H)$-module V and let M be a functor on $\mathcal{X}$ over $\mathbb{K}$. It follows from 2.3 that the $\mathbb{K}$-spaces $\mathrm{Hom}_{\mathfrak{F}}(L_{H,P(V)}, M)$ and $\mathrm{Hom}_{\mathrm{End}(H)}(P(V), M(H))$ are isomorphic. Therefore the result follows.

(2) Evaluating a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of M at H yields a series

$$0 = M_0(H) \subseteq M_1(H) \subseteq \cdots \subseteq M_n(H) = M(H)$$

of $M(H)$ considered as an $\mathrm{End}(H)$-module. Indeed, it follows from 3.3 that each

$$M_i(H)/M_{i-1}(H)$$

is a simple $\mathrm{End}(H)$-module (if nonzero), which is also a simple $\mathbb{K}\mathrm{Out}(H)$-module isomorphic to V if the simple functor M_i/M_{i-1} is isomorphic to $S_{H,V}$. The result follows. $\square$

For a functor having a unique maximal subfunctor (i.e., a quotient functor of a projective indecomposable functor) we have the following result.

Remark 4.11. If a functor M has a unique maximal subfunctor, then there is a group H, unique up to isomorphism, in $\mathcal{X}$ such that $\overline{M}(H)$ has a unique maximal $\mathbb{K}\mathrm{Out}(H)$-submodule and such that $\mathrm{Im}^M_{H,M(H)} = M$.

Proof. Suppose that M has a unique maximal subfunctor J. Let H be a minimal group of the simple functor $S = M/J$. By the definition of Im subfunctors, the $\mathrm{Im}_{H,S(H)}$ subfunctor of S is equal to

$$\left(\mathrm{Im}^M_{H,M(H)} + J\right)/J,$$

which is also equal to M/J (see 3.3). As J is the unique maximal subfunctor of M we obtain that $\mathrm{Im}^M_{H,M(H)} = M$. Moreover, it follows from 4.3 that $\overline{M}(H)$ has a unique maximal $\mathbb{K}\mathrm{Out}(H)$-submodule.

Conversely, let H be a group satisfying the required properties. Then, 4.3 implies that M has a maximal subfunctor whose simple quotient has H as a minimal group. $\square$

Furthermore, for a functor M whose subfunctor lattice is a (possibly infinite) chain (i.e., M is a uniserial functor), one may see that if $\overline{M}(H)$ and $\overline{M}(K)$ are both nonzero for some groups H and K then one of the groups H and K must be isomorphic to a section of the other.

The next result is an easy consequence of definitions and the decomposition of a transitive biset given in 2.2.

Remark 4.12. Let M be a functor, H and K be groups in $\mathcal{X}$, and let V be a $\mathbb{K}\mathrm{Out}(H)$-submodule of $M(H)$ containing $b_H(M)$.

(1) If M is a biset or an inflation functor, then

$$b_H(M) = \sum_{P<H} \mathrm{Ind}^H_P M(P) + \sum_{N \trianglelefteq H:\ N \neq 1} \mathrm{Inf}^H_{H/N} M(H/N).$$

(2) If M is a Mackey or a deflation functor, then

$$b_H(M) = \sum_{P<H} \mathrm{Ind}^H_P M(P).$$

(3) If M is a biset or a deflation functor, then

$$\mathrm{Ker}^M_{H,V}(K) = \bigcap_{A,B,f} \left\{x \in M(K)\colon\ \mathrm{Iso}^H_{A/B}(f)\,\mathrm{Def}^A_{A/B}\,\mathrm{Res}^K_A\, x \in V\right\}$$

where A ranges over all subgroups of K, and B ranges over all normal subgroups of A such that the quotient group A/B is isomorphic to H, and f ranges over all isomorphisms from A/B to H.

(4) If M is a Mackey or an inflation functor, then

$$\mathrm{Ker}^M_{H,V}(K) = \bigcap_{A,f} \bigl\{x \in M(K)\colon\ \mathrm{Iso}^H_A(f)\,\mathrm{Res}^K_A\, x \in V\bigr\}$$

where A ranges over all subgroups of K isomorphic to H, and f ranges over all isomorphisms from A to H.

5. Radicals of Burnside functors

We now use the results of the previous section to study the radicals of Burnside functors. By a functor we mean any of biset functor, inflation functor, (global) Mackey functor, or deflation functor, defined on $\mathcal{X}$ over $\mathbb{K}$.

We begin with recalling the definitions of Burnside algebras and the maps making them a functor, see [2,3,10]. For a finite group H, the set of isomorphism classes of finite H-sets form a commutative semiring under the operations disjoint union and cartesian product. The associated Grothendieck ring $\mathrm{B}_{\mathbb{Z}}(H)$ is called the Burnside ring of H. The Burnside algebra of H over $\mathbb{K}$ is the $\mathbb{K}$-algebra $\mathrm{B}_{\mathbb{K}}(H) = \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{B}_{\mathbb{Z}}(H)$. Therefore, letting V runs over representatives of the conjugacy classes of subgroups of H, then $[H/V]$ comprise (without repetition) a $\mathbb{K}$-basis of $\mathrm{B}_{\mathbb{K}}(H)$, where the notation $[H/V]$ denotes the isomorphism class of transitive H-sets whose stabilizers are H-conjugates of V. The collection of Burnside algebras form a functor with the following morphisms:

$$\mathrm{Ind}^G_H\bigl([H/V]\bigr) = [G/V], \qquad \mathrm{Inf}^G_{G/N}\bigl(\bigl[(G/N)/(V/N)\bigr]\bigr) = [G/V], \qquad \mathrm{Iso}^K_H(f)\bigl([H/U]\bigr) = \bigl[K/f(U)\bigr],$$
$$\mathrm{Def}^G_{G/N}\bigl([G/V]\bigr) = \bigl[(G/N)/(NV/N)\bigr], \qquad \mathrm{Res}^G_H\bigl([G/W]\bigr) = \sum_{HgW \subseteq G} \bigl[H/\bigl(H \cap {}^gW\bigr)\bigr].$$

The product in the algebra $\mathrm{B}_{\mathbb{K}}(G)$ of any basis elements $[G/H]$ and $[G/W]$ is given by

$$[G/H][G/W] = \sum_{HgW \subseteq G} \bigl[G/\bigl(H \cap {}^gW\bigr)\bigr],$$

which is equal to $\mathrm{Ind}^G_H \mathrm{Res}^G_H[G/W]$. Therefore, we have the next result, see [2], Section 8 of [3], and Lemma 3.3 of [9].

Remark 5.1. $M(G)$ is an ideal of the commutative algebra $\mathrm{B}_{\mathbb{K}}(G)$ for any subfunctor M of $\mathrm{B}_{\mathbb{K}}$ and any group G. In particular, a $\mathbb{K}$-linear combination of mutually orthogonal idempotents of $\mathrm{B}_{\mathbb{K}}(G)$ is in $M(G)$ if and only if each idempotent in the linear combination is in $M(G)$.

Using 4.12 we can easily obtain the Brauer quotients of Burnside functors as follows.

Remark 5.2.

(1) Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset or an inflation functor on $\mathcal{X}$. Then, $b_G(\mathrm{B}_{\mathbb{K}}) = \mathrm{B}_{\mathbb{K}}(G)$ for any $G \in \mathcal{X}$ with $G \neq 1$ and $b_1(\mathrm{B}_{\mathbb{K}}) = 0$, in particular, the Brauer quotients of $\mathrm{B}_{\mathbb{K}}$ at nontrivial groups are all zero.

(2) Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation or a Mackey functor on $\mathcal{X}$. Then,

$$b_G(\mathrm{B}_{\mathbb{K}}) = \bigoplus_{V <_G G} \mathbb{K}[G/V]$$

for any $G \in \mathcal{X}$ with $G \neq 1$ and $b_1(\mathrm{B}_{\mathbb{K}}) = 0$, in particular, the Brauer quotient of $\mathrm{B}_{\mathbb{K}}$ at any group $X \in \mathcal{X}$ is the trivial $\mathbb{K}\,\mathrm{Out}(X)$-module.

If we consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset or an inflation functor on $\mathcal{X}$ over $\mathbb{K}$ then 5.2 and 4.5 imply that $\mathrm{B}_{\mathbb{K}}$ has a unique maximal subfunctor which is $\mathrm{Ker}_{1,0}$, whose corresponding quotient is isomorphic to the simple functor $S_{1,\mathbb{K}}$. Indeed, this is a consequence of a well-known result. From the definitions of functors $L_{X,V}$ described in Section 2, we easily see that $\mathrm{B}_{\mathbb{K}}$ is isomorphic to the functor $L_{1,\mathbb{K}}$ (see [2]), and so 2.3 and 2.6 imply that $\mathrm{B}_{\mathbb{K}}$ is the projective cover of the simple functor $S_{1,\mathbb{K}}$, in particular $\mathrm{B}_{\mathbb{K}}$ has a unique maximal subfunctor. Here we investigate the maximal subfunctors of $\mathrm{B}_{\mathbb{K}}$ considered as a deflation or a Mackey functor.

Proposition 5.3. *Let $\mathbb{K}$ be of characteristic $p > 0$, and let any group in $\mathcal{X}$ be a p-group. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation (respectively, a Mackey) functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathrm{B}_{\mathbb{K}}$ has a maximal subfunctor J, then $\mathrm{B}_{\mathbb{K}}/J \cong S_{H,\mathbb{K}}$ for some $H \in \mathcal{X}$ and there is no group in $\mathcal{X}$ having a proper section (respectively, a proper subgroup) isomorphic to H.*

Proof. We give a proof for deflation functors. The same proof works also for Mackey functors. Let J be a maximal subfunctor of M where $M = \mathrm{B}_{\mathbb{K}}$, and let H be a minimal group of the simple functor M/J (which is unique up to isomorphism). For any group G, we know from 5.2 that the Brauer quotient $\overline{M}(G)$ is the trivial $\mathbb{K}\,\mathrm{Out}(G)$-module. Then 4.3 implies that

$$J = \mathrm{Ker}^M_{H,b_H(M)}, \qquad M/J \cong S_{H,\mathbb{K}}$$

and that $I + J = M$ where

$$I = \mathrm{Im}^M_{H,M(H)}.$$

Take any group K in $\mathcal{X}$. Then $I(K) + J(K) = M(K)$. It is well known from [10] that $M(K)$ is a local $\mathbb{K}$-algebra so that it has a unique maximal ideal. As $I(K)$ and $J(K)$ are ideals of $M(K)$ by 5.1, we must have that

$$M(K) = I(K) \quad \text{or} \quad M(K) = J(K).$$

Suppose for a moment that K has a proper section isomorphic to H, say $B \trianglelefteq A \leqslant K$ and $A/B \cong H$ and $|H| < |K|$. Then $M(K) \neq J(K)$, because

$$\mathrm{Iso}^H_{A/B}\,\mathrm{Def}^A_{A/B}\,\mathrm{Res}^K_A[K/K] = [H/H] \notin b_H(M)$$

which, together with 4.12, imply that $[K/K] \notin J(K)$. We now also observe that $M(K) \neq I(K)$ which finishes the proof. Indeed, since $|H| < |K|$ the group H has no sections isomorphic to K. Definitions of Im subfunctors and Brauer quotients imply then that

$$I(K) \subseteq b_K(M) \neq M(K). \qquad \square$$

The above results shows that the Burnside functor, considered as a deflation or a Mackey functor on the family of all finite p-groups over a field of characteristic $p > 0$, has no maximal subfunctors. Over arbitrary characteristics and families we have the following result.

Theorem 5.4. *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. Then, $\mathrm{B}_{\mathbb{K}}$ has a maximal subfunctor J satisfying the property that H is a minimal group of the simple functor $\mathrm{B}_{\mathbb{K}}/J$ if and only if for any $K \in \mathcal{X}$ having a subgroup isomorphic to H and for any subgroup A of K isomorphic to H the index $|N_K(A) : A|$ is not divisible by the characteristic of the field $\mathbb{K}$.*

Proof. As in the proof of 5.3, we see by using 5.2 and 4.3 that maximal subfunctors of M are precisely the subfunctors J_X with $X \in \mathcal{X}$ satisfying the property $S_X + J_X = M$ where

$$M = \mathrm{B}_{\mathbb{K}}, \qquad S_X = \mathrm{Im}^M_{X,M(X)}, \qquad J_X = \mathrm{Ker}^M_{X,b_X(M)}.$$

Moreover, if J_X is a maximal subfunctor of M then $M/J_X \cong S_{X,\mathbb{K}}$.

We know from 5.1 that $S_X(K) + J_X(K)$ is an ideal of $M(K)$ for any $K \in \mathcal{X}$. Hence, it is equal to $M(K)$ if and only if it contains the unity $[K/K]$ of the algebra $M(K)$. Furthermore, 4.12 implies that $J_X(K) = M(K)$ if K has no subgroups isomorphic to X. Consequently, the condition $S_X + J_X = M$ is equivalent to the condition $[K/K] \in S_X(K) + J_X(K)$ for all $K \in \mathcal{X}$ having a subgroup isomorphic to X.

Suppose that M has a maximal subfunctor J. Then $J = J_H$ for some $H \in \mathcal{X}$ and M/J_H is isomorphic to $S_{H,\mathbb{K}}$. Take any group K in $\mathcal{X}$ having a subgroup isomorphic to H. Then, as $[K/K] \in S_H(K) + J_H(K)$ there is an $x_K \in S_H(K)$ such that $[K/K] - x_K \in J_H(K)$, and 4.12 implies that

$$\mathrm{Iso}^H_A(f)\,\mathrm{Res}^K_A\big([K/K] - x_K\big) = [H/H] - \mathrm{Iso}^H_A(f)\,\mathrm{Res}^K_A(x_K) \in b_H(M) = \bigoplus_{V <_H H} \mathbb{K}[H/V]$$

for any subgroup A of K isomorphic to H and any isomorphism f from A to H.

We will show that we may assume

$$x_K \in \bigoplus_{U \leqslant_K K:\ U \cong H} \mathbb{K}[K/U].$$

Indeed, as $x_K \in S_H(K)$ it follows that x_K is in the sum of the spaces

$$\big[(K \times H)/L\big]M(H)$$

where $L \leqslant K \times H$ with $k_1(L) = k_2(L) = 1$. Firstly, we observe that if Y, Z, T, D, E are groups with $D \leqslant Y \times Z$ and $E \leqslant Z \times T$ then it follows easily that

$$k_2(E) \leqslant k_2(D * E) \leqslant p_2(D * E) \leqslant p_2(E),$$

in particular $|q(D * E)| \leqslant |q(E)|$. If $T = Y$ and $|q(E)| < |Y|$ then this observation implies that the product

$$\big[(Y \times Z)/D\big]\big[(Z \times Y)/E\big]$$

is in the ideal I_Y of $\mathrm{End}(Y)$ described in 2.5. Therefore, if $L \leqslant K \times H$ with $k_1(L) = k_2(L) = 1$ and $|q(L)| < |H|$ then the product

$$\mathrm{Iso}^H_A(f)\,\mathrm{Res}^K_A\big[(K \times H)/L\big]$$

is in the ideal I_H of $\mathrm{End}(H)$, so from 4.1 we get that

$$\mathrm{Iso}^H_A(f)\,\mathrm{Res}^K_A\big[(K \times H)/L\big]M(H) \subseteq I_H M \subseteq b_H(M).$$

Hence, we may assume that

$$x_K \in \sum_{B \leqslant K:\ B \cong H} \mathrm{Ind}^K_B M(B).$$

Now, if a transitive basis element $[K/U]$ of $M(K)$ appears in the decomposition of x_K then U is a subgroup of a group $B \leqslant K$ with $B \cong H$, and if

$$\mathrm{Iso}_A^H(f)\,\mathrm{Res}_A^K[K/U] = \sum_{AgU \subseteq K} \mathrm{Iso}_A^H(f)\big[A/\big(A \cap {}^gU\big)\big] \notin b_H(M)$$

then $A \cap {}^gU = A \cong H$ for some $g \in K$, and as $|U| \leqslant |H|$ we see that ${}^gU = A$ and so $U \cong H$.

Thus we may assume that

$$x_K = \sum_{U \leqslant_K K\colon U \cong H} \lambda_U [K/U]$$

for some $\lambda_U \in \mathbb{K}$. Then we see from the preceding paragraph that

$$\begin{aligned}\mathrm{Iso}_A^H(f)\,\mathrm{Res}_A^K(x_K) + b_H(M) &= \lambda_A\,\mathrm{Iso}_A^H(f)\,\mathrm{Res}_A^K\big([K/A]\big) + b_H(M)\\ &= \lambda_A\big|N_K(A) : A\big|[H/H] + b_H(M).\end{aligned}$$

As a result, we must have for any $A \leqslant K$ with $A \cong H$ that $\lambda_A|N_K(A) : A| = 1$ or $|N_K(A) : A|$ is not divisible by the characteristic of the field $\mathbb{K}$.

Conversely, suppose that the condition on the indexes are satisfied. We will show that J_H is a maximal subfunctor of M by illustrating that $S_H + J_H = M$. Indeed, for any K having a subgroup isomorphic to H if we let

$$x_K = \sum_{U \leqslant_K K\colon U \cong H} \big|N_K(U) : U\big|^{-1}[K/U] \in S_H(K),$$

then it follows from what we observed in the first part of the proof that $[K/K] - x_K \in J_H(K)$ so that $S_H + J_H = M$. Thus J_H is a maximal subfunctor of M, and clearly M/J_H is isomorphic to $S_{H,\mathbb{K}}$. □

Manipulating the proof of 5.4 one may obtain the following result.

Remark 5.5. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then, $\mathrm{B}_{\mathbb{K}}$ has a maximal subfunctor J satisfying the property that H is a minimal group of the simple functor M/J if and only if the following conditions hold:

(i) Any group in $\mathcal{X}$ having a section isomorphic to H has a subgroup isomorphic to H.
(ii) For any $G \in \mathcal{X}$ having a subgroup isomorphic to H and for any subgroup U of G isomorphic to H the index $|N_G(U) : U|$ is nonzero in the field $\mathbb{K}$.
(iii) For any group $G \in \mathcal{X}$ having a section isomorphic to H and for any section P/Q of G isomorphic to H,

$$\sum_{U \leqslant_G G\colon U \cong H} \frac{|\{PgU \subseteq G\colon (P \cap {}^gU)Q = P\}|}{|N_G(U) : U|} = 1$$

in the field $\mathbb{K}$.

We now obtain some consequences of 5.4.

It is known that any Mackey functor over any field of characteristic 0 is semisimple [15]. In the next result we show that more is true for the Burnside functor.

Corollary 5.6. *Consider the Burnside functor* $\mathrm{B}_{\mathbb{K}}$ *as a Mackey functor on* $\mathcal{X}$ *over* $\mathbb{K}$*. If* $\mathbb{K}$ *is of characteristic* $p \geqslant 0$ *and if any group in* $\mathcal{X}$ *is a* p'*-group, then* $\mathrm{B}_{\mathbb{K}}$ *is semisimple and*

$$\mathrm{B}_{\mathbb{K}} \cong \bigoplus_{H} S_{H,\mathbb{K}}$$

where H *ranges over a complete set of isomorphism classes of all groups in* $\mathcal{X}$*.*

Proof. From 5.4 and its proof, we know in this case that the maximal subfunctors of M are precisely J_H with $H \in \mathcal{X}$ where

$$M = \mathrm{B}_{\mathbb{K}}, \qquad J_H = \mathrm{Ker}^M_{H,b_H(M)},$$

and each quotient M/J_H is isomorphic to $S_{H,\mathbb{K}}$. As each $\overline{M}(H)$ is a trivial $\mathbb{K}\,\mathrm{Out}(H)$-module, it follows from 4.4 that the multiplicity of $S_{H,\mathbb{K}}$ in $M/\mathrm{Jac}(M)$ is 1. We will show that the intersection $J = \mathrm{Jac}(M)$ of all functors J_H where H ranges over all groups in $\mathcal{X}$ is 0, which completes the proof. Indeed, let K be a group in $\mathcal{X}$ and let x be an element of $J(K)$. Write x as a linear combination of transitive K-sets, say

$$x = \sum_{V \leqslant_K K} \lambda_V [K/V].$$

Take a maximal element U of the set $\{V \leqslant_K K\colon \lambda_V \neq 0\}$. Note that such a maximal element U exits unless x is zero. As $x \in J(K) \subseteq J_U(K)$, it follows from 4.12 that $\mathrm{Res}^K_U(x) \in b_U(M)$. But we see that

$$\mathrm{Res}^K_U(x) + b_U(M) = \lambda_U \big|N_K(U) : U\big| [U/U] + b_U(M),$$

and so $\mathrm{Res}^K_U(x) \in b_U(M)$ implies that $\lambda_U |N_K(U) : U| = 0$. Since K is a p'-group, λ_U must be zero, implying that $x = 0$. □

Let M be the simple deflation (respectively inflation) functor $S_{H,V}$. Considering M as a Mackey functor we may see that $M = \mathrm{Im}^M_{H,V}$ (respectively $\mathrm{Ker}^M_{H,0} = 0$). It can be deduced from 3.1 that the simple deflation functor M has a unique maximal Mackey subfunctor whose quotient is isomorphic to the simple Mackey functor $S_{H,V}$, and that the simple inflation functor M has a unique simple Mackey subfunctor isomorphic to the simple Mackey functor $S_{H,V}$. See Propositions 3.8 and 7.6 of [16]. In the case of 5.6 each quotient functor of a subfunctor of $\mathrm{B}_{\mathbb{K}}$ is semisimple as a Mackey functor, and so the next result follows. See also Section 10, especially (the proof of) 10.4.

Corollary 5.7. *Consider the Burnside functor* $\mathrm{B}_{\mathbb{K}}$ *as a deflation* (*respectively, an inflation*) *functor on* $\mathcal{X}$ *over* $\mathbb{K}$*. If* $\mathbb{K}$ *is of characteristic* $p \geqslant 0$ *and if any group in* $\mathcal{X}$ *is a* p'*-group, then composition factors of* $\mathrm{B}_{\mathbb{K}}$ *are precisely the simple deflation* (*respectively, inflation*) *functors* $S_{H,\mathbb{K}}$*, with multiplicities equal to one, where* H *ranges over a complete set of isomorphism classes of all groups in* $\mathcal{X}$*.*

Let M be the simple biset functor $S_{H,V}$. Considering M as an inflation (respectively a deflation) functor we may see that $M = \mathrm{Im}^M_{H,V}$ (respectively $\mathrm{Ker}^M_{H,0} = 0$). It can be deduced from 3.1 that the simple biset functor M has a unique maximal inflation subfunctor whose quotient is isomorphic to the simple inflation functor $S_{H,V}$, and that the simple biset functor M has a unique simple deflation subfunctor isomorphic to the simple deflation functor $S_{H,V}$. See Propositions 3.12 and 7.6 of [16]. See also Section 10 for more details. Therefore, the following is an easy consequence of 5.7.

Corollary 5.8. *(See [5, Proposition 5.5.1].) Let $\mathbb{K}$ be of characteristic $p \geqslant 0$ and let any group in $\mathcal{X}$ be a p'-group. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. If a simple functor $S_{H,V}$ appears as a composition factor of $\mathrm{B}_{\mathbb{K}}$ with multiplicity n, then $n \leqslant 1$ and $V = \mathbb{K}$, the trivial $\mathbb{K}\,\mathrm{Out}(H)$-module.*

More is known in the case of the previous result (see [5, Proposition 5.5.1]). Indeed, it is shown in [2] by using the properties of the primitive idempotents of Burnside algebras that composition factors of the Burnside functor, considered as a biset functor on all finite groups over a field of characteristic 0, are precisely the simple functors $S_{H,\mathbb{K}}$ where H ranges over some groups called b-groups. Moreover, all subfunctors of the Burnside functor, considered as a biset functor on all finite p-groups over a field of characteristic $q \neq p$, are found in [3] explicitly.

We next investigate semisimplicity of the Burnside functor.

Corollary 5.9. *Let $\mathbb{K}$ be of characteristic $p \geqslant 0$.*

(1) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathrm{B}_{\mathbb{K}}$ is semisimple, then any group in $\mathcal{X}$ is a p'-group.*
(2) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Suppose that $\mathrm{B}_{\mathbb{K}}$ is semisimple. If $p = 0$ then any group in $\mathcal{X}$ is trivial. Moreover, if $p > 0$ then p divides $|G| - 1$ for any $G \in \mathcal{X}$.*
(3) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathrm{B}_{\mathbb{K}}$ is semisimple, then any group in $\mathcal{X}$ is trivial.*
(4) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathrm{B}_{\mathbb{K}}$ is semisimple, then any group G in $\mathcal{X}$ is a cyclic group such that $\varphi(|G|)$ is not divisible by p where φ denotes the Euler totient function.*

Proof. We put $M = \mathrm{B}_{\mathbb{K}}$. Suppose that M is semisimple.

(1) It follows from 4.6 that $J = \mathrm{Ker}^M_{1,0}$ is a maximal subfunctor of M and that 1 is a minimal group of M/J. For any $G \in \mathcal{X}$ we obtain from 5.4 that $|G|$ is not divisible by p.

(2) As in the first part we see from 4.6 that M has a maximal subfunctor J and that 1 is a minimal group M/J. For any $G \in \mathcal{X}$, if we apply the condition (iii) of 5.5 to the section G/G of G then we obtain that $\frac{1}{|G|} = 1$ in $\mathbb{K}$. Therefore, the result follows.

(3) For any nontrivial group H, the Brauer quotient $\overline{M}(H)$ is zero. So, 4.6 implies that M is isomorphic to $S_{1,\mathbb{K}}$. In particular, $T = 0$ where $T = \mathrm{Ker}^M_{1,0}$ (see 3.3). Take any $G \in \mathcal{X}$ with $G \neq 1$.

If p does not divide $|G|$ then

$$x = \frac{1}{|G|}[G/1] - [G/G]$$

is nonzero element of $M(G)$, and using 4.12 we see that

$$x \in T(G) = \left\{x \in M(G)\colon\ \mathrm{Res}^G_1 x = 0\right\}.$$

If p divides $|G|$ then $x = [G/1]$ is nonzero element of $M(G)$ in $T(G)$.

(4) As in the previous part we see that $M \cong S_{1,\mathbb{K}}$. We will compare the dimensions of the $\mathbb{K}$-spaces $M(G)$ and $S_{1,\mathbb{K}}(G)$ for any $G \in \mathcal{X}$, and we use [14] to deduce the result. Take any $G \in \mathcal{X}$. Let A be the square matrix whose rows and columns are indexed by the conjugacy classes (H) of subgroups H of G, and let the entry of A in the (H)th row and in the (K)th column be the number of double cosets HgK of H and K in G. It follows from [2] that the dimension of $S_{1,\mathbb{K}}(G)$ is equal to the rank of the matrix A over $\mathbb{K}$. It is proved in [14] that the rank of A over any field of characteristic 0 is equal to the number of conjugacy classes cyclic subgroups G, and for a cyclic group G of order n it is proved in [14] that the determinant of A is equal to the product $\prod_d \varphi(d)$ where d ranges over divisors of n. The result follows. $\square$

Converses of each parts, except the second, of the previous result are all true. Indeed, the converse of the first part is 5.6, and it is clear from its justification that the converse of the fourth part is true. For the converse of the second part we may state the following.

If $\mathbb{K}$ is of characteristic $p > 0$ and if any group in $\mathcal{X}$ is an elementary abelian q-group for some prime number q such that p divides $q - 1$, then the Burnside functor $\mathrm{B}_{\mathbb{K}}$, considered as a deflation functor on $\mathcal{X}$ over $\mathbb{K}$, is semisimple. Indeed, in this case it follows from 5.7 that the composition factors of $\mathrm{B}_{\mathbb{K}}$ are precisely the simple deflation functors $S_{H,\mathbb{K}}$, with multiplicities equal to one. So it is enough to show that $S_{H,\mathbb{K}}$ appears in the head of $\mathrm{B}_{\mathbb{K}}$ for any $H \in \mathcal{X}$. This may proved easily by using 5.5.

As a biset or an inflation functor the Burnside functor defined on any family over any field is indecomposable (because the Brauer quotients of it at nontrivial groups are all equal to 0 so that it has a unique maximal subfunctor, or because we know from [2] that it is the projective cover of the simple functor $S_{1,\mathbb{K}}$). Considering it as a deflation or a Mackey functor we have the following result.

Remark 5.10. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation or a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathbb{K}$ is of characteristic $p > 0$ and if any group in $\mathcal{X}$ is a p-group, then $\mathrm{B}_{\mathbb{K}}$ is indecomposable.

Proof. Letting $M = \mathrm{B}_{\mathbb{K}}$, suppose that $M = M_1 \oplus M_2$ for some subfunctors M_1 and M_2 of M. We know from 5.1 that each $M_i(H)$ is an ideal of the algebra $M(H)$. As the dimension of $M(1)$ is 1, we may assume that $M_1(1) = M(1)$ and $M_2(1) = 0$. Let H be a minimal group of M_2. We see that $[H/1]$ is in $M_1(H)$. As $M_2(H)$ is nonzero, both of the ideals $M_1(H)$ and $M_2(H)$ of $M(H)$ are proper. Therefore, the sum of the ideals $M_1(H)$ and $M_2(H)$ cannot be equal to $M(H)$, because we know from [10] that $M(H)$ is a local algebra and so it has a unique maximal ideal. $\square$

6. Simple subfunctors and restriction kernels

Throughout this section also, by a functor we mean any of biset functor, inflation functor, (global) Mackey functor, or deflation functor, defined on $\mathcal{X}$ over $\mathbb{K}$.

We first recall the notion of the restriction kernel of a functor, see [15]. Let M be a functor and H be a group in $\mathcal{X}$, by the restriction kernel of M at H we mean the $\mathbb{K}$-module

$$\underline{M}(H) = \bigcap_{f,K} \mathrm{Ker}\big(f : M(H) \to M(K)\big)$$

where K rages over all groups in $\mathcal{X}$ having no sections isomorphic to H and f ranges in $\mathrm{Hom}(H, K)$. It is clear that $\underline{M}(H)$ is a $\mathbb{K}\,\mathrm{Out}(H)$-submodule of $M(H)$. Moreover, there is a $\mathbb{K}\,\mathrm{Out}(H)$-module isomorphism $\underline{M}(H) \cong (\overline{M^*}(H))^*$ induced by taking $\mathbb{K}$-duals, see [15]. Therefore, any result concerning Brauer quotients has a dual concerning restriction kernels. Our first aim is to collect these dual results in this section. We skip the proofs of similar results.

Remark 6.1. Let M be a functor and H be a group in $\mathcal{X}$. Then, the ideal I_H of $\mathrm{End}(H)$ described in 2.5 annihilates $\underline{M}(H)$ so that $\underline{M}(H)$ is also an $\mathrm{End}(H)$-submodule of $M(H)$ whose $\mathbb{K}\,\mathrm{Out}(H)$-submodules and $\mathrm{End}(H)$-submodules are the same.

The above result shows that the notation $\mathrm{Im}^M_{H,V}$ makes sense also for $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $\underline{M}(H)$, where $\mathrm{Im}^M_{X,W}$ subfunctors of a functor M are defined in Section 3 for $\mathrm{End}(X)$-submodules W of $M(X)$.

Theorem 6.2. *Let M be a functor and H be a group in $\mathcal{X}$. Then, the maps $J \to J(H)$ and $\mathrm{Im}^M_{H,V} \leftarrow V$ define a bijective correspondence between the smallest elements J of the set of all subfunctors I of M satisfying the property that H is a minimal group of I, and the simple $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $\underline{M}(H)$. Moreover, $\underline{M}(H) = 0$ if and only if M has no subfunctor having H as a minimal group.*

The subfunctors J mentioned in the above result may not be simple subfunctors of M.

Corollary 6.3. *Let M be a functor and H be a group in $\mathcal{X}$. Then, the maps $J \to J(H)$ and $\mathrm{Im}^M_{H,V} \leftarrow V$ define a bijective correspondence between the simple subfunctors J of M satisfying the property that H is a minimal group of J, and the simple $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $\underline{M}(H)$ satisfying for the property that*

$$\mathrm{Im}^M_{H,V} \cap \mathrm{Ker}^M_{H,0} = 0.$$

Corollary 6.4. *Let M be a functor and H be a group in $\mathcal{X}$. For any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, let n be the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $\mathrm{Soc}(\underline{M}(H))$ and let m be the multiplicity of the simple functor $S_{H,V}$ in $\mathrm{Soc}(M)$. Then, $m \leqslant n$. In particular, if $m \neq 0$ and $n = 1$ then $m = 1$.*

Proposition 6.5. *Let M be a functor and H be a group in $\mathcal{X}$. Suppose that $\underline{M}(H) \neq 0$ and that $\underline{M}(K) = 0$ for any group K in $\mathcal{X}$ having a proper section isomorphic to H. Then:*

(1) *The maps $J \to J(H)$ and $\mathrm{Im}^M_{H,V} \leftarrow V$ define a bijective correspondence between the simple subfunctors J of M satisfying the property that H is a minimal group of J, and the simple $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $\underline{M}(H)$.*
(2) *For any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, the multiplicity of V in $\mathrm{Soc}(\underline{M}(H))$ is equal to the multiplicity of $S_{H,V}$ in $\mathrm{Soc}(M)$.*

Proposition 6.6. *Let M be a semisimple functor and H be a group in $\mathcal{X}$. Then:*

(1) *The maps $J \to J(H)$ and $\mathrm{Im}^M_{H,V} \leftarrow V$ define a bijective correspondence between the simple subfunctors J of M satisfying the property that H is a minimal group of J, and the simple $\mathbb{K}\,\mathrm{Out}(H)$-submodules V of $\underline{M}(H)$.*
(2) *$\underline{M}(H)$ is a semisimple $\mathbb{K}\,\mathrm{Out}(H)$-module.*
(3) *For any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, the multiplicity of V in $\underline{M}(H)$ is equal to the multiplicity of $S_{H,V}$ in M.*

For an arbitrary functor M, which is not necessarily semisimple, let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \underline{M}(H)$$

be a composition series of the $\mathbb{K}\,\mathrm{Out}(H)$-module $\underline{M}(H)$. Letting $M_i = \mathrm{Im}^M_{H,V_i}$ for each i, we obtain a series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = \mathrm{Im}^M_{H,\underline{M}(H)}$$

of the functor M. We see for each i that $\mathrm{Im}_{H,(M_i/M_{i-1})(H)}$ subfunctor of the quotient M_i/M_{i-1} is M_i/M_{i-1}, and then we deduce by using 3.2 that M_i/M_{i-1} has a unique maximal subfunctor, namely its $\mathrm{Ker}_{H,0}$ subfunctor whose quotient is isomorphic to $S_{H,V_i/V_{i-1}}$. We then conclude that the multiplicity of $S_{H,V_i/V_{i-1}}$ in M_i/M_{i-1} is 1 because $(M_i/M_{i-1})(H) \cong V_i/V_{i-1}$. Thus, we justified the following.

Proposition 6.7. *Let M be a functor, H be a group in $\mathcal{X}$, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. Then, the multiplicity of V in $\underline{M}(H)$ is equal to the multiplicity of $S_{H,V}$ in $\mathrm{Im}^M_{H,\underline{M}(H)}$. In particular, the multiplicity of V in $\underline{M}(H)$ is less than or equal to the multiplicity of $S_{H,V}$ in M.*

Remark 6.8. Let M be a functor, H and K be groups in $\mathcal{X}$, and let V be a $\mathbb{K}\,\mathrm{Out}(H)$-submodule of $\underline{M}(H)$.

(1) If M is a biset or a deflation functor, then

$$\underline{M}(H)=\left(\bigcap_{P<H}\operatorname{Ker}\operatorname{Res}^H_P\right)\cap\left(\bigcap_{N\trianglelefteqslant H:\,N\neq 1}\operatorname{Ker}\operatorname{Def}^H_{H/N}\right).$$

(2) If M is a Mackey or an inflation functor, then

$$\underline{M}(H)=\bigcap_{P<H}\operatorname{Ker}\operatorname{Res}^H_P\,.$$

(3) If M is a biset or an inflation functor, then

$$\operatorname{Im}^M_{H,V}(K)=\sum_{A,B,f}\operatorname{Ind}^K_A\operatorname{Inf}^A_{A/B}\operatorname{Iso}^{A/B}_H(f)V$$

where A ranges over all subgroups of K, and B rages over all normal subgroups of A such that the quotient group A/B is isomorphic to H, and f ranges over all isomorphisms from H to A/B.

(4) If M is a Mackey or a deflation functor, then

$$\operatorname{Im}^M_{H,V}(K)=\sum_{A,f}\operatorname{Ind}^K_A\operatorname{Iso}^A_H(f)V$$

where A ranges over all subgroups of K isomorphic to H, and f ranges over all isomorphisms from H to A.

7. Socles of Burnside functors

In this section we try to describe simple subfunctors of the Burnside functor. For this end we need to describe simple submodules of restriction kernels. Finding restriction kernels of the Burnside functor is more difficult than finding its Brauer quotients. We find them below in a special case. We begin with the simple subfunctors parameterized by trivial modules.

Proposition 7.1. *Let $\mathbb{K}$ be of characteristic $p>0$ and let any group in $\mathcal{X}$ be a p-group. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. If $\mathrm{B}_{\mathbb{K}}$ has a simple subfunctor isomorphic to $S_{H,\mathbb{K}}$ for some H in $\mathcal{X}$, then there is no group in $\mathcal{X}$ which is a split extension of a nontrivial group by H.*

Proof. Let S be a simple subfunctor of M isomorphic to $S_{H,\mathbb{K}}$ where $M=\mathrm{B}_{\mathbb{K}}$. It follows from 6.3 that $S=\operatorname{Im}^M_{H,V}$ for some simple $\mathbb{K}\operatorname{Out}(H)$-submodule V of $\underline{M}(H)$ isomorphic to the trivial module, and that $S\cap J=0$ where $J=\operatorname{Ker}^M_{H,0}$. Assuming the existence of a $K\in\mathcal{X}$ having subgroups $1\neq N\trianglelefteqslant K\geqslant H$ with $N\cap H=1$, we will show that $S(K)\cap J(K)\neq 0$, which completes the proof.

Let $x=\operatorname{Ind}^K_H v$ where v is a nonzero element of V. Note that $x\in S(K)$ by 6.8. We will show that $x\in J(K)$ by justifying

$$\big[(H\times K)/L\big]x=0$$

for any $L\leqslant H\times K$ with $k_1(L)=k_2(L)=1$.

If $|q(L)|<|H|$ then, as in the proof of 5.4, we see that the product

$$\big[(H\times K)/L\big]\operatorname{Ind}^K_H$$

is in the ideal I_H of $\operatorname{End}(H)$ described in 2.5, and so we obtain from 6.1 that

$$\big[(H \times K)/L\big]x \in I_H v \subseteq I_H \underline{M}(H) = 0.$$

If $|q(L)| = |H|$, then 2.1 and 2.2 imply that

$$\big[(H \times K)/L\big] = \operatorname{Iso}_A^H(f)\operatorname{Res}_A^K$$

where $A = p_2(L)$ and $f : A \to H$ is an isomorphism, and that

$$\big[(H \times K)/L\big]x = \operatorname{Iso}_A^H(f)\operatorname{Res}_A^K \operatorname{Ind}_H^K v = \sum_{AgH \subseteq K} \operatorname{Iso}_A^H(f)\operatorname{Ind}_{A\cap {}^gH}^A \operatorname{Iso}_{{}^gA\cap H}^{A\cap {}^gH}(i_g)\operatorname{Res}_{A^g\cap H}^H v$$

where i_g is the conjugation by g. If A and H are not K-conjugate, then $A^g \cap H < H$ for any $g \in K$ (because $A \cong H$) and so

$$\operatorname{Res}_{A^g\cap H}^H v = 0.$$

Hence we conclude that $[(H \times K)/L]x = 0$ unless $A =_K H$. Now, let $A = H^a$ for some $a \in K$. By using 2.1 and 2.2 and using the fact that $\operatorname{Res}_X^H v = 0$ for any $X < H$, we see that

$$\big[(H \times K)/L\big]x = \big[(H \times K)/^{(1,a)}L\big]x = \operatorname{Iso}_H^H(f i_{a^{-1}})\operatorname{Res}_H^K \operatorname{Ind}_H^K v = \sum_{gH \subseteq N_K(H)} \operatorname{Iso}_H^H(f i_{a^{-1}} i_g)v,$$

implying that $[(H \times K)/L]x = 0$ because $\mathbb{K}\operatorname{Out}(H)$ acts on V trivially.

Having showed that $x \in S(K) \cap J(K)$, we finish by justifying that $x \neq 0$. Indeed, by using 2.1 and 2.2 we easily see that

$$\operatorname{Iso}_{K/N}^H \operatorname{Def}_{K/N}^K x = \operatorname{Iso}_{K/N}^H \operatorname{Def}_{K/N}^K \operatorname{Ind}_H^K v = v \neq 0,$$

and so $x \neq 0$. $\square$

One half of the following result already appeared in Lemme 2 of [6]. More explicitly, it is shown in [6] that if $\mathbb{K}$ is of characteristic $p > 0$ and G is a nontrivial p-group, then $\operatorname{Res}_P^G y = 0$ for any $P < G$ with $|G| \geqslant p^2$, and $\operatorname{Def}_{G/N}^G y = 0$ for any $N \trianglelefteq G$ with $N \neq 1$ where

$$y = [G/1] - \sum_{V \trianglelefteq G:\ |V| = p} [G/V] \in \mathrm{B}_{\mathbb{K}}(G).$$

We below show in addition that any element x of $\mathrm{B}_{\mathbb{K}}(G)$ mapped to zero under such maps Res_P^G and $\operatorname{Def}_{G/N}^G$ must be a $\mathbb{K}$-multiple of y.

Lemma 7.2. *Let $\mathbb{K}$ be of characteristic $p > 0$ and let any group in $\mathcal{X}$ be a p-group. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset or a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then,*

$$\underline{\mathrm{B}_{\mathbb{K}}}(G) = \mathbb{K}\Bigg([G/1] - \sum_{V \trianglelefteq G:\ |V| = p} [G/V]\Bigg)$$

for any $G \in \mathcal{X}$ with $|G| \neq p$, and $\underline{\mathrm{B}_{\mathbb{K}}}(G) = 0$ for any $G \in \mathcal{X}$ with $|G| = p$.

Proof. Let $G \in \mathcal{X}$ and $x \in \underline{M}(G)$ where $M = \mathrm{B}_{\mathbb{K}}$. From Lemma 7.11 of [17] we know that

$$\bigcap_{P<G} \operatorname{Ker} \operatorname{Res}_P^G \subseteq \bigoplus_{V \trianglelefteqslant G} \mathbb{K}[G/V].$$

So 6.8 implies that

$$x = \sum_{V \trianglelefteqslant G} \lambda_V [G/V]$$

for some constants λ_V, and that

$$0 = \operatorname{Def}_{G/N}^G x = \sum_{V \trianglelefteqslant G} \lambda_V \big[(G/N)/(NV/N)\big]$$

for any $N \trianglelefteqslant G$ with $N \neq 1$. In particular, the coefficient of $[(G/N)/(N/N)]$ in $\operatorname{Def}_{G/N}^G x$ is 0, implying that

$$0 = \sum_{V \trianglelefteqslant G:\ V \leqslant N} \lambda_V$$

for any $N \trianglelefteqslant G$ with $N \neq 1$.

For any $N \trianglelefteqslant G$ with $|N| = p$, we see from

$$0 = \sum_{V \trianglelefteqslant G:\ V \leqslant N} \lambda_V = \lambda_1 + \lambda_N$$

that $\lambda_N = -\lambda_1$.

We will show by induction on the order of N that $\lambda_N = 0$ for any $N \trianglelefteqslant G$ with $|N| \geqslant p^2$: Indeed, if $N \trianglelefteqslant G$ with $|N| = p^2$ then

$$0 = \sum_{V \trianglelefteqslant G:\ V \leqslant N} \lambda_V = \lambda_1 - |\mathcal{A}|\lambda_1 + \lambda_N$$

where $\mathcal{A} = \{V \trianglelefteqslant G:\ V \leqslant N,\ |V| = p\}$. We now observe that $|\mathcal{A}| \equiv 1 \pmod{p}$, which implies that $\lambda_N = 0$. Indeed, as N is normal in G, the group G acts on the set $\mathcal{B} = \{V \leqslant N:\ |V| = p\}$ by conjugation and its set of fixed elements $\mathcal{B}^G$ is equal to $\mathcal{A}$. It is well known (see, for instance, Theorem 4.8 of [13], stating that the number of subgroups of a p-group X having order equal to any fixed given number less than or equal to $|X|$ is congruent to 1 modulo p) that $|\mathcal{B}| \equiv 1 \pmod{p}$. As G is a p-group, $|\mathcal{B}| \equiv |\mathcal{B}^G| \pmod{p}$ and so $|\mathcal{A}| \equiv 1 \pmod{p}$, as desired.

We now assume that $\lambda_K = 0$ for all $K \trianglelefteqslant G$ with $p^2 \leqslant |K| \leqslant p^n$. Then, we have for any $N \trianglelefteqslant G$ with $|N| = p^{n+1}$ that

$$0 = \sum_{V \trianglelefteqslant G:\ V \leqslant N} \lambda_V = \lambda_1 - |\mathcal{C}|\lambda_1 + \left(\sum_{V \in \mathcal{D}} \lambda_V\right) + \lambda_N$$

where $\mathcal{C} = \{V \trianglelefteqslant G:\ V \leqslant N,\ |V| = p\}$ and $\mathcal{D} = \{V \trianglelefteqslant G:\ V \leqslant N,\ p^2 \leqslant |V| \leqslant p^n\}$. As in the preceding paragraph we may see that $|\mathcal{C}| \equiv 1 \pmod{p}$, and it follows from the induction hypothesis that $\lambda_V = 0$ for any $V \in \mathcal{D}$. Hence $\lambda_N = 0$.

As a result, we have just showed that if x in $\underline{M}(G)$ then $x=\lambda y$ for some λ in $\mathbb{K}$ where

$$y=[G/1]-\sum_{V\trianglelefteq G:\ |V|=p}[G/V].$$

Conversely, if $G\in\mathcal{X}$ with $|G|=p$ then $y=[G/1]-[G/G]$ and so $\mathrm{Res}^G_1 y_\lambda=\lambda[1/1]$, implying that $\underline{M}(G)=0$. Moreover, $y=[1/1]$ for $G=1$ and so $\underline{M}(1)=M(1)=\mathbb{K}y$. Let now $G\in\mathcal{X}$ with $|G|\geqslant p^2$. Then it follows from the result of [6] explained at the beginning of the present result that $\mathrm{Res}^G_P\, y=0$ for any $P<G$ and $\mathrm{Def}^G_{G/N}\, y=0$ for any $N\trianglelefteq G$ with $N\neq 1$. $\square$

Remark 7.3. Let $\mathbb{K}$ be of characteristic $p>0$ and $\mathcal{X}$ be any family of finite p-groups. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then:

(1) $\mathrm{B}_{\mathbb{K}}$ has a simple subfunctor isomorphic to $S_{1,\mathbb{K}}$.
(2) Suppose that $\mathrm{B}_{\mathbb{K}}$ has a simple subfunctor isomorphic to $S_{H,V}$ for some $H\in\mathcal{X}$ with $H\neq 1$. Then:
  (i) $V=\mathbb{K}$ and $|H|\neq p$.
  (ii) For any $K\in\mathcal{X}$ with $H\leqslant K$ and $|K:H|=p$, the element x^K_H of $\mathrm{B}_{\mathbb{K}}(K)$ is 0 where

$$x^K_H=\sum_{A\leqslant K:\ A\cong H}\mathrm{Ind}^K_A\left([A/1]-\sum_{V\trianglelefteq A:\ |V|=p}[A/V]\right)=0.$$

Proof. (1) Let S be the subfunctor of M generated by $M(1)=\underline{M}(1)=\mathbb{K}[1/1]\cong\mathbb{K}$ where $M=\mathrm{B}_{\mathbb{K}}$. That is $S=\mathrm{Im}^M_{1,M(1)}$. We see that $S(G)=\mathbb{K}[G/1]$ for any $G\in\mathcal{X}$. One may easily check that $S\cap K=0$ where $K=\mathrm{Ker}^M_{1,0}$. Then 6.3 implies that $S\cong S_{1,\mathbb{K}}$.

(2) First part follows from 7.2 and 6.3. Same results imply also that any simple subfunctor of M isomorphic to $S_{H,V}$ must be equal to $I=\mathrm{Im}^M_{H,\mathbb{K}y_H}$ and it must satisfy the property $I\cap J=0$ where

$$y_H=[H/1]-\sum_{V\trianglelefteq H:\ |V|=p}[H/V],\qquad J=\mathrm{Ker}^M_{H,0}.$$

By using 7.2 and its proof we easily see that $x^K_H\in I(K)\cap J(K)$ so that it must be zero. $\square$

The coefficient of $[K/1]$ in the element x^K_H of $\mathrm{B}_{\mathbb{K}}(K)$ defined in 7.3 is $|\{A\leqslant K:\ A\cong H\}|$, which is 1 in $\mathbb{K}$ if $\mathcal{X}$ contains (for instance) only elementary abelian p-groups. Thus, if we consider $\mathrm{B}_{\mathbb{K}}$ as a deflation functor on the family of all elementary abelian p-groups over $\mathbb{K}$ then it has a unique simple subfunctor isomorphic to $S_{1,\mathbb{K}}$. In the general case, one may see that $|\{A\leqslant K:\ A\cong H\}|$ is equal to $|\mathrm{Inj}(K,H)|/|\mathrm{Aut}(H)|$ where $\mathrm{Inj}(K,H)$ is the set of all injective group homomorphisms from K to H.

We use another result of [6] to see that the Burnside functor, considered as a biset functor on all finite p-groups over a field of characteristic $p>0$, has no simple subfunctor.

Proposition 7.4. *Let $\mathbb{K}$ be of characteristic $p>0$ and $\mathcal{X}$ be the family of all finite p-groups of order less than or equal to p^m where m is a natural number with $m\geqslant 2$. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. Then,*

$$\mathrm{Soc}(\mathrm{B}_{\mathbb{K}})\cong\bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^m in $\mathcal{X}$.

Proof. We know from 6.3 and 7.2 that simple subfunctors of M are precisely the subfunctors S_X, with $X \in \mathcal{X}$ and $|X| \neq p$, satisfying the property $S_X \cap K_X = 0$, where

$$M = \mathrm{B}_{\mathbb{K}}, \qquad S_X = \mathrm{Im}^M_{X,\underline{M}(X)}, \qquad K_X = \mathrm{Ker}^M_{X,0}.$$

We will show that $S_X \cap K_X \neq 0$ unless $|X| = p^m$: Indeed, let $p < |X| < p^m$ and choose a $Y \in \mathcal{X}$ with $X < Y$ and $|Y : X| = p$. Let

$$y_X = [X/1] - \sum_{V \trianglelefteq X:\ |V|=p} [X/V].$$

We know from 7.2 that $\underline{M}(X) = \mathbb{K} y_X$. In [6], a subfunctor F_X of M is defined and shown that F_X has a unique maximal subfunctor J_X (see Proposition 1 of [6]). We notice easily that these subfunctors F_X and J_X are indeed related to Im and Ker subfunctors studied here as follows:

$$F_X = S_X, \qquad J_X = S_X \cap K_X$$

(so that, for instance, the result of [6] mentioned above is an immediate consequence of 3.1). The result of [6] we want to use here is its Lemme 4 stating that

$$F_X(Y) = J_X(Y) \oplus \left(\sum_{N \leqslant Y:\ Y/N \cong X} \mathbb{K}\, \mathrm{Inf}^Y_{Y/N}\, y_{Y/N} \right).$$

Suppose for a moment that $S_X(Y) \cap K_X(Y) = 0$ so that $J_X(Y) = 0$. This would imply

$$S_X(Y) = \sum_{N \leqslant Y:\ Y/N \cong X} \mathbb{K}\, \mathrm{Inf}^Y_{Y/N}\, y_{Y/N},$$

which is certainly not true, because (for instance) $\mathrm{Ind}^Y_X\, y_X$ belongs to the left-hand side but not to the right-hand side. Consequently, we conclude that if S_X is a simple subfunctor of M then $|X| = 1$ or $|X| = p^m$.

We now exclude the case $|X| = 1$. Indeed, S_1 cannot be simple, because $S_1 = M$ and $y_H \in K_1(H)$ for any $|H| \in \mathcal{X}$ with $|H| \geqslant p^2$.

Finally, it follows from 6.5 and 7.2 that for any $X \in \mathcal{X}$ with $|X| = p^m$ the functors S_X are simple, and moreover $S_X \cong S_{X,\mathbb{K}}$ whose multiplicity in $\mathrm{Soc}(M)$ is equal to 1. $\square$

Let $\mathbb{K}$, $\mathcal{X}$, and $\mathrm{B}_{\mathbb{K}}$ be as in 7.4. In the case $m = 1$, not covered in 7.4, we may see that $\mathrm{B}_{\mathbb{K}}$ is isomorphic to $S_{1,\mathbb{K}}$.

The socle of the Burnside functor, considered as a Mackey functor for a fixed p-group over an algebraically closed field of characteristic $p > 0$, is studied in Nicollerat [11,12], and the socle and the restriction kernels are determined when the fixed group is taken from some classes of abelian p-groups.

8. Radical series as Mackey functors

In this section we consider the Burnside functor as a Mackey or a deflation functor on a family of finite p-groups over a field of characteristic $p > 0$. Our aim is to study the radical series of the Burnside functor. We begin with the following consequence of 3.4, see also Remark 3.11 of [16].

Remark 8.1. Let $\mathcal{Y}$ be a subfamily of $\mathcal{X}$ closed under taking subgroups, taking isomorphisms, and taking quotients. Let $S_{H,V}$ be the simple functor (i.e., any of biset, inflation, Mackey, or deflation) defined on $\mathcal{X}$. Then, its restriction to $\mathcal{Y}$ is the simple functor $S_{H,V}$ defined on $\mathcal{Y}$ if $H \in \mathcal{Y}$, and 0 otherwise.

Proposition 8.2. *Let $\mathbb{K}$ be of characteristic $p > 0$ and $\mathcal{X}$ be the family of all finite p-groups of order less than or equal to p^m where m is a natural number. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey or a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Let $J_n = \mathrm{Jac}^n(\mathrm{B}_{\mathbb{K}})$ for any natural number n. If a simple functor $S_{H,V}$ appears in the quotient J_n/J_{n+1}, then $|H| \geqslant p^{m-n}$, and in the case $|H| = p^{m-n}$ the $\mathbb{K}\,\mathrm{Out}(H)$-module V must be trivial.*

Proof. It follows from the results 4.5, 5.2 and 5.3 that

$$J_0/J_1 \cong \bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^m in $\mathcal{X}$. Thus the result is true for $n = 0$. We prove the result by induction on n. For any functor M defined on any family $\mathcal{Z}$ we use the notation $M^{\mathcal{Z}}$ to stress that we are considering M as a functor on $\mathcal{Z}$.

Assume now that the result is true for any natural number less than n. Let $S_{H,V}$ be a simple functor appearing in J_n/J_{n+1}. We want to show that $|H| \geqslant p^{m-n}$. Indeed, let $\mathcal{Y}$ be the family of all finite p-groups of order less than or equal to p^{m-n}. From the induction hypothesis, $|K| \geqslant p^{m-(n-1)}$ for any simple functor $S_{K,W}$ appearing as a composition factor of J_0/J_n. So, restricting the functors

$$\mathrm{B}_{\mathbb{K}}^{\mathcal{X}} = J_0^{\mathcal{X}} \supseteq J_n^{\mathcal{X}} \supseteq J_{n+1}^{\mathcal{X}}$$

to the family $\mathcal{Y}$, we see that

$$\mathrm{B}_{\mathbb{K}}^{\mathcal{Y}} = J_0^{\mathcal{Y}} = J_n^{\mathcal{Y}} \supseteq J_{n+1}^{\mathcal{Y}}.$$

It follows from 8.1 that $J_n^{\mathcal{Y}}/J_{n+1}^{\mathcal{Y}} \cong (J_n/J_{n+1})^{\mathcal{Y}}$ is either zero or semisimple, implying that

$$\mathrm{B}_{\mathbb{K}}^{\mathcal{Y}} = J_n^{\mathcal{Y}} \supseteq J_{n+1}^{\mathcal{Y}} \supseteq \mathrm{Jac}\big(\mathrm{B}_{\mathbb{K}}^{\mathcal{Y}}\big).$$

If $H \notin \mathcal{Y}$ then $|H| > p^{m-n}$. In the case $H \in \mathcal{Y}$, we see from 8.1 that the simple functor $S_{H,V}^{\mathcal{Y}}$ appears in $\mathrm{B}_{\mathbb{K}}^{\mathcal{Y}}/\mathrm{Jac}(\mathrm{B}_{\mathbb{K}}^{\mathcal{Y}})$, and it follows from 5.3 that $|H| = p^{m-n}$ and that V is the trivial module. □

We try to improve 8.2 in the case of Mackey functors. We first need some preliminary results. The following result will be used to show that the converse of 8.2 is true for simple functors $S_{H,\mathbb{K}}$ with $|H| = p^{m-n}$.

Lemma 8.3. *Let $\mathbb{K}$ be of characteristic $p > 0$ and let any group in $\mathcal{X}$ be a p-group. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. Then,*

$$b_G(M) \subseteq \mathrm{Ker}^M_{H,b_H(M)}(G)$$

for any groups H and G in $\mathcal{X}$, where $M = \mathrm{B}_{\mathbb{K}}$.

Proof. If G has no subgroup isomorphic to H, then 4.12 implies that

$$J(G) = M(G) \supseteq b_G(M)$$

where $J = \mathrm{Ker}^M_{H,b_H(M)}$. Now, let A be a subgroup of G isomorphic to H. For any group X, it follows from 5.2 that the transitive basis elements $[X/U]$ of $M(X)$ with $U < X$ form a basis for $b_X(M)$. For any $V < G$, the coefficient of $[H/H]$ in

$$x = \mathrm{Iso}^H_A \mathrm{Res}^G_A[G/V] = \sum_{AgV \subseteq G} \mathrm{Iso}^H_A\big[A/A \cap^g V\big]$$

is equal to the number of the elements of the set

$$\big\{AgV \subseteq G\colon\ A \cap {}^gV = A\big\} = \big\{gV \subseteq G\colon\ A \leqslant {}^gV\big\},$$

which is G/V^A, the set of the A-fixed points of G/V, on which A acts by left multiplication. As $V < G$ and as A and G be p-groups, $|G/V^A| \equiv |G/V| \equiv 0 \pmod p$. Thus, the coefficient of $[H/H]$ in x is 0 for any $V < G$, so that $x \in b_H(G)$. This shows that $[G/V] \in J(G)$ for any $V < G$ (see 4.12), finishing the proof. $\square$

Lemma 8.4. *Let $\mathbb{K}$, $\mathcal{X}$, m, and J_n be as in* 8.2. *Consider the Burnside functor $M = \mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any $G \in \mathcal{X}$ and any n with $n \leqslant m$ we have*:

(1) $J_n(G) = M(G)$ *if* $|G| \leqslant p^{m-n}$.
(2) $J_n(G) \subseteq b_G(M)$ *if* $|G| > p^{m-n}$.
(3) $J_n(G) = b_G(M)$ *if* $|G| = p^{m-n+1}$ *and* $n \geqslant 1$.
(4) $J_n(G) = b_G(M)$ *if* $|G| = p^{m-n+2}$ *and* $n \geqslant 2$.

Proof. (1) Let $\mathcal{Y}$ be the family of all p-groups of order less than or equal to p^{m-n}. From 8.2 we know that any composition factor $S_{K,W}$ of M/J_n satisfies $K \notin \mathcal{Y}$. Thus, 8.1 implies that the restriction of M/J_n to $\mathcal{Y}$ is 0, showing that $M(G) = J_n(G)$ for any $G \in \mathcal{Y}$.

(2) and (3) It follows from 4.5, 5.2 and 5.3 that

$$J_1 = \bigcap_{H \in \mathcal{X}\colon |H| = p^m} \mathrm{Ker}^M_{H,b_H(M)}.$$

Using 4.12 we see that $J_1(G) = b_G(M)$ for any G with $|G| = p^m$. Thus the result is true for $n = 1$. Assume now that the result is true for any n less than k. Take any G with $|G| > p^{m-k}$. We want to show that $J_k(G) \subseteq b_G(J_k)$. Indeed, for any $A \in \mathcal{X}$ with $|A| > p^{m-(k-1)}$ the induction hypothesis implies that

$$b_A(M) \supseteq J_{k-1}(A) \supseteq J_k(A).$$

So the result is true for k and for groups of order greater than $m - (k-1)$.

We assume now that $|G| = p^{m-k+1}$. By the first part of this result we see that

$$J_{k-1}(G) = M(G) \quad \text{and} \quad b_G(J_{k-1}) = b_G(M).$$

We will describe the maximal subfunctors of J_{k-1}, a minimal group of whose simple quotient has order $|G|$. For this end, for any $H \in \mathcal{X}$ with $|H| = |G|$ we let

$$R_H = \mathrm{Ker}^{J_{k-1}}_{H,b_H(M)} \quad \text{and} \quad S_H = \mathrm{Im}^{J_{k-1}}_{H,M(H)},$$

and we want to find those groups H satisfying the condition $S_H + R_H = J_{k-1}$ (because if an H satisfies this condition then it follows from 4.3 that J_H is a maximal subfunctor of J_{k-1}). For any $X \in \mathcal{X}$ having no subgroup isomorphic to H we know from 4.12 that

$$R_H(X) = J_{k-1}(X).$$

Moreover, if Y is any group isomorphic to H we know by the definition of Im subfunctor that

$$S_H(Y) = J_{k-1}(Y).$$

Lastly, for any group $Z \in \mathcal{X}$ having a proper subgroup isomorphic to H, as $|Z| > |H| = p^{m-k+1}$, the induction hypothesis gives that

$$J_{k-1}(Z) \subseteq b_Z(M).$$

Applying 8.3 we then see that

$$J_{k-1}(Z) \subseteq b_Z(M) \subseteq \mathrm{Ker}^M_{H,b_H(M)}(Z)$$

and so that

$$J_{k-1}(Z) \subseteq \mathrm{Ker}^M_{H,b_H(M)}(Z) \cap J_{k-1}(Z) = R_H(Z).$$

As a result, we have just shown that

$$S_H + R_H = J_{k-1}.$$

Then, from 4.3 and 5.2, the maximal subfunctors of J_{k-1} we want to describe are precisely the subfunctors R_H for any $H \in \mathcal{X}$ with $|H| = p^{m-k+1}$. Now J_k, the radical of J_{k-1}, is the intersection

$$\Bigg(\bigcap_{H \in \mathcal{X}:\ |H| = p^{m-k+1}} R_H\Bigg) \cap \Big(\bigcap R\Big)$$

where the maximal subfunctors R in the second intersection all satisfy the property that the order of a minimal group of the simple quotient J_{k-1}/R is greater than $|G|$ (see 8.2). Therefore,

$$R(G) = J_{k-1}(G).$$

Furthermore, we see by using 4.12 that

$$R_H(G) = b_G(M) \quad \text{if } G \cong H \quad \text{and} \quad R_H(G) = J_{k-1}(G) \quad \text{if } G \ncong H.$$

Consequently, $J_k(G) = b_G(M)$.

(4) Let $n \geqslant 2$ and $|G| = p^{m-n+2}$. Using the first and the third part of this result we see that the Brauer quotient of J_{n-1} at G is 0. Then, 4.3 implies that J_{n-1} has no simple quotient whose minimal group is isomorphic to G. Let $S_{K,U}$ be a simple functor appearing in J_{n-1}/J_n. We will show that $S_{K,U}(G) = 0$, implying that

$$J_n(G) = J_{n-1}(G),$$

which is equal from the third part of this result to $b_G(M)$. Indeed, $|K| \geqslant p^{m-n+1}$ from 8.2. If $|K| = p^{m-n+1}$ then arguing as in the proof of the third part of this result we see that $S_{K,U}$ is isomorphic to J_{n-1}/J where

$$J = \mathrm{Ker}^{J_{n-1}}_{K,b_K(J_{n-1})},$$

and that $J(G) = J_{n-1}(G)$ (and also that $U = \mathbb{K}$). Thus $S_{K,U}(G) = 0$ in the case $|K| = p^{n-m+1}$. Consequently, if $S_{K,U}(G) \neq 0$ then $|K| \geqslant p^{n-m+2}$ so that $G \cong K$ (because $|G| = p^{n-m+2}$ and because $S_{K,U}(G) \neq 0$ implies that K is isomorphic to a subgroup of G), which is not the case. Hence, $S_{K,U}(G) = 0$. $\square$

Reading the proof of 8.4 carefully one may easily deduce the next result.

Theorem 8.5. *Let $\mathbb{K}$, $\mathcal{X}$, m, and J_n be as in 8.2. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. For any n with $n \leqslant m$ we have:*

(1) *If $S_{H,V}$ appears in J_n/J_{n+1} then $|H| \geqslant p^{m-n}$ and $|H| \neq p^{m-n+1}$.*
(2) *If $|H| = p^{m-n}$ and $S_{H,V}$ appears in J_n/J_{n+1} then $V = \mathbb{K}$.*
(3) *For any $H \in \mathcal{X}$ with $|H| = p^{m-n}$, the multiplicity of $S_{H,\mathbb{K}}$ in J_n/J_{n+1} is 1. In particular, $S_{1,\mathbb{K}}$, whose multiplicity in $\mathrm{B}_{\mathbb{K}}$ is 1, appears in J_m/J_{m+1}.*

We notice the similarity between the above result and Theorem 7.3 of [17] where the radical series of the Burnside functor considered as a (ordinary) Mackey functor for a fixed group is studied.

We find below first few radical layers of $\mathrm{B}_{\mathbb{K}}$ which can be justified arguing as in the proof of 8.4. The multiplicities of simple functors appearing in the higher radical layers become more complicated. The first part is valid also for $\mathrm{B}_{\mathbb{K}}$ considered as a deflation functor. The notation $H \in_{\mathrm{iso}} \mathcal{X}$ in the following result means that H ranges over a complete set of isomorphism classes of all groups in $\mathcal{X}$.

Proposition 8.6. *Let $\mathbb{K}$, $\mathcal{X}$, m, n, J_n, and $\mathrm{B}_{\mathbb{K}}$ be as in 8.5. Suppose that $m \geqslant 2$. For any $H \in \mathcal{X}$ we denote by M_H the $\mathbb{K}\,\mathrm{Out}(H)$-module*

$$\bigoplus_{V \leqslant_H H:\ |H:V|=p} \mathbb{K}[H/V].$$

(1) $$J_0/J_1 \cong \bigoplus_{H \in_{\mathrm{iso}} \mathcal{X}:\ |H|=p^m} S_{H,\mathbb{K}}, \qquad J_1/J_2 \cong \bigoplus_{H \in_{\mathrm{iso}} \mathcal{X}:\ |H|=p^{m-1}} S_{H,\mathbb{K}}.$$

(2) $$J_2/J_3 \cong \Bigl(\bigoplus_{H \in_{\mathrm{iso}} \mathcal{X}:\ |H|=p^{m-2}} S_{H,\mathbb{K}}\Bigr) \oplus \Bigl(\bigoplus_{H \in_{\mathrm{iso}} \mathcal{X}:\ |H|=p^{m}} \Bigl(\bigoplus_{V} \lambda_{H,V} S_{H,V}\Bigr)\Bigr)$$

where V ranges over a complete set of isomorphism classes of simple $\mathbb{K}\,\mathrm{Out}(H)$-modules and $\lambda_{H,V}$ is the multiplicity of V in $M_H/\mathrm{Jac}(M_H)$.

As a final result in this section we find the radical series of the Burnside functor defined on cyclic p-groups. To justify it, one may use the bijective correspondence described in 4.3 and 4.5. Details are left to the reader. For any natural number n we denote by C_n the cyclic group of order n, and for any rational number s we denote by $\lfloor s \rfloor$ the largest integer less than or equal to s.

Remark 8.7. Let $\mathbb{K}$ be of characteristic $p > 0$ and $\mathcal{X}$ be the family of all cyclic p-groups of order less than or equal to p^m where m is a natural number with $m \geqslant 2$. For any natural number n we put $J_n = \mathrm{Jac}^n(\mathrm{B}_{\mathbb{K}})$.

(1) Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation or a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any natural number n with $n \leqslant m$ we have

$$J_n/J_{n+1} \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} S_{C_{p^{m-n+2k}},\mathbb{K}}.$$

(2) Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any natural number r with $1 \leqslant r \leqslant m-2$,

$$J_{m+r}/J_{m+r+1} \cong \bigoplus_{k=1}^{\lfloor (m-r)/2 \rfloor} S_{C_{p^{r+2k}},\mathbb{K}} \cong J_{m-2-r}/J_{m-1-r}.$$

Moreover, the Loewy length of $\mathrm{B}_{\mathbb{K}}$ is $2m-1$.

(3) Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any natural number r with $1 \leqslant r \leqslant m$,

$$J_{m+r}/J_{m+r+1} \cong \bigoplus_{k=0}^{\lfloor (m-r)/2 \rfloor} S_{C_{p^{r+2k}},\mathbb{K}} \cong J_{m-r}/J_{m-r+1}.$$

Moreover, the Loewy length of $\mathrm{B}_{\mathbb{K}}$ is $2m+1$.

The difference between the starting numbers of indices k in the last two parts of 8.7 is caused mainly by the difference between dimensions of the simple Mackey and the simple deflation functors parameterized by the trivial group. There are no differences between dimensions of the simple Mackey and the simple deflation functors (over characteristic $p > 0$) parameterized by the nontrivial cyclic p-groups and trivial modules, explaining the similarity between the series as Mackey and as deflation functors. It may be easily justified that as an inflation functor on cyclic p-groups the radical series of $\mathrm{B}_{\mathbb{K}}$ must be the following.

Remark 8.8. Let $\mathbb{K}$ be of characteristic $p > 0$ and $\mathcal{X}$ be the family of all cyclic p-groups. For any natural number n we put $J_n = \mathrm{Jac}^n(\mathrm{B}_{\mathbb{K}})$. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then,

$$\mathrm{B}_{\mathbb{K}}/J_1 \cong S_{1,\mathbb{K}} \quad \text{and} \quad J_n/J_{n+1} \cong nS_{C_{p^n},\mathbb{K}}$$

for any natural number n with $n \geqslant 1$.

By using 8.1 we may obtain the following as an easy consequence of 8.7 and 8.8.

Corollary 8.9. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and $\mathcal{X}$ be any family. Let G be a cyclic p-group of order p^m in $\mathcal{X}$ where m is a natural number, and V be a simple $\mathbb{K}\,\mathrm{Out}(G)$-module. Then:*

(1) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. If $S_{G,V}$ appears in $\mathrm{B}_{\mathbb{K}}$ then $V = \mathbb{K}$ and its multiplicity is equal to $m+1$.*

(2) *Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a deflation or an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. If $S_{G,V}$ appears in $\mathrm{B}_{\mathbb{K}}$ then $V = \mathbb{K}$, and its multiplicity is equal to m for $m \neq 0$, and is equal to 1 for $m = 0$.*

9. Radical and socle series as biset functors

In this section we first want to find the radical series of the Burnside functor $B_{\mathbb{K}}$ considered as a biset functor on the family of all finite abelian p-groups over a field of characteristic $p > 0$. Throughout this section, by a functor we mean a biset functor.

To find the radical series we repeatedly use the bijective correspondence given in 4.3, according to which maximal subfunctors of a functor M are of the form $\mathrm{Ker}^M_{H,V}$. For a subfunctor M of $B_{\mathbb{K}}$, it follows from 4.12 that the evaluation $\mathrm{Ker}^M_{H,V}(K)$ at a group K is the intersection of preimages of the maps of the form $\mathrm{Iso}^H_{A/B}\,\mathrm{Def}^A_{A/B}\,\mathrm{Res}^K_A$. The images of the transitive basis elements $[K/U]$ of $B_{\mathbb{K}}(K)$ under the maps of the above form are complicated. We use another basis for $B_{\mathbb{K}}(K)$ with whose elements the evaluations $\mathrm{Ker}^M_{H,V}(K)$ are easy to find.

For a p-group G and its normal subgroup N we define the element $f^G_N \in B_{\mathbb{K}}(G)$ as follows:

$$f^G_N = [G/N] - \sum_{V \trianglelefteq G:\ N \leqslant V, |V/N|=p} [G/V].$$

It is obvious that $f^G_N = \mathrm{Inf}^G_{G/N} f^{G/N}_{N/N}$, and for $|G| \neq p$ we note that f^G_1 is the basis element of the restriction kernel of $B_{\mathbb{K}}$ at G (see 7.2).

Lemma 9.1. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. For any subgroup V of G we have*

$$[G/V] = \sum_{N \leqslant G:\ N \geqslant V} f^G_N.$$

In particular, the elements f^G_N with $N \leqslant G$ form a $\mathbb{K}$-basis of the algebra $B_{\mathbb{K}}(G)$.

Proof. An elementary way of proving this result is to use a simple induction argument on the index of a subgroup V in G. Here we give another proof by the referee who suggested to define f^G_N as

$$f^G_N = \sum_{N \leqslant V \trianglelefteq G} \mu_{\trianglelefteq G}(N, V)[G/V]$$

where $\mu_{\trianglelefteq G}$ is the Möbius function of the poset of normal subgroups of G. In the case G is abelian, this is the ordinary Möbius function μ of the poset of subgroups of G. If V is a p-group, then $\mu(N, V) = 0$ if V/N is not elementary abelian, and $\mu(N, V) = (-1)^k p^{\binom{k}{2}}$ if V is elementary abelian of rank k. Hence, $\mu(N, V)$ is equal to 0 modulo p, unless $N = V$, and then $\mu(N, V) = 1$, or $|V : N| = p$, and then $\mu(N, V) = -1$. Therefore, if $\mathbb{K}$ is of characteristic $p > 0$ and G is an abelian p-group, this definition of f^G_N coincides with the definition of f^G_N given before 9.1. Moreover, with this definition of f^G_N, the result follows from the Möbius inversion theorem. $\square$

Lemma 9.2. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Then*:

(1) *For any $N \leqslant H \leqslant G$ we have*

$$\mathrm{Ind}^G_H f^H_N = \sum_{K \leqslant G:\ K \geqslant N, K\cap H = N} f^G_K.$$

(2) $\mathrm{Inf}^G_{G/M} f^{G/M}_{N/M} = f^G_N$ *for any $M \leqslant N \leqslant G$.*

(3) $\mathrm{Iso}^{G'}_G(\phi) f^G_N = f^{G'}_{\phi(N)}$ *for any $N \leqslant G$ and any isomorphism $\phi : G \to G'$.*

(4) *Let M and N be subgroups of G. Then,* $\operatorname{Def}^G_{G/M} f^G_N = f^{G/M}_{N/M}$ *if* $M \leqslant N$*, and* $\operatorname{Def}^G_{G/M} f^G_N = 0$ *if* $M \nleqslant N$.
(5) *For any subgroups H and N of G,*

$$\operatorname{Res}^G_H f^G_N = \begin{cases} f^H_{H\cap N}, & \text{if } G = HN, \\ -f^H_H, & \text{if } G \neq HN \text{ and } |G:N| = p, \\ 0, & \text{if } G \neq HN \text{ and } |G:N| \geqslant p^2. \end{cases}$$

Proof. (1) Using the definition of f^H_N and 9.1 we obtain that

$$\begin{aligned} \operatorname{Ind}^G_H f^H_N &= [G/N] - \sum_{V \leqslant H:\ N \leqslant V, |V/N|=p} [G/V] \\ &= \sum_{K \leqslant G:\ K \geqslant N} f^G_K - \sum_{V \leqslant H:\ N \leqslant V, |V/N|=p} \left(\sum_{Y \leqslant G:\ Y \geqslant V} f^G_Y \right) \\ &= \sum_{K \leqslant G:\ K \geqslant N} f^G_K - \sum_{Y \leqslant G:\ Y > N} \lambda_Y f^G_Y, \end{aligned}$$

where λ_Y is equal to the number of elements of the set

$$\{V/N \leqslant H/N:\ |V/N| = p,\ V \leqslant Y\} = \{V/N \leqslant (Y \cap H)/N:\ |V/N| = p\},$$

which is empty if $Y \cap H = N$, so that $\lambda_Y = 0$ in this case. We finish by noting that if $Y \cap H \neq N$ then $\lambda_Y \equiv 1 \pmod{p}$.

(2) and (3) Obvious.

(4) Writing $f^G_N = \operatorname{Inf}^G_{G/N} f^{G/N}_{N/N}$, and using the results 2.1 and 2.2, we see that

$$\begin{aligned} \operatorname{Def}^G_{G/M} f^G_N &= \operatorname{Def}^G_{G/M} \operatorname{Inf}^G_{G/N} f^{G/N}_{N/N} \\ &= \operatorname{Inf}^{G/M}_{(G/M)/(NM/M)} \operatorname{Iso}^{(G/M)/(NM/M)}_{(G/N)/(MN/N)}(\pi) \operatorname{Def}^{G/N}_{(G/N)/(MN/N)} f^{G/N}_{N/N} \end{aligned}$$

where π is the natural isomorphism. It follows from Lemme 2 of [6] (see the explanation given before 7.2) that

$$\operatorname{Def}^{G/N}_{(G/N)/(MN/N)} f^{G/N}_{N/N} = 0$$

if $MN/N \neq 1$, equivalently if $M \nleqslant N$. In the case $M \leqslant N$, the above equality of $\operatorname{Def}^G_{G/M} f^G_N$ becomes

$$\operatorname{Def}^G_{G/M} f^G_N = \operatorname{Inf}^{G/M}_{(G/M)/(N/M)} \operatorname{Iso}^{(G/M)/(N/M)}_{(G/N)/(N/N)}(\pi) f^{(G/N)/(N/N)}_{(N/N)/(N/N)} = f^{G/M}_{N/M}.$$

(5) As in the previous part, writing $f^G_N = \operatorname{Inf}^G_{G/N} f^{G/N}_{N/N}$, and using the results 2.1 and 2.2, we see that

$$\operatorname{Res}^G_H f^G_N = \operatorname{Res}^G_H \operatorname{Inf}^G_{G/N} f^{G/N}_{N/N} = \operatorname{Inf}^H_{H/H\cap N} \operatorname{Iso}^{H/H\cap N}_{HN/N}(\pi) \operatorname{Res}^{G/N}_{HN/N} f^{G/N}_{N/N}$$

where π is the natural isomorphism. For $|G/N| \geqslant p^2$ and $HN/N \neq G/N$, it follows from Lemme 2 of [6] that

$$\mathrm{Res}^{G/N}_{HN/N} f^{G/N}_{N/N} = 0,$$

and so $\mathrm{Res}^G_H f^G_N = 0$. Using the definitions, the remaining cases in which $|G/N| \leqslant p^2$ or $HN/N = G/N$ can be analyzed easily to finish the proof. □

We now ready to find the radical series of $\mathrm{B}_{\mathbb{K}}$. However, to facilitate reading of its proof we first state the following.

Lemma 9.3. *Let $\mathbb{K}$ be of characteristic $p > 0$ and $\mathcal{X}$ be any family of finite abelian p-groups. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. For any natural number $n \geqslant 1$ and any $G \in \mathcal{X}$ we define*

$$I_n(G) = \bigoplus_{N \leqslant G:\ |G:N| \geqslant p^{n+1}} \mathbb{K} f^G_N.$$

Then, for each natural number $n \geqslant 1$ we have:

(1) *I_n is a subfunctor of $\mathrm{B}_{\mathbb{K}}$.*
(2) *Let $G \in \mathcal{X}$. Then, $\overline{I_n}(G) \neq 0$ if and only if $|G| = p^{n+1}$. Moreover, if $|G| = p^{n+1}$ then $\overline{I_n}(G) \cong I_n(G) = \mathbb{K} f^G_1$ is the trivial $\mathbb{K}\,\mathrm{Out}(G)$-module.*
(3) *I_{n+1} is the radical of I_n, and*

$$I_n/I_{n+1} \cong \bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^{n+1} in $\mathcal{X}$.

Proof. (1) It is enough to show that I_n is invariant under the five types of basic bisets Ind, Inf, Iso, Def, and Res. It is immediate from 9.2 that I_n is invariant under the maps Inf, Iso, and Def. Let $H \leqslant G$ be groups in $\mathcal{X}$.

Take an arbitrary basis element f^H_M of $I_n(H)$, where M is a subgroup of H satisfying $|H : M| \geqslant p^{n+1}$. Applying 9.2 we write

$$\mathrm{Ind}^G_H f^H_M$$

as a sum of basis elements f^G_K of $\mathrm{B}_{\mathbb{K}}(G)$ satisfying $M \leqslant K \leqslant G$ and $K \cap H = M$. For such a K we see that

$$p^{n+1} \leqslant |H : M| = |H : K \cap H| = |HK : K| \leqslant |G : K|,$$

and so $f^G_K \in I_n(G)$. Thus,

$$\mathrm{Ind}^G_H I_n(H) \subseteq I_n(G).$$

Let f^G_N be an arbitrary basis element of $I_n(G)$, where $N \leqslant G$ and $|G : N| \geqslant p^{n+1} \geqslant p^2$ (as $n \geqslant 1$). It follows from 9.2 that if

$$\mathrm{Res}^G_H f^G_N \neq 0$$

then

$$\operatorname{Res}^G_H f^G_N = f^H_{H\cap N} \quad \text{and} \quad G = HN,$$

implying that

$$p^{n+1} \leqslant |G:N| = |HN:N| = |H:H\cap N|.$$

Thus, $\operatorname{Res}^G_H f^G_N \in I_n(H)$, so that

$$\operatorname{Res}^G_H I_n(G) \subseteq I_n(H).$$

(2) We first show that $b_G(I_n) = I_n(G)$ for any G in $\mathcal{X}$ with $|G| \geqslant p^{n+2}$. Indeed, for such a group G, it follows from 4.12 that any basis element f^G_N of $I_n(G)$ with $N \neq 1$ is in $b_G(I_n)$, because

$$f^G_N = \operatorname{Inf}^G_{G/N} f^{G/N}_{N/N} \quad \text{and} \quad f^{G/N}_{N/N} \in I_n(G/N).$$

For the basis element f^G_1 of $I_n(G)$, we choose a subgroup H of G with $|G:H| = p$, and then we see by using 9.2 that

$$\operatorname{Ind}^G_H f^H_1 = \sum_{K\leqslant G:\ K\cap H=1} f^G_K = f^G_1 + \sum_{K\leqslant G:\ K\neq 1, K\cap H=1} f^G_K.$$

As $|H| \geqslant p^{n+1}$, the element f^H_1 is in $I_n(H)$, and hence $\operatorname{Ind}^G_H f^H_1 \in b_G(I_n)$. Furthermore, for $K \leqslant G$ with $K \neq 1$ and $K \cap H = 1$, we see that

$$|G:K| \geqslant |KH:K| = |H:H\cap K| = |H| \geqslant p^{n+1},$$

which gives that $f^G_K \in I_n(G)$. For a basis element f^G_K of $I_n(G)$ with $K \neq 1$, we already observed that $f^G_K \in b_G(I_n)$. Thus, $f^G_1 \in b_G(I_n)$. Consequently, we have shown that

$$I_n(G) = b_G(I_n) \quad \text{or equivalently} \quad \overline{I_n}(G) = 0$$

for any $G \in \mathcal{X}$ with $|G| \geqslant p^{n+2}$.

Conversely, it is clear from the definition of I_n that the minimal groups of I_n are precisely the groups G in $\mathcal{X}$ with $|G| = p^{n+1}$. This shows that $b_G(I_n) = 0$, and so that

$$\overline{I_n}(G) \cong I_n(G) = \mathbb{K} f^G_1,$$

which is the trivial $\mathbb{K}\operatorname{Out}(G)$-module, for any G with $|G| = p^{n+1}$.

(3) Having found the Brauer quotients of I_n in the previous part, we apply 4.5 to conclude that the maximal subfunctors of I_n are precisely the subfunctors $\operatorname{Ker}^{I_n}_{H,0}$ where H ranges over all groups in $\mathcal{X}$ of order p^{n+1}. The radical J of I_n is the intersection of all these subfunctors $\operatorname{Ker}^{I_n}_{H,0}$. For any $G \in \mathcal{X}$ we obtain by using 4.12 that

$$J(G) = \bigcap_{H\in\mathcal{X}:\ |H|=p^{n+1}} \operatorname{Ker}^{I_n}_{H,0}(G) = \bigcap_{A,B} \operatorname{Ker}\bigl(\operatorname{Def}^A_{A/B} \operatorname{Res}^G_A : I_n(G) \to I_n(A/B)\bigr)$$

where A ranges over all subgroups of G and B ranges over all subgroups of A with $|A/B| = p^{n+1}$.

We first show that any basis element f^G_N of $I_n(G)$ with $|G:N| \geqslant p^{n+2}$ is in $J(G)$. Take any subgroups $B \leqslant A$ of G with $|A/B| = p^{n+1}$. We note that $|G:N| > p^2$. If

$$\mathrm{Def}^A_{A/B}\,\mathrm{Res}^G_A\, f^G_N \neq 0$$

then we use 9.2 to obtain that $G = AN$ and $B \leqslant A \cap N$, which is impossible according to the observation

$$p^{n+2} \leqslant |G:N| = |AN:N| = |A:A\cap N| \leqslant |A:B| = p^{n+1}.$$

Consequently, an element

$$x = \sum_{N\leqslant G:\ |G:N|\geqslant p^{n+1}} \lambda_N f^G_N$$

of $I_n(G)$ is in $J(G)$ if and only if

$$y = \sum_{N\leqslant G:\ |G:N| = p^{n+1}} \lambda_N f^G_N$$

is in $J(G)$. Take a subgroup M of G with $|G/M| = p^{n+1}$. If $y \in J(G)$ then $\mathrm{Def}^G_{G/M}\, y = 0$. Using 9.2 we then obtain

$$0 = \mathrm{Def}^G_{G/M}\, y = \lambda_M f^{G/M}_{M/M},$$

so $\lambda_M = 0$. Thus, if $y \in J(G)$ then $y = 0$. As a result, $J = I_{n+1}$.

Finally, to find the simple functors appearing in the semisimple functor I_n/I_{n+1}, we note from 4.5 that $I_n/\mathrm{Ker}^{I_n}_{H,0}$ is isomorphic to $S_{H,\mathbb{K}}$ for each maximal subfunctor, because $\overline{I_n}(H)$ is the trivial $\mathbb{K}\,\mathrm{Out}(H)$-module. Moreover, as $\overline{I_n}(X) \neq 0$ if and only if $|X| = p^{n+1}$, it follows from 4.5 that the multiplicity of $S_{H,\mathbb{K}}$ in I_n/I_{n+1} is 1 for any $H \in \mathcal{X}$ with $|H| = p^{n+1}$. This completes the proof. □

Theorem 9.4. *Let $\mathbb{K}$ be of characteristic $p > 0$ and let $\mathcal{X}$ be any family of finite abelian p-groups. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. For any natural number $k \geqslant 1$ we put $J_k = \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}})$. Then*:

(1) *For any $G \in \mathcal{X}$ and any natural number $k \geqslant 1$,*

$$J_k(G) = \bigoplus_{N\leqslant G:\ |G:N|\geqslant p^{k+1}} \mathbb{K} f^G_N.$$

(2) *$\mathrm{B}_{\mathbb{K}}/J_1 \cong S_{1,\mathbb{K}}$, and*

$$J_k/J_{k+1} \cong \bigoplus_H S_{H,\mathbb{K}},$$

for any natural number $k \geqslant 1$, where H ranges over a complete set of isomorphism classes of all groups of order p^{k+1} in $\mathcal{X}$.

Proof. The results 5.2, 4.3 and 4.5 imply that M has a unique maximal subfunctor, which is $\mathrm{Ker}^M_{1,0}$, and that $M/\mathrm{Ker}^M_{1,0} \cong S_{1,\mathbb{K}}$ where $M = \mathrm{B}_{\mathbb{K}}$. Thus, it is enough to show that the radical $J_1 = \mathrm{Ker}^M_{1,0}$ of M is equal to the subfunctor I_1 of M defined in 9.3.

For any $G \in \mathcal{X}$, it follows from 4.12 that

$$J_1(G) = \bigcap_{A \leqslant G} \operatorname{Ker}\bigl(\operatorname{Def}^A_{A/A} \operatorname{Res}^G_A : M(G) \to M(A/A)\bigr).$$

Let f^G_N be a basis element of $M(G)$ with $|G:N| \geqslant p^2$. Then, $f^G_N \in J_1(G)$, because if

$$\operatorname{Def}^A_{A/A} \operatorname{Res}^G_A f^G_N \neq 0$$

for some $A \leqslant G$ then 9.2 implies that $G = AN$ and $A \leqslant A \cap N$ (forcing that $N = G$). So, an element

$$x = \sum_{N \leqslant G} \lambda_N f^G_N$$

of $M(K)$ is in $J_1(G)$ if and only if

$$y = \lambda_G f^G_G + \sum_{N \leqslant G:\ |G:N| = p} \lambda_N f^G_N$$

is in $J_1(G)$.

Suppose now that $y \in J_1(G)$. We will show that $y = 0$, which proves that $J_1 = I_1$, and completes the proof: Indeed, we get from 9.2 that

$$0 = \operatorname{Def}^G_{G/G} y = \lambda_G f^{G/G}_{G/G},$$

and so $\lambda_G = 0$. Let M be any subgroup of G with $|G:M| = p$. As $y \in J_1(G)$ we see from 9.2 that

$$0 = \operatorname{Def}^M_{M/M} \operatorname{Res}^G_M y = -\lambda_M f^{M/M}_{M/M},$$

implying that $\lambda_M = 0$. Therefore, $y = 0$, as desired. □

Applying 8.1 we obtain the following immediate consequence of 9.4.

Corollary 9.5. *Let $\mathbb{K}$ be of characteristic $p > 0$. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on any family $\mathcal{X}$ over $\mathbb{K}$. Let G be an abelian p-group in $\mathcal{X}$ and V be a simple $\mathbb{K}\operatorname{Out}(G)$-module. Then*:

(1) *If $S_{G,V}$ appears in $\mathrm{B}_{\mathbb{K}}$, then $|G| \neq p$ and $V = \mathbb{K}$.*
(2) *If $|G| \neq p$ then the multiplicity of $S_{G,\mathbb{K}}$ in $\mathrm{B}_{\mathbb{K}}$ is 1.*

We finally proceed to obtain the socle series.

Lemma 9.6. *Let $\mathbb{K}$ be of characteristic $p > 0$ and let $\mathcal{X}$ be the family of all abelian p-groups of order less than or equal to p^m where m is a natural number with $m \geqslant 2$. Consider the Burnside functor $M = \mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. For any natural number n with $n \leqslant m-1$ and any $G \in \mathcal{X}$ we define*

$$L_n(G) = \bigoplus_{|G:N| \geqslant p^{m-n+1}} \mathbb{K} f^G_N.$$

Then, for each n we have:

(1) *L_n is a subfunctor of* $\mathsf{B}_{\mathbb{K}}$.
(2) $0 = L_0 \subset L_1 \subset \cdots \subset L_{m-1} \subset \mathsf{B}_{\mathbb{K}}$.
(3) *Let G be any group in $\mathcal{X}$. Then, $L_n(G) = 0$ if and only if $|G| \leqslant p^{m-n}$.*
(4) *$\underline{(M/L_n)}(G) = 0$ for any $G \in \mathcal{X}$ with $|G| \geqslant p^{m-n+1}$.*
(5) *$\underline{(M/L_n)}(G) \cong \mathbb{K} f_1^G$ for any $G \in \mathcal{X}$ with $|G| = p^{m-n}$ and any n with $n \leqslant m-2$.*
(6) *For any n with $n \leqslant m-2$, the socle of M/L_n is L_{n+1}/L_n and*

$$L_{n+1}/L_n \cong \bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^{m-n} in $\mathcal{X}$.

Proof. (1) We here justify that

$$\mathrm{Res}_H^G L_n(G) \subseteq L_n(H)$$

for any groups $H \leqslant G$ in $\mathcal{X}$. The invariance of L_n under the other four basic bisets Ind, Inf, Iso, and Def can be justified similarly. Let f_N^G be an arbitrary basis element of $L_n(G)$, so that $|G:N| \geqslant p^{m-n+1}$. If

$$\mathrm{Res}_H^G f_N^G \neq 0,$$

it follows from 9.2 that

$$G = HN \quad \text{or} \quad |G:N| = p.$$

The case $|G:N| = p$ can be eliminated by using the condition $n \leqslant m-1$. In the case $G = HN$ we see that

$$|H : H \cap N| = |G:N|,$$

from which we deduce by using 9.2 that

$$\mathrm{Res}_H^G f_N^G = f_{H\cap N}^H \in L_n(H).$$

(2) and (3) Obvious from the definition of L_n.
(4) Let $|G| \geqslant p^{m-n+1}$. Take an arbitrary element x of $(M/L_n)(G)$. We may write x as

$$x = \sum_{N \leqslant G:\ |G:N| \leqslant p^{m-n}} \lambda_N \big(f_N^G + L_n(G) \big)$$

for some constants $\lambda_N \in \mathbb{K}$. If $x \in \underline{(M/L_n)}(G)$, then it follows from 6.8 that

$$\mathrm{Def}_{G/K}^G\, y \in L_n(G/K)$$

for any $K \leqslant G$ with $K \neq 1$ where

$$y = \sum_{N \leqslant G:\ |G:N| \leqslant p^{m-n}} \lambda_N f_N^G,$$

because $x = y + L_n(G)$. Thus, for any such K,

$$\mathrm{Def}^G_{G/K}\, y = \sum_{N \leqslant G:\ |G:N| \leqslant p^{m-n}, K \leqslant N} \lambda_N f^{G/K}_{N/K} \in L_n(G/K) = \bigoplus_{|G/K:U/K| \geqslant p^{m-n+1}} \mathbb{K} f^{G/K}_{U/K},$$

implying that

$$\sum_{N \leqslant G:\ |G:N| \leqslant p^{m-n}, K \leqslant N} \lambda_N = 0.$$

Ranging K over all subgroups of G of indexes $1, p, p^2, \ldots, p^{m-n}$ respectively we see that each coefficient λ_N in y and so y is 0. We also note that $K \neq 1$ if $K \leqslant G$ with $|G:K| \leqslant p^{m-n}$ (as $|G| \geqslant p^{m-n+1}$) so that we may range K as above. Hence, $x = L_n(G)$ (as $y = 0$) so that the restriction kernel of M/L_n at G is 0.

(5) Let $\mathcal{Y}$ be the family of all abelian p-groups of order less than or equal to p^{m-n}. Then, from the third part of this result we see by restricting to the family $\mathcal{Y}$ that

$$(M/L_n)^{\mathcal{Y}} \cong \mathrm{B}^{\mathcal{Y}}_{\mathbb{K}}.$$

Noting that if $|G| = p^{m-n}$ then $G \in \mathcal{Y}$ and $|G| \geqslant p^2$, the result follows from 7.2.

(6) Let $\mathcal{Y}$ be as in the previous part. Note that $m - n \geqslant 2$. Let M/L_n have a simple subfunctor isomorphic to $S_{H,V}$ for some $H \in \mathcal{X}$ and module V. It follows from 6.3 and from the fourth part of this result that $H \in \mathcal{Y}$. Restricting to the family $\mathcal{Y}$ we see by using 8.1 that $(M/L_n)^{\mathcal{Y}}$ has a simple subfunctor isomorphic to $S^{\mathcal{Y}}_{H,V}$. As

$$(M/L_n)^{\mathcal{Y}} \cong \mathrm{B}^{\mathcal{Y}}_{\mathbb{K}},$$

the result 7.4 implies that $|H| = p^{m-n}$.

Now, it follows from 6.5 that the simple subfunctors of M/L_n are precisely the subfunctors T_H with $H \in \mathcal{X}$ and $|H| = p^{m-n}$ where T_H is the subfunctor of M/L_n generated by the restriction kernel

$$\underline{(M/L_n)}(H) \cong \mathbb{K} f^H_1$$

(because this restriction kernel is the trivial $\mathbb{K}\,\mathrm{Out}(H)$-module). A further consequence of 6.5 is that T_H is isomorphic to $S_{H,\mathbb{K}}$ and its multiplicity in the socle of M/L_n is 1.

We finish the proof by justifying that the sum of the simple subfunctors T_H is equal to L_{n+1}/L_n. Indeed, we see from the definition of Im subfunctors that the sum of such subfunctors T_H is equal to

$$\Bigg(L_n + \sum_{H \in \mathcal{X}:\ |H| = p^{m-n}} \mathrm{Im}^M_{H, \mathbb{K} f^H_1}\Bigg)\Big/L_n.$$

Let G be a group in $\mathcal{X}$. Take an element f^G_N in $L_{n+1}(G)$ but not in $L_n(G)$. Then $|G:N| = p^{m-n}$. As

$$f^G_N = \mathrm{Inf}^G_{G/N}\, f^{G/N}_{N/N},$$

the element f^G_N is in the subfunctor of M generated by $\underline{M}(G/N) = \mathbb{K} f^{G/N}_{N/N}$. This shows that

$$L_{n+1} \subseteq L_n + \sum_{H \in \mathcal{X}:\ |H| = p^{m-n}} \mathrm{Im}^M_{H, \mathbb{K} f^H_1}.$$

The converse inclusion can be seen easily by using 9.2. $\square$

Theorem 9.7. *Let $\mathbb{K}$ be of characteristic $p > 0$ and let $\mathcal{X}$ be the family of all abelian p-groups of order less than or equal to p^m where m is a natural number with $m \geqslant 2$. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on $\mathcal{X}$ over $\mathbb{K}$. For any natural number n we put $S_n = \mathrm{Soc}^n(\mathrm{B}_{\mathbb{K}})$. Then*:

(1) *The Loewy length of $\mathrm{B}_{\mathbb{K}}$ is m, and for any n with $n \leqslant m-1$ and any $G \in \mathcal{X}$,*

$$S_n(G) = \bigoplus_{|G:N| \geqslant p^{m-n+1}} \mathbb{K} f_N^G.$$

(2) *$\mathrm{B}_{\mathbb{K}}/S_{m-1} \cong S_{1,\mathbb{K}}$, and for any n with $n \leqslant m-2$,*

$$S_{n+1}/S_n = \bigoplus_H S_{H,V}$$

where H ranges over a complete set of isomorphism classes of all groups of order p^{m-n} in $\mathcal{X}$.

Proof. By the virtue of 9.6 we only need to show that $M/S_{m-1} \cong S_{1,\mathbb{K}}$ where $M = \mathrm{B}_{\mathbb{K}}$. This can be shown (for instance) by justifying the conditions: M/S_{m-1} is generated by its evaluation at 1, and $\mathrm{Ker}_{1,0}$ subfunctor of M/S_{m-1} is 0, and that $(M/S_{m-1})(1)$ is trivial module. Using 9.2 and definitions one may easily justify these conditions. □

If we consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as a biset functor on a family $\mathcal{X}$ of abelian p-groups over a field of characteristic $p > 0$, then the results 9.4 and 9.7 show that the radical and the socle series of $\mathrm{B}_{\mathbb{K}}$ coincide with each other, except that the socle series is defined only if one places a bound on the orders of the p-groups in $\mathcal{X}$.

10. Restriction to nonfull subcategories

Let M be a functor (i.e., any of biset, inflation, deflation, or Mackey) defined on $\mathcal{X}$, and let $\mathcal{Y}$ be a subfamily of $\mathcal{X}$ closed under taking subgroups, quotients, and isomorphisms. We may consider M as a functor defined on $\mathcal{Y}$, for which we use the notation $\downarrow_{\mathcal{Y}}^{\mathcal{X}} M$. As the morphism sets of the categories which are domains of the functors M and $\downarrow_{\mathcal{Y}}^{\mathcal{X}} M$ are the same, $\downarrow_{\mathcal{Y}}^{\mathcal{X}} M$ is the restriction of M to a full subcategory. Restricting a functor to a full subcategory is not interesting, because there is a complete analogy between restricting a functor to a full subcategory and restricting a module V of an algebra A to the module eV of the algebra eAe where e is an idempotent of A. See Remark 8.1, and see [15, Section 3].

We here study restriction of a functor to a nonfull subcategory. For example, we try to describe the structure of a given deflation functor as a Mackey functor.

To facilitate the reading we usually use the letters $\mathfrak{b}, \mathfrak{i}, \mathfrak{d}$, and $\mathfrak{m}$ for things that are related respectively to biset, inflation, deflation, and Mackey functors. For example, $\mathfrak{F}_{\mathfrak{b}}$ denotes the category of biset functors, $S_{H,V}^{\mathfrak{i}}$ denotes the simple inflation functor parameterized by the pair (H, V), and $\downarrow_{\mathfrak{m}}^{\mathfrak{d}}$ denotes the restriction functor from $\mathfrak{F}_{\mathfrak{d}}$ to $\mathfrak{F}_{\mathfrak{m}}$, that is, for any deflation functor M we denote by $\downarrow_{\mathfrak{m}}^{\mathfrak{d}} M$ the functor M considered as a Mackey functor. For another example, if M is a biset functor we use $\mathrm{Im}_{H,M(H)}^{M,\mathfrak{i}}$ to denote the $\mathrm{Im}_{H,M(H)}$ subfunctor of the inflation functor $\downarrow_{\mathfrak{i}}^{\mathfrak{b}} M$ and we use $\mathrm{Ker}_{H,0}^{M,\mathfrak{m}}$ to denote the $\mathrm{Ker}_{H,0}$ subfunctor of the Mackey functor $\downarrow_{\mathfrak{m}}^{\mathfrak{b}} M$.

The following is extracted from [16].

Lemma 10.1. *Let $\mathcal{X}$ be any family and $\mathbb{K}$ be any field.*

(1) *For any simple biset functor $S_{H,V}^{\mathfrak{b}}$ on $\mathcal{X}$ over $\mathbb{K}$, the functor $\downarrow_{\mathfrak{i}}^{\mathfrak{b}} S_{H,V}^{\mathfrak{b}}$ has a unique maximal inflation subfunctor, and its head is isomorphic to $S_{H,V}^{\mathfrak{i}}$.*

(2) *For any simple biset functor $S^{\mathfrak{b}}_{H,V}$ on $\mathcal{X}$ over $\mathbb{K}$, the functor $\downarrow^{\mathfrak{b}}_{\mathfrak{d}} S^{\mathfrak{b}}_{H,V}$ has a unique simple deflation subfunctor, and its socle is isomorphic to $S^{\mathfrak{d}}_{H,V}$.*
(3) *For any simple deflation functor $S^{\mathfrak{d}}_{H,V}$ on $\mathcal{X}$ over $\mathbb{K}$, the functor $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,V}$ has a unique maximal Mackey subfunctor, and its head is isomorphic to $S^{\mathfrak{m}}_{H,V}$.*
(4) *For any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ on $\mathcal{X}$ over $\mathbb{K}$, the functor $\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H,V}$ has a unique simple Mackey subfunctor, and its socle is isomorphic to $S^{\mathfrak{m}}_{H,V}$.*

Proof. We here give brief justifications. For details see Propositions 3.8, 3.12, and 7.6 of [16]. Let $S_1 = S^{\mathfrak{b}}_{H,V}$, $S_2 = S^{\mathfrak{i}}_{H,V}$, and $S_3 = S^{\mathfrak{d}}_{H,V}$. As H is a minimal group of each S_i, by using the factorization of a transitive biset given in 2.2 one may see that

$$\mathrm{Im}^{S_1,\mathfrak{i}}_{H,S_1(H)} = S_1, \qquad \mathrm{Ker}^{S_1,\mathfrak{d}}_{H,0} = 0, \qquad \mathrm{Im}^{S_3,\mathfrak{m}}_{H,S_2(H)} = S_3, \quad \text{and} \quad \mathrm{Ker}^{S_2,\mathfrak{m}}_{H,0} = 0.$$

We may deduce the result by using 3.2. For instance, it follows from 3.2 that $\mathrm{Ker}^{S_1,\mathfrak{i}}_{H,0}$ is the unique maximal inflation subfunctor of $S_1 = \downarrow^{\mathfrak{b}}_{\mathfrak{i}} S_1$. $\square$

It is known that the category $\mathfrak{F}_{\mathfrak{m}}$ of the Mackey functors over any field of characteristic 0 is semisimple, see [15]. Therefore, for any finite group H and any simple $\mathbb{K}\,\mathrm{Out}(H)$-module V, it follows from 10.1 that over any field of characteristic 0 we have

$$\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H,V} \cong S^{\mathfrak{m}}_{H,V} \quad \text{and} \quad \downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,V} \cong S^{\mathfrak{m}}_{H,V},$$

see [16, Theorem 3.10]. See also [15, Section 9] for a related result.

Since in a semisimple category every object is projective and injective, over a field of arbitrary characteristic we have the following slightly stronger result.

Proposition 10.2. *Let $\mathcal{X}$ be any family and $\mathbb{K}$ be any field.*

(1) *For any simple deflation functor $S^{\mathfrak{d}}_{H,V}$ on $\mathcal{X}$ over $\mathbb{K}$, if $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,V}$ is not isomorphic to $S^{\mathfrak{m}}_{H,V}$ then $S^{\mathfrak{m}}_{H,V}$ is not projective.*
(2) *For any simple inflation functor $S^{\mathfrak{i}}_{H,V}$ on $\mathcal{X}$ over $\mathbb{K}$, if $\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H,V}$ is not isomorphic to $S^{\mathfrak{m}}_{H,V}$ then $S^{\mathfrak{m}}_{H,V}$ is not injective.*

Proof. Only the first part is proved here. The second part may be proved similarly. Suppose that $S^{\mathfrak{m}}_{H,V}$ is projective. We will show that $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,V} \cong S^{\mathfrak{m}}_{H,V}$: Indeed, letting $M = \downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,V}$ we see from 10.1 that M has a unique maximal subfunctor J and that M/J is isomorphic to $S^{\mathfrak{m}}_{H,V}$. Then, there is an epimorphism

$$M \to M/J \cong S^{\mathfrak{m}}_{H,V}$$

of Mackey functors, which must split by the projectivity. Therefore, M has a simple subfunctor S isomorphic to $S^{\mathfrak{m}}_{H,V}$. As J is the unique maximal subfunctor of M, if $S \neq M$ then $S \subseteq J$. But, this is impossible because $J(H) = 0$. Hence $S = M$. $\square$

The following is an immediate consequence of 10.2, see also Proposition 4.5 of [16].

Corollary 10.3. *Let $\mathbb{K}$ be of characteristic* 0 *and $\mathcal{X}$ be any family.*

(1) *Let M be an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any simple inflation functor $S^{\mathfrak{i}}_{H,V}$, the multiplicity of $S^{\mathfrak{i}}_{H,V}$ in M is equal to the multiplicity of the simple Mackey functor $S^{\mathfrak{m}}_{H,V}$ in $\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M$.*

(2) *Let M be a deflation functor on $\mathcal{X}$ over $\mathbb{K}$. Then, for any simple deflation functor $S^{\mathfrak{d}}_{H,V}$, the multiplicity of $S^{\mathfrak{d}}_{H,V}$ in M is equal to the multiplicity of the simple Mackey functor $S^{\mathfrak{m}}_{H,V}$ in $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M$.*

We have also the following result related to 10.2.

Proposition 10.4. *Let $\mathbb{K}$ be of characteristic $p \geqslant 0$, and let $\mathcal{X}$ be any family of p'-groups such that there are finitely many groups, up to isomorphism, in $\mathcal{X}$. For any $H \in \mathcal{X}$ we have the following isomorphism of functors on $\mathcal{X}$ over $\mathbb{K}$:*

$$\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H,\mathbb{K}} \cong S^{\mathfrak{m}}_{H,\mathbb{K}} \cong \downarrow^{\mathfrak{d}}_{\mathfrak{m}} S^{\mathfrak{d}}_{H,\mathbb{K}}.$$

Proof. We will show that $\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H,\mathbb{K}} \cong S^{\mathfrak{m}}_{H,\mathbb{K}}$, the second isomorphism may be shown similarly. Consider a composition series

$$\mathrm{B}_{\mathbb{K}} = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

of the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. (A composition series exists by the condition on $\mathcal{X}$.) We know from 5.6 that $\mathrm{B}_{\mathbb{K}}$ is semisimple as a Mackey functor on $\mathcal{X}$ over $\mathbb{K}$. In particular, each simple inflation functor M_i/M_{i+1} must be semisimple as Mackey functors. Then, it follows from part (4) of 10.1 that if $M_i/M_{i+1} \cong S^{\mathfrak{i}}_{H_i,V_i}$ then

$$\downarrow^{\mathfrak{i}}_{\mathfrak{m}} S^{\mathfrak{i}}_{H_i,V_i} \cong S^{\mathfrak{m}}_{H,V_i}.$$

Therefore, the series

$$\mathrm{B}_{\mathbb{K}} = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

is also a composition series of $\mathrm{B}_{\mathbb{K}}$ as a Mackey functor. Finally, we use again the result 5.6 to deduce that there must be an i for which $M_i/M_{i+1} \cong S^{\mathfrak{i}}_{H,\mathbb{K}}$, finishing the proof. $\square$

For restriction of biset functors to inflation functors, using part (1) of 10.1 we may imitate the proof of 10.2 to obtain the result: if $\downarrow^{\mathfrak{b}}_{\mathfrak{i}} S^{\mathfrak{b}}_{H,V}$ is not isomorphic to $S^{\mathfrak{i}}_{H,V}$ then $S^{\mathfrak{i}}_{H,V}$ is not projective. Therefore, it may be useful to determine projective simple inflation functors. Indeed, if a simple functor $S_{H,V}$ is projective then 2.3 implies that $S_{H,V} = L_{H,V}$ and that the module V is projective as $\mathrm{End}(H)$-module and hence as $\mathbb{K}\mathrm{Out}(H)$-module (because the ideal I_H of $\mathrm{End}(H)$ described in 2.5 annihilates V). However, the following result suggests that most of simple inflation functors are not projective.

In the following result we denote by e^G_H the primitive idempotent of the Burnside algebra $\mathrm{B}_{\mathbb{K}}(G)$ indexed by the conjugacy class of the subgroup H of G. See Section 8 of [3] for more details about primitive idempotents of $\mathrm{B}_{\mathbb{K}}(G)$.

Proposition 10.5. *Let $\mathbb{K}$ be of characteristic $p \geqslant 0$ and q be a prime number with $q \neq p$. Consider the Burnside functor $\mathrm{B}_{\mathbb{K}}$ as an inflation functor on $\mathcal{X}$ over $\mathbb{K}$. For any natural number k we put $J_k = \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}})$ and $S_k = \mathrm{Soc}(\mathrm{B}_{\mathbb{K}})$.*

(1) *If $\mathcal{X}$ is the family of all finite q-groups, then*

$$J_k = \bigoplus_{H \leqslant G:\ |H| \geqslant q^k} \mathbb{K}e^G_H \quad \text{and} \quad J_k/J_{k+1} \cong \bigoplus_{H} S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order q^k in $\mathcal{X}$.

(2) *If $\mathcal{X}$ is the family of all q-groups of order less than or equal to q^m where m is a natural number, then the Loewy length of $\mathrm{B}_{\mathbb{K}}$ is $m+1$, and for any k with $0 \leqslant k \leqslant m+1$ we have*

$$S_k = J_{m+1-k} = \bigoplus_{H \leqslant G\colon |H| \geqslant q^{m+1-k}} \mathbb{K}e_H^G \quad \textit{and} \quad S_{k+1}/S_k \cong \bigoplus_H S_{H,\mathbb{K}}$$

where H ranges over a complete set of isomorphism classes of all groups of order q^{m-k} in $\mathcal{X}$.

Proof. We will prove the first part, second part can be proved similarly. For any k and any G in $\mathcal{X}$ we define

$$I_k(G) = \bigoplus_{H \leqslant G\colon |H| \geqslant q^k} \mathbb{K}e_H^G.$$

Using the images of primitive idempotents e_H^G of $B_{\mathbb{K}}(G)$ under the maps Ind, Inf, Iso, and Res (see Lemma (8.1) in [3]) one may see that each I_k is a subfunctor of $\mathrm{B}_{\mathbb{K}}$. Next, we may observe that the Brauer quotient of I_k at a group G is zero unless $|G| = q^k$. It follows from 4.5 that maximal subfunctors of I_k are precisely the functors $R_K = \mathrm{Ker}_{K,0}^{I_k}$ where K is any group of order q^k, and it follows that I_k/R_K is isomorphic to $S_{K,\mathbb{K}}$ and that the multiplicity of $S_{K,\mathbb{K}}$ in the head of I_k is 1. It then follows from 4.12 that the evaluation $J(G)$ of the radical J of I_k at any $G \in \mathcal{X}$ satisfies

$$J(G) = \bigcap_{K \leqslant G\colon |K| = q^k} R_K = \bigcap_{K \leqslant G\colon |K| = q^k} \bigl\{x \in I_k(G)\colon \mathrm{Res}_K^G x = 0\bigr\}.$$

We may see by using Lemma (8.1) of [3] that $J = I_{k+1}$. As $I_0 = \mathrm{B}_{\mathbb{K}}$ the result follows. $\square$

Let S be a simple functor which is projective. For any functor M, as the spaces $\mathrm{Hom}_{\mathfrak{F}}(S, M)$ and $\mathrm{Hom}_{\mathfrak{F}}(S, \mathrm{Soc}(M))$ are isomorphic it follows that S does not appear in $M/\mathrm{Soc}(M)$. Now, in the case of 10.5 we know that each simple inflation functor $S_{H,\mathbb{K}}^{\mathfrak{i}}$ with $|H| < q^m$ appears in $\mathrm{B}_{\mathbb{K}}/\mathrm{Soc}(\mathrm{B}_{\mathbb{K}})$ so that they are not projective.

We finally study restrictions of the functors $L_{X,V}$ defined in Section 2. It is shown in Proposition 3.12 of [16] that if H is an abelian group and V is a simple $\mathbb{K}\,\mathrm{Out}(H)$-module then the biset functor $L_{H,V}^{\mathfrak{b}}$ has a unique maximal inflation subfunctor. We now observe that there is a similar result about the deflation functor $L_{H,V}^{\mathfrak{d}}$, and obtain some consequences.

Lemma 10.6. *Let $\mathbb{K}$ be of characteristic $p \geqslant 0$, and $\mathcal{X}$ be any family, and $H \in \mathcal{X}$ be an abelian p'-group, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. Then, on $\mathcal{X}$ over $\mathbb{K}$, the functor $\downarrow_{\mathfrak{m}}^{\mathfrak{d}} L_{H,V}^{\mathfrak{d}}$ has a unique maximal Mackey subfunctor, and its head is isomorphic to $S_{H,V}^{\mathfrak{m}}$.*

Proof. Letting $L = L_{H,V}^{\mathfrak{d}}$, we first show that L is generated as a Mackey functor by its value $L(H)$ at H, that is $L = \mathrm{Im}_{H,L(H)}^{L,\mathfrak{m}}$: Indeed, by its definition it is clear that L is generated as a deflation functor by its value $L(H)$ at H, that is $L = \mathrm{Im}_{H,L(H)}^{L,\mathfrak{d}}$. Thus, for any $G \in \mathcal{X}$ we see that $L(G)$ is equal to the sum of the spaces of the form

$$\bigl[(G \times H)/M\bigr] \otimes_{\mathrm{End}_{\mathfrak{d}}(H)} V = \mathrm{Ind}_R^G \mathrm{Iso}_{P/Q}^R \mathrm{Def}_{P/Q}^P \mathrm{Res}_P^H \otimes_{\mathrm{End}_{\mathfrak{d}}(H)} V$$

where M ranges over subgroups of $G \times H$ with $k_1(M) = 1$, and where $R = p_1(M)$, $P = p_2(M)$, and $k_2(M) = Q$. As P/Q is a section of the abelian group H, it follows from the duality of abelian groups that there is a subgroup K of H such that K is isomorphic to P/Q. Taking an isomorphism $f : R \to K$, as H is an abelian p'-group we see by using 2.1 that

$$\frac{1}{|H:K|}\,\mathrm{Iso}_K^R\big(f^{-1}\big)\,\mathrm{Res}_K^H\,\mathrm{Ind}_K^H\,\mathrm{Iso}_R^K(f)$$

is the identity of the morphism set $\mathrm{End}_{\mathfrak{d}}(R)$. Therefore,

$$\mathrm{Ind}_R^G\,\mathrm{Iso}_{P/Q}^R\,\mathrm{Def}_{P/Q}^P\,\mathrm{Res}_P^H\otimes_{\mathrm{End}_{\mathfrak{d}}(H)}V$$

is equal to

$$\mathrm{Ind}_R^G\left(\frac{1}{|H:K|}\,\mathrm{Iso}_K^R\big(f^{-1}\big)\,\mathrm{Res}_K^H\,\mathrm{Ind}_K^H\,\mathrm{Iso}_R^K(f)\right)\mathrm{Iso}_{P/Q}^R\,\mathrm{Def}_{P/Q}^P\,\mathrm{Res}_P^H\otimes_{\mathrm{End}_{\mathfrak{d}}(H)}V,$$

which is equal to

$$\frac{1}{|H:K|}\,\mathrm{Ind}_R^G\,\mathrm{Iso}_K^R\big(f^{-1}\big)\,\mathrm{Res}_K^H\otimes_{\mathrm{End}_{\mathfrak{d}}(H)}\mathrm{Ind}_K^H\,\mathrm{Iso}_R^K(f)\,\mathrm{Iso}_{P/Q}^R\,\mathrm{Def}_{P/Q}^P\,\mathrm{Res}_P^H\,V.$$

As the ideal $I_H^{\mathfrak{d}}$ of $\mathrm{End}_{\mathfrak{d}}(H)$ described in 2.5 annihilates V, we see that if

$$\big[(G\times H)/M\big]\otimes_{\mathrm{End}_{\mathfrak{d}}(H)}V\neq 0$$

then $|P/Q|=|q(M)|=H$, implying that $k_2(M)=1$. This shows that L is generated as a Mackey functor by its value $L(H)$ at H, that is $L=\mathrm{Im}_{H,L(H)}^{L,\mathfrak{m}}$, as desired.

Now, it follows from 3.2 that $I=\mathrm{Ker}_{H,0}^{L,\mathfrak{m}}$ is the unique maximal Mackey subfunctor of L. We finish the proof by observing that the Mackey functor L/I is isomorphic to $S_{H,V}^{\mathfrak{m}}$: Indeed, if $L(G)\neq 0$, we observed in the above paragraph that there is a subgroup M of $G\times H$ with $k_1(M)=1$ such that $|q(M)|=|H|$, implying that H is isomorphic to a section of G. As $(L/I)(H)\cong V$, it follows that H is a minimal group of the simple Mackey functor L/I and that L/I is isomorphic to $S_{H,V}^{\mathfrak{m}}$. □

In the proof of 10.6 it is shown that if $L_{H,V}^{\mathfrak{d}}(G)\neq 0$ then H is isomorphic to a section of G. Under the assumptions of 10.6 we may also justify in a similar way that if the value $L_{H,V}^{\mathfrak{i}}(G)$ of the inflation functor $L_{H,V}^{\mathfrak{i}}$ is nonzero at a group G then H is isomorphic to a section of G. See also Lemma (9.1) of [3] for the related result about the biset functor $L_{H,V}^{\mathfrak{b}}$.

Proposition 10.7. *Let $\mathbb{K}$ be of characteristic* 0*, and $\mathcal{X}$ be any family, and $H\in\mathcal{X}$ be an abelian group, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. Then, on $\mathcal{X}$ over $\mathbb{K}$, we have that $L_{H,V}^{\mathfrak{d}}\cong S_{H,V}^{\mathfrak{d}}$.*

Proof. As $\mathbb{K}$ is of characteristic 0, the category of Mackey functors is semisimple, see [15]. Thus, 10.6 and part (3) of 10.1 imply respectively that

$$\downarrow_{\mathfrak{m}}^{\mathfrak{d}}L_{H,V}^{\mathfrak{d}}\cong S_{H,V}^{\mathfrak{m}}\quad\text{and}\quad\downarrow_{\mathfrak{m}}^{\mathfrak{d}}S_{H,V}^{\mathfrak{d}}\cong S_{H,V}^{\mathfrak{m}},$$

so that $\downarrow_{\mathfrak{m}}^{\mathfrak{d}}L_{H,V}^{\mathfrak{d}}\cong\downarrow_{\mathfrak{m}}^{\mathfrak{d}}S_{H,V}^{\mathfrak{d}}$. Then, the result follows because $S_{H,V}^{\mathfrak{d}}$ is a (the unique simple) quotient of the deflation functor $L_{H,V}^{\mathfrak{d}}$, see Section 2. □

Corollary 10.8. *Let $\mathbb{K}$ be of characteristic* 0*, and $\mathcal{X}$ be any family, and $H\in\mathcal{X}$ be an abelian group, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. For any deflation functor M on $\mathcal{X}$ over $\mathbb{K}$, the multiplicity of $S_{H,V}^{\mathfrak{d}}$ in the socle of M is equal to the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $\underline{M}(H)$.*

Proof. Let m be the multiplicity of $S^{\mathfrak{d}}_{H,V}$ in $\mathrm{Soc}(M)$, and let n be the multiplicity of V in $\underline{M}(H)$ (which is a semisimple $\mathbb{K}\,\mathrm{Out}(H)$-module because $\mathbb{K}$ is of characteristic 0), and let r be the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $M_0(H)$ defined as

$$M_0(H)=\{x\in M(H)\colon\ I^{\mathfrak{d}}_H x=0\}$$

where $I^{\mathfrak{d}}_H$ is the ideal of $\mathrm{End}_{\mathfrak{d}}(H)$ described in 2.5.

As $I^{\mathfrak{d}}_H$ annihilates V, the image of any $\mathrm{End}_{\mathfrak{d}}(H)$-module homomorphism $V\to M(H)$ is in $M_0(H)$, so that the $\mathbb{K}$-spaces $\mathrm{Hom}_{\mathrm{End}_{\mathfrak{d}}(H)}(V,M(H))$ and $\mathrm{Hom}_{\mathbb{K}\,\mathrm{Out}(H)}(V,M_0(H))$ are isomorphic. Then, using 10.7 and the adjointness of the pair $(L_{H,-},e_H)$ given in 2.3 we obtain the isomorphisms of the following $\mathbb{K}$-spaces:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{F}_{\mathfrak{d}}}\big(S^{\mathfrak{d}}_{H,V},\mathrm{Soc}(M)\big)&\cong\mathrm{Hom}_{\mathfrak{F}_{\mathfrak{d}}}\big(S^{\mathfrak{d}}_{H,V},M\big)\\&\cong\mathrm{Hom}_{\mathfrak{F}_{\mathfrak{d}}}\big(L^{\mathfrak{d}}_{H,V},M\big)\\&\cong\mathrm{Hom}_{\mathrm{End}_{\mathfrak{d}}(H)}\big(V,M(H)\big)\\&\cong\mathrm{Hom}_{\mathbb{K}\,\mathrm{Out}(H)}\big(V,M_0(H)\big).\end{aligned}$$

This shows that $m=r$, because it follows from 3.8 that the endomorphism algebra of the deflation functor $S^{\mathfrak{d}}_{H,V}$ is isomorphic to the endomorphism algebra of the $\mathbb{K}\,\mathrm{Out}(H)$-module V.

On the other hand, it is clear from part (1) of 6.8 that $\underline{M}(H)\subseteq M_0(H)$. So, $n\leqslant r$.

Finally, the equality $m=n$ follows by the virtue of 6.4 stating that $m\leqslant n$. □

Applying the notion of the dual of a functor (see [2] for details), one may obtain the following result that is the dual of 10.8.

Corollary 10.9. *Let $\mathbb{K}$ be of characteristic 0, and $\mathcal{X}$ be any family, and $H\in\mathcal{X}$ be an abelian group, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module. For any inflation functor M on $\mathcal{X}$ over $\mathbb{K}$, the multiplicity of $S^{\mathfrak{i}}_{H,V}$ in the head of M is equal to the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module $\overline{M}(H)$.*

Corollary 10.10. *Let $\mathbb{K}$ be of characteristic 0, and $\mathcal{X}$ be any family, and $H\in\mathcal{X}$ be an abelian group, and V be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module.*

(1) *For any deflation functor M on $\mathcal{X}$ over $\mathbb{K}$, the multiplicity of $S^{\mathfrak{d}}_{H,V}$ in $M/\mathrm{Soc}(M)$ is equal to the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module*

$$\underline{\big(\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M\big)}(H)/\underline{M}(H)$$

where $\underline{(\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M)}(H)$ is the restriction kernel of the Mackey functor $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M$ at H and $\underline{M}(H)$ is the restriction kernel of the deflation functor M at H.

(2) *For any inflation functor M on $\mathcal{X}$ over $\mathbb{K}$, the multiplicity of $S^{\mathfrak{i}}_{H,V}$ in $\mathrm{Jac}(M)$ is equal to the multiplicity of V in the $\mathbb{K}\,\mathrm{Out}(H)$-module*

$$b_H(M)/b_H\big(\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M\big)$$

which is the kernel of the canonical epimorphism from the Brauer quotient $\overline{(\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M)}(H)$ of the Mackey functor $\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M$ at H to the Brauer quotient $\overline{M}(H)$ of the inflation functor at H.

Proof. We only prove the first part, the second part may be proved similarly. Let $S = S^{\mathfrak{d}}_{H,V}$, and n be the multiplicity of S in M, and m be the multiplicity of S in $\mathrm{Soc}(M)$. Then, $n - m$ is the multiplicity of S in $M/\mathrm{Soc}(M)$.

It follows from 10.3 that n is equal to the multiplicity of the simple Mackey functor $S^{\mathfrak{m}}_{H,V}$ in $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M$, which is equal by the virtue of 6.6 to the multiplicity of the simple $\mathbb{K}\mathrm{Out}(H)$-module V in the restriction kernel $\underline{(\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M)}(H)$ of the Mackey functor $\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M$ at H (because $\mathbb{K}$ is of characteristic 0 and any Mackey functor over $\mathbb{K}$ is semisimple).

On the other hand, we know from 10.8 that m is equal to the multiplicity of the simple $\mathbb{K}\mathrm{Out}(H)$-module V in the restriction kernel $\underline{M}(H)$ of the deflation functor M at H.

Since $\underline{M}(H)$ is a subset of $\underline{(\downarrow^{\mathfrak{d}}_{\mathfrak{m}} M)}(H)$ (see 6.8), the result follows. □

As a last result we obtain the following semisimplicity criterion for an inflation or a deflation functor defined on abelian groups over a field of characteristic 0.

Corollary 10.11. *Let $\mathbb{K}$ be of characteristic 0 and $\mathcal{X}$ be any family of abelian groups.*

(1) *A deflation functor M on $\mathcal{X}$ over $\mathbb{K}$ is semisimple if and only if*

$$\left(\bigcap_{P<H} \mathrm{Ker}\big(\mathrm{Res}^H_P : M(H) \to M(P)\big)\right) \subseteq \left(\bigcap_{N\leqslant H:\ N\neq 1} \mathrm{Ker}\big(\mathrm{Def}^H_{H/N} : M(H) \to M(H/N)\big)\right)$$

for any $H \in \mathcal{X}$.

(2) *An inflation functor M on $\mathcal{X}$ over $\mathbb{K}$ is semisimple if and only if*

$$\left(\sum_{N\leqslant H:\ N\neq 1} \mathrm{Inf}^H_{H/N} M(H/N)\right) \subseteq \left(\sum_{P<H} \mathrm{Ind}^H_P M(P)\right)$$

for any $H \in \mathcal{X}$.

Proof. Here we prove the second part, the first part may be proved similarly. Let M be an inflation functor. Then, M is semisimple if and only if $\mathrm{Jac}(M) = 0$, which is equivalent by part (2) of 10.10 to the condition

$$b_H(M)/b_H\big(\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M\big) = 0.$$

The result follows from 4.12, stating that

$$b_H(M) = \sum_{P<H} \mathrm{Ind}^H_P M(P) + \sum_{N\leqslant H:\ N\neq 1} \mathrm{Inf}^H_{H/N} M(H/N),$$

$$b_H\big(\downarrow^{\mathfrak{i}}_{\mathfrak{m}} M\big) = \sum_{P<H} \mathrm{Ind}^H_P M(P). \qquad \square$$

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