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# Quantum dimensions and fusion rules of the VOA $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$

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In this article, we determine quantum dimensions and fusion rules for the orbifold code VOA $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$. As our main result, we show that all irreducible $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$-modules are simple current modules if the $\mathbb{F}_4$-code $\mathcal{C}$ is self-dual.



**Contents**

1. Introduction . . . 310
    1.1. Table of notations . . . 312
2. Preliminaries and basic properties . . . 313
3. The VOAs $V_{L_{\mathcal{C}\times\mathcal{D}}}$ and $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$ . . . 317
4. Quantum dimensions of irreducible $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$-modules . . . 321
5. Fusion rules . . . 329
References . . . 347

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## 1. Introduction

Let $V$ be a vertex operator algebra (VOA) and $G$ a finite automorphism subgroup. The fixed point subspace $V^G$ of $G$ in $V$ forms a vertex operator subalgebra (subVOA). This subalgebra is called the orbifold subVOA and the study of the fixed point subVOA $V^G$ and its representation theory is often referred to as *orbifold theory*. There are many interesting and important examples. For instance, Frenkel–Lepowsky–Meurman Moonshine VOA $V^\natural$ [16] was constructed as an extension of the fixed point subVOA $V_\Lambda^\theta$ by its simple module $V_\Lambda^{T,\theta}$, where $V_\Lambda$ is the lattice VOA associated to the Leech lattice, $\theta$ is a lift of the $-1$ isometry of $\Lambda$ and $V_\Lambda^T$ is the unique irreducible $\theta$-twisted module of $V_\Lambda$ and $V_\Lambda^{T,\theta}$ is fixed point of $\theta$ in $V_\Lambda^T$. It was also conjectured [13] that one can obtain a similar construction of the Moonshine VOA $V^\natural$ by using a fixed-point free isometry of the Leech lattice with prime order. In fact, orbifold theory is an important tool for constructing new VOAs from a known one [13,28,30,32]. Unfortunately, orbifold theory is very difficult to study since the structure of the fixed point subVOA $V^G$ is often very complicated.

When $V = V_L$ is a lattice VOA and $\theta$ is an order 2 automorphism induced by the $-1$ isometry of $L$, the orbifold subVOA $V_L^\theta$ has been studied extensively [1–3,10,15]. It is known that $V_L^\theta$ is $C_2$-cofinite and rational [1,4,15,36] and any irreducible $V_L^\theta$-module is contained in some irreducible $V$-module or irreducible $\theta$-twisted module of $V$ [2,10]. Moreover, the fusion rules for $V_L^\theta$ were completely determined in [3]. Nevertheless, very little is known about the VOA $V^G$ in general even when $G$ is a cyclic group.

Motivated by the study of certain $W_3$-algebras and the $3A$-elements of the Monster simple group [9,12,21,22,27], an orbifold VOA $V_{\sqrt{2}A_2}^\tau$ was studied, where $\sqrt{2}A_2$ denotes the $\sqrt{2}$ times of the root lattice of type $A_2$ and $\tau$ is a lift of a fixed point free isometry of order 3 in the automorphism group of $\sqrt{2}A_2$. In [34], all irreducible modules of $V_{\sqrt{2}A_2}^\tau$ are classified by using certain $W_3$-algebras. It was also shown that $V_{\sqrt{2}A_2}^\tau$ is $C_2$-cofinite and rational. Moreover, the fusion rules among some irreducible modules were partially determined in [33,35]. On the other hand, certain extensions of the VOA $(V_{\sqrt{2}A_2}^\tau)^{\otimes \ell}$ were studied in [22,35]. In particular, some integral lattice $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$ is constructed as an extension of the lattice $(\sqrt{2}A_2)^{\oplus \ell}$ from an $\mathbb{F}_4$-code $\boldsymbol{\mathcal{C}}$ and an $\mathbb{Z}_3$-code $\boldsymbol{\mathcal{D}}$. Moreover, the irreducible modules for the orbifold VOA $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$ were studied in [22,35]. Under certain assumptions (that $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ are self-dual and the minimal weight of $\boldsymbol{\mathcal{C}}$ is $\geq 4$), Tanabe–Yamada [35] showed that all irreducible modules for $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$ are simple current modules and the fusion ring of $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$ is isomorphic to the group ring of an elementary abelian group of order $3^2$.

The purpose of this article is to determine the fusion rules for all irreducible $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$-modules for arbitrary $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$. The main tool is an explicit embedding of the subVOA $(V_{\sqrt{2}A_2}^\tau)^{\otimes \ell}$ into $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$ and the decomposition of $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$ as a direct sum of irreducible $(V_{\sqrt{2}A_2}^\tau)^{\otimes \ell}$-modules. The quantum dimensions and the fusion rules for irreducible $V_{\sqrt{2}A_2}^\tau$-modules, which are determined in [5], also play fundamental roles in our calculations. Usually, there are two major steps for determining the fusion rules. First, one has to obtain some lower bounds for the fusion rules, which is usually achieved by explicit constructions of certain intertwining operators. The second step is to show that the bounds obtained in Step 1 is tight by using Frenkel–Zhu $A(V)$-bimodule theory [17,25]. Nevertheless, it is extremely difficult to determine the structures of the $A(V)$-bimodule $A(M)$ for an irreducible module $M$. Therefore, determining the fusion rules is often very difficult. Recently, a new theory on quantum dimensions has been developed in [7]. Along with other results, it was shown that the quantum dimension is multiplicative with respect to the fusion product if the underlining VOA is $C_2$-cofinite, rational, self-dual and of CFT-type. Since the quantum dimension is defined as a certain limit associated to the characters of modules (cf. Definition 2.2), it is often computable and it provides an alternative method for obtaining certain bounds for the fusion rules. In fact, the fusion rules for irreducible $V^{\tau}_{\sqrt{2}A_2}$-modules are obtained by using quantum dimensions in [5] (see [6] for more examples).

In this article, we will compute the quantum dimensions for all irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules and determine the fusion rules for arbitrary $\mathcal{C}$ and $\mathcal{D}$. In particular, we show that all irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are simple current modules if the $\mathbb{F}_4$-code $\mathcal{C}$ is self-dual. Moreover, the fusion ring for $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ is isomorphic to a group ring of an elementary abelian 3-group and the set of all inequivalent irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules forms a quadratic space over $\mathbb{Z}_3$ if $\mathcal{C}$ is self-dual. The main results are as follows:

**Main Theorem 1** *(See Theorem 4.14 and Section 3 for the notations). Let $\mathcal{C}$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and let $\mathcal{D}$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Let $\mathcal{C}^{\perp}_{\equiv_\tau}$ denote the set of all $\tau$-orbits in $\mathcal{C}^{\perp}$. The quantum dimensions of irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are as follows.*

(i) $\operatorname{qdim}_{V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}} V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon] = 1$;
(ii) $\operatorname{qdim}_{V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}} = 3$;
(iii) $\operatorname{qdim}_{V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}} V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|}$,

*where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\mathbf{0} \neq \boldsymbol{\lambda} + \mathcal{C} \in \mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}$, $\boldsymbol{\eta} \in \mathcal{D}^{\perp} \bmod \mathcal{D}$ and $\boldsymbol{\delta} + \mathcal{D} \in \mathcal{D}^{\perp}/\mathcal{D}$.*

*In particular, when $\mathcal{C}$ is self-dual, we have $\mathcal{C}^{\perp}/\mathcal{C} = 0$, $|\mathcal{C}| = 2^\ell$ and the quantum dimensions for all irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are 1.*

**Main Theorem 2** *(See Section 5). Let $\mathcal{C}$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and let $\mathcal{D}$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Then the fusion rules for irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are as follows.*

(i) $V_{L_{\mathcal{C}\times(\boldsymbol{\delta}^1+\mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C}\times(\boldsymbol{\delta}^2+\mathcal{D})}}[\varepsilon^2] = V_{L_{\mathcal{C}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\mathcal{D})}}[\varepsilon^1+\varepsilon^2]$;
(ii) $V_{L_{\mathcal{C}\times(\boldsymbol{\delta}^1+\mathcal{D})}}[\varepsilon^1] \times V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}^2+\mathcal{D})}} = V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\mathcal{D})}}$;
(iii) $V_{L_{\mathcal{C}\times(\boldsymbol{\delta}^1+\mathcal{D})}}[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon^2] = V^{T,\boldsymbol{\delta}^2-i\boldsymbol{\delta}^1}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[i\varepsilon^1+\varepsilon^2]$;
(iv) $V_{L_{(\boldsymbol{\lambda}^1+\mathcal{C})\times(\boldsymbol{\delta}^1+\mathcal{D})}} \times V_{L_{(\boldsymbol{\lambda}^2+\mathcal{C})\times(\boldsymbol{\delta}^2+\mathcal{D})}} = \sum_{h=0}^{2} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\mathcal{C})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\mathcal{D})}}$;

(v) $V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\varepsilon] = \sum_{\rho=0}^{2} V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\rho];$

(vi)
$$\begin{aligned}&V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^1}(\tau)[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^2}(\tau^2)[\varepsilon^2]\\ &= V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1-\varepsilon^2] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\boldsymbol{\mathcal{C}}^{\perp}_{\equiv_\tau} \bmod \boldsymbol{\mathcal{C}}} V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\boldsymbol{\mathcal{D}})}};\end{aligned}$$

(vii)
$$\begin{aligned}&V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^1}(\tau^i)[\varepsilon_1] \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^2}(\tau^i)[\varepsilon_2]\\ &= \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d}+(-1)^{\ell}\Xi(\ell-\varepsilon)}{3} V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,-(\boldsymbol{\eta}^1+\boldsymbol{\eta}^2)}(\tau^{2i})[\varepsilon-\varepsilon_1-\varepsilon_2].\end{aligned}$$

*In the above,* $\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}}, \boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^{\perp}/\boldsymbol{\mathcal{D}}$, $\mathbf{0} \neq \boldsymbol{\lambda}+\boldsymbol{\mathcal{C}}, \boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}}, \boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^{\perp}_{\equiv_\tau} \bmod \boldsymbol{\mathcal{C}}$, $\boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \boldsymbol{\mathcal{D}}^{\perp} \bmod \boldsymbol{\mathcal{D}}$ *and* $\varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3$. *The function* $\Xi : \mathbb{Z} \to \{-1, 2\}$ *is defined by* $\Xi(n) = 2$ *if* $3$ *divides* $n$ *and* $\Xi(n) = -1$ *if* $n$ *is not a multiple of* $3$.

This article is organized as follows. In Section 2, we review some basic properties of the VOA $V_{\sqrt{2}A_2}^{\tau}$ and the notion of quantum dimensions. In Section 3, we review a construction of the integral lattice $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$ from some $\mathbb{F}_4$ and $\mathbb{Z}_3$-codes. Some basic facts about the lattice VOA $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$ and its $\mathbb{Z}_3$-orbifold $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{\tau}$ will also be recalled. In Section 4, we compute the quantum dimensions of the orbifold VOA $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{\tau}$. In Section 5, we compute the fusion rules among irreducible $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{\tau}$-modules.

### *1.1. Table of notations*

| Notation | Explanation |
|---|---|
| $\mathbb{Z}_+$ | non-negative integers. |
| $\boldsymbol{\mathcal{C}}$ | an $\mathbb{F}_4$-code of length $\ell$. |
| $\boldsymbol{\mathcal{D}}$ | a $\mathbb{Z}_3$-code of length $\ell$. |
| $W_{\boldsymbol{\mathcal{C}}}(X,Y)$ | the Hamming weight enumerator of $\boldsymbol{\mathcal{C}}$. |
| $W'_{\boldsymbol{\mathcal{C}}}(X,Y)$ | $W'_{\boldsymbol{\mathcal{C}}}(X,Y) := \frac{1}{3}(W_{\boldsymbol{\mathcal{C}}}(X,Y) - X^{\ell})$. |
| $S_\varepsilon$ | $S_\varepsilon := \{\boldsymbol{x} := (x_1,\cdots,x_\ell) \in \mathbb{Z}_3^{\ell} \mid \sum x_i \equiv \varepsilon \bmod 3\}$. |
| $W_\varepsilon(X,Y)$ | $W_\varepsilon(X,Y) := \sum_{\boldsymbol{x}\in S_\varepsilon} X^{\ell-\mathrm{wt}(\boldsymbol{x})}Y^{\mathrm{wt}(\boldsymbol{x})}$, the weight enumerator of the set $S_\varepsilon$. |
| $\boldsymbol{\gamma}, \boldsymbol{\delta}, \cdots$ | codewords of length $\ell$. We denote $\boldsymbol{\gamma} = (\gamma_1,\cdots,\gamma_\ell)$, etc. |
| $\boldsymbol{\mathcal{C}}^{\perp}$ | the dual code of $\boldsymbol{\mathcal{C}}$. |
| $\boldsymbol{\mathcal{C}}^{\perp}_{\equiv_\tau}$ | the set of all $\tau$-orbits in $\boldsymbol{\mathcal{C}}^{\perp}$. |
| $\boldsymbol{x}\cdot\boldsymbol{y}$ | inner product of codewords $\boldsymbol{x}$ and $\boldsymbol{y}$. |
| $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$ | the lattice associated to codes $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$. |
| $K^{\perp}$ | the dual lattice of the lattice $K$. |
| $\xi$ | $e^{2\pi i/3}$, the cubic root of unity. |
| $L$ | $L = \sqrt{2}A_2$. |
| $L^{(i,j)}$ | cosets of $L$ in $L^{\perp}$. |
| $\omega$ | a root of $x^2+x+1=0$ over $\mathbb{F}_2$. We denote $\mathbb{F}_4 = \{0,1,\omega,\bar{\omega}\}$. |
| $\tau$ | a fixed point free isometry of the root lattice $A_2$ of order 3. We also use $\tau$ to denote its extension to larger lattices or corresponding VOAs. |

| Notation | Explanation |
|---|---|
| $V_L^{T,i}(\tau^j)$ | $\tau^j$-twisted $V_L$-modules. |
| $V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}$ | irreducible $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules, |
| $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}}(\tau^j)$ | $\tau^j$-twisted $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules, |
| $U[i]$ | $U[i] := \{x \in U \mid \tau x = \xi^i x\}$ for $i = 0, 1, 2$. |
| $M'$ | the contragredient dual of the module $M$. |
| $I_V\binom{M^3}{M^1,\ M^2}$ | the space of intertwining operators of $V$-modules of type $\binom{M^3}{M^1,\ M^2}$. |
| $N_V\binom{M^3}{M^1,\ M^2}$ | $N_V\binom{M^3}{M^1,\ M^2} = \dim I_V\binom{M^3}{M^1,\ M^2}$. |
| $N_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{\_}{\_,\ \_}$ | $N_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{\_}{\_,\ \_} = \dim I_{V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}}\binom{\_}{\_,\ \_}$, |
| $N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{\_}{\_,\ \_}$ | $N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{\_}{\_,\ \_} = \dim I_{V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}}\binom{\_}{\_,\ \_}$, |
| $N_\otimes\binom{\_}{\_,\ \_}$ | $N_\otimes\binom{\_}{\_,\ \_} = \dim I_{(V_L^\tau)^{\otimes\ell}}\binom{\_}{\_,\ \_}$, |
| $N_\circ^\tau\binom{\_}{\_,\ \_}$ | $N_\circ^\tau\binom{\_}{\_,\ \_} = \dim I_{V_L^\tau}\binom{\_}{\_,\ \_}$. |
| $T[\varepsilon], T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon], T^{\boldsymbol{\eta}}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon]$ | $T[\varepsilon] = T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon] := V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau)[\varepsilon]; T^{\boldsymbol{\eta}}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon] := V^{T,\boldsymbol{\eta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau)[\varepsilon]$, |
| $\check{T}[\varepsilon], \check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon], \check{T}^{\boldsymbol{\eta}}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon]$ | $\check{T}[\varepsilon] = \check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon] := V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^2)[\varepsilon]; \check{T}^{\boldsymbol{\eta}}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon] := V^{T,\boldsymbol{\eta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^2)[\varepsilon]$, |
| $S[\varepsilon]$ | $S[\varepsilon] := V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}[\varepsilon]$. |
| $R_M(v, g, h; z)$ | the trace function of the module $M$, |
| $M(g, h; z), Z_M(g, h; z)$ | $M(g, h; z) = Z_M(g, h; z) := R_M(\mathbf{1}, g, h; z)$, |
| $M(z)$ | $M(z) := Z_M(\mathrm{id}, \mathrm{id}; z)$. |
| $A.B$, $A{:}B$, | extension of normal subgroup $A$ by quotient $B$, split extension, nonsplit |
| $A{\cdot}B$ | extension, respectively. |
| $N_G(A), C_G(A)$ | the normalizer and centralizer of $A$ in $G$, |
| $Stab(X)$ | the stabilize of $X$. |

## 2. Preliminaries and basic properties

***The VOAs $V_{\sqrt{2}A_2}$ and $V^\tau_{\sqrt{2}A_2}$*** In this paragraph, we review some facts about the orbifold VOA $V^\tau_{\sqrt{2}A_2}$ [22,34]. For general background concerning lattice VOA, we refer to [16,26].

Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. Then $\langle \alpha_i, \alpha_i\rangle = 2$ and $\langle \alpha_i, \alpha_j\rangle = -1$ if $i \neq j$, $i, j \in \{0, 1, 2\}$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ be the lattice spanned by $\beta_1$ and $\beta_2$. Then $L$ is isometric to $\sqrt{2}A_2$.

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ denote the Galois field of four elements, where $\omega$ is a root of $x^2 + x + 1 = 0$ over $\mathbb{F}_2$. We adopt the similar notation as in [22,12] and denote the cosets of $L$ in the dual lattice $L^\perp = \{\alpha \in \mathbb{Q} \otimes_\mathbb{Z} L \mid \langle \alpha, L\rangle \subset \mathbb{Z}\}$, as follows:

$$L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$

$$L_0 = L, \quad L_1 = \frac{\beta_2}{2} + L, \quad L_\omega = \frac{\beta_0}{2} + L, \quad L_{\bar{\omega}} = \frac{\beta_1}{2} + L, \tag{2.1}$$

and

$$L^{(i,j)} = L_i + L^j,$$

for $i \in \mathbb{F}_4$ and $j \in \mathbb{Z}_3$. Then, $L^{(i,j)}, i \in \mathbb{F}_4, j \in \mathbb{Z}_3 = \{0, 1, 2\}$ are all the cosets of $L$ in

$L^\perp$ and $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. It is shown in [14] that there are exactly 12 isomorphism classes of irreducible $V_L$-modules, which are given by $V_{L(i,j)}$, $i \in \mathbb{F}_4$ and $j \in \mathbb{Z}_3$.

Consider the isometry $\tau : L \to L$ defined by $\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1$. Then $\tau$ is fixed point free of order three and can be lifted naturally to an automorphism of $V_L$ by mapping $a^1(-n_1)\cdots a^k(-n_k)e^b \longmapsto (\tau a^1)(-n_1)\cdots(\tau a^k)(-n_k)e^{\tau b}$. By abuse of notation, we also use $\tau$ to denote the lift. Note that $\tau$ also acts as an isometry on the dual lattice $L^\perp$ and induces an action on the Fock space $V_{L^\perp} = S(\mathfrak{h}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L^\perp\}$ (see [22, Section 4]).

For the orbifold VOA $V_L^\tau$, there are two types of irreducible modules – the untwisted type (those constructed from irreducible $V_L$-modules) and the twisted type (those constructed from irreducible $\tau$ or $\tau^2$-twisted $V_L$-modules).

First we recall the notion of $\tau$-conjugate modules. Let $W$ be an irreducible $V_L$-module. The $\tau$-conjugate module $(W \circ \tau, Y_{W\circ\tau}(\ ,z))$ is defined as follows: $W \circ \tau = W$ as a vector space and the vertex operator $Y_{W\circ\tau}(u,z) = Y_W(\tau u, z)$, for $u \in V_L$. An irreducible module $W$ is said to be $\tau$-stable if $W \circ \tau \cong W$. For any $\tau$-stable $V_L$-module $U$, we denote

$$U[\varepsilon] = \{u \in U \mid \tau u = \exp(2\pi\sqrt{-1}\varepsilon/3)u\}, \quad \varepsilon = 0,1,2.$$

Note also that the automorphism $\tau$ induces an action on the set of all inequivalent irreducible $V_L$-modules by the $\tau$-conjugation. By the definition of $\tau$, it is easy to show that $V_{L(i,j)} \circ \tau \cong V_{L(\bar{\omega} i,j)}$ for any $i \in \mathbb{F}_4$ and $j \in \mathbb{Z}_3$ (cf. [22, Section 4]). Therefore, $V_{L(i,j)}$ is $\tau$-stable if and only if $i = 0$.

For irreducible twisted modules, a general construction has been given in [8]. Moreover, it was shown in [22] that there are exactly three irreducible $\tau$-twisted $V_L$-modules and three irreducible $\tau^2$-twisted $V_L$-modules, up to isomorphism. They are denoted by $V_L^{T,j}(\tau)$ or $V_L^{T,j}(\tau^2)$ for $j = 0,1,2$. We will follow the notation in [35, Section 3]. Let $\langle\kappa_n\rangle$ be a cyclic group of order $n$ with generator $\kappa_n$ and $\xi_n := \exp(2\pi\sqrt{-1}/n)$.

Let $1 \to \langle\kappa_{36}\rangle \to \hat{L}_\tau \to L \to 1$ and $1 \to \langle\kappa_{36}\rangle \to \hat{L}_{\tau^2} \to L \to 1$ be central extensions of $L$ associated with the bilinear forms $c_2$ and $c_2'$ given in [35, (2.8) and (2.9)]. Let $K = \{a\tau^{-1}(a) \mid a \in \hat{L}_\tau\}$ and $K' = \{a\tau^{-2}(a) \mid a \in \hat{L}_{\tau^2}\}$. Then

$$V_L^{T,j}(\tau) = S[\tau] \otimes T_{\chi_j} \quad \text{and} \quad V_L^{T,j}(\tau^2) = S[\tau^2] \otimes T'_{\chi_j},$$

where $T_{\chi_j}$ (resp. $T'_{\chi_j}$) is the one-dimensional irreducible module of $\hat{L}_\tau/K$ (resp. $\hat{L}_{\tau^2}/K'$) affording the character $\chi_j$ such that $\chi_j(\kappa_3 e^{\beta_1}) = \xi_3^j$ (see also Remark 3.6). By definition, there is a natural action of $\tau$ on $S[\tau]$ and $S[\tau^2]$ (cf. [8,35]). As in [35, Section 3], we define the action of $\tau$ on $T_{\chi_j}$ (resp. $T'_{\chi_j}$) as a scalar $\xi_3^{2\,\mathrm{wt}(j)}$ (resp. $\xi_3^{\mathrm{wt}(j)}$). We also denote $V_L^{T,j}(\tau^i)[\varepsilon] = \{u \in V_L^{T,j}(\tau^i) \mid \tau u = \xi^\varepsilon u\}$ for $\varepsilon = 0,1,2$.

In [34], the irreducible modules for the orbifold VOA $V_L^\tau$ are classified and the following result is proved.

**Proposition 2.1** *([34]). The VOA $V_L^\tau$ is a simple, rational, $C_2$-cofinite, and of CFT type. There are exactly* 30 *inequivalent irreducible $V_L^\tau$-modules. They are given as follows.*

(i) $V_{L^{(0,j)}}[\varepsilon]$ *for* $j, \varepsilon = 0, 1, 2$.
(ii) $V_{L^{(\omega,j)}}$ *for* $j = 0, 1, 2$.
(iii) $V_L^{T,j}(\tau^i)[\varepsilon]$ *for* $i = 1, 2$ *and* $j, \varepsilon = 0, 1, 2$.

*The conformal weights of these modules are given by (see [34, (5,10)]):*

$$\mathrm{wt}\, V_{L^{(0,j)}}[\varepsilon] \in \frac{2j^2}{3} + \mathbb{Z}, \quad \mathrm{wt}\, V_L^{T,j}(\tau^i)[\varepsilon] \in \frac{10 - 3(j^2+\varepsilon)}{9} + \mathbb{Z}, \text{ for } i = 1, 2, j, \varepsilon \in \mathbb{Z}_3.$$

*Note that the conformal weights of these modules are positive except* $V_L^\tau$ *itself.*

***Quantum dimension*** We now review the notion of quantum dimension introduced by Dong et al. [7]. Let $V$ be a VOA of central charge $c$ and let $M = \oplus_{n\in\mathbb{Z}_+} M_{\lambda+n}$ be a $V$-module, where $\lambda$ is the lowest conformal weight of $M$. The *character* of $M$ is defined as

$$\mathrm{ch}\, M(q) := q^{\lambda - c/24} \sum_{n\in\mathbb{Z}_+} \dim M_{\lambda+n} q^n,$$

where $q = e^{2\pi\sqrt{-1}z}$ and $z$ is in the complex upper half-plane $\mathbb{H}$. It is proved in [37] and [11] that $\mathrm{ch}\, M(q)$ converges to a holomorphic function on the domain $|q| < 1$ if $V$ is $C_2$-cofinite.

The following notion of quantum dimension is introduced by Dong et al. [7].

**Definition 2.2.** Suppose $\mathrm{ch}\, V(q)$ and $\mathrm{ch}\, M(q)$ exist. The *quantum dimension of $M$ over $V$* is defined as

$$\mathrm{qdim}_V\, M := \lim_{y\to 0^+} \frac{\mathrm{ch}\, M(\sqrt{-1}y)}{\mathrm{ch}\, V(\sqrt{-1}y)}, \tag{2.2}$$

where $y$ is a positive real number.

From now on, we will omit the variable $q$ and write the character $\mathrm{ch}\, M(q)$ as $\mathrm{ch}\, M$ instead. Fundamental properties of quantum dimension are also proved in their paper.

**Proposition 2.3** *([7, Section 4]). Let $V$ be a simple, rational, $C_2$-cofinite VOA of CFT-type and $V \cong V'$, the contragredient dual of $V$. Moreover, the conformal weights of irreducible $V$-modules are positive, except $V$ itself. Let $W, W^1, W^2$ be $V$-modules. Then*

(i) $\mathrm{qdim}_V\, W \geq 1$.
(ii) $\mathrm{qdim}_V$ *is multiplicative, that is* $\mathrm{qdim}_V(W^1 \times W^2) = \mathrm{qdim}_V\, W^1 \cdot \mathrm{qdim}_V\, W^2$, *where* $W^1 \times W^2$ *denotes the fusion product.*
(iii) *A $V$-module $W$ is a simple current if and only if* $\mathrm{qdim}_V\, W = 1$.
(iv) $\mathrm{qdim}_V\, W = \mathrm{qdim}_V\, W'$, *where $W'$ is the contragredient dual of $W$.*

**Remark 2.4.** Recall that an irreducible $V$-module $M$ is a *simple current* module if and only if for every irreducible $V$-module $W$, $M \times W$ exists and is also an irreducible $V$-module.

Quantum dimensions of irreducible $V_L^\tau$-modules are computed in [5].

**Proposition 2.5** *([5, Theorems 3.5 and 3.10]). We have*

(i) $\operatorname{qdim}_{V_L^\tau} V_{L^{(0,j)}}[\varepsilon] = 1$ *for* $j, \varepsilon = 0, 1, 2$.
(ii) $\operatorname{qdim}_{V_L^\tau} V_{L^{(\omega,j)}} = 3$ *for* $j = 0, 1, 2$.
(iii) $\operatorname{qdim}_{V_L^\tau} V_L^{T,j}(\tau^i)[\varepsilon] = 2$ *for* $i = 1, 2$ *and* $j, \varepsilon = 0, 1, 2$.

***Trace functions**, $S$**-matrix and Verlinde formula*** We review Dong–Li–Mason's theory on trace functions [11]. Let $V$ be a rational VOA and $g, h \in \operatorname{Aut} V$ be commuting automorphisms of finite orders. Let $M$ be a $g$-twisted $h$-stable $V$-module. There exists a linear isomorphism $\varphi(h)$ of $M$ such that $\varphi(h)Y_M(u, z) = Y_M(hu, z)\varphi(h)$.

For a homogeneous $v \in V$ with $L(1)v = 0$, we define the trace function

$$R_M(v, g, h; z) := \operatorname{tr}_M \varphi(h)o(v)q^{L(0)-c/24} = q^{\lambda - c/24} \sum_{n \in \frac{1}{|g|}\mathbb{Z}_+} \operatorname{tr}_{M_{\lambda+n}} o(v)\varphi(h)q^n,$$

where $o(v)$ is the degree zero operator of $v$, $\lambda$ is the conformal weight of $M$, $c$ is the central charge of $V$ and $q = e^{(2\pi\sqrt{-1}z)}$.

**Proposition 2.6** *([11, Theorems 5.4 and 8.7]). Let $C_1(g, h)$ be the $\mathbb{C}$-vector space*

$$C_1(g, h) := \operatorname{Span}_{\mathbb{C}}\{R_M(v, g, h; z) \mid M \textit{ is a } g\textit{-twisted } h\textit{-stable } V\textit{-module}\}.$$

*Then (i) $C_1(g, h)$ has a basis:*

$$\{R_M(v, g, h; z) \mid M \textit{ is an irreducible } g\textit{-twisted } h\textit{-stable } V\textit{-module}\}.$$

*(ii) Modular invariance: Let $R_M(v, g, h; z) \in C_1(g, h)$ and $\Gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then we have $R_M(v, g, h; \Gamma \circ z) \in C_1(g, h) \circ \Gamma$ in the sense that*

$$R_M(v, g, h; \frac{az + b}{cz + d}) \in C_1(g^a h^c, g^b h^d).$$

*In fact, if $M$ is a $g$-twisted $h$-stable $V$-module, then*

$$R_M(v, g, h; \frac{az + b}{cz + d}) = \sum S_N^{(g,h)} R_N(v, g, h; z),$$

*where $N$ runs over all irreducible $g^a h^c$-twisted $g^b h^d$-stable $V$-modules, and the coefficients $S_N^{(g,h)}$ are independent of $v$.*

In particular, when $g = h = \mathrm{id}$, $\Gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $V = M^0, \cdots, M^m$ are all inequivalent irreducible $V$-modules, we have

$$R_{M^i}(v, \mathrm{id}, \mathrm{id}; -\frac{1}{z}) = \sum_{j=0}^{m} S_{i,j} R_{M^j}(v, \mathrm{id}, \mathrm{id}; z). \tag{2.3}$$

For simplicity, we denote

$$M(g, h; z) = Z_M(g, h; z) := R_M(\mathbf{1}, g, h; z), \tag{2.4}$$

and

$$M(z) := Z_M(\mathrm{id}, \mathrm{id}; z) = \mathrm{ch}\, M(z). \tag{2.5}$$

**Definition 2.7.** The matrix $S = (S_{i,j})$ defined in Equation (2.3) is called the $S$-matrix.

**Theorem 2.8** *([19]). Let $V$ be a rational and $C_2$-cofinite simple VOA of CFT type and assume $V \cong V'$. Let $S = (S_{i,j})_{i,j=0}^m$ be the $S$-matrix as defined in (2.3). Then*

(i) *$(S^{-1})_{i,j} = S_{i,j'} = S_{i',j}$, and $S_{i',j'} = S_{i,j}$, where $i', j'$ denote indexes of the duals $(M^i)'$ and $(M^j)'$.*
(ii) *$S$ is symmetric and $S^2 = (\delta_{i,j'})$.*
(iii) *$N_{i,j}^k = \sum_{s=0}^m \frac{S_{j,s} S_{i,s} S_{s,k}^{-1}}{S_{0,s}}$, where $N_{i,j}^k = \dim I_V\binom{M^k}{M^i,\ M^j}$.*
(iv) *The $S$-matrix diagonalizes the fusion matrix $N(i) = (N_{i,j}^k)_{j,k=0}^m$ with diagonal entries $\frac{S_{i,s}}{S_{0,s}}$, for $i, s = 0, \cdots, m$. More explicitly, $S^{-1} N(i) S = \mathrm{diag}(\frac{S_{i,s}}{S_{0,s}})_{s=0}^m$. In particular, $S_{0,s} \neq 0$ for $s = 0, \cdots, m$.*

**Proposition 2.9** *([7, Lemma 4.2]). Let $V$ be a simple, rational and $C_2$-cofinite VOA of CFT type. Let $M^0,\ M^1, \cdots,\ M^d$ be as before with the corresponding conformal weights $\lambda_i > 0$ for $0 < i \leq d$. Then $0 < \mathrm{qdim}_V\, M^i < \infty$ for any $0 \leq i \leq d$. Moreover, we have*

$$\mathrm{qdim}_V\, M^i = \frac{S_{i,0}}{S_{0,0}}. \tag{2.6}$$

## 3. The VOAs $V_{L_{\mathcal{C}\times\mathcal{D}}}$ and $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$

In this section, we will review some properties about the lattice VOA $V_{L_{\mathcal{C}\times\mathcal{D}}}$ and the orbifold VOA $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$.

***$\mathbb{Z}_3$ and $\mathbb{F}_4$-codes*** We first review the coding theory concerned in this paper. All codes mentioned in this paper are linear codes. From now on, we fix $\ell \in \mathbb{N}$. We also use a boldface lowercase letter $\boldsymbol{x}$ to denote a vector or a sequence of length $\ell$ and its $i$-th coordinate is denoted by $x_i$. That is $\boldsymbol{x} = (x_1, \cdots, x_\ell)$.

**Definition 3.1.** Let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_\ell)$ be a codeword of length $\ell$, its *support* is defined to be $\mathrm{Supp}(\boldsymbol{\lambda}) = \{i \mid \lambda_i \neq 0\}$. The cardinality of $\mathrm{Supp}(\boldsymbol{\lambda})$, denoted by $\mathrm{wt}(\boldsymbol{\lambda})$, is called the *(Hamming) weight* of $\boldsymbol{\lambda}$. A code $\mathbf{C}$ is said to be *even* if $\mathrm{wt}(\boldsymbol{\lambda})$ is even for every $\boldsymbol{\lambda} \in \mathbf{C}$.

Let $\boldsymbol{\mathcal{S}}$ be a subset of codewords of length $\ell$. The *(Hamming) weight enumerator* of $\boldsymbol{\mathcal{S}}$ is defined to be

$$W_{\boldsymbol{\mathcal{S}}}(X, Y) = \sum_{\boldsymbol{\lambda} \in \boldsymbol{\mathcal{S}}} X^{\ell - \mathrm{wt}(\boldsymbol{\lambda})} Y^{\mathrm{wt}(\boldsymbol{\lambda})}, \tag{3.1}$$

which is a homogeneous polynomial of degree $\ell$.

We consider the inner products for codes over $\mathbb{F}_4$ and $\mathbb{Z}_3$ as follows. For codes over $\mathbb{F}_4$, we use the Hermitian inner product, i.e.,

$$\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{\ell} x_i \bar{y}_i, \qquad \text{for } \boldsymbol{x} = (x_1, \cdots, x_\ell),\ \boldsymbol{y} = (y_1, \cdots, y_\ell) \in \mathbb{F}_4^\ell,$$

where $\bar{x} = x^2$ is the *conjugate* of $x \in \mathbb{F}_4$. For $\mathbb{Z}_3$-codes, we use the usual Euclidean inner product:

$$\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{\ell} x_i y_i \qquad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_3^\ell.$$

**Definition 3.2.** Let $\mathcal{K} = \mathbb{F}_4$ or $\mathbb{Z}_3$. For a $\mathcal{K}$-code $\boldsymbol{\mathcal{S}}$ of length $\ell$ with the inner product given as above, we define its dual code by $\boldsymbol{\mathcal{S}}^\perp = \{\boldsymbol{\lambda} \in \mathcal{K}^\ell \mid \boldsymbol{\lambda} \cdot \boldsymbol{\mu} = 0 \text{ for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{S}}\}$. A $\mathcal{K}$-code $\boldsymbol{\mathcal{S}}$ is said to be *self-orthogonal* if $\boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{S}}^\perp$ and *self-dual* if $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^\perp$.

**Remark 3.3.** By [18, Theorem 1.4.10], an $\mathbb{F}_4$-code $\boldsymbol{\mathcal{C}}$ is even if and only if $\boldsymbol{\mathcal{C}}$ is Hermitian self-orthogonal. Note that the underlying "additive" group structure of $\mathbb{F}_4$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, an even $\mathbb{F}_4$-code $\boldsymbol{\mathcal{C}}$ is also an even "additive" $\mathbb{Z}_2 \times \mathbb{Z}_2$ code. Moreover, $\boldsymbol{\mathcal{C}}$ is $\tau$-invariant since it is $\mathbb{F}_4$-linear. In [34], even $\tau$-invariant $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes are used. Instead of the Hermitian inner product, they used the trace Hermitian inner product defined by $\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{\ell} x_i \bar{y}_i + \bar{x}_i y_i$.

In the notation of [31], codes $\boldsymbol{\mathcal{C}}$ in our setting belong to the family $4^H$, while codes in Tanabe and Yamada's setting belong to the family $4^{H+}$. If the code $\boldsymbol{\mathcal{C}}$ is also linear, then its dual $\boldsymbol{\mathcal{C}}^\perp$ in $4^{H+}$ coincides with the dual of $\boldsymbol{\mathcal{C}}$ in $4^H$. In other words, $\boldsymbol{\mathcal{C}}$ is self-orthogonal in $4^H$ if and only if $\boldsymbol{\mathcal{C}}$ is self-orthogonal in $4^{H+}$. Therefore, these two notions are essentially the same and almost all theorems we proved in this paper have analogous statements in their setting.

***The lattice $L_{\boldsymbol{\mathcal{C}} \times \boldsymbol{\mathcal{D}}}$ and the VOAs $V_{L_{\boldsymbol{\mathcal{C}} \times \boldsymbol{\mathcal{D}}}}$ and $V_{L_{\boldsymbol{\mathcal{C}} \times \boldsymbol{\mathcal{D}}}}^\tau$*** From now on, we let $\boldsymbol{\mathcal{C}}$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and let $\boldsymbol{\mathcal{D}}$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. First we review a construction of an even lattice from the codes $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ (see [22,35]). For $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_\ell) \in \mathbb{F}_4^\ell$ and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_\ell) \in \mathbb{Z}_3^\ell$, we define

$$L_{\boldsymbol{\lambda}\times\boldsymbol{\delta}} := \{(x_1, \ldots, x_\ell) \in (L^\perp)^{\oplus\ell} \mid x_i \in L^{(\lambda_i,\delta_i)}, i = 1, \ldots, \ell\}.$$

For subsets $\boldsymbol{P} \subset \mathbb{F}_4^\ell$ and $\boldsymbol{Q} \subset \mathbb{Z}_3^\ell$, we define

$$L_{\boldsymbol{P}\times\boldsymbol{Q}} := \bigcup_{\boldsymbol{\lambda}\in\boldsymbol{P},\boldsymbol{\delta}\in\boldsymbol{Q}} L_{\boldsymbol{\lambda}\times\boldsymbol{\delta}} \subset (L^\perp)^{\oplus\ell}.$$

Let $\tau$ act diagonally on $(L^\perp)^{\oplus\ell}$. It induces an action on $V_{(L^\perp)^{\oplus\ell}}$.

We will determine the quantum dimensions and the fusion rules for the irreducible modules of the orbifold VOA $V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$ in this paper.

**Proposition 3.4** *([35, Lemmas 2.5, 2.6]). Let $\boldsymbol{\mathcal{C}}$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and $\boldsymbol{\mathcal{D}}$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Then the subset $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$ is an even sublattice of $(L^\perp)^{\oplus\ell}$. Moreover, the dual lattice $(L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}})^\perp = L_{\boldsymbol{\mathcal{C}}^\perp\times\boldsymbol{\mathcal{D}}^\perp}$.*

**Proposition 3.5** *([14,8,35]). Let $\boldsymbol{\mathcal{C}}$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and $\boldsymbol{\mathcal{D}}$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Let $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$ be the lattice VOA associated to $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$. Then we have the following.*

(i) *The set of all inequivalent irreducible $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules is given by*

$$\{V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}} \mid \boldsymbol{\lambda}+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^\perp/\boldsymbol{\mathcal{C}}, \boldsymbol{\delta}+\boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}\}.$$

(ii) *We have $V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}} \circ \tau \cong V_{L_{(\tau^{-1}(\boldsymbol{\lambda})+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}$.*

(iii) *For $i = 1, 2$, there are exactly $\left|\boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}\right|$ inequivalent irreducible $\tau^i$-twisted $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules. They are represented by $(V^{T,\boldsymbol{\eta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i), Y^{\tau^i})$ for $\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}^\perp \bmod \boldsymbol{\mathcal{D}}$.*

**Remark 3.6.** We briefly review the construction of $\tau$-twisted modules $V^{T,\boldsymbol{\eta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau)$. For general references, the readers may refer to [23,8]. We follow the notations of [35, Section 3]. For any positive integer $n$, let $\langle\kappa_n\rangle$ be a cyclic group of order $n$ with generator $\kappa_n$ and $\xi_n := \exp(2\pi\sqrt{-1}/n)$. Let $1 \to \langle\kappa_{36}\rangle \to \hat{L}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}},\tau} \to L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}} \to 1$ be the central extension of $L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}$ with associated bilinear forms given in [35, (2.8)]. Let $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_\ell) \in \mathbb{Z}_3^\ell$ and define a homomorphism $\psi_{\boldsymbol{\eta}} : \hat{L}_{\boldsymbol{\mathcal{C}}\times\mathbf{0},\tau} \to \mathbb{C}^\times$ such that (i) $\psi_{\boldsymbol{\eta}}(\kappa_{36}) = \xi_{36}$, (ii) $\psi_{\boldsymbol{\eta}}$ is 1 on $K_0 := \{a\tau(a)^{-1} \mid a \in \hat{L}_{\boldsymbol{\mathcal{C}}\times\mathbf{0},\tau}\}$, and (iii) $\psi_{\boldsymbol{\eta}}(\kappa_3 e^{\beta_i^{(s)}}) = \xi_3^{\eta_s}$, where $1 \le s \le \ell$ and $\beta_i^{(s)} = (0, \cdots, 0, \overset{s\text{-th}}{\beta_i}, 0, \cdots, 0) \in L^\ell$.

Let $\mathbb{C}_{\psi_{\boldsymbol{\eta}}}$ be a one dimensional $\hat{L}_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}$-module affording the character $\psi_{\boldsymbol{\eta}}$ and let

$$T_{\psi_{\boldsymbol{\eta}}} := \mathbb{C}[\hat{L}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}},\tau}] \otimes_{\mathbb{C}[\hat{L}_{\boldsymbol{\mathcal{C}}\times\mathbf{0},\tau}]} \mathbb{C}_{\psi_{\boldsymbol{\eta}}}$$

be the $\hat{L}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}},\tau}$-modules induced from $\mathbb{C}_{\psi_{\boldsymbol{\eta}}}$.

Set $\widehat{\mathfrak{h}}[\tau]$ be the $\tau$-twisted affine Lie algebra and $S[\tau]$ be the induced $\widehat{\mathfrak{h}}[\tau]$-module

$$S[\tau] = U(\widehat{\mathfrak{h}}[\tau]) \otimes_{\widehat{\mathfrak{h}}[\tau]^+ \oplus \widehat{\mathfrak{h}}[\tau]^0} \mathbb{C}.$$

The $\tau$-twisted $V_{L_{\mathcal{C}\times\mathcal{D}}}$-module is defined in [35, (3.24)] as

$$V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau) = S[\tau] \otimes T_{\psi_{\boldsymbol{\eta}}},$$

for $\boldsymbol{\eta} \in D^{\perp}$ and $\tau$ acts on $T_{\psi_{\boldsymbol{\eta}}}$ as a scalar $\xi^{2\,\mathrm{wt}(\boldsymbol{\eta})}$. As given in [35, (3.27)],

$$V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau) \cong \bigoplus_{\boldsymbol{\gamma}\in\mathcal{D}} V^{T,\boldsymbol{\eta}-\boldsymbol{\gamma}}_{L^{\oplus \ell}}(\tau) \qquad \text{as a } \tau\text{-twisted } V_{L^{\oplus\ell}}\text{-module.}$$

We have similar descriptions for $\tau^2$-twisted $V_{L_{\mathcal{C}\times\mathcal{D}}}$-modules. We consider a central extension

$$1 \to \langle \kappa_{36} \rangle \to \hat{L}_{\mathcal{C}\times\mathcal{D},\tau^2} \to L_{\mathcal{C}\times\mathcal{D}} \to 1$$

associated with the bilinear form $c_2'$ given in [35, (2.9)]. One can then construct a class of irreducible $\hat{L}_{\mathcal{C}\times\mathcal{D},\tau^2}$-modules $T'_{\psi_{\boldsymbol{\eta}}}$ for any $\boldsymbol{\eta} \in D^{\perp}$. Then

$$V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2) = S[\tau^2] \otimes T'_{\psi_{\boldsymbol{\eta}}}$$

and we assume that $\tau$ acts on $T'_{\psi_{\boldsymbol{\eta}}}$ as a scalar $\xi^{\mathrm{wt}(\boldsymbol{\eta})}$, i.e., $\tau^2$ acts on $T'_{\psi_{\boldsymbol{\eta}}}$ as a scalar $\xi^{2\,\mathrm{wt}(\boldsymbol{\eta})}$.

Since $\tau$ acts trivially on $\mathcal{D}$, by Proposition 3.5, $V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}} \cong V_{L_{(\boldsymbol{\lambda}'+\mathcal{C})\times(\boldsymbol{\delta}'+\mathcal{D})}}$ as $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules if and only if (1) $\boldsymbol{\lambda}+\mathcal{C}$ and $\boldsymbol{\lambda}'+\mathcal{C}$ belong to the same $\tau$-orbit of $\mathcal{C}^{\perp}$; and (2) $\boldsymbol{\delta}+\mathcal{D} = \boldsymbol{\delta}'+\mathcal{D}$ in $\mathcal{D}^{\perp}/\mathcal{D}$. Let $\mathcal{C}^{\perp}_{\equiv\tau}$ denote the set of all $\tau$-orbits in $\mathcal{C}^{\perp}$. Then

$$\{V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon], V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}} \mid \varepsilon \in \mathbb{Z}_3, \mathbf{0} \neq \boldsymbol{\lambda}+\mathcal{C} \in \mathcal{C}^{\perp}_{\equiv\tau} \bmod \mathcal{C}, \boldsymbol{\delta}+\mathcal{D} \in \mathcal{D}^{\perp}/\mathcal{D}\}$$

is a set of inequivalent irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules, which are obtained from the irreducible (untwisted) $V_{L_{\mathcal{C}\times\mathcal{D}}}$-modules.

It is usually very difficult to classify all irreducible modules of an orbifold VOA. Recently, Miyamoto gave a classification in the $\mathbb{Z}_3$-orbifold case.

**Proposition 3.7** *([29,30]). Let $V$ be a rational VOA of CFT-type. Assume $V \cong V'$, its contragredient dual. Let $\sigma$ be an automorphism of $V$ of order three. If the fixed point subVOA $V^{\sigma}$ is $C_2$-cofinite, then $V^{\sigma}$ is rational. Moreover, every irreducible $V^{\sigma}$-module is a submodule of some $\sigma^j$-twisted $V$-module for some $j$.*

**Proposition 3.8.** *The VOAs $V_{L_{\mathcal{C}\times\mathcal{D}}}$ and $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ are simple, rational, $C_2$-cofinite VOAs of CFT type and are isomorphic to their contragredient dual, respectively.*

**Proof.** Assertions about lattice VOA are well-known. By [24, Corollary 3.2], we know $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ is self-dual. It is proved in [35, Theorem 7.10] that $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ is a simple, $C_2$-cofinite VOA of CFT type. Together with Proposition 3.7, we get the rationality of $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$. □

**Remark 3.9.** Since $V^{\tau}_{L}$ is $C_2$-cofinite and rational by [34] and $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}} \supset (V^{\tau}_{L})^{\otimes\ell}$, it also follows from [4] that $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ is $C_2$-cofinite and from [20, Theorem 3.5] that $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ is rational.

By Proposition 3.7, we can classify irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules.

**Proposition 3.10.** *An irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-module must belong to one of the following types.*

$$\text{(i)}\quad V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon],\qquad \text{(ii)}\quad V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}},\qquad \text{(iii)}\quad V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon],$$

*where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $\mathbf{0} \neq \boldsymbol{\lambda} + \mathcal{C} \in \mathcal{C}^{\perp}_{\equiv\tau} \bmod \mathcal{C}$, $\boldsymbol{\eta} \in \mathcal{D}^{\perp} (\mathrm{mod}\, \mathcal{D})$ and $\boldsymbol{\delta} + \mathcal{D} \in \mathcal{D}^{\perp}/\mathcal{D}$. In particular, there is no modules of the second type (ii) if $\mathcal{C}$ is self-dual.*

It is easy to check the following lemma.

**Lemma 3.11.** *We have* $\#\{\mathbf{0} \neq \boldsymbol{\gamma} \in \mathcal{C}^{\perp}_{\equiv\tau} \bmod \mathcal{C}\} = \left(\left|\mathcal{C}^{\perp}/\mathcal{C}\right| - 1\right)/3.$

In this paper, our calculations depend heavily on the decomposition of the irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules as $(V^{\tau}_{L})^{\otimes\ell}$-modules.

**Proposition 3.12** *([22],[35, Theorem 3.13]). As modules of $(V^{\tau}_{L})^{\otimes\ell}$, we have the following decomposition.*

$$\text{(i)}\quad V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon] \cong V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\mathcal{D})}}; \tag{3.2a}$$

$$\text{(ii)}\quad V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon] \cong \bigoplus_{\boldsymbol{\delta}\in\mathcal{D}} \left( \bigoplus_{e_1+\cdots+e_\ell\equiv\varepsilon \bmod 3} V^{T,\eta_1-i\delta_1}_{L}(\tau^i)[e_1] \otimes \cdots \otimes V^{T,\eta_\ell-i\delta_\ell}_{L}(\tau^i)[e_\ell] \right), \tag{3.2b}$$

*where $\boldsymbol{\delta}, \boldsymbol{\eta} \in \mathcal{D}^{\perp}$ and $\mathcal{C}_{\equiv\tau}$ denotes the set of all $\tau$-orbits in $\mathcal{C}$.*

**Remark 3.13.** By these decompositions and Proposition 2.1, we know the conformal weights of irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are positive except $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ itself.

## 4. Quantum dimensions of irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules

In this section, we compute the quantum dimensions of irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules. We will first consider the case when $\mathcal{D} = \{\mathbf{0}\}$ is the trivial $\mathbb{Z}_3$-code. Results in this case are summarized in Theorem 4.13. Results for the general case are given in Theorem 4.14. In addition, we verify one conjecture about global dimensions proposed by Dong et. al. for the VOA $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$ in this section.

***Weight enumerators*** We first define several (generalized) weight enumerators.

**Definition 4.1.** For $\varepsilon = 0, 1, 2$, let $S_\varepsilon := \{\boldsymbol{x} := (x_1, \cdots, x_\ell) \in \mathbb{Z}_3^\ell \mid \sum x_i \equiv \varepsilon \bmod 3\}$.

We denote the weight enumerator of $S_\varepsilon$ by $W_\varepsilon(X,Y)$, i.e.,

$$W_\varepsilon(X,Y) := \sum_{\boldsymbol{x}\in S_\varepsilon} X^{\ell-\mathrm{wt}(\boldsymbol{x})}Y^{\mathrm{wt}(\boldsymbol{x})}, \tag{4.1}$$

where $\mathrm{wt}(\boldsymbol{x})$ denotes the Hamming weight of $\boldsymbol{x}$ (cf. Definition 3.1).

We also consider a weight enumerator induced from an $\mathbb{F}_4$-code $\mathcal{C}$.

**Definition 4.2.** Let $\mathcal{C}$ be an $\mathbb{F}_4$-code and let $W_{\mathcal{C}}(X,Y)$ be its Hamming weight enumerator. We define

$$W'_{\mathcal{C}}(X,Y) := \frac{1}{3}(W_{\mathcal{C}}(X,Y) - X^\ell). \tag{4.2}$$

**Remark 4.3.** Note that $W_\varepsilon(X,Y), W'_{\mathcal{C}}(X,Y)$ are homogeneous polynomials in $X, Y$ of the same degree $\ell$.

**Lemma 4.4.** *We have $W'_{\mathcal{C}}(1,1) = (|\mathcal{C}|-1)/3$ and $W_\varepsilon(1,1) = 3^{\ell-1}$. Moreover, the self-orthogonal $\mathbb{F}_4$-code $\mathcal{C}$ is self-dual if and only if $W'_{\mathcal{C}}(1,1) = \frac{2^\ell-1}{3}$.*

**Proof.** First we note that $W_{\mathcal{C}}(1,1) = |\mathcal{C}|$ is equal to the number of elements in $\mathcal{C}$; hence

$$W'_{\mathcal{C}}(1,1) = \frac{W_{\mathcal{C}}(1,1)-1}{3} = \frac{|\mathcal{C}|-1}{3}.$$

It is clear that $S_\varepsilon = (\varepsilon, 0, \ldots, 0) + S_0$ for any $\varepsilon = 1, 2$. Therefore, $|S_1| = |S_2| = |S_0|$. Note also that $W_\varepsilon(1,1) = |S_\varepsilon|$ for any $\varepsilon = 0, 1$, or 2 and $|S_0| + |S_1| + |S_2| = 3^\ell$. Hence we have $W_\varepsilon(1,1) = 3^{\ell-1}$ for any $\varepsilon = 0, 1, 2$.

Since $\mathcal{C}$ is self-orthogonal, we know $\mathcal{C}^\perp \supset \mathcal{C}$ and $\dim \mathcal{C}^\perp + \dim \mathcal{C} = \ell$. Therefore, $|\mathcal{C}| \le 2^\ell$ and the equality holds if and only if $\mathcal{C}$ is self-dual. The lemma now follows. $\square$

The following lemmas explain why we introduce these weight enumerators. By Proposition 3.12, the module $V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon]$ admits a decomposition of $(V_L^\tau)^{\otimes\ell}$-modules as

$$V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon] \cong V_{L_{\mathbf{0}\times\boldsymbol{\delta}}}[\varepsilon] \oplus \bigoplus_{\mathbf{0}\neq\gamma\in\mathcal{C}_{\equiv\tau}} V_{L_{\gamma\times\boldsymbol{\delta}}}, \tag{4.3}$$

where $\boldsymbol{\delta} \in \mathbb{Z}_3^\ell$ and $\mathcal{C}_{\equiv\tau}$ denotes the set of all orbits of $\tau$ in $\mathcal{C}$. In particular, when $\mathcal{C} = \{\mathbf{0}\}$ and $\boldsymbol{\delta} = \mathbf{0}$, we have $V_{L_{\mathbf{0}\times\mathbf{0}}}[0] \cong \left(V_{L^{\oplus\ell}}\right)^\tau$, which should not be confused with the subVOA $(V_L^\tau)^{\otimes\ell} \subsetneq \left(V_{L^{\otimes\ell}}\right)^\tau$.

**Lemma 4.5.** *Let $Z_0(q) := \mathrm{ch}\, V_L[0]$ and $Z_1(q) := \mathrm{ch}\, V_L[1] = \mathrm{ch}\, V_L[2]$. Then the character of $V_{L_{\mathbf{0}\times\mathbf{0}}}[\varepsilon]$ is given by $\mathrm{ch}\, V_{L^{\oplus\ell}}[\varepsilon] = W_\varepsilon(Z_0(q), Z_1(q))$, for $\varepsilon = 0, 1, 2$.*

**Proof.** For $\varepsilon = 0, 1, 2$, we have a decomposition of $(V_L^\tau)^{\otimes\ell}$-modules:

$$V_{L^{\oplus\ell}}[\varepsilon] = \bigoplus_{\sum r_i \equiv \varepsilon \bmod 3} V_L[r_1] \otimes \cdots \otimes V_L[r_\ell]. \tag{4.4}$$

We also know $\mathrm{ch}\,(V_L[r_1] \otimes \cdots \otimes V_L[r_\ell]) = \mathrm{ch}\, V_L[r_1] \times \cdots \times \mathrm{ch}\, V_L[r_\ell] = Z_0^{\ell-r} Z_1^r$, where $r$ is the weight of $\boldsymbol{r} := (r_1, \cdots, r_\ell) \in \mathbb{Z}_3^\ell$.

Recall that $W_\varepsilon(X, Y) := \sum_{\boldsymbol{x}\in S_\varepsilon} X^{\ell-\mathrm{wt}(\boldsymbol{x})} Y^{\mathrm{wt}(\boldsymbol{x})}$ (cf. Definition 4.1). We have

$$\mathrm{ch}\, V_{L^{\oplus\ell}}[\varepsilon] = \sum_{\sum r_i \equiv \varepsilon \bmod 3} \mathrm{ch}\, V_L[r_1] \times \cdots \times \mathrm{ch}\, V_L[r_\ell] = \sum_{\boldsymbol{r}\in S_\varepsilon} Z_0^{\ell-\mathrm{wt}(\boldsymbol{r})} Z_1^{\mathrm{wt}(\boldsymbol{r})} = W_\varepsilon(Z_0, Z_1)$$

as desired. □

**Lemma 4.6.** *Let $Y_0(q) := \mathrm{ch}\, V_{L^{(0,0)}}$ and $Y_1(q) := \mathrm{ch}\, V_{L^{(1,0)}}$. We have the character*

$$\mathrm{ch}\left(\bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}}\right) = W'_{\mathcal{C}}(Y_0, Y_1),$$

**Proof.** We first note that $Y_1(q) = \mathrm{ch}\, V_{L^{(x,0)}}$ for $x = 1, \omega, \bar{\omega} \in \mathbb{F}_4$. Let $\mathbf{0} \neq \boldsymbol{\gamma} \in \mathcal{C}_{\equiv\tau}$. Then

$$\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}} = \prod_i \mathrm{ch}\, V_{L^{(\gamma_i,0)}} = Y_0^{\ell-\mathrm{wt}(\boldsymbol{\gamma})} Y_1^{\mathrm{wt}(\boldsymbol{\gamma})}.$$

We know the $\tau$-orbit of $\boldsymbol{\gamma}$ is the set $\{\boldsymbol{\gamma}, \omega\boldsymbol{\gamma}, \omega^2\boldsymbol{\gamma}\}$, where $\omega\boldsymbol{\gamma} := (\omega\gamma_1, \cdots, \omega\gamma_\ell)$. Note that $\omega\gamma_i = 0$ if and only if $\gamma_i = 0$. This means $\mathrm{wt}\,\boldsymbol{\gamma} = \mathrm{wt}\,\tau\boldsymbol{\gamma}$ and hence

$$\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}} = \mathrm{ch}\, V_{L_{\omega\boldsymbol{\gamma}\times\mathbf{0}}} = \mathrm{ch}\, V_{L_{\omega^2\boldsymbol{\gamma}\times\mathbf{0}}}.$$

By Definition 4.2, we have

$$\mathrm{ch}\left(\bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}}\right) = \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} \mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}} = \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} Y_0^{\ell-\mathrm{wt}(\boldsymbol{\gamma})} Y_1^{\mathrm{wt}(\boldsymbol{\gamma})} = W'_{\mathcal{C}}(Y_0, Y_1) \tag{4.5}$$

as desired. □

**Proposition 4.7.** *For $\varepsilon = 0, 1, 2$, we have*

$$\operatorname{ch} V_{L_{\mathcal{C}\times\mathbf{0}}}[\varepsilon] = W_\varepsilon(Z_0, Z_1) + W'_{\mathcal{C}}(Y_0, Y_1),$$

**Proof.** This proposition follows directly from Equation (4.3) and Lemmas 4.5 and 4.6. □

***Quantum dimensions of $V^\tau_{L_{\mathcal{C}\times\mathbf{0}}}$-modules*** We first compute the quantum dimensions of irreducible $V^\tau_{L_{\mathcal{C}\times\mathbf{0}}}$-modules in the case that the code $\mathcal{D} = \{\mathbf{0}\} \subset \mathbb{Z}_3^\ell$ is the trivial code. Note that in this case $\mathcal{D}^\perp = \mathbb{Z}_3^\ell$.

The idea is easy; we observe that

$$\operatorname{qdim}_{V^\tau_{L_{\mathcal{C}\times\mathbf{0}}}} M = \frac{\operatorname{qdim}_{(V_L^\tau)^{\otimes\ell}} M}{\operatorname{qdim}_{(V_L^\tau)^{\otimes\ell}} V^\tau_{L_{\mathcal{C}\times\mathbf{0}}}}, \quad \text{for any } V^\tau_{L_{\mathcal{C}\times\mathbf{0}}}\text{-module } M. \tag{4.6}$$

Then using the decompositions given in Proposition 3.12, we can express both enumerator and denominator in terms of the weight enumerators we introduced before.

**Proposition 4.8.** *For $\varepsilon \in \mathbb{Z}_3$ and $\boldsymbol{\delta} \in \mathbb{Z}_3^\ell$, the irreducible $V_{L_{\mathcal{C}\times\mathbf{0}}}[0]$-module $V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon]$ has the quantum dimension one.*

**Proof.** Fix $0 \le \varepsilon \le 2$ and $\boldsymbol{\delta} \in \mathbb{Z}_3^\ell$. Let

$$Z(q) := \frac{\operatorname{ch} V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon]}{\operatorname{ch} V_{L_{\mathcal{C}\times\mathbf{0}}}[0]} = \frac{\operatorname{ch} V_{L_{\mathbf{0}\times\boldsymbol{\delta}}}[\varepsilon] + \sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} \operatorname{ch} V_{L_{\boldsymbol{\gamma}\times\boldsymbol{\delta}}}}{\operatorname{ch} V_{L_{\mathbf{0}\times\mathbf{0}}}[0] + \sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} \operatorname{ch} V_{L_{\boldsymbol{\gamma}\times\mathbf{0}}}}$$

$$= \frac{\sum_{\boldsymbol{r}\in S_\varepsilon} \operatorname{ch} V_{L^{(0,\delta_1)}}[r_1] \times\cdots\times \operatorname{ch} V_{L^{(0,\delta_1)}}[r_\ell] + \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} \operatorname{ch} V_{L^{(r_1,\delta_1)}} \times\cdots\times \operatorname{ch} V_{L^{(r_\ell,\delta_\ell)}}}{\sum_{\boldsymbol{r}\in S_0} \operatorname{ch} V_L[r_1] \times\cdots\times \operatorname{ch} V_L[r_\ell] + \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} \operatorname{ch} V_{L^{(r_1,0)}} \times\cdots\times \operatorname{ch} V_{L^{(r_\ell,0)}}}.$$

Dividing both denominator and numerator by $(\operatorname{ch} V_L[0])^\ell$, we get

$$Z(q) = \frac{\sum_{\boldsymbol{r}\in S_\varepsilon} \frac{\operatorname{ch} V_{L^{(0,\delta_1)}}[r_1]}{\operatorname{ch} V_L[0]} \times\cdots\times \frac{\operatorname{ch} V_{L^{(0,\delta_1)}}[r_\ell]}{\operatorname{ch} V_L[0]} + \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} \frac{\operatorname{ch} V_{L^{(r_1,\delta_1)}}}{\operatorname{ch} V_L[0]} \times\cdots\times \frac{\operatorname{ch} V_{L^{(r_\ell,\delta_\ell)}}}{\operatorname{ch} V_L[0]}}{\sum_{\boldsymbol{r}\in S_0} \frac{\operatorname{ch} V_L[r_1]}{\operatorname{ch} V_L[0]} \times\cdots\times \frac{\operatorname{ch} V_L[r_\ell]}{\operatorname{ch} V_L[0]} + \frac{1}{3}\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}} \frac{\operatorname{ch} V_{L^{(r_1,0)}}}{\operatorname{ch} V_L[0]} \times\cdots\times \frac{\operatorname{ch} V_{L^{(r_\ell,0)}}}{\operatorname{ch} V_L[0]}}.$$

Recalling the quantum dimensions of $V_L[0]$-modules given in Proposition 2.5, we have

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}[0]} V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon] = \lim_{y\to 0^+} Z(q) = \frac{W_\varepsilon(1,1) + W'_{\mathcal{C}}(1,1)}{W_0(1,1) + W'_{\mathcal{C}}(1,1)} = \frac{3^{\ell-1} + W'_{\mathcal{C}}(1,1)}{3^{\ell-1} + W'_{\mathcal{C}}(1,1)} = 1$$

as desired. □

**Proposition 4.9.** *Let $\varepsilon = 0, 1, 2$ and $\boldsymbol{\eta} \in \mathbb{Z}_3^\ell$. The irreducible $V_{L_{\mathcal{C}\times\mathbf{0}}}[0]$-module $V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathbf{0}}}(\tau^i)[\varepsilon]$ has the quantum dimension $2^\ell/|\mathcal{C}|$.*

**Proof.** By Proposition 3.12, an irreducible $V_{L_{\mathcal{C}\times\mathbf{0}}}[0]$-module of twisted type admits a decomposition of $(V_L^{\tau})^{\otimes \ell}$-modules:

$$V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] \cong \bigoplus_{\boldsymbol{e}\in S_\varepsilon} V_L^{T,\eta_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T,\eta_\ell}(\tau^i)[e_\ell]. \tag{4.7}$$

By Proposition 2.5, we have $\operatorname{qdim}_{V_L[0]} V_L^{T,j}(\tau^i)[k] = 2$, for any $j, k = 0, 1, 2$. Therefore,

$$\begin{aligned}\operatorname{qdim}_{(V_L^{\tau})^{\otimes \ell}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] &= \lim_{y\to 0^+} \frac{\sum_{\boldsymbol{e}\in S_\varepsilon} \operatorname{ch} V_L^{T,\eta_1}(\tau^i)[e_1] \times \cdots \times \operatorname{ch} V_L^{T,\eta_\ell}(\tau^i)[e_\ell]}{(\operatorname{ch} V_L[0])^\ell} \\ &= \sum_{\boldsymbol{e}\in S_\varepsilon} \lim_{y\to 0^+} \frac{\operatorname{ch} V_L^{T,\eta_1}(\tau^i)[e_1]}{\operatorname{ch} V_L[0]} \cdots \frac{\operatorname{ch} V_L^{T,\eta_\ell}(\tau^i)[e_\ell]}{\operatorname{ch} V_L[0]} \\ &= W_\varepsilon(2,2).\end{aligned}$$

Since $W_\varepsilon$ are homogeneous polynomials of degree $\ell$, we have

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = \frac{W_\varepsilon(2,2)}{W_0(1,1) + W'_{\mathcal{C}}(3,3)} = \frac{2^\ell W_\varepsilon(1,1)}{W_0(1,1) + 3^\ell W'_{\mathcal{C}}(1,1)}. \tag{4.8}$$

Now by Proposition 4.7 and Lemma 4.4 we have

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = \frac{2^\ell \cdot 3^{\ell-1}}{3^{\ell-1} + 3^{\ell-1}(|\mathcal{C}| - 1)} = \frac{2^\ell}{|\mathcal{C}|}$$

as desired. □

**Remark 4.10.** Note that $\frac{2^\ell}{|\mathcal{C}|} = \sqrt{|\mathcal{C}^\perp/\mathcal{C}|}$ since $|\mathcal{C}^\perp| \cdot |\mathcal{C}| = \mathbb{F}_4^\ell = (2^\ell)^2$.

**Corollary 4.11.** *Let $\mathcal{C}$ be a self-dual $\mathbb{F}_4$-code. Then all irreducible $V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}$-modules are simple current modules.*

**Proof.** If $\mathcal{C}$ is self-dual, then $V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}$ has only two types of irreducible modules. Moreover,

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|} = 1,$$

by the self-duality of $\mathcal{C}$. That means all irreducible modules of the type $V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]$ are simple current modules. By Proposition 4.8, the irreducible modules of the type $V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon]$ are simple current modules, also. □

Now suppose $\mathcal{C}$ is self-orthogonal but not self-dual. Then the quantum dimension of the $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$-module $V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]$ is strictly greater than 1. In addition, $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$ has irreducible modules of the type $V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}$.

**Proposition 4.12.** *Let* $\boldsymbol{\lambda}+\mathcal{C}\in\mathcal{C}^{\perp}/\mathcal{C}$ *and* $\boldsymbol{\delta}\in\mathbb{Z}_3^{\ell}$*, we have* $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}=3$*.*

**Proof.** By definition,

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}=\lim_{y\to 0^+}\frac{\operatorname{ch} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}}{\operatorname{ch} V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}}=\lim_{y\to 0^+}\frac{\sum_{\boldsymbol{\mu}\in\mathcal{C}}\operatorname{ch} V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mu})\times\boldsymbol{\delta}}}}{\operatorname{ch} V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}}.$$

Dividing both the denominator and the numerator by $(\operatorname{ch} V_L[0])^{\ell}$ and using the fact that $\operatorname{qdim}_{V_L[0]} V_{L^{(i,j)}}=3$ for any $i\in\mathbb{F}_4\setminus\{0\}, j\in\mathbb{Z}_3$, we have

$$\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}=\frac{|\mathcal{C}|\cdot 3^{\ell}}{3^{\ell-1}+3^{\ell-1}(|\mathcal{C}|-1)}=3$$

as desired. □

To summarize, we have the theorem.

**Theorem 4.13.** *The quantum dimensions for irreducible* $V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}$*-modules are as follows.*

(i) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\boldsymbol{\delta}}}[\varepsilon]=1$;
(ii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times\boldsymbol{\delta}}}=3$;
(iii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]=\frac{2^{\ell}}{|\mathcal{C}|}$,

*where* $i=1,2$, $\varepsilon\in\mathbb{Z}_3$, $\mathbf{0}\neq\boldsymbol{\lambda}+\mathcal{C}\in\mathcal{C}^{\perp}_{\equiv\tau} \bmod \mathcal{C}$ *and* $\boldsymbol{\eta},\boldsymbol{\delta}\in\mathbb{Z}_3^{\ell}$.

***Quantum dimension of*** $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$***-modules*** We now deal with the general case. Let $\mathcal{D}$ be a self-orthogonal $\mathbb{Z}_3$-code. The basic idea is to express the characters of $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$-modules in terms of the characters of $V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}$-modules.

**Theorem 4.14.** *The quantum dimensions of irreducible* $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$*-modules are as follows.*

(i) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]=1$;
(ii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}}=3$;
(iii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]=\frac{2^{\ell}}{|\mathcal{C}|}$,

*where* $i=1,2$, $\varepsilon\in\mathbb{Z}_3$, $\mathbf{0}\neq\boldsymbol{\lambda}+\mathcal{C}\in\mathcal{C}^{\perp}_{\equiv\tau} \bmod \mathcal{C}$, $\boldsymbol{\eta}\in\mathcal{D}^{\perp} \bmod \mathcal{D}$ *and* $\boldsymbol{\delta}+\mathcal{D}\in\mathcal{D}^{\perp}/\mathcal{D}$.

**Proof.** (i) For the module $V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]$, we have a decomposition of $(V_L^{\tau})^{\otimes\ell}$-modules:

$$V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\cong V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\oplus\bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\mathcal{D})}}. \tag{4.9}$$

Although the characters $\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\Delta}}$ may vary as $\Delta$ varies in $\boldsymbol{\mathcal{D}}$, we still have

$$\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\Delta)}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \prod_{i=1}^{\ell} \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L(\gamma^i,\boldsymbol{\delta}^i+\Delta^i)}}{\mathrm{ch}\, V_L^\tau} = \prod_{i=1}^{\ell} \mathrm{qdim}\, V_{L(\gamma^i,\boldsymbol{\delta}^i+\Delta^i)} = \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\boldsymbol{\delta}}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}},$$

for all $\Delta \in \boldsymbol{\mathcal{D}}$. This implies

$$\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{\sum_{\Delta\in\boldsymbol{\mathcal{D}}} \mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\Delta)}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = |\boldsymbol{\mathcal{D}}| \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\gamma}\times\boldsymbol{\delta}}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}}. \tag{4.10}$$

Similarly, we have

$$\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\Delta)}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\mathbf{0}\times\boldsymbol{\delta}}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}}, \text{ for all } \Delta \in \boldsymbol{\mathcal{D}}.$$

Therefore,

$$\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{\sum_{\Delta\in\boldsymbol{\mathcal{D}}} \mathrm{ch}\, V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\Delta)}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = |\boldsymbol{\mathcal{D}}| \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\mathbf{0}\times\boldsymbol{\delta}}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}}. \tag{4.11}$$

Thus by (4.9), (4.10) and (4.11) we know

$$\begin{aligned} &\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\mathbf{0}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon] + \mathrm{ch} \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\boldsymbol{\mathcal{C}}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} \\ &= \lim_{y\to 0^+} \frac{|\boldsymbol{\mathcal{D}}|\, \mathrm{ch}\, V_{L_{\mathbf{0}\times\boldsymbol{\delta}}}[\varepsilon] + |\boldsymbol{\mathcal{D}}|\, \mathrm{ch} \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\boldsymbol{\mathcal{C}}_{\equiv\tau}} V_{L_{\boldsymbol{\gamma}\times\boldsymbol{\delta}}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{|\boldsymbol{\mathcal{D}}|\, \mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\delta}}}[\varepsilon]}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathrm{qdim}_{V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon] = \lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{\mathrm{ch}\, V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} = \lim_{y\to 0^+} \frac{\frac{1}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} \mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{\frac{1}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} \mathrm{ch}\, V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} \\ &= \lim_{y\to 0^+} \frac{|\boldsymbol{\mathcal{D}}|\, \mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\delta}}}[\varepsilon]}{|\boldsymbol{\mathcal{D}}|\, \mathrm{ch}\, V_{L_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}}[0]} = \mathrm{qdim}_{V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}}} V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\delta}}}[\varepsilon] = 1. \end{aligned}$$

(ii) By the similar arguments as (i), we have

$$\lim_{y\to 0^+} \frac{\mathrm{ch}\, V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{\mathrm{ch}\left(\bigoplus_{\Delta\in\boldsymbol{\mathcal{D}}} V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\Delta)}}\right)}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}} = \lim_{y\to 0^+} \frac{|\boldsymbol{\mathcal{D}}|\, \mathrm{ch}\, V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\delta}}}}{\mathrm{ch}\,(V_L^\tau)^{\otimes\ell}};$$

hence $\mathrm{qdim}_{V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}} = \mathrm{qdim}_{V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}}} V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\delta}}} = 3$.

(iii) By Proposition 3.12, we have the decomposition of $V^{T,\boldsymbol{\eta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon]$ as $(V_L^\tau)^{\otimes\ell}$-modules:

$$V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] \cong \bigoplus_{\boldsymbol{\gamma}\in\mathcal{D}} \bigoplus_{\boldsymbol{e}\in S_\varepsilon} V_L^{T,\eta_1-i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T,\eta_\ell-i\gamma_\ell}(\tau^i)[e_\ell].$$

Fix $\boldsymbol{e}\in\mathbb{Z}_3^\ell$; the characters $\operatorname{ch} V_L^{T,\eta_1-i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T,\eta_\ell-i\gamma_\ell}(\tau^i)[e_\ell]$ are all the same for any $(\gamma_1,\cdots,\gamma_\ell)\in\mathcal{D}$. Thus,

$$\begin{aligned}\operatorname{ch} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] &= |\mathcal{D}| \bigoplus_{\boldsymbol{e}\in S_\varepsilon} \operatorname{ch}\left(V_L^{T,\eta_1-i\gamma_1}(\tau^i)[e_1] \otimes \cdots \otimes V_L^{T,\eta_\ell-i\gamma_\ell}(\tau^i)[e_\ell]\right)\\ &= |\mathcal{D}| \operatorname{ch} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\eta}(\tau^i)[\varepsilon].\end{aligned}$$

As before, we have $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = \operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = 2^\ell/|\mathcal{C}|$. □

***Global dimension*** Let $V$ be a VOA with only finitely many irreducible modules, the *global dimension* of V [7] is defined as

$$\operatorname{glob}(V) := \sum_{M\in\operatorname{Irr}(V)} \operatorname{qdim}(M)^2. \tag{4.12}$$

Assume $G$ is a finite subgroup of $\operatorname{Aut}(V)$, it is conjectured in [7] that

$$|G|^2 \operatorname{glob}(V) = \operatorname{glob}(V^G).$$

We will verify this conjecture in our case, i.e., $V = V_{L_{\mathcal{C}\times\mathcal{D}}}$ and $G = \langle\tau\rangle$.

Since all irreducible $V_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are simple currents, we have

$$\operatorname{glob}(V_{L_{\mathcal{C}\times\mathcal{D}}}) = \left|\mathcal{C}^\perp/\mathcal{C}\right| \left|\mathcal{D}^\perp/\mathcal{D}\right| \cdot 1^2.$$

The global dimension of $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}$ will be computed below. We count the number of irreducibles that have the same quantum dimensions.

(i) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon] = 1$. There are $\left|\mathcal{D}^\perp/\mathcal{D}\right| \cdot 3$ irreducible modules of this type.
(ii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}} = 3$ if $\mathbf{0} \neq \boldsymbol{\lambda}+\mathcal{C} \in \mathcal{C}^\perp_{\equiv\tau} \bmod \mathcal{C}$. There are $\left|\mathcal{D}^\perp/\mathcal{D}\right| \cdot \frac{\left|\mathcal{C}^\perp/\mathcal{C}\right|-1}{3}$ irreducible modules of this type.
(iii) $\operatorname{qdim}_{V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] = \frac{2^\ell}{|\mathcal{C}|}$. There are $\left|\mathcal{D}^\perp/\mathcal{D}\right| \cdot 3\cdot 2$ irreducible modules of this type.

Note that $\left(2^\ell/\,|\mathcal{C}|\,\right)^2 = \left|\mathcal{C}^\perp/\mathcal{C}\right|$. Therefore,

$$\operatorname{glob} V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau} = \left|\mathcal{D}^\perp/\mathcal{D}\right| \left(3 + \frac{\left|\mathcal{C}^\perp/\mathcal{C}\right|-1}{3}\cdot 3^2 + 6\left|\mathcal{C}^\perp/\mathcal{C}\right|\right) = 9\left|\mathcal{C}^\perp/\mathcal{C}\right| \left|\mathcal{D}^\perp/\mathcal{D}\right|.$$

Hence we have $\operatorname{glob}(V_{L_{\mathcal{C}\times\mathcal{D}}})\cdot 3^2 = \operatorname{glob}(V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau})$. This verified the conjecture of Dong, Jiao and Xu in this special case.

## 5. Fusion rules

In this section, we compute the fusion rules of $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules. The next three propositions are crucial to our calculations.

**Proposition 5.1** *([35, Proposition 4.5]). Let $\varepsilon, \varepsilon_1, \varepsilon_2, j, j_1, j_2, k \in \mathbb{Z}_3$ and $i = 1, 2$. Then*

(i) $V_{L(0,j_1)}[\varepsilon_1] \times V_{L(0,j_2)}[\varepsilon_2] = V_{L(0,j_1+j_2)}[\varepsilon_1 + \varepsilon_2]$;
(ii) $V_{L(0,j_1)}[\varepsilon] \times V_{L(c,j_2)} = V_{L(c,j_1+j_2)}$;
(iii) $V_{L(\omega,j_1)} \times V_{L(\omega,j_2)} = \sum_{\rho=0}^{2} V_{L(0,j_1+j_2)}[\rho] + 2V_{L(\omega,j_1+j_2)}$;
(iv) $V_{L(0,j)}[\varepsilon_1] \times V_L^{T,k}(\tau^i)[\varepsilon_2] = V_L^{T,k-ij}(\tau^i)[i\varepsilon_1 + \varepsilon_2]$;
(v) $V_{L(\omega,j)} \times V_L^{T,k}(\tau^i)[\varepsilon] = \sum_{\rho=0}^{2} V_L^{T,k-ij}(\tau^i)[\rho]$.

**Proposition 5.2.** [5] *We have the following fusion rules among irreducible $V_L^{\tau}$-modules of twisted type.*

(i) $V_L^{T,i}(\tau^l)[\varepsilon] \times V_L^{T,j}(\tau^l)[\varepsilon'] = V_L^{T,-(i+j)}(\tau^{2l})[-(\varepsilon+\varepsilon')] + V_L^{T,-(i+j)}(\tau^{2l})[2-(\varepsilon+\varepsilon')]$;
(ii) $V_L^{T,i}(\tau)[\varepsilon] \times V_L^{T,j}(\tau^2)[\varepsilon'] = V_{L(0,i+2j)}[\varepsilon + 2\varepsilon'] + V_{L(\omega,i+2j)}$,

*where $l \in \{1, 2\}$, $i, j, \varepsilon, \varepsilon' \in \{0, 1, 2\}$.*

Let $M^1, M^2$ and $M^3$ be $V$-modules. Denote $I_V\binom{M^3}{M^1,\ M^2}$ the space of intertwining operators of $V$-modules of type $\binom{M^3}{M^1,\ M^2}$, and $N_V\binom{M^3}{M^1,\ M^2} := \dim I_V\binom{M^3}{M^1,\ M^2}$.

Let $\mathcal{M}$ be the set of all irreducible $V$-modules up to isomorphism. We write $\sum_{M\in\mathcal{M}} S_M M \geq \sum_{\mathcal{M}} T_M M$ when $S_M \geq T_M$ for all $M \in \mathcal{M}$.

**Proposition 5.3** *([3, Proposition 2.9]). Let $V$ be a vertex operator algebra and let $M^1$, $M^2$, $M^3$ be $V$-modules among which $M^1$ and $M^2$ are irreducible. Suppose that $U$ is a vertex operator subalgebra of $V$ (with the same Virasoro element) and that $N^1$ and $N^2$ are irreducible $U$-submodules of $M^1$ and $M^2$, respectively. Then the restriction map from $I_V\binom{M^3}{M^1,\ M^2}$ to $I_U\binom{M^3}{N^1,\ N^2}$ is injective. In particular,*

$$\dim I_V\binom{M^3}{M^1,\ M^2} \leq \dim I_U\binom{M^3}{N^1,\ N^2}. \tag{5.1}$$

In our case, we consider the following chain of subVOAs:

$$V_{L_{\mathcal{C}\times\mathcal{D}}} \supset V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}} \supset V^{\tau}_{L_{\mathcal{C}\times 0}} \supset (V_L^{\tau})^{\otimes \ell}.$$

For simplicity, we denote

$$N_{\mathcal{C}\times\mathcal{D}}\binom{-}{-,\ -} = \dim I_{V_{L_{\mathcal{C}\times\mathcal{D}}}}\binom{-}{-,\ -}, \qquad N^{\tau}_{\mathcal{C}\times\mathcal{D}}\binom{-}{-,\ -} = \dim I_{V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}}\binom{-}{-,\ -},$$

$$N_\otimes\begin{pmatrix} - \\ -,\ - \end{pmatrix} = \dim I_{(V_L^\tau)^{\otimes \ell}}\begin{pmatrix} - \\ -,\ - \end{pmatrix}, \qquad N_\circ^\tau\begin{pmatrix} - \\ -,\ - \end{pmatrix} = \dim I_{V_L^\tau}\begin{pmatrix} - \\ -,\ - \end{pmatrix}.$$

The basic idea is to use Proposition 5.3 and the quantum dimensions of $V_L^\tau$-modules to show that many fusion coefficients are zero. This gives some inequalities on fusion rules. Next by using quantum dimensions, we show that these inequalities are actually equalities.

Let $\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}}, \boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}}, \boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^\perp/\boldsymbol{\mathcal{C}}$, $\boldsymbol{\delta}+\boldsymbol{\mathcal{D}}, \boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}}, \boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}$, $\boldsymbol{\eta}, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \boldsymbol{\mathcal{D}}^\perp \bmod \boldsymbol{\mathcal{D}}$ and $\varepsilon, \varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3$. We will compute fusion rules separately in the following cases:

- (I) Fusion rules of the form $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon] \times M$ for any irreducible module $M$ (see Proposition 5.4);
- (II) Fusion rules of the form $V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}$ (see Proposition 5.6);
- (III) Fusion rules of the form $V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})}} \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon]$ (see Proposition 5.7);
- (IV) Fusion rules of the form $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^1}(\tau)[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^2}(\tau^2)[\varepsilon^2]$ (see Proposition 5.8);
- (V) Fusion rules of the form $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^1}(\tau^i)[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\eta}^2}(\tau^i)[\varepsilon^2]$ (See Proposition 5.13).

We start with Case (I).

**Proposition 5.4.** *We have the following fusion rules.*

$$\text{(i)} \qquad V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon^2] = V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon^1+\varepsilon^2]; \tag{5.2a}$$

$$\text{(ii)} \qquad V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} = V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}; \tag{5.2b}$$

$$\text{(iii)} \qquad V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\varepsilon^2] = V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^{T,\boldsymbol{\delta}^2-i\boldsymbol{\delta}^1}(\tau^i)[i\varepsilon^1+\varepsilon^2], \tag{5.2c}$$

*where* $\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}}, \boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}$, $\mathbf{0} \neq \boldsymbol{\lambda}+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^\perp_{\equiv\tau} \bmod \boldsymbol{\mathcal{C}}$, $i=1,2$ *and* $\varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3$.

**Proof.** (i) By Proposition 2.3 and Theorem 4.14, $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^i+\boldsymbol{\mathcal{D}})}}[\varepsilon^i]$ are simple currents for $i=1,2$; therefore the fusion product $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon^2]$ is irreducible.

Recall the fusion rules of $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules:

$$1 = N_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}},\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \end{pmatrix}.$$

By restricting to $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$-modules and using Proposition 5.3, we have

$$\begin{aligned} 1 &\le N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1],\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon^2] \end{pmatrix} \\ &= \sum_{\varepsilon=0}^{2} N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon] \\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1],\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon^2] \end{pmatrix}. \end{aligned}$$

Therefore, we know $V_{L_{\mathcal{C}\times(\delta^1+\mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C}\times(\delta^2+\mathcal{D})}}[\varepsilon^2] = V_{L_{\mathcal{C}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon]$, for some $\varepsilon \in \mathbb{Z}_3$. For simplicity, we let $M^i := V_{L_{\mathcal{C}\times(\delta^i+\mathcal{D})}}[\varepsilon^i]$ and $M := V_{L_{\mathcal{C}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon]$.

Recall the decompositions (cf. Proposition 3.12)

$$V_{L_{\mathcal{C}\times(\delta+\mathcal{D})}}[\varepsilon] = V_{L_{\mathbf{0}\times(\delta+\mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0}\neq\gamma\in\mathcal{C}_{\equiv\tau}} V_{L_{\gamma\times(\delta+\mathcal{D})}};$$

$$V_{L_{\mathbf{0}\times\delta}}[\varepsilon] = \bigoplus_{\boldsymbol{e}\in S_\varepsilon} V_{L^{(0,\delta_1)}}[e_1] \otimes \cdots \otimes V_{L^{(0,\delta_\ell)}}[e_\ell].$$

For $i = 1, 2$, we fix an irreducible $(V_L^\tau)^{\otimes\ell}$-submodule

$$N^i := V_{L^{(0,\delta_1^i)}}[e_1^i] \otimes \cdots \otimes V_{L^{(0,\delta_\ell^i)}}[e_\ell^i] \subset V_{L_{\mathbf{0}\times(\delta^i+\mathcal{D})}}[\varepsilon^i] \subset M^i$$

for some $\boldsymbol{e}^i := (e_1^i, \cdots, e_\ell^i) \in S_{\varepsilon_i}$. Since

$$M := V_{L_{\mathcal{C}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon] \cong V_{L_{\mathbf{0}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon] \oplus \bigoplus_{\mathbf{0}\neq\gamma\in\mathcal{C}_{\equiv\tau}} V_{L_{\gamma\times(\delta^1+\delta^2+\mathcal{D})}}, \tag{5.3}$$

we have the fusion coefficients

$$1 = N^\tau_{\mathcal{C}\times\mathcal{D}}\binom{M}{M^1,\ M^2} \leq N_\otimes\binom{V_{L_{\mathbf{0}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon]}{N^1,\ N^2} + \sum_{\mathbf{0}\neq\gamma\in\mathcal{C}_{\equiv\tau}} N_\otimes\binom{V_{L_{\gamma\times(\delta^1+\delta^2+\mathcal{D})}}}{N^1,\ N^2}. \tag{5.4}$$

Next we will show that

$$N_\otimes\binom{V_{L_{\gamma\times(\delta^1+\delta^2+\mathcal{D})}}}{N^1,\ N^2} = 0 \quad \text{for all } \mathbf{0}\neq\gamma\in\mathcal{C}_{\equiv\tau}.$$

Note that

$$N_\otimes\binom{V_{L_{\gamma\times(\delta^1+\delta^2+\mathcal{D})}}}{N^1,\ N^2} = \sum_{\Delta\in\mathcal{D}}\prod_{k=1}^{\ell} N^\tau_\circ\binom{V_{L^{(\gamma_k,\delta_k^1+\delta_k^2+\Delta_k)}}}{V_{L^{(0,\delta_k^1)}}[e_k^1],\ V_{L^{(0,\delta_k^2)}}[e_k^2]}. \tag{5.5}$$

Since $\gamma \neq \mathbf{0}$, we have $\gamma_h \neq 0$ for some $1 \leq h \leq \ell$ and hence

$$N^\tau_\circ\binom{V_{L^{(\gamma_h,\delta_h^1+\delta_h^2+\Delta_h)}}}{V_{L^{(0,\delta_h^1)}}[e_h^1],\ V_{L^{(0,\delta_h^2)}}[e_h^2]} = 0.$$

This proves our claim and Equation (5.4) becomes

$$1 \leq N_\otimes\binom{V_{L_{\mathbf{0}\times(\delta^1+\delta^2+\mathcal{D})}}[\varepsilon]}{N^1,\ N^2}. \tag{5.6}$$

Set $(e_1^i, \cdots, e_\ell^i) = (\varepsilon^i, 0, \cdots, 0)$ for $i = 1, 2$. Then we have

$$\begin{aligned}
1 \le N_{\otimes}\binom{V_{L_{\mathbf{0}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{N^1,\ N^2} &= \sum_{\Delta\in\boldsymbol{\mathcal{D}}} N_{\otimes}\binom{V_{L_{\mathbf{0}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\Delta)}}[\varepsilon]}{N^1,\ N^2} \\
&= \sum_{\Delta\in\boldsymbol{\mathcal{D}}} N_{\otimes}\binom{\oplus_{\boldsymbol{r}\in S_\varepsilon} V_{L^{(0,\delta_1^1+\delta_1^2+\Delta_1)}}[r_1]\otimes\cdots\otimes V_{L^{(0,\delta_\ell^1+\delta_\ell^2+\Delta_\ell)}}[r_\ell]}{V_{L^{(0,\delta_1^1)}}[\varepsilon^1]\otimes V_{L^{(0,\delta_2^1)}}[0]\otimes\cdots\otimes V_{L^{(0,\delta_\ell^1)}}[0],\ N^2} \\
&= \sum_{\substack{\boldsymbol{r}\in S_\varepsilon\\ \Delta\in\boldsymbol{\mathcal{D}}}} \left(N_\circ^\tau\binom{V_{L^{(0,\delta_1^1+\delta_1^2+\Delta_1)}}[r_1]}{V_{L^{(0,\delta_1^1)}}[\varepsilon^1],\ V_{L^{(0,\delta_1^2)}}[\varepsilon^2]}\prod_{k=2}^{\ell} N_\circ^\tau\binom{V_{L^{(0,\delta_k^1+\delta_k^2+\Delta_k)}}[r_k]}{V_{L^{(0,\delta_k^1)}}[0],\ V_{L^{(0,\delta_k^2)}}[0]}\right).
\end{aligned}$$

By Proposition 5.1, if $(r_2,\cdots,r_\ell)\neq(0,\cdots,0)$,

$$N_\circ^\tau\binom{V_{L^{(0,\delta_1^1+\delta_1^2+\Delta_1)}}[r_1]}{V_{L^{(0,\delta_1^1)}}[\varepsilon^1],\ V_{L^{(0,\delta_1^2)}}[\varepsilon^2]}\prod_{k=2}^{\ell} N_\circ^\tau\binom{V_{L^{(0,\delta_k^1+\delta_k^2+\Delta_k)}}[r_k]}{V_{L^{(0,\delta_k^1)}}[0],\ V_{L^{(0,\delta_k^2)}}[0]} = 0$$

Thus only $\boldsymbol{r}=(r_1,0,\cdots,0)\in S_\varepsilon$ contributes a nonzero summand. Therefore

$$\begin{aligned}
1 \le N_{\otimes}&\binom{V_{L_{\mathbf{0}\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{N^1,\ N^2} \\
&= \sum_{\Delta\in\boldsymbol{\mathcal{D}}} \left(N_\circ^\tau\binom{V_{L^{(0,\delta_1^1+\delta_1^2+\Delta_1)}}[r_1]}{V_{L^{(0,\delta_1^1)}}[\varepsilon^1],\ V_{L^{(0,\delta_1^2)}}[\varepsilon^2]}\prod_{k=2}^{\ell} N_\circ^\tau\binom{V_{L^{(0,\delta_k^1+\delta_k^2+\Delta_k)}}[0]}{V_{L^{(0,\delta_k^1)}}[0],\ V_{L^{(0,\delta_k^2)}}[0]}\right).
\end{aligned}$$

Since $\boldsymbol{r}\in S_\varepsilon$, we must have $r_1=\varepsilon=\varepsilon^1+\varepsilon^2$. This proves (i).

(ii) We know the fusion coefficient of $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules:

$$1 = N_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}}{V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}},\ V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}}.$$

By restricting to $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}^\tau$-modules, we have

$$1 \le N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}}{V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1],\ V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}}.$$

Since $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]$ is a simple current module, we have

$$V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} = V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]\times V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}.$$

This proves (ii).

(iii) Since $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]$ is a simple current module, $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]\times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2]$ is also an irreducible $V^\tau_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-module and

$$\operatorname{qdim}\left(V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]\times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2]\right) = \operatorname{qdim}\left(V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2]\right) = \frac{2^\ell}{|\boldsymbol{\mathcal{C}}|}$$

Therefore, the fusion product $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2]$ is either $V^{T,\boldsymbol{\delta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^j)[\varepsilon]$ for some $\boldsymbol{\delta}+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^\perp/\boldsymbol{\mathcal{C}}, \varepsilon \in \mathbb{Z}_3$ and $j = 1,2$ or $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]$ if $|\boldsymbol{\mathcal{C}}| = 2^\ell$.

Assume that $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2] = V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]$. Then we have

$$\begin{aligned}
&V_{L_{\boldsymbol{\mathcal{C}}\times(-\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[-\varepsilon^1] \times \left(V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2]\right) \\
=&V_{L_{\boldsymbol{\mathcal{C}}\times(-\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[-\varepsilon^1] \times V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon],
\end{aligned}$$

and hence by (5.2a)

$$V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2] = V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}[0] \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2] = V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}-\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon-\varepsilon^1],$$

which is a contradiction. Therefore, $V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2] = V^{T,\boldsymbol{\delta}^3}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^j)[\varepsilon^3]$, for some $j = 1,2$, $\boldsymbol{\delta}^h + \boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}$, $\varepsilon^h \in \mathbb{Z}_3$, for $h = 1,2,3$.

Similar to (i), we pick the following irreducible $(V_L^\tau)^{\otimes\ell}$-modules

$$\begin{aligned}
&V_{L^{(0,\delta^1_1)}}[e^1_1] \otimes \cdots \otimes V_{L^{(0,\delta^1_\ell)}}[e^1_\ell] \subset V_{L_{\mathbf{0}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1]; \\
&V_L^{T,\delta^2_1}(\tau^i)[e^2_1] \otimes \cdots \otimes V_L^{T,\delta^2_\ell}(\tau^i)[e^2_\ell] \subset V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2];
\end{aligned}$$

of $M^i$ for some $\boldsymbol{e}^h := (e^h_1, \cdots, e^h_\ell) \in S_{\varepsilon_h}$, $h = 1,2$.

By Proposition 5.3,

$$\begin{aligned}
1 &= N^\tau_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V^{T,\boldsymbol{\delta}^3}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^j)[\varepsilon^3] \\ V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}}[\varepsilon^1],\ V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon^2] \end{pmatrix} \\
&\le N^\circ\begin{pmatrix} \bigoplus_{\boldsymbol{e}^3\in S_{\varepsilon_3}} V_L^{T,\delta^3_1}(\tau^j)[e^3_1] \otimes \cdots \otimes V_L^{T,\delta^3_\ell}(\tau^j)[e^3_\ell] \\ V_{L^{(0,\delta^1_1)}}[e^1_1] \otimes \cdots \otimes V_{L^{(0,\delta^1_\ell)}}[e^1_\ell],\ V_L^{T,\delta^2_1}(\tau^i)[e^2_1] \otimes \cdots \otimes V_L^{T,\delta^2_\ell}(\tau^i)[e^2_\ell] \end{pmatrix} \\
&= \sum_{\boldsymbol{e}^3\in S_{\varepsilon_3}} \prod_{k=1}^{\ell} N^\tau_\circ \begin{pmatrix} V_L^{T,\delta^3_k}(\tau^j)[e^3_k] \\ V_{L^{(0,\delta^1_1)}}[e^1_k],\ V_L^{T,\delta^2_k}(\tau^i)[e^2_k] \end{pmatrix}.
\end{aligned}$$

If $j \neq i$, then Proposition 5.1 gives $1 \le 0$, a contradiction. Therefore $j = i$. If there exists $1 \le k \le \ell$ such that $\delta^3_k \neq \delta^2_k - i\delta^1_k$ or $e^3_k \neq ie^1_k + e^2_k$, again Proposition 5.1 gives $1 \le 0$, a contradiction. Therefore, we must have $\delta^3_k = \delta^2_k - i\delta^1_k$ and $e^3_k = ie^1_k + e^2_k$ for all $k$. This gives $\boldsymbol{\delta}^3 = \boldsymbol{\delta}^2 - i\boldsymbol{\delta}^1$ and $\varepsilon_3 \equiv \sum_{k=1}^{\ell} e^3_k = \sum_{k=1}^{\ell} ie^1_k + e^2_k \equiv i\varepsilon^1 + \varepsilon^2 \bmod 3$. This completes the proof. $\square$

Using the above proposition, we can find the contragredient dual of irreducible modules. Recall there are natural isomorphisms between the following fusion rules: For $V$-modules $A, B$ and $C$, we have

$$N_V\binom{C}{A,\ B} = N_V\binom{C}{B,\ A} = N_V\binom{B'}{A,\ C'}.$$

**Proposition 5.5.** *The contragredient duals of irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules are listed below.*

(i) $\left(V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\right)' = V_{L_{\mathcal{C}\times(-\boldsymbol{\delta}+\mathcal{D})}}[-\varepsilon]$;
(ii) $\left(V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(\boldsymbol{\delta}+\mathcal{D})}}\right)' = V_{L_{(-\boldsymbol{\lambda}+\mathcal{C})\times(-\boldsymbol{\delta}+\mathcal{D})}} = V_{L_{(\boldsymbol{\lambda}+\mathcal{C})\times(-\boldsymbol{\delta}+\mathcal{D})}}$;
(iii) $\left(V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon]\right)' = V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[\varepsilon]$.

**Proof.** It is discussed in Proposition 3.8 that $\left(V_{L_{\mathcal{C}\times\mathcal{D}}}[0]\right)' \cong V_{L_{\mathcal{C}\times\mathcal{D}}}[0]$ and $V_{L_{\mathcal{C}\times\mathcal{D}}}[0]$ is self-dual. We also have the fusion rule:

$$1 = N^{\tau}_{\mathcal{C}\times\mathcal{D}}\binom{V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]}{V_{L_{\mathcal{C}\times\mathcal{D}}}[0],\ V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]} = N^{\tau}_{\mathcal{C}\times\mathcal{D}}\binom{\left(V_{L_{\mathcal{C}\times\mathcal{D}}}[0]\right)'}{\left(V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\right)',\ V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]}.$$

Since $\operatorname{qdim} M = \operatorname{qdim} M'$ for any module $M$, we may assume $\left(V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\right)' \cong V_{L_{\mathcal{C}\times(\boldsymbol{\delta}'+\mathcal{D})}}[\varepsilon']$ for some $\boldsymbol{\delta}', \varepsilon'$ by Proposition 5.4. Using Equation (5.2a), we must have

$$\left(V_{L_{\mathcal{C}\times(\boldsymbol{\delta}+\mathcal{D})}}[\varepsilon]\right)' = V_{L_{\mathcal{C}\times(-\boldsymbol{\delta}+\mathcal{D})}}[-\varepsilon].$$

This proves (i). Similarly, using Equation (5.2b) we have (ii).

(iii) We take a different approach. We first consider the contragredient dual of an irreducible $V_L^{\tau}$-modules of twisted type. Note that $V_L^{\tau}$ is self-dual. Let $i, \varepsilon \in \mathbb{Z}_3$, then

$$1 = N^{\tau}_{\circ}\binom{V_L^{T,i}(\tau^j)[\varepsilon]}{V_L^{\tau},\ V_L^{T,i}(\tau^j)[\varepsilon]} = N^{\tau}_{\circ}\binom{V_L^{\tau}}{V_L^{T,i}(\tau^j)[\varepsilon],\ \left(V_L^{T,i}(\tau^j)[\varepsilon]\right)'}.$$

By fusion rules of $V_L^{\tau}$-modules, we must have $\left(V_L^{T,i}(\tau^j)[\varepsilon]\right)' = V_L^{T,i}(\tau^{2j})[\varepsilon]$.

Now consider the decomposition of $(V_L^{\tau})^{\otimes\ell}$-modules:

$$V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon] \cong \bigoplus_{\boldsymbol{\delta}\in\mathcal{D}} \bigoplus_{e_1+\cdots+e_\ell\equiv\varepsilon \bmod 3} V_L^{T,\eta_1-i\delta_1}(\tau^i)[e_1]\otimes\cdots\otimes V_L^{T,\eta_\ell-i\delta_\ell}(\tau^i)[e_\ell].$$

Taking its contragredient dual as $(V_L^{\tau})^{\otimes\ell}$-modules, we have

$$\left(V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon]\right)' \cong \bigoplus_{\boldsymbol{\delta}\in\mathcal{D}} \bigoplus_{e_1+\cdots+e_\ell\equiv\varepsilon \bmod 3} V_L^{T,\eta_1-i\delta_1}(\tau^{2i})[e_1]\otimes\cdots\otimes V_L^{T,\eta_\ell-i\delta_\ell}(\tau^{2i})[e_\ell].$$

Since $V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[\varepsilon]$ is the only irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-module admitting the above decomposition of $(V_L^{\tau})^{\otimes\ell}$-modules, we must have $\left(V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon]\right)' \cong V^{T,\boldsymbol{\eta}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[\varepsilon]$. $\square$

**Case (II):** $V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}$

**Proposition 5.6.** *We have the fusion rules*

$$V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} = \bigoplus_{h=0}^{2} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}},$$

*where* $0 \neq \boldsymbol{\lambda}^i + \boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^\perp/\boldsymbol{\mathcal{C}}$ *and* $\boldsymbol{\delta}^i + \boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^\perp/\boldsymbol{\mathcal{D}}$ *for* $i = 1, 2$.

**Proof.** Fix $0 \le h \le 2$, and we have the fusion rules of $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules:

$$\begin{aligned} 1 &= N_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \\ V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}},\ V_{L_{(\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \end{pmatrix} \\ &\le N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \\ V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}},\ V_{L_{(\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \end{pmatrix}. \end{aligned}$$

Since $\omega^h\boldsymbol{\lambda}^2 + \boldsymbol{\mathcal{C}}$, $0 \le h \le 2$, are identical in $\boldsymbol{\mathcal{C}}^\perp_{\equiv\tau} \bmod \boldsymbol{\mathcal{C}}$, there is an isomorphism of $V^{\tau}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules

$$V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \cong V_{L_{(\omega\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \cong V_{L_{(\omega^2\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}.$$

Therefore,

$$1 \le N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\begin{pmatrix} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \\ V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}},\ V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} \end{pmatrix},$$

for all $0 \le h \le 2$.

Since $\boldsymbol{\lambda}^1 + \omega^h\boldsymbol{\lambda}^2 + \boldsymbol{\mathcal{C}}$, $0 \le h \le 2$, are distinct in $\boldsymbol{\mathcal{C}}^\perp_{\equiv\tau} \bmod \boldsymbol{\mathcal{C}}$, by counting quantum dimensions, we can prove

$$V_{L_{(\boldsymbol{\lambda}^1+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V_{L_{(\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} = \bigoplus_{h=0}^{2} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}.$$

This completes the proof.

Recall that $\operatorname{qdim}_{V^{\tau}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}} = 3$ for any $0 \neq \boldsymbol{\lambda}+\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\delta}+\boldsymbol{\mathcal{D}}$. Note also that if $\boldsymbol{\lambda}^1 + \omega^h\boldsymbol{\lambda}^2 = 0$ for some $h$, then the module $V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}$ is not irreducible and admits a decomposition of irreducible modules of $V^{\tau}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}$-modules:

$$V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}} = \sum_{\varepsilon=0}^{2} V_{L_{(\boldsymbol{\lambda}^1+\omega^h\boldsymbol{\lambda}^2+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}})}}[\varepsilon];$$

nevertheless, $\operatorname{qdim}_{V^{\tau}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}} V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}$ is still 3. $\square$

**Case (III):** $V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon]$

**Proposition 5.7.** *We have*

$$\text{(i)} \qquad V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}} \times V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0] = \bigoplus_{\rho=0}^{2} V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho]; \tag{5.7a}$$

$$\text{(ii)} \qquad V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}})}} \times V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\varepsilon] = \bigoplus_{\rho=0}^{2} V^{T,\boldsymbol{\delta}^2}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho], \tag{5.7b}$$

*where* $\boldsymbol{\delta}^1+\boldsymbol{\mathcal{D}}, \boldsymbol{\delta}^2+\boldsymbol{\mathcal{D}} \in \boldsymbol{\mathcal{D}}^{\perp}/\boldsymbol{\mathcal{D}}$, $\mathbf{0} \neq \boldsymbol{\gamma}+\boldsymbol{\mathcal{C}} \in \boldsymbol{\mathcal{C}}^{\perp}_{\equiv\tau} \bmod \boldsymbol{\mathcal{C}}$.

**Proof.** (i) Similar to Proposition 5.4(iii), it is straightforward to verify that

$$0 = N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V^{T,\boldsymbol{\delta}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^j)}{V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}},\ V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0]},$$

when (1) $i=j$ and $\boldsymbol{\delta}\neq\mathbf{0}$ or (2) $i\neq j$. By Proposition 5.4, Proposition 5.5 and Proposition 5.6, we also have

$$N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V_{L_{\boldsymbol{\mathcal{C}}\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[\varepsilon]}{V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}},\ V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0]} = N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^{2i})[0]}{V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}},\ V_{L_{\boldsymbol{\mathcal{C}}\times(-\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}[-\varepsilon]} = 0,$$
$$N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V_{L_{(\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}}{V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}},\ V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0]} = N^{\tau}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}\binom{V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^{2i})[0]}{V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}},\ V_{L_{(-\boldsymbol{\lambda}+\boldsymbol{\mathcal{C}})\times(-\boldsymbol{\delta}+\boldsymbol{\mathcal{D}})}}} = 0.$$

Therefore

$$V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}} \times V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0] = \bigoplus_{\rho=0}^{2} n_\rho V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho], \tag{5.8}$$

for some $n_\rho \in \mathbb{N}$. Multiply Equation (5.8) by $V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}[h]$, $h=1,2$, we have

$$\left(V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}[h] \times V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}}\right) \times V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0] = V_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}[h] \times \bigoplus_{\rho=0}^{2} n_\rho V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho].$$

By Proposition 5.4, the left hand side is equal to

$$V_{L_{(\boldsymbol{\gamma}+\boldsymbol{\mathcal{C}})\times\boldsymbol{\mathcal{D}}}} \times V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[0] = \bigoplus_{\rho=0}^{2} n_\rho V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho],$$

while the right hand side is $\bigoplus_{\rho=0}^{2} n_\rho V^{T,\mathbf{0}}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}}(\tau^i)[\rho+h]$; thus, we have

$$\bigoplus_{\rho=0}^{2} n_\rho V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[\rho] = \bigoplus_{\rho=0}^{2} n_\rho V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[\rho+h],$$

for all $0 \leq h \leq 2$. This gives $n_0 = n_1 = n_2$. Finally, by comparing the quantum dimensions of both sides of (5.8), we have $3(2^\ell/|\mathcal{C}|) = (n_0+n_1+n_2)(2^\ell/|\mathcal{C}|)$ and hence $n_0 = n_1 = n_2 = 1$. This proves (i).

(ii) By Proposition 5.4, we have

$$V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\varepsilon] = V_{L_{\mathcal{C}\times((-1)^i\boldsymbol{\delta}^2+\mathcal{D})}}[(-1)^{i+1}\varepsilon] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[0].$$

Therefore, by (i)

$$\begin{aligned}
& V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times(\boldsymbol{\delta}^1+\mathcal{D})}} \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\varepsilon] \\
&= V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times(\boldsymbol{\delta}^1+\mathcal{D})}} \times \left(V_{L_{\mathcal{C}\times((-1)^i\boldsymbol{\delta}^2+\mathcal{D})}}[(-1)^{i+1}\varepsilon] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[0]\right) \\
&= V_{L_{\mathcal{C}\times((-1)^i\boldsymbol{\delta}^2+\mathcal{D})}}[(-1)^{i+1}\varepsilon] \times \left(V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times(\boldsymbol{\delta}^1+\mathcal{D})}} \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[0]\right) \\
&= V_{L_{\mathcal{C}\times((-1)^i\boldsymbol{\delta}^2+\mathcal{D})}}[(-1)^{i+1}\varepsilon] \times \bigoplus_{\rho=0}^{2} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[\rho] \\
&= \bigoplus_{\rho=0}^{2} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,-i(-1)^i\boldsymbol{\delta}^2}(\tau^i)[i(-1)^{i+1}\varepsilon+\rho] = \bigoplus_{\rho=0}^{2} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\rho-\varepsilon] = \bigoplus_{\rho=0}^{2} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\delta}^2}(\tau^i)[\rho].
\end{aligned}$$

This completes the proof. $\square$

**Case (IV):** $V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^2}(\tau^2)[\varepsilon^2]$

**Proposition 5.8.** *We have the fusion rules:*

$$\text{(i)} \qquad V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau)[0] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^2)[0] = V_{L_{\mathcal{C}\times\mathcal{D}}}[0] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}}; \tag{5.9a}$$

$$\begin{aligned}
\text{(ii)} \quad V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^2}(\tau^2)[\varepsilon^2] &= V_{L_{\mathcal{C}\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\mathcal{D})}}[\varepsilon^1-\varepsilon^2] \\
&\oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\mathcal{D})}}.
\end{aligned} \tag{5.9b}$$

*In particular, if $\mathcal{C}$ is self-dual, then we have*

$$V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}^2}(\tau^2)[\varepsilon^2] = V_{L_{\mathcal{C}\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\mathcal{D})}}[\varepsilon^1-\varepsilon^2].$$

**Proof.** (i) By Proposition 5.5, we have

$$N^{\tau}_{\mathcal{C}\times\mathcal{D}}\left(\begin{matrix} V_{L_{\mathcal{C}\times\mathcal{D}}}[\varepsilon] \\ V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau)[0],\ V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^2)[0] \end{matrix}\right) = N^{\tau}_{\mathcal{C}\times\mathcal{D}}\left(\begin{matrix} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau)[0] \\ V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau)[0],\ V_{L_{\mathcal{C}\times\mathcal{D}}}[2\varepsilon] \end{matrix}\right).$$

By Proposition 5.4,

$$N^{\tau}_{\mathcal{C}\times\mathcal{D}}\begin{pmatrix} V_{L_{\mathcal{C}\times\mathcal{D}}}[\varepsilon] \\ V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0],\ V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[0] \end{pmatrix} = \begin{cases} 1 & \text{if } \varepsilon = 0; \\ 0 & \text{if } \varepsilon = 1, 2. \end{cases}$$

Similarly, by Proposition 5.5 and Proposition 5.7, we have

$$N^{\tau}_{\mathcal{C}\times\mathcal{D}}\begin{pmatrix} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}} \\ V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0],\ V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[0] \end{pmatrix} = N^{\tau}_{\mathcal{C}\times\mathcal{D}}\begin{pmatrix} V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0] \\ V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0],\ V_{L_{(-\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}} \end{pmatrix} = 1.$$

Therefore,

$$V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[0] \geq V_{L_{\mathcal{C}\times\mathcal{D}}}[0] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}}.$$

Recall that

$$\operatorname{qdim}\left(V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[0]\right) = \left(\frac{2^{\ell}}{|\mathcal{C}|}\right)^2 = \left|\mathcal{C}^{\perp}/\mathcal{C}\right|; \qquad \operatorname{qdim} V_{L_{\mathcal{C}\times\mathcal{D}}}[0] = 1.$$

Moreover,

$$\operatorname{qdim}\left(\bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}}\right) = \#\{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}\} \cdot 3.$$

By Lemme 3.11,

$$\#\{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}\} = \frac{1}{3}\left(\left|\mathcal{C}^{\perp}/\mathcal{C}\right| - 1\right);$$

therefore we have

$$\begin{aligned} &\operatorname{qdim}\left(V_{L_{\mathcal{C}\times\mathcal{D}}}[0] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}}\right) = \left|\mathcal{C}^{\perp}/\mathcal{C}\right| \\ &= \operatorname{qdim}\left(V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[0] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[0]\right). \end{aligned}$$

This proves (i).

(ii) By Proposition 5.4, we have

$$V^{T,\boldsymbol{\eta}^i}_{L_{\mathcal{C}\times(\boldsymbol{\gamma}^i+\mathcal{D})}}(\tau^i)[\varepsilon^i] = V_{L_{\mathcal{C}\times((-1)^i\boldsymbol{\eta}^i+\mathcal{D})}}[(-1)^{i+1}\varepsilon^i] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[0].$$

Therefore,

$$\begin{aligned}
&V^{T,\boldsymbol{\eta}^1}_{L_{\mathcal{C}\times(\boldsymbol{\gamma}^1+\mathcal{D})}}(\tau)[\varepsilon^1] \times V^{T,\boldsymbol{\eta}^2}_{L_{\mathcal{C}\times(\boldsymbol{\gamma}^2+\mathcal{D})}}(\tau^2)[\varepsilon^2] \\
&= V_{L_{\mathcal{C}\times(-\boldsymbol{\eta}^1+\mathcal{D})}}[\varepsilon^1] \times V_{L_{\mathcal{C}\times(\boldsymbol{\eta}^2+\mathcal{D})}}[-\varepsilon^2] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathbf{0}}}(\tau)[0] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathbf{0}}}(\tau^2)[0] \\
&= V_{L_{\mathcal{C}\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\mathcal{D})}}[\varepsilon^1-\varepsilon^2] \times \left( V_{L_{\mathcal{C}\times\mathbf{0}}}[0] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}} \right) \\
&= V_{L_{\mathcal{C}\times(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\mathcal{D})}}[\varepsilon^1-\varepsilon^2] \oplus \bigoplus_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{C}^{\perp}_{\equiv_\tau} \bmod \mathcal{C}} V_{L_{(\boldsymbol{\eta}^2-\boldsymbol{\eta}^1+\boldsymbol{\gamma}+\mathcal{C})\times\mathcal{D}}}.
\end{aligned}$$

This proves (ii). □

**Case (V):** $V^{T,\boldsymbol{\delta}^1}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon^1] \times V^{T,\boldsymbol{\delta}^2}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[\varepsilon^2]$

We first consider the case $\boldsymbol{\delta}^1 = \boldsymbol{\delta}^2 = \mathbf{0}$ and $\varepsilon^1 = \varepsilon^2 = 0$.

By the similar analysis as in the previous few cases, we can show quickly that many fusion coefficients are zero. In fact, we have this proposition.

**Proposition 5.9.** *We have fusion rules:*

$$\begin{aligned}
&V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[0] \times V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^i)[0] \\
&= x_i V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[0] + y_i V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[1] + z_i V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^{2i})[2],
\end{aligned} \tag{5.10}$$

*for some $x_i, y_i, z_i \in \mathbb{N}$. Moreover, we have*

$$x_i + y_i + z_i = 2^{\ell-2d} \tag{5.11}$$

*where $i = 1, 2$ and $d = \dim \mathcal{C}$. Note that $2^{\ell-2d} = \sqrt{\left|\mathcal{C}^{\perp}/\mathcal{C}\right|}$.*

**Proof.** By Proposition 4.14, all irreducible $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$-modules of twisted type have quantum dimensions $2^{\ell-2d}$. By computing the quantum dimensions of terms in (5.10), we know $x_i + y_i + z_i = 2^{\ell-2d}$, for $i = 1, 2$. □

***The case that $\mathcal{D}$ is self-dual*** Let $\xi = e^{(2\pi\sqrt{-1})/3}$ be a primitive cubic root of unity.

**Notation 5.10.** We define a function $\Xi : \mathbb{Z} \to \{-1, 2\}$ by $\Xi(n) := \xi^n + \xi^{2n}$, for $n \in \mathbb{Z}$. Note that $\Xi(n) = 2$ if $n \equiv 0 \mod 3$ and $\Xi(n) = -1$ otherwise.

For simplicity, we denote

$$T[j] := V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau)[j]; \qquad \check{T}[j] := V^{T,\mathbf{0}}_{L_{\mathcal{C}\times\mathcal{D}}}(\tau^2)[j]; \qquad S[j] := V_{L_{\mathcal{C}\times\mathcal{D}}}[j].$$

**Proposition 5.11.** *Let $\mathcal{D}$ be a self-dual $\mathbb{Z}_3$-code of length $\ell$, then we have*

$$
\begin{aligned}
T[0]\times T[0] &= \frac{2^{\ell-2d}+\Xi(\ell)}{3}\check{T}[0]+\frac{2^{\ell-2d}+\Xi(\ell+2)}{3}\check{T}[1]+\frac{2^{\ell-2d}+\Xi(\ell+1)}{3}\check{T}[2]\\
&= \sum_{\varepsilon=0,1,2}\frac{2^{\ell-2d}+\Xi(\ell-\varepsilon)}{3}\check{T}[\varepsilon].
\end{aligned}
$$

**Proof.** We mimic the proof of [30, Lemma 18].

Let $V$ denote the lattice VOA $V_{L_{\mathcal{C}\times\mathcal{D}}}$. Since $\mathcal{D}$ is self-dual, $V$ is the only $\tau$-stable irreducible module. Moreover, $V$ has exactly one $\tau^i$-twisted module for each $i=1,2$. We denote them by $T$ and $\check{T}$, respectively. Let

$$
W^i := V[i], \qquad W^{3+i} := T[i], \qquad W^{6+i} := \check{T}[i],
$$

for $i=0,1,2$. Then $W^j, (j=0,\cdots,8)$, are irreducible $V^\tau$-modules. Note that there are also irreducible $V^\tau$-modules which are not $\tau$-stable, but we won't need them in the proof.

Let $C_1(g,h)$ be the vector space generated by trace functions of $g$-twisted and $h$-stable $V$-modules. By Proposition 2.6, we know that the modular transformation $\Gamma: z\mapsto \frac{-1}{z}$ maps $C_1(g,h)$ to $C_1(h,g^{-1})$. In particular, $\Gamma$ sends $C_1(\tau,\tau^j)$ to $C_1(\tau^j,\tau^2)$ for $j=0,1,2$.

Recall that the trace function $Z_T(\tau,1;\frac{-1}{z})\in C_1(1,\tau^2)$ which is spanned by $Z_V(1,1;z)$. Therefore, we can write

$$
Z_T(\tau,1;\frac{-1}{z}) = \lambda_1 Z_V(1,1;z), \qquad \text{for some } \lambda_1\in\mathbb{C}.
$$

Denote $W^i(g,h,z)=Z_{W^i}(g,h;z)$ for any $i$. Then we have

$$
W^3(\frac{-1}{z})+W^4(\frac{-1}{z})+W^5(\frac{-1}{z}) = \lambda_1\big(W^0(z)+W^1(z)+W^2(z)\big). \tag{5.12}
$$

Similarly, using $Z_T(\tau,\tau^j;\frac{-1}{z})\in C_1(\tau^j,\tau^2)$ for $j=1,2$, we can write

$$
\begin{aligned}
&W^3(1,\tau,\frac{-1}{z})+W^4(1,\tau,\frac{-1}{z})+W^5(1,\tau,\frac{-1}{z})\\
&=\mu_1\Big(W^3(1,\tau^2;z)+W^4(1,\tau^2;z)+W^5(1,\tau^2;z)\Big);\\
&W^3(1,\tau^2,\frac{-1}{z})+W^4(1,\tau^2,\frac{-1}{z})+W^5(1,\tau^2,\frac{-1}{z})\\
&=\mu_2\Big(W^6(1,\tau^2;z)+W^7(1,\tau^2;z)+W^8(1,\tau^2;z)\Big),
\end{aligned} \tag{5.13}
$$

for some $\mu_1,\mu_2\in\mathbb{C}$.

We can define a linear isomorphism $\varphi(\tau^j)$ as following: $\varphi(\tau^j) = \xi^{ij}$ on $W^{3+i}$ and $W^{6+i}$. Therefore we can rewrite the equation (5.13) as

$$\begin{aligned} & W^3(\tau, 1; \frac{-1}{z}) + \xi W^4(\tau, 1; \frac{-1}{z}) + \xi^2 W^5(\tau, 1; \frac{-1}{z}) \\ &= \mu_1 \Big( W^3(\tau, 1; z) + \xi^2 W^4(\tau, 1; z) + \xi W^5(\tau, 1; z) \Big); \\ & W^3(\tau, 1; \frac{-1}{z}) + \xi^2 W^4(\tau, 1; \frac{-1}{z}) + \xi W^5(\tau, 1; \frac{-1}{z}) \\ &= \mu_2 \Big( W^6(\tau^2, 1; z) + \xi^2 W^7(\tau^2, 1; z) + \xi W^8(\tau^2, 1; z) \Big). \end{aligned} \tag{5.14}$$

Solving equations (5.12) and (5.14), we know

$$\begin{aligned} W^3(\frac{-1}{z}) &= \frac{\lambda_1}{3}\big(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)\big) + \frac{\mu_1}{3}\big(W^3(z) + \xi^2 W^4(z) + \xi W^5(z)\big) \\ &\quad + \frac{\mu_2}{3}\big(W^6(z) + \xi W^7(z) + \xi^2 W^8(z)\big), \\ W^4(\frac{-1}{z}) &= \frac{\lambda_1}{3}\big(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)\big) + \frac{\mu_1}{3}\big(\xi^2 W^3(z) + \xi W^4(z) + W^5(z)\big) \\ &\quad + \frac{\mu_2}{3}\big(\xi W^6(z) + \xi^2 W^7(z) + W^8(z)\big), \\ W^5(\frac{-1}{z}) &= \frac{\lambda_1}{3}\big(W^0(z) + \xi^2 W^1(z) + \xi W^2(z)\big) + \frac{\mu_1}{3}\big(\xi W^3(z) + W^4(z) + \xi^2 W^5(z)\big) \\ &\quad + \frac{\mu_2}{3}\big(\xi^2 W^6(z) + W^7(z) + \xi W^8(z)\big). \end{aligned}$$

In other words, the rows $S_{i,j}$ for $i = 3, 4, 5$ are given by

$$\frac{1}{3}\begin{pmatrix} \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \mu_1 & \xi^2\mu_1 & \xi\mu_1 & \mu_2 & \xi\mu_2 & \xi^2\mu_2 & 0 & \cdots & 0 \\ \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \xi^2\mu_1 & \xi\mu_1 & \mu_1 & \xi\mu_2 & \xi^2\mu_2 & \mu_2 & 0 & \cdots & 0 \\ \lambda_1 & \xi^2\lambda_1 & \xi\lambda_1 & \xi\mu_1 & \mu_1 & \xi^2\mu_1 & \xi^2\mu_2 & \mu_2 & \xi\mu_2 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $S_{0,0}^2 \mathrm{glob}(V^\tau) = 1$, we know $S_{0,0}^2 \cdot 9 \left|\mathcal{C}^\perp/\mathcal{C}\right| \left|\mathcal{D}^\perp/\mathcal{D}\right| = 1$, $S_{0,i}/S_{0,0} = \operatorname{qdim} W^i$, and

$$\operatorname{qdim} M^i = \begin{cases} 1, & \text{if } i = 0, 1, 2 \\ \frac{2^\ell}{|\mathcal{C}|}, & \text{if } 3 \le i \le 8. \end{cases}$$

This gives $S_{0,0} = S_{0,1} = S_{0,2} = \frac{\pm 2^{2d-\ell}}{3}$ and $\lambda_1 = 3S_{0,h} = \pm 1$ for $3 \le h \le 8$.

By Verlinde formula [19], the fusion rules are given by

$$N_{3,3}^{6} = \frac{3\cdot(\lambda_1/3)^3}{S_{0,0}} + \frac{3\big((\mu_1/3)^3+(\mu_2/3)^3\big)}{(\lambda_1/3)} = \frac{\pm\big(2^{\ell-2d}+\mu_1^3+\mu_2^3\big)}{3},$$
$$N_{3,3}^{7} = \frac{\pm\big(2^{\ell-2d}+\xi^2\mu_1^3+\xi\mu_2^3\big)}{3},$$
$$N_{3,3}^{8} = \frac{\pm\big(2^{\ell-2d}+\xi\mu_1^3+\xi^2\mu_2^3\big)}{3}.$$

Using (5.11), we get $N_{3,3}^6+N_{3,3}^7+N_{3,3}^8 = 2^{\ell-2d}$. Therefore, we know $S_{0,0}=2^{2d-\ell}$ and

$$N_{3,3}^{6} = \frac{2^{\ell-2d}+\mu_1^3+\mu_2^3}{3},\quad N_{3,3}^{7} = \frac{2^{\ell-2d}+\xi^2\mu_1^3+\xi\mu_2^3}{3},\quad N_{3,3}^{8} = \frac{2^{\ell-2d}+\xi\mu_1^3+\xi^2\mu_2^3}{3}.$$

Since $S_{3,6}=1$, we have $9=3\lambda_1^2+6\mu_1\mu_2$, and hence $\mu_1\mu_2=1$.

By Proposition 2.1 and the decomposition given in Proposition 3.12, we have

$$\mathrm{wt}\, V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^i)[\varepsilon] \in \ell/9 - 1/3\Big(\sum_{\boldsymbol{e}\in S_\varepsilon} e_i\Big)+\mathbb{Z} = -\varepsilon/3+\ell/9+\mathbb{Z}, \qquad \text{for } i=1,2.$$

By considering the characters and the above $S$-matrix, we have

$$\begin{aligned}
Z_V(1,\tau;z) &= \mathrm{ch}(W^0)+\xi\,\mathrm{ch}(W^1)+\xi^2\,\mathrm{ch}(W^2),\\
Z_V(1,\tau;-1/z) &= \lambda_1\{\mathrm{ch}(W^3)+\mathrm{ch}(W^4)+\mathrm{ch}(W^5)\},\\
Z_V(1,\tau;-1/(z+1)) &= e^{2\pi\sqrt{-1}N/24}\cdot e^{2\pi\sqrt{-1}\ell/9}\lambda_1\{\mathrm{ch}(W^3)+\xi^2\,\mathrm{ch}(W^4)\\
&\quad +\xi\,\mathrm{ch}(W^5)\},\\
Z_V(1,\tau;-1/((-1/z)+1)) &= e^{2\pi\sqrt{-1}N/24}\cdot e^{2\pi\sqrt{-1}\ell/9}\lambda_1\mu_1\{\mathrm{ch}(W^3)+\xi\,\mathrm{ch}(W^4)\\
&\quad +\xi^2\,\mathrm{ch}(W^5)\},
\end{aligned}$$

where $N=2\ell$ is the rank of the lattice $L_{\mathcal{C}\times\mathcal{D}}$. On the other hand, since

$$\begin{aligned}
&Z_V(1,\tau;-1/((-1/z)+1))\\
&= Z_V\Big(1,\tau;-1-\frac{1}{z-1}\Big)\\
&= e^{-2\pi\sqrt{-1}N/24}Z_V(1,\tau,-1/(z-1))\\
&= e^{-4\pi\sqrt{-1}N/24}\cdot e^{-2\pi\sqrt{-1}\ell/9}\lambda_1\{\mathrm{ch}(W^3)+\xi\,\mathrm{ch}(W^4)+\xi^2\,\mathrm{ch}(W^5)\},
\end{aligned}$$

we have $\mu_1\cdot e^{6\pi\sqrt{-1}N/24}\cdot e^{4\pi\sqrt{-1}\ell/9}=1$. Since $N=2\ell$ and $\ell$ is a multiple of 4, we know $8|N$ and $\mu_1=e^{-4\pi\sqrt{-1}\ell/9}$. Using $\mu_1\mu_2=1$, we have $\mu_1^3=\xi^\ell$ and $\mu_2^3=\xi^{2\ell}$. This gives

$$\begin{aligned}
&T[0] \times T[0] \\
&= \frac{2^{\ell-2d} + \xi^{\ell} + \xi^{2\ell}}{3}\check{T}[0] + \frac{2^{\ell-2d} + \xi^{\ell+2} + \xi^{2\ell+1}}{3}\check{T}[1] + \frac{2^{\ell-2d} + \xi^{\ell+1} + \xi^{2\ell+2}}{3}\check{T}[2] \\
&= \frac{2^{\ell-2d} + \Xi(\ell)}{3}\check{T}[0] + \frac{2^{\ell-2d} + \Xi(\ell+2)}{3}\check{T}[1] + \frac{2^{\ell-2d} + \Xi(\ell+1)}{3}\check{T}[2],
\end{aligned}$$

and we have completed the proof. □

***General case*** Recall the decomposition given in Proposition 3.12:

$$V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^i)[\varepsilon] \cong \bigoplus_{\boldsymbol{\gamma}\in\mathcal{D}} V_{L_{\mathcal{C}\times\mathbf{0}}}^{T,\boldsymbol{\eta}-i\boldsymbol{\gamma}}(\tau^i)[\varepsilon]. \tag{5.15}$$

In the following, we will denote

$$T_{\mathcal{C}\times\mathcal{D}}^{\boldsymbol{\eta}}[\varepsilon] := V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau)[\varepsilon], \qquad \check{T}_{\mathcal{C}\times\mathcal{D}}^{\boldsymbol{\eta}}[\varepsilon] := V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}}(\tau^2)[\varepsilon];$$

in addition, we let

$$T_{\mathcal{C}\times\mathcal{D}}[\varepsilon] := V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau)[\varepsilon], \qquad \check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon] := V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\mathbf{0}}(\tau^2)[\varepsilon].$$

We also let $\mathbf{0}$ be the trivial $\mathbb{Z}_3$-code of various length depending on context.

**Proposition 5.12.** *Let $\mathcal{B}$ be a self-dual $\mathbb{F}_4$-code of length 2. Then*

$$T_{\mathcal{B}\times\mathbf{0}}[0] \times T_{\mathcal{B}\times\mathbf{0}}[0] = \check{T}_{\mathcal{B}\times\mathbf{0}}[2].$$

**Proof.** In this case, all irreducible modules are simple currents. It suffices to find the non-zero fusion rules. Let $\mathcal{B}^2 := \mathcal{B} \oplus \mathcal{B}$ be a self-dual code of length 4 and let $\mathcal{S}$ be a self-dual $\mathbb{Z}_3$-code of length 4. By Proposition 5.11, we know $T_{\mathcal{B}^2\times\mathcal{S}}[0] \times T_{\mathcal{B}^2\times\mathcal{S}}[0] = \check{T}_{\mathcal{B}^2\times\mathcal{S}}[1]$.

Considering the subVOA $V_{L_{\mathcal{B}^2\times\mathbf{0}}}^{\tau} \subset V_{L_{\mathcal{B}^2\times\mathcal{S}}}^{\tau}$, we have the decomposition of $V_{L_{\mathcal{B}^2\times\mathbf{0}}}^{\tau}$-modules

$$T_{\mathcal{B}^2\times\mathcal{S}}[\varepsilon] = \bigoplus_{\boldsymbol{\eta}\in\mathcal{S}} T_{\mathcal{B}^2\times\mathbf{0}}^{\boldsymbol{\eta}}[\varepsilon], \qquad \check{T}_{\mathcal{B}^2\times\mathcal{S}}[\varepsilon] = \bigoplus_{\boldsymbol{\eta}\in\mathcal{S}} \check{T}_{\mathcal{B}^2\times\mathbf{0}}^{\boldsymbol{\eta}}[\varepsilon].$$

By Proposition 5.3,

$$1 = N\binom{\check{T}_{\mathcal{B}^2\times\mathcal{S}}[1]}{T_{\mathcal{B}^2\times\mathcal{S}}[0],\ T_{\mathcal{B}^2\times\mathcal{S}}[0]} \le N\binom{\oplus_{\boldsymbol{\eta}\in\mathcal{S}}\check{T}_{\mathcal{B}^2\times\mathbf{0}}^{\boldsymbol{\eta}}[1]}{T_{\mathcal{B}^2\times\mathbf{0}}[0],\ T_{\mathcal{B}^2\times\mathbf{0}}[0]} = N\binom{\check{T}_{\mathcal{B}^2\times\mathbf{0}}[1]}{T_{\mathcal{B}^2\times\mathbf{0}}[0],\ T_{\mathcal{B}^2\times\mathbf{0}}[0]},$$

where the last equality follows from Proposition 5.9. Since $\mathcal{B}^2$ is self-dual, we have

$$T_{\mathcal{B}^2\times\mathbf{0}}[0] \times T_{\mathcal{B}^2\times\mathbf{0}}[0] = \check{T}_{\mathcal{B}^2\times\mathbf{0}}[1].$$

Now consider the subVOA $V_{L_{\mathcal{B}\times\mathbf{0}}}^{\tau} \otimes V_{L_{\mathcal{B}\times\mathbf{0}}}^{\tau} \subset V_{L_{\mathcal{B}^2\times\mathbf{0}}}^{\tau}$ and the decomposition

$$T_{\mathcal{B}^2\times\mathbf{0}}[\varepsilon] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathcal{B}\times\mathbf{0}}[\varepsilon_0] \otimes T_{\mathcal{B}\times\mathbf{0}}[\varepsilon-\varepsilon_0]$$

of $V_{L_{\mathcal{B}\times\mathbf{0}}}^{\tau} \otimes V_{L_{\mathcal{B}\times\mathbf{0}}}^{\tau}$-modules, where $\otimes$ denotes the tensor product of vector spaces. Then

$$\begin{aligned} 1 = N\binom{\check{T}^{\boldsymbol{\eta}}_{\mathcal{B}^2\times\mathbf{0}}[1]}{T_{\mathcal{B}^2\times\mathbf{0}}[0],\ T_{\mathcal{B}^2\times\mathbf{0}}[0]} &\le N\binom{\bigoplus_{\varepsilon_0=0,1,2} \check{T}_{\mathcal{B}\times\mathbf{0}}[\varepsilon_0]\otimes \check{T}_{\mathcal{B}\times\mathbf{0}}[1-\varepsilon_0]}{T_{\mathcal{B}\times\mathbf{0}}[0]\otimes T_{\mathcal{B}\times\mathbf{0}}[0],\ T_{\mathcal{B}\times\mathbf{0}}[0]\otimes T_{\mathcal{B}\times\mathbf{0}}[0]} \\ &= \sum_{\varepsilon_0=0,1,2} N\binom{\check{T}_{\mathcal{B}\times\mathbf{0}}[\varepsilon_0]}{T_{\mathcal{B}\times\mathbf{0}}[0],\ T_{\mathcal{B}\times\mathbf{0}}[0]} N\binom{\check{T}_{\mathcal{B}\times\mathbf{0}}[1-\varepsilon_0]}{T_{\mathcal{B}\times\mathbf{0}}[0],\ T_{\mathcal{B}\times\mathbf{0}}[0]}. \end{aligned} \tag{5.16}$$

Since $\mathcal{B}$ is self-dual, only one of the fusion rules $N\binom{\check{T}_{\mathcal{B}\times\mathbf{0}}[\varepsilon_0]}{T_{\mathcal{B}\times\mathbf{0}}[0],\ T_{\mathcal{B}\times\mathbf{0}}[0]}$, $(\varepsilon_0 = 0,1,2)$ is non-zero. The inequality (5.16) then implies that $N\binom{\check{T}_{\mathcal{B}\times\mathbf{0}}[\varepsilon_0]}{T_{\mathcal{B}\times\mathbf{0}}[0],\ T_{\mathcal{B}\times\mathbf{0}}[0]} = \delta_{\varepsilon_0,2}$. This completes the proof. □

**Proposition 5.13.** *Let $\mathcal{C}$ and $\mathcal{D}$ be self-orthogonal codes of length $\ell$ and let $\Xi$ be defined as in Notation 5.10.*

(i) *If the length $\ell$ is even, then $T[0]\times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d}+\Xi(\ell-\varepsilon)}{3}\check{T}[\varepsilon]$.*
(ii) *If the length $\ell$ is odd, then $T[0]\times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d}-\Xi(\ell-\varepsilon)}{3}\check{T}[\varepsilon]$.*

*As a summary, we have*

$$T[0]\times T[0] = \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d}+(-1)^{\ell}\Xi(\ell-\varepsilon)}{3}\check{T}[\varepsilon].$$

*It also implies*

$$\begin{aligned} &V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_1}(\tau^i)[\varepsilon_1] \times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_2}(\tau^i)[\varepsilon_2] \\ &= \sum_{\varepsilon=0,1,2} \frac{2^{\ell-2d}+(-1)^{\ell}\Xi(\ell-\varepsilon)}{3} V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,-(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)}(\tau^{2i})[\varepsilon-\varepsilon_1-\varepsilon_2]. \end{aligned}$$

**Proof.** (i) First we assume $\ell$ is a multiple of 4. Then there exists a self-dual $\mathbb{Z}_3$ code $\mathcal{S}$ of length $\ell$. Restricting to $V_{L_{\mathcal{C}\times\mathbf{0}}}^{\tau}$-modules, we know

$$N\binom{\check{T}_{\mathcal{C}\times\mathcal{S}}[\varepsilon]}{T_{\mathcal{C}\times\mathcal{S}}[0],\ T_{\mathcal{C}\times\mathcal{S}}[0]} \le N\binom{\oplus_{\boldsymbol{\eta}\in\mathcal{S}}\check{T}^{\boldsymbol{\eta}}_{\mathcal{C}\times\mathbf{0}}[\varepsilon]}{T_{\mathcal{C}\times\mathbf{0}}[0],\ T_{\mathcal{C}\times\mathbf{0}}[0]} = N\binom{\check{T}_{\mathcal{C}\times\mathbf{0}}[\varepsilon]}{T_{\mathcal{C}\times\mathbf{0}}[0],\ T_{\mathcal{C}\times\mathbf{0}}[0]}, \tag{5.17}$$

for every $\varepsilon = 0,1,2$. On the other hand, we know from (5.11) that

$$\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[0]} = \sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0]},$$

Therefore, the inequality in (5.17) must attain equality and we prove (i) when $\boldsymbol{\mathcal{D}}$ is the trivial code of length divisible by 4.

Now let $\boldsymbol{\mathcal{D}}$ be a self-orthogonal code of length $\ell$. Similarly, we have

$$N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0]} \leq N\binom{T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0]}.$$

The same argument as in the case for $\boldsymbol{\mathcal{D}} = \mathbf{0}$ shows

$$N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0]} = N\binom{T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\mathbf{0}}[0]}.$$

This implies

$$N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0]} = N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[\varepsilon]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{S}}}[0]},$$

and proves (i) by Proposition 5.11 when $\ell$ is a multiple of 4.

Now assume $\ell \equiv 2 \mod 4$. Let $\boldsymbol{\mathcal{B}}$ be a self-dual $\mathbb{F}_4$-code of length 2. Then $\boldsymbol{\mathcal{C}} \oplus \boldsymbol{\mathcal{B}}$ is a self-orthogonal code of length divisible by 4 and $(\boldsymbol{\mathcal{D}} \oplus \mathbf{0})$ is a self-orthogonal code of the same length. Restricting to $V^{\tau}_{L_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}} \otimes V^{\tau}_{L_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}} \subset V^{\tau}_{L_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}}$, we know

$$T_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[\varepsilon] = \bigoplus_{\varepsilon_0=0,1,2} T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon_0] \otimes T_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[\varepsilon - \varepsilon_0].$$

Moreover,

$$\begin{aligned} &N\binom{\check{T}_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[\varepsilon]}{T_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[0],\ T_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[0]} \\ &\leq \bigoplus_{\varepsilon_0=0,1,2} N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon-\varepsilon_0]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0]} N\binom{\check{T}_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[\varepsilon_0]}{T_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[0],\ T_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[0]}. \end{aligned}$$

By Proposition 5.12, we know $T_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[0] \times T_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[0] = \check{T}_{\boldsymbol{\mathcal{B}}\times\mathbf{0}}[2]$; therefore the above inequality becomes

$$N\binom{\check{T}_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[\varepsilon]}{T_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[0],\ T_{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})\times(\boldsymbol{\mathcal{D}}\oplus\mathbf{0})}[0]} \leq N\binom{\check{T}_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[\varepsilon-2]}{T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0],\ T_{\boldsymbol{\mathcal{C}}\times\boldsymbol{\mathcal{D}}}[0]}.$$

On the other hand, $\sqrt{\left|\frac{(\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}})^{\perp}}{\boldsymbol{\mathcal{C}}\oplus\boldsymbol{\mathcal{B}}}\right|} = \sqrt{\left|\frac{\boldsymbol{\mathcal{C}}^{\perp}}{\boldsymbol{\mathcal{C}}}\right|}$ and hence by (5.11)

$$\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[\varepsilon]}{T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0],\ T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0]} = \sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}.$$

Therefore, we must have

$$N\binom{\check{T}_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[\varepsilon]}{T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0],\ T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0]} = N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon-2]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}.$$

Note that $\mathcal{C}\oplus\mathcal{B}$ has length $\ell+2$, thus we have

$$\begin{aligned} &N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} = N\binom{\check{T}_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[\varepsilon+2]}{T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0],\ T_{(\mathcal{C}\oplus\mathcal{B})\times(\mathcal{D}\oplus\mathbf{0})}[0]} \\ &= \frac{2^{\ell-2d}+\Xi(\ell+2-\varepsilon-2)}{3} = \frac{2^{\ell-2d}+\Xi(\ell-\varepsilon)}{3}. \end{aligned}$$

This proves (i) when $\ell \equiv 2 \mod 4$.

Now assume $\ell$ is odd, let $\mathcal{C}_e := \mathcal{C}\oplus\mathbf{0}$ and $\mathcal{D}_e := \mathcal{D}\oplus\mathbf{0}$ be self-orthogonal codes of even length $\ell+1$. Restricting to the subVOA $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}\otimes V_L^{\tau}$, we have decomposition of $V_{L_{\mathcal{C}\times\mathcal{D}}}^{\tau}\otimes V_L^{\tau}$-modules

$$T_{\mathcal{C}_e\times\mathcal{D}_e}[0] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathcal{C}\times\mathcal{D}}[\varepsilon_0]\otimes T_{\mathbf{0}\times\mathbf{0}}[-\varepsilon_0].$$

By Proposition 5.2, we have the following fusion rules of $V_L^{\tau}$:

$$T_{\mathbf{0}\times\mathbf{0}}[0]\times T_{\mathbf{0}\times\mathbf{0}}[0] = T_{\mathbf{0}\times\mathbf{0}}[0] + T_{\mathbf{0}\times\mathbf{0}}[2].$$

Therefore,

$$\begin{aligned} N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[0]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} &\le N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[0]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} + N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[1]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}, \\ N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[1]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} &\le N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[1]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} + N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[2]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}, \\ N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[2]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} &\le N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[0]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} + N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[2]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}. \end{aligned} \tag{5.18}$$

This gives

$$\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[\varepsilon]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} \le 2\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}.$$

From (5.11) we know

$$\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[\varepsilon]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} = 2^{\ell+1-2d} = 2\cdot 2^{\ell-2d}$$
$$= 2\sum_{\varepsilon=0,1,2} N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[\varepsilon]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}.$$

Therefore the inequalities in (5.18) must attain equalities. This gives

$$\begin{aligned} N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[0]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} &= 2^{\ell-2d} - \left(N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[1]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} + N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[2]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]}\right) \\ &= 2^{\ell-2d} - N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[1]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} \\ &= 2^{\ell-2d} - \frac{2^{\ell+1-2d}+\Xi(\ell+1-1)}{3} \\ &= \frac{2^{\ell-2d}-\Xi(\ell)}{3}. \end{aligned}$$

Similarly, we have

$$N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[1]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} = 2^{\ell-2d} - N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[2]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} = \frac{2^{\ell-2d}-\Xi(\ell+1-2)}{3},$$
$$N\binom{\check{T}_{\mathcal{C}\times\mathcal{D}}[2]}{T_{\mathcal{C}\times\mathcal{D}}[0],\ T_{\mathcal{C}\times\mathcal{D}}[0]} = 2^{\ell-2d} - N\binom{\check{T}_{\mathcal{C}_e\times\mathcal{D}_e}[0]}{T_{\mathcal{C}_e\times\mathcal{D}_e}[0],\ T_{\mathcal{C}_e\times\mathcal{D}_e}[0]} = \frac{2^{\ell-2d}-\Xi(\ell+1)}{3}.$$

This proves (ii). By Proposition 5.4, it follows immediately that

$$\begin{aligned} &V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_1}(\tau)[\varepsilon_1]\times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_2}(\tau)[\varepsilon_2] \\ &= \sum_{\varepsilon=0,1,2}\frac{2^{\ell-2d}+(-1)^{\ell}\Xi(\ell-\varepsilon)}{3}V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,-(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)}(\tau^2)[\varepsilon-\varepsilon_1-\varepsilon_2]. \end{aligned}$$

The corresponding statements for $V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_1}(\tau^2)[\varepsilon_1]\times V_{L_{\mathcal{C}\times\mathcal{D}}}^{T,\boldsymbol{\eta}_2}(\tau^2)[\varepsilon_2]$ can be proved by the similar arguments. □

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