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Characteristic polynomials of symmetric matrices over the univariate polynomial ring

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ABSTRACT

Viewing a bivariate polynomial $f \in \mathbb{R}[x,t]$ as a family of univariate polynomials in t parametrized by real numbers x, we call f *real rooted* if this family consists of monic polynomials with only real roots. If f is the characteristic polynomial of a symmetric matrix with entries in $\mathbb{R}[x]$, it is obviously real rooted. In this article the converse is established, namely that every real rooted bivariate polynomial is the characteristic polynomial of a symmetric matrix over the univariate real polynomial ring. As a byproduct we present a purely algebraic proof of the Helton–Vinnikov Theorem which solved the 60 year old Lax conjecture on the existence of definite determinantal representation of ternary hyperbolic forms.

Introduction

Given a monic polynomial $f \in A[t]$ over a commutative ring A we call a square matrix $M \in \mathrm{Mat}_n A$ a *spectral representation of f over A* if f is the characteristic polynomial of M, i.e., $f = \det(tI_n - M)$. The main result of this paper is the following

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Theorem 1. *Let $f \in \mathbb{R}[x,t]$ be real rooted, i.e., monic in t and for all $a \in \mathbb{R}$ the univariate polynomial $f(a,t) \in \mathbb{R}[t]$ has only real roots. Then f admits a symmetric spectral representation over $\mathbb{R}[x]$, i.e., there exists $M \in \mathrm{Sym}_n \mathbb{R}[x]$ such that $f = \det(tI_n - M)$.*

Symmetric spectral representations as certificates of real rootedness

Given a commutative ring A, it is generally a difficult problem to characterize those monic polynomials $f \in A[t]$ that admit a symmetric spectral representation over A. As noted above, in the case where A is the polynomial ring $\mathbb{R}[x]$ there is an obvious necessary condition, namely that f is real rooted. In other words, this condition means that for every homomorphism $\mathbb{R}[x] \to \mathbb{R}$ the image of f in $\mathbb{R}[t]$ (under coefficient-wise application) has only real roots. The following generalization of this property is shared by all characteristic polynomials of symmetric matrices over any commutative ring A: We call $f \in A[t]$ *real rooted over* A if f is monic and for all ring homomorphisms from A to any real closed field R the image of f in $R[t]$ has only roots in R. In the case $A = \mathbb{R}[x]$ it suffices to check homomorphisms to $\mathbb{R}$ and hence this is indeed a generalization, see Remark 3.2.

Now it is natural to ask about the converse: Which *real rooted* polynomials admit a symmetric spectral representation, or some related, possibly weaker, representation that manifests the real rootedness?

The following characterization of real rooted polynomials over fields is due to Krakowski [16]: If K is any field of characteristic different from 2 then $f \in K[t]$ is real rooted over K if and only if a power of f admits a symmetric spectral representation over K. See also [15] for a generalization and some lower and upper bounds on the exponent needed.

A useful reformulation of the existence of symmetric spectral representations has been given by Bender [3], generalizing a result of Latimer and MacDuffee [22], who established a correspondence between equivalence classes of spectral representations of a polynomial f over the ring of integers $\mathbb{Z}$ and ideal classes in $\mathbb{Z}[t]/(f)$. Bender's observation in [3] serves as an inspiration for the present work as it did for Bass, Estes and Guralnick who proved in [2] that if A is a Dedekind domain and $f \in A[t]$ real rooted, then f divides the characteristic polynomial of a symmetric matrix over A. In other words this means that all roots of f are eigenvalues of a symmetric matrix. Using this result the eigenvalues of adjacency matrices of regular graphs are characterized.

For a slightly smaller class of polynomials, their result can be further extended: A monic polynomial over A is *strictly real rooted* if for any homomorphism $A \to R$ to a real closed field R all roots of the image of f in $R[t]$ lie in R *and are simple*. Kummer recently showed in [19] that for any integral domain A every strictly real rooted polynomial $f \in A[t]$ divides the characteristic polynomial of a symmetric matrix.

The first result towards classification of polynomials that admit symmetric spectral representations without additional factor is also due to Bender [4]: If K is a number field and $f \in K[t]$ is real rooted over K with an odd degree factor, then f admits a symmetric spectral representation over K. It makes essential use of Hasse's local global principle for quadratic forms. A geometric counterpart of this number theoretic theorem holds without restriction: If K is a univariate function field over $\mathbb{R}$ then any real rooted polynomial over K admits a symmetric spectral representation. This follows from Krüskemper's work on scaled trace forms [17]. Ultimately it is a consequence of another local global principle for quadratic forms, due to Witt. See [9, Lemma 1.5] and also [13, Theorem 3.16] for a more direct proof using an argument by Leep.

The main result of the present paper, Theorem 1, can be read as a strengthening of two of the aforementioned ones: In contrast to the general case of Dedekind domains in [2], no additional factors are required. Moreover, it is a denominator free version of the case of the real rational function field: If the coefficients of the polynomial in question are denominator free, then it admits a *denominator free* symmetric spectral representation. From this it is easy to deduce the version with denominators, using transformations of the form $f \mapsto a^{-d} f(at)$. However, our main argument relies less on the theory of quadratic forms but rather on classical theory of divisors on algebraic curves.

The previous results reveal the exceptionality of the case $\mathbb{R}[x]$ over which the class of real rooted polynomials consists *exactly* of the characteristic polynomials of symmetric matrices. In fact, this seems to be essentially the only known nontrivial example of a ring, that is not a field and for which these two classes of polynomials coincide.

Application to hyperbolic polynomials

Closely related to spectral representations are linear determinantal representations of forms in several variables $F \in \mathbb{R}[x_1, \dots, x_\ell]$. These are linear pencils of the form

$$L = A_1 x_1 + \cdots + A_\ell x_\ell \quad (A_i \in \mathrm{Mat}_n(\mathbb{R}),\ n = \deg F)$$

with determinant F. We apply our main result to obtain linear symmetric determinantal representations of ternary hyperbolic forms. A homogeneous polynomial $F \in \mathbb{R}[x_1, \dots, x_\ell]$ is called *hyperbolic* with respect to some direction $e \in \mathbb{R}^\ell$ if $F(e) > 0$ and all real lines in this direction intersect the projective hypersurface defined by F in real points only, i.e., for all $a \in \mathbb{R}^\ell$ the univariate polynomial $F(te - a) \in \mathbb{R}[t]$ has only real roots.

We show that a consequence of Theorem 1 is the following famous result.

Theorem 2 *(Helton–Vinnikov). Let $F \in \mathbb{R}[x, y, z]$ be a form of degree n that is hyperbolic with respect to $e \in \mathbb{R}^3$. Then there exists a real symmetric matrix pencil $L = Ax + By + Cz$ $(A, B, C \in \mathrm{Sym}_n \mathbb{R})$ such that $L(e)$ is positive definite and $F = \det L$.*

The original proof can be found in [14]. It relies on transcendental tools from algebraic geometry such as theta functions on the Jacobian of a Riemann surface. In sharp contrast, our treatment involves only purely algebraic ingredients. The statement of Theorem 2 has been conjectured by Lax in 1958, see [21]. Its solution also settled a question of Parrilo and Sturmfels in [28] for the characterization of the plane convex semi-algebraic sets that are the feasible sets of semidefinite programming, so-called spectrahedra, and their minimal descriptions.

What has become known as the Generalized Lax Conjecture asserts a higher dimensional variant of the Helton–Vinnikov Theorem, namely that every *hyperbolicity cone* is *spectrahedral*. See [34] for a survey on related problems, including more precise formulations of the Generalized Lax Conjecture. An algebraic reformulation is that every hyperbolic polynomial divides the determinant of a definite real symmetric matrix pencil in such a way, that its hyperbolicity cone is contained in the hyperbolicity cone of the other factor. A weaker variant, without the condition on the additional factor has been proved in [20]. Further investigation of the relation to previously formulated certificates for real rootedness seems worthwhile.

A slightly weaker version of Theorem 2 (see Section 7) has been used in the celebrated proof of the Kadison–Singer Conjecture by Marcus, Spielman and Srivastava[2] in [24] and by Speyer in [32] to give another proof of the affirmative answer to Horn's problem on eigenvalues of sums of Hermitian matrices.

The case of constant coefficients

Our proof is an adaptation of the following simple construction with constant coefficients. Suppose $f \in \mathbb{R}[t]$ is a monic polynomial with real simple roots $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. We are going to find a real symmetric spectral representation $M \in \operatorname{Sym}_n \mathbb{R}$ of f without computing the roots of f. To this end we define the n-dimensional $\mathbb{R}$-algebra $B := \mathbb{R}[t]/(f)$ and the vector space endomorphism μ of B that is given by multiplication by $\bar{t} = t + (f)$. As is easily verified, f is the characteristic polynomial of μ. Moreover, μ is obviously self-adjoint with respect to the trace form

$$\begin{aligned} \tau \colon B \times B &\to \mathbb{R} \\ (\bar{g}, \bar{h}) &\mapsto \operatorname{Tr}_{B|\mathbb{R}}(\bar{g}\bar{h}) = \sum_i gh(\lambda_i) \end{aligned}$$

which is positive definite and hence admits an orthonormal basis $\mathcal{B}$. Now the representing matrix M of μ with respect to $\mathcal{B}$ is symmetric and its characteristic polynomial is f.

Besides basic field operations this construction only involves taking square roots of positive real numbers in the orthonormalization step.

The obvious obstacle to generalizing this construction to the coefficient ring $A := \mathbb{R}[x]$ instead of $\mathbb{R}$ is the non-existence of an orthonormal basis of the trace form. Although for every real rooted polynomial $f \in A[t]$ the trace form τ of $B := A[t]/(f)$ over A is

[2] Using general properties of hyperbolic polynomials, Brändén shows in [6] how to avoid the use of determinantal representations as in [24].

positive semidefinite in all real points (viewing τ as a family of forms parametrized by x), it does in general not admit an orthonormal basis. This is due to the fact that it becomes singular at those (possibly non-real) points $a \in \mathbb{C}$ where $f(a,t) \in \mathbb{C}[t]$ has multiple roots. These correspond to the ramification points of the curve $\mathcal{C} = \operatorname{Spec} B$ defined by f under the projection onto the x-axis, $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} A$. These ramification points will play a central role in what follows.

Reader's guide

After collecting some definitions, notations, conventions and general facts in Section 1, we have a look at a few properties of *trace forms* in Section 2. Among these is a variant of Bender's result about modified trace forms that are unimodular, i.e., regular everywhere. These modifications can be expressed in terms of certain factorizations of the so-called *codifferent ideal.* See Lemma 2.1 for a precise statement. Also, we recall how *ramification* is encoded in the codifferent ideal, see Lemma 2.3. Section 3 contains some general observations on real rooted polynomials and their interplay with trace forms. An important consequence is *non-reality* of the above mentioned ramification points, Corollary 3.5. The main result of Section 4 is Corollary 4.2, characterizing 2-divisibility of classes of ideals in the *narrow class group* in terms of their real prime factors. This is combined in Section 5 with the non-reality of ramification to conclude that we can modify the trace form as described in Section 2, leading to a proof of Theorem 1. Section 6 is concerned with the growth behavior of eigenvalues of symmetric matrices. Applied to polynomial matrices this gives a degree bound for their entries in terms of the coefficients of their characteristic polynomial. We use it to derive the Helton–Vinnikov Theorem 2 from our main result. Section 7 outlines how the proof of Theorem 1 can be simplified to obtain slightly weaker representations, namely complex Hermitian instead of real symmetric ones.

1. Preliminaries

In this section we list some definitions, notations and conventions as well as some of the basic facts that are used throughout the text. Since our methods are purely algebraic we will not make use of any topological properties of the fields of real and complex numbers. Accordingly, throughout the paper one can replace $\mathbb{R}$ and $\mathbb{C}$ by any real closed field and its algebraic closure, respectively.

Notions from commutative algebra

Let A be a commutative ring, which will always be assumed to have a unit.

(1) $\operatorname{Mat}_n A$ and $\operatorname{Sym}_n A$ are the sets of $n \times n$ matrices and symmetric matrices, respectively.

(2) An extension $B|A$ is *finite*, if B is finitely generated as an A-module.
(3) $\operatorname{Spec} A$ and $\operatorname{Sper} A$ are the spectrum and real spectrum of A, respectively. For the definition of the real spectrum and a general reference on real algebraic geometry see [23] (also [1] or [27]).
(4) For $\mathfrak{p} \in \operatorname{Spec} A$ we denote by $k(\mathfrak{p}) := \operatorname{Quot}(A/\mathfrak{p})$ the residue field of $\mathfrak{p}$.
(5) A prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ is *real* if $k(\mathfrak{p})$ is formally real, i.e., admits an ordering.
(6) A symmetric bilinear form $\beta\colon M \times M \to A$ on an A module M is *unimodular* if M is isomorphic to its own dual via β, i.e., the induced map

$$M \to \operatorname{Hom}_A(M, A)$$
$$a \mapsto \beta(a, \cdot)$$

is an isomorphism.
(7) If B is an A-algebra, free of finite rank as an A-module, then the *trace form of* $B|A$ is the symmetric bilinear form

$$\tau_{B|A}\colon B \times B \to A$$
$$(a, b) \mapsto \operatorname{Tr}_{B|A}(ab)$$

where for $x \in B$ the trace $\operatorname{Tr}_{B|A}(x)$ is the trace of the A-endomorphism of B that is given by multiplication by x.
(8) Let $f \in A[t]$ be a monic polynomial and $B := A[t]/(f)$. The *Hermite matrix* of f is the representing matrix H of $\tau_{B|A}$ with respect to the standard basis $1, \bar{t}, \dots, \bar{t}^{n-1}$ of B.

For the present paper, the importance of the trace form lies in the following well-known classical result on real root counting which goes back to the work of Sturm, Hermite and Sylvester. A proof can be found in [5, Theorem 4.58].

Lemma 1.1. *Let K be an ordered field with real closure R and $f \in K[t]$ monic. Then the signature of the trace form of $K[t]/(f)$ over K is the number of distinct roots of f that lie in R.*

Now let A be a Dedekind domain. For basic theory of Dedekind domains we refer to [31, Chapter I].

(1) By $\mathcal{I}_A$ we denote the group of nonzero fractional A-ideals. It is freely generated by the nonzero elements of $\operatorname{Spec} A$.
(2) The *class group* of A, denoted by $\operatorname{Cl} A$, is the quotient of $\mathcal{I}_A$ modulo the subgroup of principal ideals.

(3) For conceptual reasons we also define the finer *narrow class group* $\mathrm{Cl}_+ A$ to be the quotient of $\mathcal{I}_A$ modulo the subgroup of those principal ideals that are generated by a sum of squares.
(4) If $\mathfrak{p}$ is a nonzero prime ideal of A we denote the $\mathfrak{p}$-adic valuation of the field of fractions of A by $v_\mathfrak{p}$ and for a fractional A-ideal I we write

$$v_\mathfrak{p}(I) := \min\{ v_\mathfrak{p}(a) \mid a \in I \}$$

for the multiplicity of $\mathfrak{p}$ in the prime ideal factorization of I.

Unimodular forms over polynomial rings

We will make essential use of the following special feature of univariate polynomial rings over fields, generalizing the well-known fact that they have only constant units.

Theorem 1.2 *(Harder). Let k be a field of characteristic different from 2 and M a free $k[x]$-module of rank n. Then any unimodular bilinear form β on M admits an orthogonal basis $q_1, \dots, q_n$. Moreover, for every such orthogonal basis we have*

$$\beta(q_i, q_i) \in k^\times.$$

Proof. See for example [30, Theorem 6.3.3]. □

2. Scaled trace forms and the codifferent

The trace form of a finite ring extension is in general not unimodular. This is the main obstacle to finding an orthonormal basis in the proof of our Theorem 1. Lemma 2.1 shows how one can overcome this by scaling the trace form appropriately. The relation to ramification can be found in Lemma 2.3.

The complementary module

Let $B|A$ be a finite extension of integral domains and assume the extension of their respective fields of fractions $L|K$ is separable. For an A-submodule M of L we denote by

$$M' := \{ x \in L \mid \mathrm{Tr}_{L|K}(xM) \subseteq A \}$$

the *complementary module* of M and by $\Delta(B|A) := B'$ the *codifferent* of $B|A$, which is a fractional B-ideal.

The following is a variant of Bender's observation in [3].

Lemma 2.1. *Let $B|A$ be a finite extension of integral domains with separable extension $L|K$ of their respective fields of fractions. Further, let $c \in L^\times$ and I be an A-submodule of L that generates L as a K-vector space. We define the scaled trace form*

$$\begin{aligned} \beta \colon L \times L &\to K \\ (a, b) &\mapsto \operatorname{Tr}_{L|K}(abc). \end{aligned}$$

(a) The restriction of β to I takes its values in A and is unimodular, if and only if $cI = I'$.
(b) If $BI \subseteq I$ then I' coincides with the ideal quotient

$$(\Delta(B|A) : I) = \{\, x \in L \mid xI \subseteq \Delta(B|A)\,\}.$$

(c) If B is a Dedekind domain and I is a fractional B-ideal then β restricts to a unimodular form on I if and only if $cI^2 = \Delta(B|A)$.

Proof. (a) Since $\tau_{L|K}$ is regular and I generates L as a K-vector space, the map

$$\begin{aligned} I' &\to \operatorname{Hom}_A(I, A) \\ x &\mapsto \tau_{L|K}(x, \cdot) \end{aligned}$$

is an isomorphism. This means via $\tau_{L|K}$ we can identify the complementary module I' with the dual module of I. So via the scaled trace form β the dual of I becomes the scaled complementary module $c^{-1}I'$. Further, β is unimodular on I if and only if I coincides with its own dual, i.e., if $I = c^{-1}I'$.

Part (b) follows immediately from the definition and (c) is just a combination of (a) and (b) using the fact that if B is a Dedekind domain then I is invertible and the ideal quotient $(\Delta(B|A) : I)$ can thus be written as $\Delta(B|A)I^{-1}$. $\square$

Remark 2.2. The codifferent ideal and the role of the scaling factor in (a) of the previous lemma become more concrete in the case of primitive ring extensions. For this we use a lemma often attributed to Euler [31, Lemma III.6.2]: Let A be an integral domain and let $f \in A[t]$ be monic with only simple roots, $f' := \partial f / \partial t$ its formal derivative and define $B := A[t]/(f)$. Then the scaled trace form

$$\begin{aligned} \beta \colon B \times B &\to A \\ (a, b) &\mapsto \operatorname{Tr}_{B|A}\left(\frac{ab}{f'(\bar{t})}\right) \end{aligned}$$

is well defined and unimodular. In particular $\Delta(B|A) = \left(\frac{1}{f'(\bar{t})}\right)$. However, if f is non-linear then this form will be totally indefinite,[3] since by Rolle's Theorem, the derivative changes sign between two consecutive simple real roots. Therefore, also β does (in general) not admit an orthonormal basis.

The codifferent encodes ramification

Roughly speaking, the next lemma states that the support of the codifferent contains only ramified primes. A more precise statement about the ramification index is know as Dedekind's Different Theorem, see [26, Theorem III.2.6]. A proof of the following can also be found in [31, Theorem III.5.1], but since it is short we include it for self-containedness.

Lemma 2.3. *Let $B|A$ be a finite extension of Dedekind domains and let $\mathfrak{p}$ be a nonzero prime ideal of A such that $\mathfrak{q}|\mathfrak{p}$ is unramified and $k(\mathfrak{q})|k(\mathfrak{p})$ is separable for all primes $\mathfrak{q}$ of B lying above $\mathfrak{p}$. Then none of the latter appears in the prime ideal factorization of $\Delta(B|A)$, i.e., $v_{\mathfrak{q}}(\Delta(B|A)) = 0$ for all $\mathfrak{q} \in \mathcal{I}_B$ lying above $\mathfrak{p}$.*

Proof. By considering the localization at $\mathfrak{p}$ it suffices to assume that A is a discrete valuation ring with maximal ideal $\mathfrak{p}$ and prove that $\Delta(B|A) = B$.

By Lemma 2.1 this is equivalent to the trace form $\tau_{B|A}$ being unimodular. Since A is a discrete valuation ring it suffices to show that $\tau_{B|A}$ becomes regular modulo the maximal ideal, i.e., that the trace form

$$\tau_{B|A} \otimes k(\mathfrak{p}) = \tau_{B\otimes k(\mathfrak{p})|k(\mathfrak{p})}$$

of the residue ring extension is regular. Let $\mathfrak{p}B = \prod_i \mathfrak{q}_i$ be the prime ideal decomposition of $\mathfrak{p}B$. By assumption the $\mathfrak{q}_i$ are pairwise distinct and therefore, coprime. This means $B \otimes k(\mathfrak{p}) = B/\mathfrak{p}B = \prod_i k(\mathfrak{q}_i)$. Now we see that the trace form of $B \otimes k(\mathfrak{p})$ over $k(\mathfrak{p})$ is regular, since it is the orthogonal sum of the trace forms of the separable extensions $k(\mathfrak{q}_i)|k(\mathfrak{p})$. $\square$

3. Real rooted polynomials and the trace form

In this section we collect some basic properties of real rooted polynomials. In particular their interplay with trace forms is used to show absence of real ramification, see Corollary 3.5.

Let A be a commutative ring and $f \in A[t]$ monic. Recall that f is *real rooted over* A, if for every ring homomorphism $A \to R$ to a real closed field R the image of f in $R[t]$ has only roots in R. For systematic reasons we want to replace homomorphisms into real closed fields by points in the real spectrum $\operatorname{Sper} A$ of A. For $P \in \operatorname{Sper} A$ with support

[3] I.e. indefinite at every point in the real spectrum.

$\mathfrak{p} \in \operatorname{Spec} A$ denote by $R(P)$ the real closure of the (ordered) residue field $k(\mathfrak{p})$ of P and by $f_P := f \otimes 1 \in A[t] \otimes R(P) = R(P)[t]$ the coefficient-wise evaluation of f at P.

(1) We say f is *real rooted in* P if all roots of f_P lie in $R(P)$ and accordingly f is *real rooted in* $U \subseteq \operatorname{Sper} A$ if f is real rooted in every point of U.
(2) In this sense f is real rooted over A if it is real rooted in $\operatorname{Sper} A$.

From Lemma 1.1 on real root counting we immediately get the following

Corollary 3.1. *Let A be a commutative ring, $f \in A[t]$ monic, $B = A[t]/(f)$ and $\tau := \tau_{B|A}$ the trace form of $B|A$. Then f is real rooted in $P \in \operatorname{Sper} A$ if and only if $\tau \otimes_A R(P)$ is positive semidefinite.*

Remark 3.2. (a) From the previous corollary it follows that the set $U \subseteq \operatorname{Sper} A$ of points where f is real rooted consists exactly of those points where all the principal minors of the Hermite matrix of f are nonnegative. In particular, it is a basic closed subset of $\operatorname{Sper} A$ with respect to the Harrison topology.

(b) For $A = \mathbb{R}[x_1, \dots, x_\ell]$ we view $\mathbb{R}^\ell$ as a subset of $\operatorname{Sper} A$. Then for $a \in \mathbb{R}^\ell$ a polynomial $f \in A[t]$ is real rooted in a if $f_a = f(a,t) \in \mathbb{R}[t]$ has only real roots.

The set of points $\mathbb{R}^\ell$ and the set of orderings $\operatorname{Sper} \mathbb{R}(x_1, \dots, x_\ell)$ of the rational function field are both dense in $\operatorname{Sper} A$. This follows essentially from Tarski's Transfer Principle [23, Theorem 2.4.3] and from the Baer–Krull correspondence [23, Section 1.5], respectively. In particular real rootedness of f in $\mathbb{R}^\ell$, $\operatorname{Sper} \mathbb{R}[x_1, \dots, x_\ell]$ and $\operatorname{Sper} \mathbb{R}(x_1, \dots, x_\ell)$ are all equivalent.

We make use of the following special local case of this transfer argument, which can be treated completely elementarily.

Lemma 3.3. *Let $f \in \mathbb{R}[x,t]$ be real rooted in a neighborhood of the origin of $\mathbb{R} \subseteq \operatorname{Sper} \mathbb{R}[x]$. Then f is real rooted over the field of Laurent series $\mathbb{R}((x))$.*

Proof. Let $H \in \operatorname{Sym}_n \mathbb{R}[x]$ be any representing matrix of the trace form of $\mathbb{R}[x,t]/(f)$ over $\mathbb{R}[x]$, e.g. the Hermite matrix of f. Using Corollary 3.1 we get that $H(a)$ is positive semidefinite for all a in some neighborhood of 0 and we want to conclude that H is positive semidefinite with respect to both orderings of $\mathbb{R}((x))$. To see this in an elementary way we diagonalize H as a quadratic form over $\mathbb{R}(x)$. Then the resulting diagonal entries are nonnegative rational functions on $(-\varepsilon, \varepsilon)$ for some $\varepsilon \in \mathbb{R}_{>0}$ and thus lie in the preordering generated by $\varepsilon + x$ and $\varepsilon - x$, which is the set of elements of the form $\sigma_0 + \sigma_1(\varepsilon + x) + \sigma_2(\varepsilon - x)$, where the σ_i are sums of squares of elements in $\mathbb{R}(x)$. So they are also nonnegative with respect to the two orderings of $\mathbb{R}((x))$ since both make $\varepsilon \pm x$ positive. $\square$

Lemma 3.4. *Over $\mathbb{R}((x))$ every real rooted polynomial splits into linear factors.*

Proof. Any finite field extension of $\mathbb{R}((x))$ either contains $\mathbb{C}$ or is of the form $\mathbb{R}((x^{\frac{1}{e}}))$. Both have nonreal embeddings into the algebraic closure of $\mathbb{R}((x))$ unless $e = 1$. That means if $f \in \mathbb{R}((x))[t]$ is real rooted and irreducible over $\mathbb{R}((x))$ then it must be of degree one. □

As a consequence we get the absence of real ramification that we need for the factorization of the codifferent in the proof of our main result. Similar results can be found in [8, Corollary to Lemma 4.1], [2, Theorem 6.2] and a higher dimensional generalization in [18, Theorem 2.19].

Corollary 3.5. *Let $f \in \mathbb{R}[x,t]$ be irreducible and define $K := \mathbb{R}(x)$ and $L := K[t]/(f)$. If f is real rooted in a neighborhood of $a \in \mathbb{R}$ then the $(x - a)$-adic valuation of K is unramified in $L|K$.*

Proof. Let f be real rooted in a neighborhood of a which we can assume to be the origin. By Lemma 3.3 it is also real rooted over $\mathbb{R}((x))$. Let v be the x-adic valuation of $\mathbb{R}(x)$ and w an extension of v to L. Then the completion L_w is a factor in $L \otimes \mathbb{R}((x))$ which must be of degree one by Lemma 3.4, i.e., $L_w = \mathbb{R}((x))$. In particular, $w|v$ is unramified. □

4. Squares in the narrow class group

Recall that the narrow class group of a Dedekind domain A is the ideal group $\mathcal{I}_A$ modulo the subgroup of principal ideals generated by a sum of squares. In Corollary 4.2 we characterize the squares in the narrow class group of a smooth affine curve over $\mathbb{R}$, which is the essential step in finding positive definite unimodular scaled trace forms in the proof of Theorem 1. This characterization is a consequence of the 2-divisibility of the class group of a smooth affine curve over $\mathbb{C}$.

Theorem 4.1. *If A is a Dedekind domain that is a finitely generated $\mathbb{C}$-algebra, then its class group is divisible.*

Proof. The class group $\operatorname{Cl} A$ is a quotient of the degree zero part $\operatorname{Cl}^0 K$ of the divisor class group of the univariate function field $K = \operatorname{Quot} A$ over $\mathbb{C}$.

A direct, algebraic proof of the divisibility of $\operatorname{Cl}^0 K$ is due to Frey [10] and holds even in positive characteristic. More geometric arguments rely on the Jacobian of the smooth curve corresponding to the function field K. See, e.g. [11, Section 2.2] for a classical analytic treatment or [25, p. 42] for an approach using Weil's algebraic generalization. □

Corollary 4.2. *Let A be a Dedekind domain that is a finitely generated $\mathbb{R}$-algebra and $J \in \mathcal{I}_A$ a fractional A-ideal. Then the class of J is a square in the narrow class group $\operatorname{Cl}_+ A$ if and only if all real prime ideals appear in J with even order. In other words there exists $I \in \mathcal{I}_A$ and a sum of squares $c \in \operatorname{Quot} A$ such that $J = cI^2$ if and only if $2|v_{\mathfrak{p}}(J)$ for every nonzero real $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. The last condition is clearly necessary, since in general the value of a sum of squares under any real valuation is divisible by 2, see [27, Exercise 1.4.10]. For the converse let $v_{\mathfrak{p}}(J)$ be even for every real prime $\mathfrak{p}$. Multiplying J by an appropriate product of even powers of real prime ideals we can even assume that J is a product of nonreal prime ideals and their inverses. It thus suffices to show that the class of every nonreal prime ideal is a square in $\mathrm{Cl}_+ A$.

If $\mathbb{C} \subseteq A$ then $\mathrm{Cl}_+ A = \mathrm{Cl}\, A$ and the claim follows directly from Theorem 4.1. Assume now that -1 is not a square in A and hence the finitely generated $\mathbb{C}$-algebra $B := A \otimes_{\mathbb{R}} \mathbb{C}$ is again a Dedekind domain the class group of which is divisible, again by Theorem 4.1.

Now let $\mathfrak{p} \in \mathcal{I}_A$ be a nonreal prime ideal. We want to show that its class in $\mathrm{Cl}_+ A$ is a square. The norm of an element of B is a sum of two squares in A. Therefore, the ideal norm map $\mathrm{N}_{B|A}$ induces a homomorphism $\mathrm{Cl}\, B \to \mathrm{Cl}_+ A$. Using the 2-divisibility of $\mathrm{Cl}\, B$ it thus suffices to show that $\mathfrak{p}$ is the norm of an ideal in B. Since $\mathfrak{p}$ is nonreal we have $k(\mathfrak{p}) = \mathbb{C}$. Hence for $\mathfrak{q} \in \mathcal{I}_B$ lying above $\mathfrak{p}$ the extension $k(\mathfrak{q})$ of $k(\mathfrak{p})$ is trivial, so the residue degree $f_{\mathfrak{q}|\mathfrak{p}} = [k(\mathfrak{q}) : k(\mathfrak{p})]$ is 1 and we get $\mathrm{N}_{B|A}(\mathfrak{q}) = \mathfrak{p}^{f_{\mathfrak{q}|\mathfrak{p}}} = \mathfrak{p}$, as desired.

More concretely this means that $\mathfrak{p}$ corresponds to a pair of conjugate points on the affine curve $\mathrm{Spec}\, B \otimes \mathbb{C}$. Hence $\mathfrak{p}$ factors over $\mathbb{C}$ into a product of two conjugate prime ideals. The norm of each of these two factors equals $\mathfrak{p}$. $\square$

5. Proof of the Main Theorem

Now we have collected all the necessary ingredients to prove Theorem 1. Let $f \in \mathbb{R}[x,t]$ be real rooted, i.e., f is monic in t and $f(a,t)$ has only real roots for all $a \in \mathbb{R}$. To prove that f is the characteristic polynomial of a symmetric matrix over $\mathbb{R}[x]$ we may assume that f is irreducible. Otherwise we find a symmetric spectral representation of each of its irreducible factors and compose them to a block diagonal matrix which then gives a symmetric spectral representation of f.

We fix the following notation:

- $n = \deg_t f$,
- $A = \mathbb{R}[x]$ the coordinate ring of the real affine line $\mathbb{A}^1_{\mathbb{R}}$,
- $K = \mathbb{R}(x)$ its function field,
- $L = K[t]/(f)$ the function field of the plane affine curve $\mathcal{C}$ defined by f,
- B the integral closure of A in L, i.e., the coordinate ring of the normalization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$,

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \qquad & \begin{array}{ccc} B & \xrightarrow{\subseteq} & L \\ \big| & & \big|\, n \\ A & \xrightarrow{\subseteq} & K \end{array} \\ \big\downarrow & & \\ \mathbb{A}^1_{\mathbb{R}} & & \end{array}$$

- $\tau = \tau_{L|K}$ the trace form of $L|K$.
- $\Delta = \Delta(B|A)$ the codifferent of $B|A$.

Since f is real rooted in every point $a \in \mathbb{R}$, the extension $B|A$ is unramified in all real primes of A by Corollary 3.5. Therefore, $v_{\mathfrak{q}}(\Delta) = 0$ for all real primes $\mathfrak{q} \in \mathcal{I}_B$ by Lemma 2.3. Using Corollary 4.2 it now follows that the class of Δ in the narrow class group $\mathrm{Cl}_+ B$ is a square, i.e., there exists a sum of squares $c \in L^\times$ and a fractional ideal $I \in \mathcal{I}_B$ such that $cI^2 = \Delta$. By Lemma 2.1(c) the scaled trace form

$$\begin{aligned} \beta\colon I \times I &\to A \\ (a,b) &\mapsto \mathrm{Tr}_{L|K}(abc) \end{aligned}$$

is well-defined and unimodular. Since A is a principal ideal domain and I is finitely generated and torsion free as an A-module, it is already free. Now by Harder's Theorem 1.2 we can orthogonalize it with nonzero real numbers on the diagonal. These must be positive as follows easily from Lemma 1.1 since c is a sum of squares. This means (I, β) admits an orthonormal basis $\mathcal{B}$.

Let μ denote multiplication by $\bar{t}$, viewed as an endomorphism of the K-vector space L. Its characteristic polynomial is f. Since any A-basis of I is also a K-basis of L, the restriction of μ to I has characteristic polynomial f as well.

Since μ is obviously self-adjoint with respect to β, its representing matrix $M \in \mathrm{Mat}_n A$ with respect to the orthonormal basis $\mathcal{B}$ of I is symmetric, hence M is a symmetric spectral representation of f over A, as desired. □

Remark 5.1. If the curve $\mathcal{C}$ defined by f is smooth, the coordinate ring $\mathbb{R}[x,t]/(f)$ is integrally closed and therefore, coincides with B in the above proof. It is not hard to see that in this case every symmetric spectral representation of f arises in the way pointed out above. We can even describe their equivalence classes in terms of pairs (I, c) with $cI^2 = \Delta$ and c a sum of squares, where equivalence of representations is induced by the action of the orthogonal group. For a more precise statement see [13, Theorem 3.22]. If the curve is not smooth, then $\mathbb{R}[x,t]/(f)$ is not integrally closed. The symmetric spectral representations that are produced in the proof of Theorem 1 are those that extend to homomorphisms from the integral closure B of $\mathbb{R}[x,t]/(f)$ to $\mathrm{Sym}_n \mathbb{R}[x]$. However, it is not clear which representations of f extend to B in this case.

6. Symmetric matrices and real valuations

The size of the entries of a symmetric matrix over $\mathbb{R}$ can be bounded in terms of its eigenvalues and hence in terms of the coefficients of its characteristic polynomial. We give a valuation theoretic analogue of this observation. Applied to the degree valuation this shows that the Helton–Vinnikov Theorem 2 follows from Theorem 1.

In the following let v be a real valuation on K, i.e., the residue field $\overline{K}$ is formally real. Let $M \in \mathrm{Mat}_n K$. Let $v(M)$ denote the minimal value of the entries of M. We obtain an obvious lower bound on the values of the coefficients of its characteristic polynomial

$f = \det(tI_n - M) = \sum_i a_i t^i \in K[t]$ since each a_i is homogeneous of degree $n-i$ in the entries of M. In particular we have

$$v(a_i) \geq (n-i)v(M).$$

If the matrix is symmetric then this bound is sharp, i.e., we have equality for at least one i:

Proposition 6.1. *Let $M \in \operatorname{Sym}_n K$ be nonzero and $f = \det(tI_n - M) = \sum_i a_i t^i \in K[t]$ $(a_i \in K)$. Then*

$$v(M) = \min_{0 \leq i < n} \frac{v(a_i)}{n-i}.$$

In particular the right hand side lies in the value group of v.

Proof. Let $a \in K^\times$ be an entry of M with minimal value, i.e., $v(a) = v(M)$. We rescale M and f so that both lie in the valuation ring of v. So we define

$$M_0 := a^{-1}M$$

and

$$f_0 := \det(tI_n - M_0) = a^{-n}\det(atI_n - M) = a^{-n}f(at) = \sum_i \frac{a_i}{a^{n-i}} t^i.$$

Since the residue field $\overline{K}$ is formally real and $\overline{M_0} \in \operatorname{Mat}_n \overline{K}$ is symmetric and nonzero it cannot be nilpotent. By the Cayley–Hamilton Theorem at least one coefficient of its characteristic polynomial other than the leading one must be nonzero. So there exists $i < n$ such that $\overline{\frac{a_i}{a^{n-i}}}$ is nonzero and hence

$$v(a_i) = v(a^{n-i}) = (n-i)v(M)$$

as claimed. $\square$

Applying this to the case where v is the degree valuation on $K = \mathbb{R}(x)$, i.e., $v = -\deg$, we immediately obtain the following

Corollary 6.2. *Let $M \in \operatorname{Sym}_n \mathbb{R}[x]$ and $f = \det(tI_n - M) \in \mathbb{R}[x,t]$ its characteristic polynomial. If the total degree of f is n, then M is linear, i.e., its entries have at most degree one.*

Using this it becomes easy to derive the Helton–Vinnikov Theorem from our main result.

Proof of Theorem 2. After rescaling F and e and applying a linear change of variables we can assume that $F(e) = 1$ and $e = (0, 0, 1)$. Then the dehomogenization $f := F(x, 1, t) \in \mathbb{R}[x, t]$ is real rooted and thus admits a symmetric spectral representation $M \in \operatorname{Sym}_n \mathbb{R}[x]$ by Theorem 1. The condition that f is of total degree n forces the entries of M to be linear, by Corollary 6.2. This means M is of the form $M_1 x + M_0$ for some $M_0, M_1 \in \operatorname{Sym}_n \mathbb{R}$. Homogenizing again we see that F is the determinant of the real symmetric pencil $L := I_n z - M_0 y - M_1 x$. Moreover, $L(e) = I_n$ is positive definite. $\square$

7. Hermitian spectral representations

Finally, we want to sketch how the above procedure can be simplified to produce complex Hermitian instead of real symmetric spectral representations of real rooted polynomials. As usual a matrix $M \in \operatorname{Mat}_n \mathbb{C}[x]$ is *Hermitian* if it equals its conjugate transpose, where conjugation refers to coefficient-wise complex conjugation of the entries. The process of producing Hermitian representations becomes considerably more elementary, since it does not depend on Theorem 4.1, the divisibility of the class group.

By the same argument provided in the previous section as well as the appropriate reformulation of Proposition 6.1 this weaker result can be used to prove existence of definite linear Hermitian determinantal representations of hyperbolic polynomials. This result has been obtained previously by Dubrovin [8] and Vinnikov [33]. Further elementary proofs can be found in [29] and [12].

To produce Hermitian representations we replace the symmetric bilinear trace form of L over K in the proof of Theorem 1 by the Hermitian trace form of $\widetilde{L} := L \otimes_{\mathbb{R}} \mathbb{C}$ over $\widetilde{K} := \mathbb{C}(x)$ which is given by

$$\widetilde{\tau}\colon \widetilde{L} \times \widetilde{L} \to \widetilde{K}$$
$$(a, b) \mapsto \operatorname{Tr}_{\widetilde{L}|\widetilde{K}}(a^* b)$$

where $*$ denotes the induced complex conjugation on $\widetilde{L}$. The crucial difference now is the required factorization of the codifferent $\widetilde{\Delta}$ of $\widetilde{B} := B \otimes \mathbb{C}$ over $\widetilde{A} := \mathbb{C}[x]$. Namely $\widetilde{\Delta}$ is already a Hermitian square, i.e., there exists a fractional $\widetilde{B}$-ideal I such that $I^* I = \widetilde{\Delta}$. After a slight modification of Lemma 2.1 we see that $\widetilde{\tau}$ restricts to a unimodular positive definite Hermitian form on I. Using a generalization of Theorem 1.2 found in [7, Proposition 6] it, therefore, admits an orthonormal basis. Now we can proceed as before to get a Hermitian spectral representation of f. A more detailed explanation can be found in [13].

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