A computational solution to a question by Beauville on the invariants of the binary quintic

A. Abdesselam

LAGA, Institut Galilée, CNRS UMR 7539, Université Paris XIII, 99 Avenue J.B. Clément,
F93430 Villetaneuse, France

Abstract

We obtain an alternative proof of an injectivity result by Beauville for a map from the moduli space of quartic del Pezzo surfaces to the set of conjugacy classes of certain subgroups of the Cremona group. This amounts to showing that a projective configuration of five distinct unordered points on the line can be reconstructed from its five projective four-point subconfigurations. This is done by reduction to a question in the classical invariant theory of the binary quintic, which is solved by computer-assisted methods. More precisely, we show that six specific invariants of degree 24, the construction of which was explained to us by Beauville, generate all invariants whose degree is divisible by 48.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Invariant theory; Binary forms; Tschirnhaus transformations; del Pezzo surfaces; Pencils of quadrics

1. Introduction

Throughout this article, our base field will be $\mathbb{C}$. Let Cr denote the Cremona group of birational transformations of $\mathbb{P}^2$. To any element S in the moduli space of quartic del Pezzo surfaces, one can naturally associate an element G_S in the set of conjugacy classes of subgroups isomorphic to $(\mathbb{Z}/2)^4$ inside Cr. This construction was considered in the

recent work of Beauville [5]; among the results he proves therein, one finds the following statement [5, Proposition 4.2].

Proposition 1. *The map $S \to G_S$ is injective.*

In the mentioned article, this result was obtained by an elegant geometric argument, using an idea of Iskovskikh [17]. However, in an earlier version of the same work [4], the weaker statement of generic injectivity was obtained by a radically different approach, with a flavor of classical invariant theory. The purpose of the present article is to push this second approach to completion, and show that it leads to a strengthening of Proposition 1, which is Theorem 13 below. Our proof however is a *computer-assisted* one, since it relies on rather heavy calculations using the MAPLE software.

In this classical invariant theoretic setting, the quite pretty 'reconstruction problem' that needs to be solved is the following. Let $\Lambda = \{\lambda_1, \ldots, \lambda_5\}$ be a set of five distinct unordered points on $\mathbb{P}^1$, and consider the quintic

$$\mathcal{R} \stackrel{\text{def}}{=} \prod_{1 \leqslant l \leqslant 5} (z - j_l), \tag{1}$$

where j_l is the well-known j-invariant of the four-point subset

$$\{\lambda_1, \ldots, \hat{\lambda}_l, \ldots, \lambda_5\}.$$

Question 2. Does the quintic $\mathcal{R}$ uniquely determine the SL_2 orbit of Λ? In other words, can one reconstruct the projective configuration of a five-point set on the line from the projective configurations of its four-point subsets?

Remark 3. Most of the difficulty here stems from the lack of any ordering information, as well as from the possibility of deforming each of the five four-point pictures by a priori *unrelated* homographies.

Theorem 13 below gives an affirmative answer to this question, and also implies Proposition 1. Indeed, by considering a homogenized version of the quintic $\mathcal{R}$, one is naturally led to the construction of six invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$ of the binary quintic corresponding to the quintuple Λ, all of degree 24. Question 2 is then solved by reduction to the following one.

Question 4. Is there a strictly positive integer d_0, divisible by 24, such that for all multiples d of d_0, all invariants of the binary quintic which have degree d can be polynomially expressed in terms of the invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$?

Note that there are 7 linearly independent invariants in degree 24, therefore if such a d_0 exists it has to be no smaller than 48. Theorem 12 below shows that $d_0 = 48$ indeed does the job, providing a positive answer to Question 4. Trying to understand the intriguing

rather high degree at which this phenomenon first occurs was our primary motivation for the present work.

More precise statements of our results as well as the detailed explanation of the steps in our calculations will be given in Section 3, after the necessary material from the classical invariant theory of binary forms is recalled in Section 2. In Section 4, we will briefly relate our results with Beauville's. Finally, Section 5 will outline some suggestions for further work.

2. Preliminaries on the classical invariant theory of binary forms

2.1. Covariants, invariants and symmetric functions of root differences

The following material is classical. However, it is recalled here firstly for the convenience of the reader, and secondly in order to fix the numerical normalization of the invariants we will be considering.

A binary form of order p is a homogeneous polynomial

$$F(\mathbf{x}) = \sum_{i=0}^{p} a_i x_1^{p-i} x_2^i \tag{2}$$

of degree p in the variables $\mathbf{x} \stackrel{\text{def}}{=} (x_1, x_2)$. A matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

in GL_2 acting on the variables by

$$\mathbf{x} \to g\mathbf{x} = (g_{11}x_1 + g_{12}x_2, g_{21}x_1 + g_{22}x_2),$$

induces a transformation $F \to gF$ on the coefficients of the binary form F, by forcing the equality

$$(gF)(\mathbf{x}) \stackrel{\text{def}}{=} F(g^{-1}\mathbf{x}).$$

A covariant of F, of degree d, order r and weight ω, is a polynomial $C(F, \mathbf{x}) = C(a_0, \ldots, a_p; x_1, x_2)$, homogeneous of total degree d in $a_0, \ldots, a_p$, and homogeneous of total degree r in x_1, x_2, such that for any g in GL_2,

$$C(gF, g\mathbf{x}) = (\det g)^{-\omega} C(F, \mathbf{x}). \tag{3}$$

One has a simple relation between p, r, d, and ω:

$$dp = 2\omega + r. \tag{4}$$

An invariant $I = I(a_0, \ldots, a_p) = I(F)$ simply is a covariant of order zero. For two pairs of variables $b = (b_1, b_2)$ and $c = (c_1, c_2)$ which can be thought of as the homogeneous coordinates of two generic points in $\mathbb{P}^1$, following the elegant classical notation, we write

$$(bc) \stackrel{\text{def}}{=} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

In terms of its homogeneous roots $\xi_1, \ldots, \xi_p$, the form F can therefore be written as

$$F(\mathbf{x}) = (\mathbf{x}\xi_1) \cdots (\mathbf{x}\xi_p).$$

Now one has the following classical result (see [8, p. 97] or [21]).

Proposition 5.

(1) *Every symmetric polynomial in the pairs of variables $\xi_1, \ldots, \xi_p$ which is a linear combination of expressions of the form*

$$\prod_{\substack{1 \leqslant i, j \leqslant p \\ i \neq j}} (\xi_i \xi_j)^{k_{ij}} \times \prod_{1 \leqslant i \leqslant p} (\mathbf{x}\xi_i)^{l_i},$$

where the k's and the l's are nonnegative integers satisfying

$$\sum_{1 \leqslant i \leqslant p} l_i = r, \qquad \sum_{\substack{1 \leqslant i, j \leqslant p \\ i \neq j}} k_{ij} = \omega,$$

and

$$\forall i, \qquad \sum_{\substack{1 \leqslant j \leqslant p \\ j \neq i}} (k_{ij} + k_{ji}) + l_i = d,$$

is an (irrational) expression for a covariant of F, of degree d, order r and weight ω.
(2) *Conversely any covariant C of F can be so written.*

Note that the proposition has an obvious generalization to the case of simultaneous covariants of more than one form. For example, if one considers two binary forms

$$F(\mathbf{x}) = \sum_{i=0}^{p} a_i x_1^{p-i} x_2^i = (\mathbf{x}\xi_1) \cdots (\mathbf{x}\xi_p)$$

and

$$G(\mathbf{x}) = \sum_{i=0}^{q} b_i x_1^{q-i} x_2^i = (\mathbf{x}\eta_1) \cdots (\mathbf{x}\eta_q),$$

the resultant, which is a joint invariant of F and G, is

$$\text{Res}(F, G) \overset{\text{def}}{=} \prod_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant q}} (\xi_i \eta_j) \tag{5}$$

$$= \begin{vmatrix} a_0 & \dots & \dots & a_p & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_0 & \dots & \dots & a_p \\ b_0 & \dots & \dots & b_q & & 0 \\ & \ddots & & & \ddots & \\ 0 & & b_0 & \dots & \dots & b_q \end{vmatrix} \tag{6}$$

the usual Sylvester $(p + q) \times (p + q)$-determinant formula. Likewise the discriminant of a form F is by definition the invariant

$$\text{Disc}(F) \overset{\text{def}}{=} \prod_{1 \leqslant i < j \leqslant p} (\xi_i \xi_j)^2 = \frac{(-1)^{\frac{p(p-1)}{2}}}{p^{p-2}} \text{Res}\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right). \tag{7}$$

We now need to recall the classical notion of transvectant (or the "Übereinander-schiebung" of [9, §1]), which allows the formation of new covariants from old ones, and the formulation of quick yet *precise* definitions for those used in Section 3. If F is a binary form of order p and G a binary form of order q, the kth transvectant of F and G is

$$(F, G)_k \overset{\text{def}}{=} \frac{(p - k)!(q - k)!}{p!q!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k F}{\partial x_1^{k-i} \partial x_2^i} \frac{\partial^k G}{\partial x_1^i \partial x_2^{k-i}}. \tag{8}$$

2.2. Invariants of the binary quartic

The ring of invariants of a generic binary form F of order p as in (2) is denoted by $\mathbb{C}[a_0, \dots, a_p]^{SL_2}$ or simply $\mathbb{C}[F]^{SL_2}$. It is given the grading by the degree in the coefficients of F. The graded component of degree d is denoted by $\mathbb{C}[F]_d^{SL_2}$. For a binary quartic, more conveniently written

$$Q(\mathbf{x}) = q_0 x_1^4 + 4q_1 x_1^3 x_2 + 6q_2 x_1^2 x_2^2 + 4q_3 x_1 x_2^3 + q_4 x_2^4, \tag{9}$$

the ring of invariants has been know since the time of Boole and Cayley. Following [24, p. 189] it can be described as

$$\mathbb{C}[Q]^{SL_2} = \mathbb{C}[S, T], \tag{10}$$

where

$$S(Q) \overset{\text{def}}{=} \frac{1}{2}(Q, Q)_4 = q_0 q_4 - 4q_1 q_3 + 3q_2^2 \tag{11}$$

is of degree 2 and weight 4, and

$$T(Q) \stackrel{\text{def}}{=} \frac{1}{6}\big(Q, (Q, Q)_2\big)_4 = q_0 q_2 q_4 + 2q_1 q_2 q_3 - q_2^3 - q_0 q_3^2 - q_1^2 q_4 \qquad (12)$$

is of degree 3 and weight 6; besides, S and T are algebraically independent. One also has the weight 12 invariant

$$\text{Disc}(Q) = 2^8 \big(S(Q)^3 - 27T(Q)^2\big) \qquad (13)$$

as is readily checked on the canonical form written with obvious notation

$$Q(\mathbf{x}) = (\mathbf{x}0)(\mathbf{x}1)(\mathbf{x}\infty)(\mathbf{x}\lambda). \qquad (14)$$

The classical j-invariant of the four-point set in $\mathbb{P}^1$ corresponding to the roots of Q is

$$j(Q) \stackrel{\text{def}}{=} \frac{S(Q)^3}{S(Q)^3 - 27T(Q)^2} = \frac{4}{27} \times \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \qquad (15)$$

2.3. Invariants of the binary quintic

For a binary quintic F, the description of the ring of invariants was completed by Hermite [13] building on the previous work of Cayley and Sylvester. Again according to [24, pp. 227–234], one can describe it as

$$\mathbb{C}[F]^{SL_2} = \mathbb{C}[J, K, L, H]/\text{Relation}, \qquad (16)$$

where the invariants J, K, L, and H are respectively of degree 4, 8, 12, and 18; and there is a unique relation between them in degree 36 expressing H^2 in terms of J, K, and L. More precisely, one can make the following choices for the generators. First, define the covariants

$$C_1 \stackrel{\text{def}}{=} (F, F)_4, \qquad (17)$$

$$C_2 \stackrel{\text{def}}{=} (F, C_1)_2, \qquad (18)$$

$$C_3 \stackrel{\text{def}}{=} (C_2, C_2)_2, \qquad (19)$$

$$C_4 \stackrel{\text{def}}{=} (C_2, C_1)_2. \qquad (20)$$

Now the invariants are defined as

$$J \stackrel{\text{def}}{=} -\frac{1}{2}(C_1, C_1)_2, \qquad (21)$$

$$K \stackrel{\text{def}}{=} \frac{1}{8}(C_1, C_3)_2, \qquad (22)$$

$$L \stackrel{\text{def}}{=} \frac{1}{96} (C_3, C_3)_2, \tag{23}$$

$$H \stackrel{\text{def}}{=} -\frac{1}{384} \big((C_4, C_3)_1, (C_1, C_4)_1\big)_1. \tag{24}$$

Note that the full-fledged *Cartesian expressions* for these invariants as linear combinations of monomials in the coefficients of F are quite complicated. Indeed, J, K, L, H respectively have 12, 68, 228, and 848 terms. In order to calculate with invariants of the quintic F, we will sometimes find it convenient to use the Sylvester canonical form

$$F(\mathbf{x}) = ux_1^5 + vx_2^5 - w(x_1 + x_2)^5. \tag{25}$$

Indeed, every form F in the affine open set $\{L \neq 0\}$ can be written as the sum of the fifth powers of three nonproportional linear forms. By taking these points in the dual $\mathbb{P}^1$ to 0, 1, and ∞, one sees that such an F is in the SL_2 orbit of a form as in (25). The reason for this is that

$$L = -\frac{1}{2^4.3^5} \operatorname{Disc}\big(\operatorname{Can}(F)\big), \tag{26}$$

where the canonizant of F is $\operatorname{Can}(F) \stackrel{\text{def}}{=} -C_2$, or in classical symbolic notation [2, §2],

$$\operatorname{Can}(F) = (ab)^2(ac)^2(bc)^2 a_{\mathbf{x}} b_{\mathbf{x}} c_{\mathbf{x}}. \tag{27}$$

The above linear forms correspond to the distinct linear factors of $\operatorname{Can}(F)$ (see, e.g., [24, pp. 153–156] or [20]). The point of this discussion is that any identity in the ring $\mathbb{C}[F]^{SL_2}$ can be checked by specialization to this canonical form. The fundamental invariants will then be given by the remarkably simple expressions:

$$J = (uv + uw + vw)^2 - 4uvw(u + v + w), \tag{28}$$

$$K = u^2v^2w^2(uv + uw + vw), \tag{29}$$

$$L = u^4v^4w^4, \tag{30}$$

$$H = u^5v^5w^5(u - v)(u - w)(v - w). \tag{31}$$

Remark 6. The latter explain our choice of numerical normalization in (21)–(24). The explanation of the construction scheme we used based on the covariants $C_1, \dots, C_4$, is that it is the most straightforward way to build, as a 'Lego game,' the 'Feynman diagrammatic' expression of the four invariants (see [1, §6] and [18, p. 120]). The sums over 'Wick contractions' involved in each of the transvectant operations produce, up to symmetry, only one graph. Also note that (28)–(31) exactly agree with Salmon's conventions [24], except for the Hermite invariant H which differs in sign and notation. The invariants given by Gordan [9, §9] are different from the ones we used here.

The unique relation, which can easily be checked using (28)–(31), is

$$16H^2 = -432L^3 - 72L^2KJ + 8LK^3 - 2LK^2J^2 + L^2J^3 + K^4J. \tag{32}$$

The dimension of a graded component of degree d which is divisible by 4 can easily be calculated by solving for the nonnegative solutions of an elementary diophantine equation. Indeed, because of the relation (32) one simply has to count the monomials in the algebraically independent invariants J, K, L, with the given degree. In sum,

$$\dim\big(\mathbb{C}[F]_d^{SL_2}\big) = \nu(0) + \nu(1) + \cdots + \nu\left(\frac{d}{4}\right), \tag{33}$$

where

$$\nu(k) \stackrel{\text{def}}{=} \begin{cases} \lfloor \frac{k}{6} \rfloor, & \text{if } 6 | (k-1), \\ \lfloor \frac{k}{6} \rfloor + 1, & \text{otherwise.} \end{cases} \tag{34}$$

For d a multiple of 24, and letting $l = \frac{d}{24}$, this simplifies to

$$\dim\big(\mathbb{C}[F]_d^{SL_2}\big) = 3l^2 + 3l + 1. \tag{35}$$

3. The invariant theory computations

3.1. The basic construction

We now proceed to the definition of the homogeneous version $\bar{\mathcal{R}}$ of the quintic $\mathcal{R}$: a construction due to Beauville. In terms of the homogeneous roots $\lambda_1, \dots, \lambda_5$ in $\mathbb{P}^1$, write

$$F(\mathbf{x}) = (\mathbf{x}\lambda_1) \cdots (\mathbf{x}\lambda_5) \tag{36}$$

and define the five quartics $Q_1, \dots, Q_5$ by

$$Q_i(\mathbf{x}) = (\mathbf{x}\lambda_1) \cdots \widehat{(\mathbf{x}\lambda_i)} \cdots (\mathbf{x}\lambda_5). \tag{37}$$

Now introduce a new variable z and let

$$\bar{\mathcal{R}} \stackrel{\text{def}}{=} \prod_{i=1}^{5} \big((S(Q_i)^3 - 27T(Q_i)^2)z - S(Q_i)^3 \big) \tag{38}$$

$$= \sum_{i=0}^{5} \mathcal{B}_i(F) z^{5-i} \tag{39}$$

which defines the expressions $\mathcal{B}_0, \dots, \mathcal{B}_5$.

Lemma 7. $\mathcal{B}_0, \ldots, \mathcal{B}_5$ *are homogeneous polynomial invariants of F, of degree* 24.

Proof. This is a straightforward application of Proposition 5. To get the degree, one first calculates the weight by counting the bracket factors $(\lambda_i \lambda_j)$: $\omega = 5 \times 12 = 60$, and then uses (4) to obtain $d = \frac{2 \times 60}{5} = 24$. $\square$

3.2. The main results

The most crucial step in this article is the following exact determination of the invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$ in terms of J, K, L.

Proposition 8.

$$\mathcal{B}_0(F) = \frac{5^{15}}{2^{40}} \left\{ -2^{21}.K^3 + 2^{14}.3.K^2 J^2 - 2^7.3.K J^4 + J^6 \right\}, \tag{40}$$

$$\mathcal{B}_1(F) = \frac{5^{16}}{2^{35}.3^3} \left\{ 2^{16}.7.K^3 - 2^{10}.23.K^2 J^2 + 2^2.71.K J^4 - J^6 \right\}, \tag{41}$$

$$\mathcal{B}_2(F) = \frac{5^{16}}{2^{30}.3^6} \left\{ 2^{11}.5^3.LKJ - 2^4.5^3.LJ^3 - 2^{15}.3.K^3 + 2^7.11.13.K^2 J^2 \right. \\ \left. - 3.131.K J^4 + 2.J^6 \right\}, \tag{42}$$

$$\mathcal{B}_3(F) = \frac{5^{16}}{2^{25}.3^9} \left\{ -2^{11}.5^4.L^2 - 2^9.3.5^3.LKJ + 2.5^3.11.LJ^3 + 2^9.17.K^3 \right. \\ \left. - 2^2.23.37.K^2 J^2 + 3^5.K J^4 - 2.J^6 \right\}, \tag{43}$$

$$\mathcal{B}_4(F) = \frac{5^{16}}{2^{22}.3^{12}} \left\{ -2^5.3^2.5^3.LKJ - 5^3.29.LJ^3 - 2^7.11.K^3 - 7^2.83.K^2 J^2 \right. \\ \left. - 2^2.59.K J^4 + 2^2.J^6 \right\}, \tag{44}$$

$$\mathcal{B}_5(F) = \frac{5^{15}}{2^{15}.3^{15}} \left\{ 3^3.K^3 - 3^3.K^2 J^2 + 3^2.K J^4 - J^6 \right\}. \tag{45}$$

Remark 9. It is clear, by construction, that

$$\mathcal{B}_0(F) = \prod_{i=1}^{5} \left(2^{-8} \operatorname{Disc}(Q_i) \right) = 2^{-40}. \operatorname{Disc}(F)^3 \tag{46}$$

which can be compared, as a consistency check with (40) rewritten as

$$\mathcal{B}_0(F) = 2^{-40} \left[5^5 \left(J^2 - 128K \right) \right]^3. \tag{47}$$

Indeed, one can verify, with the help of MAPLE, that

$$\operatorname{Disc}(F) = 5^5 \left(J^2 - 128K \right). \tag{48}$$

Computer-assisted proof of the proposition. The argument relies on noticing that the construction of $\bar{\mathcal{R}}$ is a particular instance of a quartic *Tschirnhaus transformation* of a quintic equation. Since one already knows that $\mathcal{B}_0, \dots, \mathcal{B}_5$ are homogeneous polynomials of degree 24 in the coefficients of the quintic

$$F(\mathbf{x}) = a_0 x_1^5 + a_1 x_1^4 x_2 + a_2 x_1^3 x_2^2 + a_3 x_1^2 x_2^3 + a_4 x_1 x_2^4 + a_5 x_2^5; \tag{49}$$

one can safely dehomogenize by letting $a_0 = 1$. We also dehomogenize with respect to the variables x_1, x_2 by letting $x_1 = x$ and $x_2 = 1$. With a harmless abuse of notation, the quintic F becomes the monic polynomial

$$F(x) = x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = (x - \lambda_1) \cdots (x - \lambda_5). \tag{50}$$

Now the Q_i become

$$Q_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{5} (x - \lambda_i) = \frac{F(x)}{x - \lambda_i}. \tag{51}$$

In terms of a root λ (or rather a new variable which will *later* be specialized to such root), the corresponding quartic is given, after explicitly performing the Euclidean division, as in (9) by

$$Q_\lambda(x) = q_0 x^4 + 4q_1 x^3 + 6q_2 x^2 + 4q_3 x + q_4, \tag{52}$$

where

$$q_0 = 1, \tag{53}$$

$$q_1 = \frac{1}{4}(\lambda + a_1), \tag{54}$$

$$q_2 = \frac{1}{6}(\lambda^2 + a_1 \lambda + a_2), \tag{55}$$

$$q_3 = \frac{1}{4}(\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3), \tag{56}$$

$$q_4 = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4. \tag{57}$$

Using the expressions (11) and (12) for the quartic invariants S and T, one substitutes these values in

$$\phi(\lambda) \stackrel{\text{def}}{=} (S(Q_\lambda)^3 - 27T(Q_\lambda)^2)z - S(Q_\lambda)^3. \tag{58}$$

This is, a priori, a polynomial in λ of degree 12 (i.e., the weight of the isobaric expression $(S^3 - 27T^2)z - S^3$ in the q's). We now perform the Euclidean division of $\phi(\lambda)$ by

$$F(\lambda) = \lambda^5 + a_1 \lambda^4 + a_2 \lambda^3 + a_3 \lambda^2 + a_4 \lambda + a_5, \tag{59}$$

and call the remainder $\bar{\phi}(\lambda)$. Since the initially generic λ is going to be specialized to a root of F, one will have

$$\bar{\mathcal{R}} = \prod_{i=1}^{5} \phi(\lambda_i) = \prod_{i=1}^{5} \bar{\phi}(\lambda_i). \tag{60}$$

By the Poisson product formula this boils down to

$$\bar{\mathcal{R}} = \mathrm{Res}(F, \bar{\phi}) \tag{61}$$

the resultant of two polynomials in λ: $F(\lambda)$ of degree 5, and $\bar{\phi}(\lambda)$ of degree 4. This is calculated using the Sylvester determinant formula as in (6). One obtains $\bar{\mathcal{R}}$ as a polynomial in $a_1, \ldots, a_5$, and z. The invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$ are extracted as the coefficients of the powers of z. Now one rehomogenizes by performing the substitutions

$$(a_1, \ldots, a_5) \rightarrow \left(\frac{a_1}{a_0}, \ldots, \frac{a_5}{a_0} \right),$$

and multiplying by a_0^{24} to get the Cartesian expressions of $\mathcal{B}_0, \ldots, \mathcal{B}_5$ as homogeneous polynomials in the coefficients of the original binary quintic F. Finally one is reduced to a question of linear algebra, that of decomposing these invariants in terms of the basis of the degree 24 component of the ring $\mathbb{C}[F]^{SL_2}$ given by the following monomials in the algebraically independent invariants J, K, L:

$$L^2, \ LKJ, \ LJ^3, \ K^3, \ K^2J^2, \ KJ^4, \ J^6.$$

To make life easier for MAPLE we did so by first specializing to the canonical form (25), and then solving the linear system in $\mathbb{C}[u, v, w]$. The result of these computer calculations is the statement of the proposition. $\quad\square$

Remark 10. The computationally costly step in this derivation is the resultant calculation with specialized coefficients in terms of $a_1, \ldots, a_5$ and z. It took 6 minutes and 37 seconds on a 2×450 MHz SUN UltraSparc-II workstation running Version 9.5 of MAPLE.

Our next computational result is the following.

Proposition 11. *The* 21 *polynomials* $\mathcal{B}_i^2$, $0 \leqslant i \leqslant 5$, *and* $\mathcal{B}_i\mathcal{B}_j$, $0 \leqslant i < j \leqslant 5$, *linearly generate the component of degree* 48 *in the ring of invariants* $\mathbb{C}[F]^{SL_2}$.

Proof. A linear basis of this vector space is given by the 19 monomials in J, K, L of that degree. We simply calculated the 19×21 matrix of coefficients, in this basis, for the 21 given polynomials; and we checked, with the help of MAPLE, that the matrix has full rank. $\quad\square$

We can now state the main result of this article, which is the solution to Question 4.

Theorem 12. *For every integer $d > 0$ which is a multiple of $d_0 = 48$, all invariants of the quintic F, of degree d, can be written as polynomials in the invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$.*

Proof. Now that Proposition 11 has been established, all one needs to do is show that for any $d = 48k$, where $k \geqslant 1$ is an integer, every monomial $L^{\alpha_1} K^{\alpha_2} J^{\alpha_3}$ of degree $d = 12\alpha_1 + 8\alpha_2 + 4\alpha_3$ can be written as a product of monomials of degree 48. This is done by induction on k. For $k = 1$, this is a tautology. Noting that $3\alpha_1 + 2\alpha_2 + \alpha_3 = 12k$, let us perform the Euclidean division of α_1 by 4, α_2 by 6, and α_3 by 12:

$$
\begin{aligned}
\alpha_1 &= 4\beta_1 + \gamma_1, && 0 \leqslant \gamma_1 \leqslant 3; \\
\alpha_2 &= 6\beta_2 + \gamma_2, && 0 \leqslant \gamma_2 \leqslant 5; \\
\alpha_3 &= 12\beta_3 + \gamma_3, && 0 \leqslant \gamma_3 \leqslant 11.
\end{aligned}
\tag{62}
$$

Clearly, the degree of $L^{4\beta_1} K^{6\beta_2} J^{12\beta_3}$ is a multiple of 48, and therefore so is that of $L^{\gamma_1} K^{\gamma_2} J^{\gamma_3}$. If both triplets $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ are different from $(0, 0, 0)$, we are done by induction.

If $(\gamma_1, \gamma_2, \gamma_3) = (0, 0, 0)$, then

$$
L^{\alpha_1} K^{\alpha_2} J^{\alpha_3} = \left(L^4\right)^{\beta_1} \left(K^6\right)^{\beta_2} \left(J^{12}\right)^{\beta_3}
\tag{63}
$$

is of the required form.

If $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$, then by the inequalities (62),

$$
d = 12\gamma_1 + 8\gamma_2 + 4\gamma_3 \leqslant 120.
\tag{64}
$$

But $48 | d$ and the case $d = 48$ has been dealt with; so we are left with the case where $d = 96$. Since $0 \leqslant \gamma_1 \leqslant 4$, $0 \leqslant \gamma_2 \leqslant 6$ and $3\gamma_1 + 2\gamma_2 + \gamma_3 = 24$, one can write

$$
L^{\alpha_1} K^{\alpha_2} J^{\alpha_3} = L^{\gamma_1} K^{\gamma_2} J^{\gamma_3} = \left(L^{\gamma_1} J^{12-3\gamma_1}\right)\left(K^{\gamma_2} J^{12-2\gamma_2}\right)
\tag{65}
$$

which is the required decomposition. $\square$

4. The reconstruction problem and the relation to del Pezzo surfaces

The presentation here closely follows, notation included, that of Beauville [4,5].

4.1. The reconstruction problem

Let $V \stackrel{\text{def}}{=} (\mathbb{P}^1)^5 \setminus \Delta$ where Δ is the big diagonal. One has two commuting actions on V given by that of SL_2 and that of the symmetric group $\mathfrak{S}_5$. Let $J : V \to (\mathbb{P}^1)^5$ be the map

which to a quintuple $(\lambda_1, \ldots, \lambda_5)$ associates the quintuple $(j_1, \ldots, j_5)$ where j_l is the j-invariant, as in (15), of the four-point set $\{\lambda_1, \ldots, \hat{j}_l, \ldots, j_5\}$. The map is $\mathfrak{S}_5$-equivariant and factors through the quotient $P \overset{\text{def}}{=} V/SL_2$; i.e., one has a commutative diagram:

$$
\begin{array}{ccc}
P & \overset{J}{\longrightarrow} & \left(\mathbb{P}^1\right)^5 \\
\downarrow & & \downarrow \\
P/\mathfrak{S}_5 & \overset{\bar{J}}{\longrightarrow} & \operatorname{Sym}^5\left(\mathbb{P}^1\right).
\end{array}
\tag{66}
$$

The solution to Question 2 is the following.

Theorem 13. *The map $\bar{J}$ is injective.*

Proof. Consider two elements p_1 and p_2 of $P/\mathfrak{S}_5$ which map by $\bar{J}$ to the same element of $\operatorname{Sym}^5(\mathbb{P}^1)$. These correspond to two binary quintics F_1 and F_2, defined up to a multiplicative constant. By hypothesis, the corresponding quintics $\bar{\mathcal{R}}$ have the same roots, i.e.,

$$
\forall i, \ 0 \leqslant i \leqslant 5, \quad \frac{\mathcal{B}_i(F_1)}{\mathcal{B}_0(F_1)} = \frac{\mathcal{B}_i(F_2)}{\mathcal{B}_0(F_2)};
\tag{67}
$$

or what is the same

$$
\forall i, \ 0 \leqslant i \leqslant 5, \quad \frac{\mathcal{B}_i(F_1)}{\operatorname{Disc}(F_1)^3} = \frac{\mathcal{B}_i(F_2)}{\operatorname{Disc}(F_2)^3}.
\tag{68}
$$

Now we claim that every expression $J^\alpha K^\beta L^\gamma H^\delta . \operatorname{Disc}(F)^{-\epsilon}$ of degree 0 where α, β, γ, δ, and ϵ are nonnegative integers, takes the same value for F_1 and F_2. Indeed, one has

$$
4\alpha + 8\beta + 12\gamma + 18\delta - 8\epsilon = 0
\tag{69}
$$

therefore $4|18\delta$ so δ is even. Using the relation (32) one can get rid of the invariant H. Now

$$
J^\alpha K^\beta L^\gamma . \operatorname{Disc}(F)^{-\epsilon} = \left(\operatorname{Disc}(F)^\rho J^\alpha K^\beta L^\gamma\right) \operatorname{Disc}(F)^{-(\epsilon+\rho)},
\tag{70}
$$

where $\rho \overset{\text{def}}{=} 6\lceil \frac{\epsilon}{6} \rceil - \epsilon \geqslant 0$. Expressing $\operatorname{Disc}(F)$ on the left in terms of J, and K, one is reduced to the case of an expression $J^\alpha K^\beta L^\gamma . \operatorname{Disc}(F)^{-\epsilon}$ where the degree of $J^\alpha K^\beta L^\gamma$ is divisible by 48. The claim now is a consequence of Theorem 12 and (68).

Now following [23, Chapter 5], F_1 and F_2 can be seen as elements of the open affine set

$$
U_{1,5} \overset{\text{def}}{=} \left\{ \begin{array}{c} \text{binary quintics without} \\ \text{repeated linear factors} \end{array} \right\}
\tag{71}
$$

equipped with the natural GL_2 action. It is well known (see, e.g., [23, Corollary 5.24]) that $U_{1,5} \to U_{1,5}/GL_2$ is a good geometric quotient. The elements of the coordinate ring $\mathbb{C}[F, \mathrm{Disc}(F)^{-1}]^{GL_2}$ of the latter separate the GL_2 orbits. Using the description of $\mathbb{C}[F]^{SL_2}$ recalled in Section 2, these elements are finite linear combinations of expressions $J^\alpha K^\beta L^\gamma H^\delta . \mathrm{Disc}(F)^{-\epsilon}$ as above. Now the claim which we have just proved entails: F_1 and F_2 are in the same GL_2 orbit. Therefore the corresponding points p_1 and p_2 in $P/\mathfrak{S}_5$ are the same. $\quad\square$

4.2. The relation to del Pezzo surfaces and the Cremona group

We now come full-circle by explaining how Theorem 13 provides an alternative proof of Proposition 1. The discussion will be quite brief, since much more detail can be found in [4,5] for the specifics of the situation, and [3] as well as [12, Lecture 22] for the standard prerequisites on quartic del Pezzo surfaces. Such a surface S is usually seen as the blow up of $\mathbb{P}^2$ at five points in general position. The linear system of cubics through these five points embeds S as a complete intersection of two quadrics in $\mathbb{P}^4$. By choosing an appropriate coordinate system in the latter one can take these quadrics to be given by equations $Q_\infty = 0$ and $Q_0 = 0$ where

$$Q_\infty \overset{\text{def}}{=} \sum_{i=1}^{5} X_i^2 \quad \text{and} \quad Q_0 \overset{\text{def}}{=} \sum_{i=1}^{5} \lambda_i X_i^2. \tag{72}$$

There is a canonical subgroup of automorphisms of S, isomorphic to $(\mathbb{Z}/2)^4$, which is the one generated by the involutions σ_l mapping $(X_1 : \ldots : X_l : \ldots : X_5)$ to $(X_1 : \ldots : -X_l : \ldots : X_5)$. This descends, via the birational map from S to $\mathbb{P}^2$ corresponding to the blow up, to a subgroup G_S isomorphic to $(\mathbb{Z}/2)^4$ inside the Cremona group Cr, or rather to a conjugacy class of such. This is the construction given by Beauville for the map in Proposition 1.

Now note that the moduli space of (nonsingular) quartic del Pezzo surfaces S is the same as that of binary quintics without repeated linear factors, or more precisely the space we denoted earlier by $P/\mathfrak{S}_5$. This correspondence is given by the consideration of the pencil $Q_\infty \lambda - Q_0$ which is singular exactly when λ belongs to the five point set $\{\lambda_1, \ldots, \lambda_5\}$ (see [3] for a very thorough treatment). From the knowledge of the conjugacy class G_S one can recover the isomorphism class of the normalized fixed point locus, i.e., the normalization of the union of the nonrational curves in $\mathbb{P}^2$ which are fixed by an element of G_S. At the level of the surface S, this means that one can recover the data of the j-invariants of the five elliptic curves obtained as the intersection of S with each of the hyperplanes $X_l = 0$. This is the same as the unordered collection of the j_l's as in (1). As a result Theorem 13 implies Proposition 1.

Remark 14. We did not try to see if the nice geometric method used by Beauville in [5] could be refined in order to obtain Theorem 13, or Theorem 12 (at least with unspecified d_0). This might be an interesting point to elaborate upon in view of the generalization proposed in Section 5.1 below.

5. A shopping list

One of the 'raisons d'être' of experiment in natural sciences is to spur new theoretical investigations. Accordingly, we would be very happy to see the experimental mathematical result obtained in this article initiate some search for theoretical understanding, whatever modest. We can already see different questions arise from this work which might variously interest the communities of algebraic geometers, combinatorial/computational algebraists, and representation theorists. We will organize these questions accordingly.

5.1. Algebraic geometry

Very loosely speaking, our Theorem 13 can be recast with vast although probably not maximal generality as the following. Consider a projective variety X equipped with the action of a reductive group G. One can try to mimic the construction of Beauville's map $\bar{J}$ and analyze the injectivity of

$$\text{``}(\text{Sym}^p X)/G \to \left(\prod_I \big((\text{Sym}^{|I|} X)/G \big) \right) \Big/ \mathfrak{S}_p, \text{''} \tag{73}$$

where I ranges through the $\binom{p}{q}$ subsets of cardinality q in $\{1, \ldots, p\}$. One would have to do some work even in order to give a clean formulation of the question, in particular with regard to the analogue of the big diagonal Δ one needs to remove and related stability issues; this is why we put quotes. In particular, if only for esthetic reasons, one might want to investigate the case of SL_{n+1} acting on $\mathbb{P}^n$, or an invariant theoretic interrelation of Chow varieties of zero-cycles in $\mathbb{P}^n$ of different degrees. A special situation with binary forms, where the precise formulation of the problem is straightforward is the reconstruction problem for a binary p-ic from the j-invariants of its four-root subsets. It is somewhat the natural one-dimensional projective analogue of similar questions in distance geometry and rigidity theory (see, e.g., [6]) where one tries to determine a Euclidean configuration of points from mutual distances. Indeed, in the Euclidean situation *one* modulus corresponds to two-point subsets, whereas here it corresponds to four-point subsets.

5.2. Combinatorial/computational algebra

A natural problem, under this heading, is to reduce the computations which we have done (especially the ones in Proposition 8) to human proportions. This might well be needed in order to tackle the next open case of reconstructing the binary sextic from the quartics it contains. Indeed, one would have to identify 16 invariants of degree 60 one of which is the 6th power of the discriminant. When doing explicit calculations with invariants of binary forms, one essentially has the following tools and combinations thereof to choose from:

(1) Cartesian expressions,
(2) canonical forms,
(3) symmetric functions of root differences,
(4) the symbolic method.

In our opinion, the most interesting is the one we did not use in this article, i.e., the last one. One would need to *invariantively* rephrase the proof of Proposition 8, i.e., keeping the SL_2-equivariance explicit throughout. We believe the methods to do that are already available in the classical literature (see, e.g., [7,10,11,14]) waiting to be carefully studied anew by computational and combinatorial algebraists. Maybe a word of caution for those who would be willing to do so is in order. They will find, in addition to the common vicissitudes of research life, three practical obstacles specific to this task, and pertaining to

(1) physical accessibility of the literature,
(2) mathematical accessibility of its contents,
(3) language barrier.

Fortunately, removing the first obstacle is well under way thanks to the highly commendable efforts of the retrodigitalization projects throughout the world. The second one is no obstacle at all, if only psychological. Indeed, we explained in [1,2] a minimally acrobatic way of making rigorous mathematical sense out of the symbolic method as used by classical masters such as P. Gordan. The third problem is serious and requires a generous, volunteer-based translation effort following the example for instance set by Ackerman and Hermann [16], or Cox and Rojas [22]. Since one should follow one's own advice, let us announce a forthcoming translation into English by K. Hoechsmann, with commentary by the present author, of the classical masterpiece [9].

5.3. Representation theory

Modern practice in algebraic geometry does not encourage the writing of equations in coordinates. The successes obtained in conformity with this ideological choice can hardly be argued against; the resulting achievements are among the greatest of 20th century mathematics. However, it is sometimes *necessary* to calculate with coordinates, especially in view of the currently growing importance of computational/combinatorial algebra. It is therefore essential, when it is required, to try to do so wisely; and in this respect, there is much to be learned from the 19th century mathematicians. To continue on what we said in the previous subsection, one has to realize that from the mere use of Cartesian expressions one is automatically breaking SL-invariance and, perhaps unwittingly, doing toric geometry. More appropriate to calculations in the realm of projective geometry is the symbolic method which *explicitly* preserves the SL-equivariance. Concerning the latter, there are a few questions arising as to what is the representation theoretic interpretation of our Theorem 12.

Let $S_d(\cdot)$ denote the dth symmetric power of an SL_2 representation; if no argument is indicated it means that of the defining vector space $\mathbb{C}^2$ which is also identified with its dual. It is not hard to rephrase our Theorem 12 as the surjectivity of an SL_2-equivariant map

$$S_{2k}\big(S_5\big(\big[S_6(S_4)\big]^{SL_2}\big)\big) \to \big(S_{48k}(S_5)\big)^{SL_2}.$$

One can then ask if one could remove the restriction to SL_2-invariant subspaces. Indeed the construction of the invariants $\mathcal{B}_0, \ldots, \mathcal{B}_5$ is susceptible of many variations and twists.

For instance, one can do it not only for invariants but also for *covariants* since Proposition 5 works equally well for them. This ties in with one of the main themes of the article [1], as well as a rather mysterious 'devissage' property of classical invariants alluded to in [25, pp. 114–118]. As an exercise we leave to the reader, and as an illustration of the point we are making, one can do the following construction. Take the covariant $C(Q; \mathbf{x}) = (Q, (Q, Q)_2)_1$ of the quartic Q; and similarly to the description of $\bar{\mathcal{R}}$, consider the expression

$$\prod_{i=1}^{5} C(Q_i; \xi_i)$$

which reintroduces the missing root by specializing $\mathbf{x}$. Now one can check that this is a nonzero numerical multiple of the invariant H. This gives a somewhat less 'out-of-the-blue' derivation for the root-difference expression found by Hermite for his own invariant H (see [15] and, e.g., [19] for related recent work).

Acknowledgments

References

[1] A. Abdesselam, J. Chipalkatti, Brill–Gordan loci, transvectants and an analogue of the Foulkes conjecture, preprint, math.AG/0411110, 2004.
[2] A. Abdesselam, J. Chipalkatti, The bipartite Brill–Gordan locus and angular momentum, preprint, math.AG/0502542, 2005.
[3] D. Avritzer, R. Miranda, Stability of pencils of quadrics in $\mathbb{P}^4$, Bol. Soc. Mat. Mexicana (3) 5 (2) (1999) 281–300.
[4] A. Beauville, p-Elementary subgroups of the Cremona group, preprint, math.AG/0502123 v1, 2005.
[5] A. Beauville, p-Elementary subgroups of the Cremona group, preprint, http://math1.unice.fr/~beauvill/pubs/pp.pdf, 2005.
[6] C. Borcea, I. Streinu, The number of embeddings of minimally rigid graphs, Discrete Comput. Geom. 31 (2) (2004) 287–303.

[7] F. Brioschi, Sur la transformation des équations algébriques, C. R. Acad. Sci. 124 (1897) 661–665 (GA).
[8] E.B. Elliott, An Introduction to the Algebra of Quantics, second ed., Oxford Univ. Press, Oxford, 1913.
[9] P. Gordan, Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist, J. Reine Angew. Math. 69 (1868) 323–354 (GDZ).
[10] P. Gordan, Über die Invarianten binärer Formen bei höheren Transformationen, J. Reine Angew. Math. 71 (1870) 164–194 (GDZ).
[11] P. Gordan, Über die Bildung der Resultante zweier Gleichungen, Math. Ann. 3 (1871) 355–414 (GDZ).
[12] J. Harris, Algebraic Geometry. A First Course, Grad. Texts in Math., Springer-Verlag, New York, 1992.
[13] C. Hermite, Sur le théorie des fonctions homogènes à deux indéterminées, Cambridge and Dublin Math. J. 9 (1854) 172–217 (UM).
[14] C. Hermite, Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré, C. R. Acad. Sci. 46 (1856) 961–967 (GA, UM).
[15] C. Hermite, Sur l'invariant du 18$^{\text{e}}$ ordre des formes du cinquième degré et sur le role qu'il joue dans la résolution de l'équation du cinquième degré, extrait de deux lettres de M. Hermite à l'éditeur, J. Reine Angew. Math. 59 (1861) 304–305 (GDZ, UM).
[16] D. Hilbert, Hilbert's Invariant Theory Papers, translated from the German by Michael Ackerman, with comments by Robert Hermann, Lie Groups: History, Frontiers and Applications, vol. 8, Math. Sci. Press, Brookline, MA, 1978.
[17] V.A. Iskovskikh, Rational surfaces with a pencil of rational curves and with positive square of the canonical class, Math. USSR Sb. 12 (1970) 93–117.
[18] A.B. Kempe, On the application of Clifford's graphs to ordinary binary quantics, Proc. London Math. Soc. 17 (1885) 107–121.
[19] H. Kraft, A result of Hermite and equations of degree 5 and 6, J. Algebra 297 (2006) 236–253.
[20] J.P.S. Kung, Canonical forms of binary forms: Variations on a theme of Sylvester, in: Invariant Theory and Tableaux, Minneapolis, MN, 1988, in: IMA Vol. Math. Appl., vol. 19, Springer-Verlag, New York, 1990, pp. 46–58.
[21] J.P.S. Kung, G.-C. Rota, The invariant theory of binary forms, Bull. Amer. Math. Soc. 10 (1) (1984) 27–85.
[22] F. Minding, On the determination of the degree of an equation obtained by elimination, translated from the German and with a commentary by D. Cox and J.M. Rojas, in: Topics in Algebraic Geometry and Geometric Modeling, in: Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 351–362.
[23] S. Mukai, An Introduction to Invariants and Moduli, translated from the 1998 and 2000 Japanese editions by W.M. Oxbury, Cambridge Stud. Adv. Math., vol. 81, Cambridge Univ. Press, Cambridge, 2003.
[24] G. Salmon, Higher Algebra, fifth ed., 1885, reprinted by Chelsea, New York, 1964.
[25] J.J. Sylvester, On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, with three appendices, Amer. J. Math. 1 (1878) 64–125 (JS).