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Journal of Algebra 310 (2007) 880–902

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

The G-biliaison class of symmetric determinantal schemes

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Received 31 May 2005

Available online 28 December 2006

Communicated by Luchezar L. Avramov

Abstract

We consider a family of schemes, that are defined by minors of a homogeneous symmetric matrix with polynomial entries. We assume that they have maximal possible codimension, given the size of the matrix and of the minors that define them. We show that these schemes are G-bilinked to a linear variety of the same dimension. In particular, they can be obtained from a linear variety by a finite sequence of ascending G-biliaisons on some determinantal schemes. We describe the biliaisons explicitly in the proof of Theorem 2.3. In particular, it follows that these schemes are glicci.

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Keywords: G-biliaison; G-liaison; Glicci scheme; Arithmetically Cohen–Macaulay scheme; Arithmetically Gorenstein scheme; Minor; Symmetric matrix

Introduction

A main open question in liaison theory consists of deciding whether every arithmetically Cohen–Macaulay scheme is in the same G-liaison class of a complete intersection (glicci). The Theorem of Gaeta says that every arithmetically Cohen–Macaulay scheme of codimension 2 belongs to the CI-liaison class of a complete intersection. This result was generalized by Kleppe et al. [16], where they proved that every standard determinantal scheme is in

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the (even) G-liaison class of a complete intersection of the same codimension. Hartshorne strengthened their result in [14], proving that every standard determinantal scheme is in the G-biliaison class of a complete intersection. In [6], Casanellas and Miró-Roig proved that any arithmetically Cohen–Macaulay divisor on a rational normal scroll surface is glicci. In her PhD thesis [4], M. Casanellas generalized this result to arithmetically Cohen–Macaulay divisors on a rational normal scroll (see also [3] for a statement of the same result). It easily follows from the main theorem in the paper of Watanabe [21] that every arithmetically Gorenstein scheme of codimension 3 is licci (i.e. it belongs to the CI-liaison class of a complete intersection). Moreover, M. Casanellas, E. Drozd, and R. Hartshorne recently proved that every arithmetically Gorenstein scheme is glicci, regardless of its codimension (see [5, Theorem 7.1]). We are still far from being able to answer in full generality to the question of whether any arithmetically Cohen–Macaulay scheme is glicci. As far as we know, all the work in this direction deals with specific families of arithmetically Cohen–Macaulay schemes. In this note we consider a family of schemes whose saturated ideal is generated by minors of a fixed size of a symmetric matrix with polynomial entries. We prove that these schemes are G-bilinked to a complete intersection. In particular, they are glicci.

In the first section, we introduce the family of schemes that will be the object of our study. Their defining ideals are generated by the minors of a fixed size of a symmetric matrix with polynomial entries. We assume that the schemes have the highest possible codimension, for a fixed size of the matrix and of the minors that define them. We call a scheme of this kind *symmetric determinantal*. We observe that symmetric determinantal schemes do not exist for any given codimension, in fact they can only have codimension $\binom{b}{2}$, for some $b \geq 2$. Examples of symmetric determinantal schemes are complete intersections of admissible codimension, the Veronese surface in \mathbb{P}^5 , and some standard and good determinantal schemes (see Examples 1.6–1.8). In the first section we also introduce the concept of *almost-symmetric determinantal scheme*. Symmetric and almost-symmetric determinantal schemes are arithmetically Cohen–Macaulay. This follows from a result of Kutz [19]. In Theorem 1.18 we discuss when an almost-symmetric determinantal scheme is generically complete intersection, by giving an equivalent condition and a sufficient condition. In Theorem 1.21 we state the analogous result for symmetric determinantal schemes. In Theorem 1.22 we give an upper bound on the height of the ideal of minors of size $t \times t$ of a symmetric $m \times m$ matrix modulo the ideal of minors of size $t \times t$ of the same matrix that do not involve the last row.

Proposition 2.1 in Section 2 clarifies the connection between symmetric and almost-symmetric determinantal schemes. For each symmetric determinantal scheme X we produce an almost-symmetric determinantal scheme Y such that X is a generalized divisor on Y , Y is arithmetically Cohen–Macaulay and generically complete intersection. We also construct another symmetric determinantal scheme X' that is a generalized divisor on Y . Theorem 2.3 is the main result of this paper: any symmetric determinantal scheme belongs to the G-biliaison class of a linear variety. All the divisors involved in the G-biliaisons are symmetric determinantal, and the G-biliaisons are performed on almost-symmetric determinantal schemes.

We wish to emphasize the analogy from the point of view of liaison theory between the family of symmetric determinantal schemes and the family of standard determinantal schemes. See [18] for the definition of standard and good determinantal schemes. See also

[16] for some of their properties, mainly in relation with liaison theory. In [16], Kleppe et al. prove that standard determinantal schemes are glicci. Their argument is constructive, meaning that following the proof of Theorem 3.6 of [16] one can write down explicitly all the links. In [14], Hartshorne proves that standard determinantal schemes are in the same G-biliaison class of a linear variety. He shows that the G-bilinks constructed in [16] can indeed be regarded as elementary G-biliaisons. Our main theorem is analogous to the main theorem in [14], and the G-bilinks that one obtains following our proof are in the spirit of [16].

After submission of this paper, the author was informed by Kleppe and Miró-Roig that they independently proved that the ideal generated by the submaximal minors of a symmetric square matrix is glicci. They work under the assumption that the codimension of the ideal is 3 (hence maximal). The result is unpublished, and it is part of the work in progress [17]. Notice that the ideal of submaximal minors of a square matrix that is not symmetric is Gorenstein (hence glicci) whenever it has maximal codimension 4.

1. Symmetric and almost-symmetric determinantal schemes

Let X be a scheme in $\mathbb{P}^n = \mathbb{P}_K^n$, where K is an algebraically closed field. We assume that the characteristic of K is different from 2. Let I_X be the saturated homogeneous ideal corresponding to X in the polynomial ring $R = K[x_0, x_1, \dots, x_n]$. We denote by \mathfrak{m} the homogeneous irrelevant maximal ideal of R , $\mathfrak{m} = (x_0, x_1, \dots, x_n)$. For an ideal $I \subset R$, we denote by $H_*^0(I)$ the saturation of I with respect to the maximal ideal \mathfrak{m} .

Let $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of X . Let Y be a scheme that contains X . We denote by $\mathcal{I}_{X|Y}$ the ideal sheaf of X restricted to Y , i.e. the quotient sheaf $\mathcal{I}_X/\mathcal{I}_Y$. For $i \geq 0$, we let $H_*^i(\mathbb{P}^n, \mathcal{I}) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}(t))$ denote the i th cohomology module of the sheaf \mathcal{I} on \mathbb{P}^n . We write just $H_*^0(\mathcal{I})$, when the ambient space \mathbb{P}^n is clearly defined.

Notation 1.1. Let $I \subseteq R$ be a homogeneous ideal. We let $\mu(I)$ denote the cardinality of a set of minimal generators of I .

In this paper we deal with schemes whose saturated ideals are generated by minors of matrices with polynomial entries. We refer the reader to [18] for the definition of standard and good determinantal schemes.

Definition 1.2. Let M be a matrix of size $m \times m$ with entries in R . We say that M is t -homogeneous if the minors of M of size $s \times s$ are homogeneous polynomials for all $s \leq t$. We say that M is homogeneous if it is m -homogeneous.

We will always consider t -homogeneous matrices. Moreover, we will regard symmetric matrices up to invertible linear transformations that preserve their symmetry, and almost symmetric matrices up to invertible linear transformations that preserve the property of being almost-symmetric. We regard all matrices up to changes of coordinates. See Definition 1.9 for the definition of almost-symmetric matrix.

Definition 1.3. Let $X \subset \mathbb{P}^n$ be a scheme. We say that X is *symmetric determinantal* if:

- (1) there exists a symmetric t -homogeneous matrix M of size $m \times m$ with entries in R , such that the saturated ideal of X is generated by the minors of size $t \times t$ of M , $I_X = I_t(M)$, and
- (2) X has codimension $\binom{m-t+2}{2}$.

Remark 1.4. For any scheme X satisfying requirement (1) of Definition 1.3, we have

$$\text{codim}(X) \leq \binom{m-t+2}{2}.$$

See [15, Theorem 2.1] for a proof of this fact.

The remark shows that symmetric determinantal schemes have highest possible codimension among the schemes defined by minors of a symmetric matrix, for a given size of the matrix and of the minors. Notice also that by requiring that $I_t(M)$ is the defining ideal of a scheme X , we are requiring that the ideal of $t \times t$ minors of M is saturated.

It is worth emphasizing that symmetric determinantal schemes do not occur for every codimension. However, complete intersections are a special case of symmetric determinantal schemes for the codimensions for which symmetric determinantal schemes do exist.

Remark 1.5. Symmetric determinantal schemes do not exist for every codimension. In fact, we have symmetric determinantal schemes of codimension c if and only if c is of the form $\binom{b}{2}$, for some integer $b \geq 2$.

Complete intersections are an easy example of symmetric determinantal schemes, for each admissible codimension.

Example 1.6. Let X be a complete intersection of codimension $\binom{b}{2}$. Then

$$I_X = (F_{ij} \mid 1 \leq i \leq j \leq b-1) = I_1(M)$$

where $M = (F'_{ij})$, $F'_{ij} = F_{ij}$ if $i \leq j$ and $F'_{ij} = F_{ji}$ if $i \geq j$. The matrix M is symmetric of size $(b-1) \times (b-1)$. Its entries, i.e. its minors of size 1 define X , and the codimension of X is $\binom{b}{2}$. Hence X is symmetric determinantal.

The Veronese surface $V \subset \mathbb{P}^5$ is an example of a symmetric determinantal scheme that is not a complete intersection.

Example 1.7. Let $V \subset \mathbb{P}^5$ be the Veronese surface. The saturated ideal of V is minimally generated by the (distinct) minors of size two by two of a symmetric matrix of indeterminates of size three by three:

$$I_V = I_2 \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}.$$

The Veronese surface has codimension $3 = \binom{3-2+2}{2}$.

Proposition 2.1 in [19] provides an example of a symmetric determinantal scheme in \mathbb{P}^n for $n = \binom{m+1}{2}$ and for each $t \leq m$.

Example 1.8. For any fixed $m \geq 1$, and for any choice of t such that $1 \leq t \leq m$, let $n = \binom{m+1}{2}$. Let $X \subset \mathbb{P}^n$ be the symmetric determinantal scheme whose saturated ideal is generated by the minors of size $t \times t$ of the symmetric matrix of indeterminates of size $m \times m$:

$$I_X = I_t \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & & \vdots \\ x_{1,m} & x_{2,m} & \cdots & x_{m,m} \end{bmatrix}.$$

From [19, Proposition 2.1] we have that X has

$$\operatorname{codim}(X) = \operatorname{depth}(I_X) = \binom{m-t+2}{2},$$

therefore X is arithmetically Cohen–Macaulay and symmetric determinantal.

In his dissertation [7], A. Conca studied the ideals of Example 1.8, and in fact a larger family of ideals generated by minors of matrices of indeterminates. He computed the Gröbner basis of the ideals of Example 1.8 with respect to a diagonal monomial order. As a consequence, he was able to compute some of the invariants related to the Poincaré series of these rings. He also showed that the schemes corresponding to these ideals are reduced, irreducible and normal, and he characterized the arithmetically Gorenstein ones among them. See [7, Section 4] for more details.

Complete intersection schemes of codimension $\binom{b}{2}$ for some b are good determinantal schemes that are also symmetric determinantal (as observed in Example 1.6). Notice however that symmetric determinantal schemes are not a subfamily of standard or good determinantal schemes. For example, the Veronese surface in \mathbb{P}^5 is a symmetric determinantal scheme, but it is not standard determinantal (see [10, Proposition 6.7]). Moreover, in [10, Propositions 6.7 and 6.17] we provide a large class of examples of symmetric determinantal schemes that are not standard determinantal. They include the schemes of Example 1.8. Other examples are analyzed in [9].

Definition 1.9. Let O be a matrix of size $(m-1) \times m$. We say that O is *almost symmetric* if the submatrix of O consisting of the first $m-1$ columns is symmetric.

Definition 1.10. Let $Y \subset \mathbb{P}^n$ be a scheme. We say that Y is *almost-symmetric determinantal* if:

- (1) there exists an almost-symmetric t -homogeneous matrix O of size $(m-1) \times m$ with entries in R , such that the saturated ideal of Y is generated by the minors of size $t \times t$ of O , $I_Y = I_t(O)$.
- (2) Y has codimension $\binom{m-t+2}{2} - 1$.

Notice that in analogy with the case of symmetric determinantal schemes, the ideal of $t \times t$ minors of O is saturated since it is the ideal associated to a projective scheme. Moreover, we require almost-symmetric determinantal schemes to have highest possible codimension among the schemes defined by minors of an almost-symmetric matrix, for a given size of the matrix and of the minors.

Remark 1.11. For any scheme Y satisfying requirement (1) of Definition 1.10, we have

$$\operatorname{codim}(Y) \leqslant \binom{m-t+2}{2} - 1.$$

See, for example, the paper of Kutz [19]. A. Conca showed in his PhD dissertation that if O is an almost-symmetric matrix of indeterminates, then (see [7, Proposition 4.6.2])

$$\operatorname{codim}(Y) = \binom{m-t+2}{2} - 1.$$

The previous remark provides us with an example of almost-symmetric determinantal schemes.

Example 1.12. Let O be an almost-symmetric matrix of indeterminates of size $(m-1) \times m$. Let $n = \binom{m+1}{2}$. For any choice of $1 \leqslant t \leqslant m-1$, let $Y_t \subseteq \mathbb{P}^n$ be the scheme whose saturated ideal is generated by the minors of size $t \times t$ of M . Then it follows from [7, Proposition 4.6.2] that

$$\operatorname{codim}(Y_t) = \binom{m-t+2}{2} - 1.$$

Hence Y_t is an almost-symmetric determinantal scheme.

Similarly to the case of symmetric determinantal schemes, almost-symmetric determinantal schemes do not exist for any codimension. This is clear from part (2) of Definition 1.10. Notice also that complete intersections of codimension $\binom{b}{2} - 1$ for some $b \geqslant 3$ are almost-symmetric determinantal.

Example 1.13. Every complete intersection of codimension $\binom{b}{2} - 1$ for some $b \geqslant 3$ is an almost-symmetric determinantal scheme. In fact, let $Y \subset \mathbb{P}^n$ be a complete intersection of codimension $\binom{b}{2} - 1$. Notice that $\binom{b}{2} - 1 = \binom{b-1}{2} + b - 2$. Let

$$I_Y = (F_{i,j}, G_k \mid 1 \leqslant i \leqslant j \leqslant b-2, 1 \leqslant k \leqslant b-2)$$

be a minimal system of generators of the ideal of Y . Then I_Y is generated by the entries of the almost-symmetric matrix

$$O = \begin{bmatrix} F_{1,1} & F_{1,2} & \cdots & F_{1,b-2} & G_1 \\ F_{1,2} & F_{2,2} & \cdots & F_{2,b-2} & G_2 \\ \vdots & \vdots & & \vdots & \vdots \\ F_{1,b-2} & F_{2,b-2} & \cdots & F_{b-2,b-2} & G_{b-2} \end{bmatrix},$$

hence Y is almost-symmetric determinantal.

Remark 1.14. The family of almost-symmetric determinantal schemes does not coincide with the family of symmetric determinantal schemes, since by Remarks 1.4 and 1.11 it follows that in general they have different codimensions.

Almost-symmetric determinantal schemes are not a subfamily of standard or good determinantal schemes. The schemes of Example 1.12 are a family of almost-symmetric determinantal schemes that are not standard determinantal whenever $1 < t < m$.

Cohen–Macaulayness of symmetric and almost-symmetric determinantal schemes was proved by R. Kutz in [19, Theorem 1]. We state here a special case of this result, as we will need it in this section.

Theorem 1.15 (Kutz). *Symmetric and almost-symmetric determinantal schemes are arithmetically Cohen–Macaulay.*

We now establish some further properties of almost-symmetric determinantal schemes that will be needed in this paper. We use the notation of Definition 1.10.

We start by observing that a scheme defined by the $t \times t$ minors of a t -homogeneous matrix is a complete intersection only when it is generated by the entries of the matrix, or by its determinant (in the case of a square matrix). In [11], Goto has shown that the ideal of $t \times t$ minors of a symmetric matrix of indeterminates of size $m \times m$ is Gorenstein if and only if $t = 1$ or $m - t$ is even. However, here we need a result that applies to almost-symmetric matrices, and in general to matrices with polynomial entries.

Lemma 1.16. *Let M be a t -homogeneous symmetric matrix of size $m \times m$, or a t -homogeneous almost-symmetric matrix of size $(m - 1) \times m$, $1 \leq t \leq m - 1$. Let M have entries in R or in R_P for some prime P , and assume that M has no invertible entries. If $I_t(M)$ is a complete intersection, then $t = 1$.*

Proof. We know that the thesis is true for $M = (x_{ij})$ a generic symmetric or almost-symmetric matrix. In fact, in both cases a minimal system of generators of M is given by the minors of the $t \times t$ submatrices of M whose diagonal is on or above the diagonal of M . Comparing the number of such minors and the codimension of $I_t(M)$, we conclude that they agree if and only if $t = 1$.

Consider now the general case when $M = (F_{ij})$ has entries in R or R_P and $I_t(M)$ is a complete intersection. Let $I = (x_{ij} - F_{ij})_{ij} \subseteq R[x_{ij}]$ and let $S = R[x_{ij}]/I$. Let $N = (x_{ij})$

be a symmetric or almost-symmetric matrix of indeterminates of the same size as M . If we tensor a minimal free resolution of $I_t(N)$ by S , we obtain a free resolution for $I_t(M)$. This follows from [2, Theorem 3.5]. The resolution that we obtain for $I_t(M)$ is minimal, since we construct it from a minimal free resolution by substituting the indeterminates x_{ij} with F_{ij} . Therefore, there can be no invertible entry in any of the maps. Then $I_t(M)$ is a complete intersection if and only if $I_t(N)$ is a complete intersection, and the thesis follows. \square

Definition 1.17. Let $X \subset \mathbb{P}^n$ be a scheme. X is *generically complete intersection* if the localization $(I_X)_P$ is a complete intersection for every P minimal associated prime of I_X .

X is *generically Gorenstein*, abbreviated G_0 , if the localization $(I_X)_P$ is a Gorenstein ideal for every P minimal associated prime of I_X .

The next theorem will be used in the proof of Theorem 2.3.

Theorem 1.18. Let Y be an almost-symmetric determinantal scheme with defining matrix O , $I_Y = I_t(O)$. Let $c = \binom{m-t+2}{2} - 1$ be the codimension of Y . The following are equivalent:

- (1) Y is generically complete intersection.
- (2) $\text{ht } I_{t-1}(O) \geq c + 1$.

Let N be the symmetric matrix obtained from O by deleting the last column. The two equivalent conditions are verified if $\text{ht } I_{t-1}(N) = c + 1$.

Proof. (1) \Rightarrow (2). Since $I_t(O) \subseteq I_{t-1}(O)$, then $\text{ht } I_{t-1}(O) \geq c$. Therefore it suffices to show that $\text{ht } I_{t-1}(O) \neq c$. By contradiction, assume that there exists a minimal associated prime P of $I_{t-1}(O)$ of height c . Then P is also a minimal associated prime of $I_t(O)$. Let

$$\begin{aligned} \varphi: \mathbb{F} &\rightarrow \mathbb{G} \\ v &\mapsto Ov \end{aligned}$$

be the map induced by O . Here v is a column vector whose entries are polynomials, \mathbb{F} and \mathbb{G} are free R -modules of ranks m and $m - 1$, respectively. By [2, Proposition 16.3], we have that the map φ_P that we obtain from φ after localizing at the prime ideal P is an isomorphism on a direct summand R_P^s of \mathbb{F}_P , \mathbb{G}_P for some $s \leq t - 1$. We let s be maximal with this property. The localization O_P of O at P can be reduced after invertible row and column operations to the form

$$O_P = \begin{bmatrix} I_s & 0 \\ 0 & B \end{bmatrix},$$

where I_s is an identity matrix of size $s \times s$, 0 represents a matrix of zeroes, and B is a matrix of size $(m-s) \times (m-1-s)$ that has no invertible entries. By assumption, $I_t(O)_P \subseteq R_P$ is a complete intersection ideal. Since

$$I_t(O)_P = I_t(O_P) = I_{t-s}(B)$$

and B has no invertible entries, it follows by Lemma 1.16 that $t-s=1$, that is $s=t-1$. But then

$$I_{t-1}(O)_P = I_{t-1}(O_P) = R_P,$$

that contradicts the assumption that $P \supsetneq I_{t-1}(O)$.

(2) \Rightarrow (1). Let P be a minimal associated prime of $I_t(O)$. Then

$$\text{ht } P = c < c+1 \leq \text{ht } I_{t-1}(O)$$

so $P \not\supseteq I_{t-1}(O)$. Let

$$\begin{aligned} \varphi: \mathbb{F} &\rightarrow \mathbb{G} \\ v &\mapsto Ov \end{aligned}$$

be the map induced by O . Here v is a column vector whose entries are polynomials, \mathbb{F} and \mathbb{G} are free R -modules of ranks m and $m-1$, respectively. By [2, Proposition 16.3] we have that the map φ_P that we obtain from φ after localizing at the prime ideal P is an isomorphism on a direct summand R_P^{t-1} of $\mathbb{F}_P, \mathbb{G}_P$. Hence, the localization O_P of O at P can be reduced, after elementary row and column operations, to the form

$$O_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B \end{bmatrix},$$

where I_{t-1} is an identity matrix of size $(t-1) \times (t-1)$, 0 represents a matrix of zeroes, and B is a matrix of size $(m+1-t) \times (m-t)$. We claim that B is an almost-symmetric matrix. This is clear if the symmetric part of O_P contains an invertible minor of size $(t-1) \times (t-1)$. If instead we have an invertible $(t-1) \times (t-1)$ minor of O that contains the last column, we can write

$$O_P = \begin{bmatrix} I_{t-2} & 0 \\ 0 & B \end{bmatrix} \quad \text{where } B = \begin{bmatrix} b_{ij} & 1 \\ & 0 \\ & \vdots \\ & 0 \end{bmatrix}, \quad b_{ij} = b_{ji}, \quad 1 \leq i, j \leq m+1-t.$$

In fact, up to invertible row and column operations that preserve the almost-symmetry of B , we can assume that B has a symmetric block of maximal size and an invertible entry in the last column. Then

$$B = \begin{bmatrix} 0 \dots 0 & 1 \\ & 0 \\ & \vdots \\ b_{ij} & \\ & 0 \end{bmatrix}, \quad 2 \leq i \leq m+1-t, \quad 1 \leq j \leq m+1-t,$$

so it contains an almost-symmetric block of size $(m+1-t) \times (m-t)$. Summarizing, if we let $B' = (b_{ij})_{2 \leq i \leq m+1-t, 1 \leq j \leq m+1-t}$ we have that

$$O_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B' \end{bmatrix}.$$

This completes the proof of our claim.

The localization of $I_t(O)$ at the prime ideal P is

$$I_t(O)_P = I_t(O_P) = I_1(B)$$

thus it is generated by the entries of B . Since B is an almost-symmetric matrix, we have

$$\mu(I_t(O)_P) \leq \binom{m+1-t}{2} + (m-t) = \binom{m+2-t}{2} - 1 = c = \text{ht } I_t(O)_P.$$

Then $I_t(O)$ is locally generated by a regular sequence at all the minimal associated primes, i.e. Y is generically complete intersection.

Assume now that $\text{ht } I_{t-1}(N) = c+1$. Since $I_{t-1}(N) \subseteq I_{t-1}(O)$, then

$$c+1 = \text{ht } I_{t-1}(N) \leq \text{ht } I_{t-1}(O).$$

Then the two equivalent conditions hold. \square

Remark 1.19. The condition that $I_{t-1}(N) = c+1$ means that Y contains a symmetric determinantal subscheme X' of codimension 1, whose defining ideal is $I_{X'} = I_{t-1}(N)$. Notice that whenever this is the case, Y is generically complete intersection, hence it is G_0 . Under this assumption we have a concept of generalized divisor on Y (see [12–14] about generalized divisors). Then X' is a generalized divisor on Y . Theorem 1.18 proves that the existence of such a subscheme X' of codimension 1 guarantees that Y is locally a complete intersection. Notice the analogy with standard determinantal schemes: a standard determinantal scheme Y is good determinantal if and only if it is locally a complete intersection, if and only if it contains a standard determinantal subscheme of codimension 1, whose defining matrix is obtained by deleting a column from the defining matrix of Y .

The next example shows that in general the condition that $\text{ht } I_{t-1}(N) = c+1$ is stronger than the two equivalent conditions of Theorem 1.18.

Example 1.20. Let $R = k[x_0, x_1, x_2, x_3]$, and let

$$O = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \end{bmatrix}.$$

$I_2(O)$ is a Cohen–Macaulay ideal of height 2, $I_1(O) = (x_0, x_1, x_2, x_3)$ has height $4 \geq \text{ht } I_2(O) + 1$. Deleting the last column of O yields the matrix

$$N = \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}.$$

$I_1(N) = (x_0, x_1)$ is an ideal of height $2 < 2 + 1$. Notice that $I_1(N)$ does not change, even if we perform invertible row and column operations on O that preserve its almost symmetric structure, before deleting the last column.

Now assume that O is obtained from a homogeneous symmetric matrix M by deleting the last row. Let M be of the form

$$M = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_3 & x_2 \end{bmatrix}.$$

Notice that if we apply chosen invertible row and column operations that preserve the symmetry of M , we obtain a matrix

$$M = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2x^2 & x_1 + x_0 + ax_3 & x_2 + x_3 + ax_2 \\ x_1 + x_0 + ax_3 & x_0 & x_3 \\ x_2 + x_3 + ax_2 & x_3 & x_2 \end{bmatrix}$$

for any choice of $a \neq 0$. If we now delete the last row we obtain a new matrix O'

$$O' = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2x_2 & x_1 + x_0 + ax_3 & x_2 + x_3 + ax_2 \\ x_1 + x_0 + ax_3 & x_0 & x_3 \end{bmatrix}.$$

Deleting the last column of O' yields the matrix

$$N' = \begin{bmatrix} 2x_0 + 2x_1 + 2ax_2 + 2ax_3 + a^2x_2 & x_1 + x_0 + ax_3 \\ x_1 + x_0 + ax_3 & x_0 \end{bmatrix}$$

and $I_1(N') = (x_0, x_1 + ax_3, a(2x_2 + ax_2))$. Then $I_1(N')$ has height 3 for any $a \neq 0$.

Summarizing the example, $I_2(O)$ has maximal height, $\text{ht } I_2(O) < \text{ht } I_1(O)$, $I_1(M)$ has maximal height, and $I_1(N)$ does not have maximal height. However, after applying to M a general transformation that preserves its symmetry, we obtain O' and N' with the property that both $I_2(O')$ and $I_1(N')$ have maximal height.

The following is the analogous of Theorem 1.18 for symmetric determinantal schemes. We will not need it in the sequel, but we wish to emphasize that a result of this kind holds. The proof is very similar to that of the previous theorem, so we omit it.

Theorem 1.21. *Let X be a symmetric determinantal scheme with defining matrix M , $I_X = I_t(M)$. Let $c + 1 = \binom{m-t+2}{2}$ be the codimension of X . The following are equivalent:*

- (1) X is generically complete intersection.
- (2) $\text{ht } I_{t-1}(M) \geq c + 2$.

Let O be the almost-symmetric matrix obtained from M by deleting the last row. If $\text{ht } I_{t-1}(O) \geq c + 2$, then the two equivalent conditions are verified.

Finally, we prove a result in the lines of the Eisenbud–Evans generalized principal ideal theorem (see [8]) and of its generalization by Bruns (see [1]). The result is not new for an arbitrary matrix M , but the estimate on the height can be sharpened in the case of symmetric matrices. We essentially follow the proof of [1, Theorem 2].

Theorem 1.22. *Let M be a t -homogeneous symmetric matrix of size $m \times m$ with entries in R . Assume that M has no invertible entries. Let O be the matrix obtained from M by deleting the last row, after applying invertible generic row and column operations to M which preserve its symmetry and t -homogeneity. Then*

$$\text{ht } I_t(M)/I_t(O) \leq 1.$$

Proof. Let

$$M = \begin{bmatrix} F_{1,1} & \cdots & F_{1,m-1} & F_{1,m} \\ \vdots & & \vdots & \vdots \\ F_{1,m-1} & \cdots & F_{m-1,m-1} & F_{m-1,m} \\ F_{1,m} & \cdots & F_{m-1,m} & F_{m,m} \end{bmatrix}$$

and

$$O = \begin{bmatrix} F_{1,1} & \cdots & F_{1,m-1} & F_{1,m} \\ \vdots & & \vdots & \vdots \\ F_{1,m-1} & \cdots & F_{m-1,m-1} & F_{m-1,m} \end{bmatrix}.$$

Consider the matrix

$$L = \begin{bmatrix} F_{1,1} & \cdots & F_{1,m-1} & F_{1,m} & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ F_{1,m-1} & \cdots & F_{m-1,m-1} & F_{m-1,m} & 0 \\ F_{1,m} & \cdots & F_{m-1,m} & F_{m,m} & -1 \end{bmatrix}.$$

Notice that L and M are related in the same way as φ and φ' in the proof of [1, Theorem 2]. We regard the matrices over the ring $S = R/I_t(O)$. L defines a morphism $\psi: S^m \rightarrow S^{m+1}$ where the images of a basis of S^m are given by the rows of L . Similarly, M defines a morphism $\varphi: S^m \rightarrow S^m$. Let $\mathcal{M} := \text{Coker } \psi$ and $\mathcal{M}' := \text{Coker } \varphi$. Then $\mathcal{M}' \cong \mathcal{M}/e_{m+1}$,

where e_1, \dots, e_{m+1} denote the elements of the standard basis of S^{m+1} . For an $x \in \mathcal{M}$, we define

$$\mathcal{M}^*(x) := \{f(x) : f \in \text{Hom}_S(\mathcal{M}, S)\}.$$

As shown in [1, Theorem 2], we have that

$$I_t(M)/I_t(O) \subseteq \mathcal{M}^*(e_{m+1}).$$

Following the proof of [1, Theorem 2], one can show that $\text{ht } \mathcal{M}^*(e_{m+1}) \leq m - t + 1$. However, we claim that in the special case of a symmetric matrix M , one has the sharper bound $\text{ht } \mathcal{M}^*(e_{m+1}) \leq 1$.

Since $e_{m+1} = F_{1,m}e_1 + \dots + F_{m,m}e_m$, we have

$$\mathcal{M}^*(e_{m+1}) = \{f(F_{1,m}e_1 + \dots + F_{m,m}e_m) : f \in \text{Hom}_S(\mathcal{M}, S)\}.$$

For each $f \in \text{Hom}_S(\mathcal{M}, S)$, we have

$$f(F_{1,m}e_1 + \dots + F_{m,m}e_m) = F_{1,m}f(e_1) + \dots + F_{m,m}f(e_m).$$

Let $f_i := f(e_i) \in S$. Assume by contradiction that $\text{ht } \mathcal{M}^*(e_{m+1}) \geq 2$. Then we can find $f, g \in \text{Hom}_S(\mathcal{M}, S)$ such that $f(e_{m+1})$ and $g(e_{m+1})$ form a regular sequence in S . For all $i = 1, \dots, m-1$ we have

$$f\left(\sum_{j=1}^m F_{i,j}e_j\right) = \sum_{j=1}^m F_{i,j}f_j = 0$$

therefore

$$f_m F_{i,m} = - \sum_{j=1}^{m-1} F_{i,j} f_j.$$

Analogously,

$$g_m F_{i,m} = - \sum_{j=1}^{m-1} F_{i,j} g_j.$$

Hence

$$g_m f(e_{m+1}) = \sum_{i=1}^m F_{i,m} f_i g_m = f_m g_m F_{m,m} - \sum_{\substack{i=1, \dots, m-1 \\ j=1, \dots, m-1}} f_i g_j F_{i,j}$$

and since M is symmetric

$$g_m f(e_{m+1}) = f_m g(e_{m+1}).$$

By assumption $f(e_{m+1})$ and $g(e_{m+1})$ form a regular sequence in S , so $g_m = h g(e_{m+1})$ and $f_m = h f(e_{m+1})$ for some $h \in S$. Assume that $f_m, g_m \neq 0$. So $h \neq 0$ and

$$f_m = h f(e_{m+1}) = \sum_{i=1}^m h F_{i,m} f_i = \sum_{i \in I} h F_{i,m} f_i \quad (1)$$

where $I \subseteq \{1, \dots, m\}$ is the set of indexes of the summands that effectively contribute to the sum. In other words, if $f = (f_1, \dots, f_m)$ and $\sum_{i \notin I} F_{i,m} f_i = 0$ then $f(e_{m+1}) = \phi(e_{m+1})$ where $\phi = (\phi_1, \dots, \phi_m)$ with $\phi_i = f_i$ if $i \in I$ and $\phi_i = 0$ if $i \notin I$. So we can replace f with ϕ . This proves that we can assume without loss of generality that since $f_m \neq 0$, then $m \in I$. Hence the term $F_{m,m} f_m$ appears in the sum (1). Comparing degrees in (1), we get

$$\deg(f_m) \geq \deg(h) + \deg(F_{m,m}) + \deg(f_m) \geq \deg(F_{m,m}) + \deg(f_m),$$

hence $\deg(F_{m,m}) \leq 0$. By the assumption that M has no invertible entries, $F_{m,m} = 0$. But this is a contradiction: since we are allowing generic invertible row and column operations that preserve the symmetry of M we can always assume that $F_{m,m} \neq 0$ unless all the entries in the last row and column of M are zero. However, in that case $I_t(O) = I_t(M)$ and the thesis is trivially verified.

We still need to analyze the case when $f_m = g_m = 0$. We have

$$f \left(\sum_{j=1}^m F_{i,j} e_j \right) = \sum_{j=1}^{m-1} F_{i,j} f_j = 0,$$

therefore

$$f_{m-1} F_{i,m-1} = - \sum_{j=1}^{m-2} F_{i,j} f_j.$$

Analogously,

$$g_{m-1} F_{i,m-1} = - \sum_{j=1}^{m-2} F_{i,j} g_j.$$

Hence, proceeding as in the previous case,

$$f_{m-1} g(e_{m+1}) = g_{m-1} f(e_{m+1})$$

and either $f_{m-1}, g_{m-1} \neq 0$ or $f_{m-1} = g_{m-1} = 0$. In the first case, we can conclude as above, in the second case we obtain

$$f_{m-2}g(e_{m+1}) = g_{m-2}f(e_{m+1}).$$

We can keep iterating this reasoning until either $f_i, g_i \neq 0$ for some i , or $f = g = 0$. In both cases we get a contradiction. \square

2. Biliaison of symmetric determinantal schemes

In this section we prove that symmetric determinantal schemes are in the same G-biliaison class of a complete intersection of the same codimension. We start by proving that any symmetric determinantal scheme is a divisor on an almost-symmetric determinantal scheme. This result will be used in the proof of Theorem 2.3.

Proposition 2.1. *Let X be a symmetric determinantal scheme of codimension $c + 1$. Assume that the ideal of X is generated by the $t \times t$ minors of a t -homogeneous matrix M , $I_X = I_t(M)$. Let O be the matrix obtained from M by deleting the last row (after performing generic invertible row and column operations that preserve the symmetry of M). Let N be the matrix obtained from O by deleting the last column. Then:*

- N is a t -homogeneous symmetric matrix. It defines a symmetric determinantal scheme X' of codimension $c + 1$, with $I_{X'} = I_{t-1}(N)$.
- O is a t -homogeneous almost-symmetric matrix. It defines an almost-symmetric determinantal scheme Y of codimension c , with $I_Y = I_t(O)$. Y is an arithmetically Cohen–Macaulay, generically complete intersection scheme.

Proof. Let N be the matrix obtained from M by deleting a row and the corresponding column, after performing generic invertible row and column operations on M that preserve its symmetry. Then N is symmetric, and by Remark 1.4 $\text{ht } I_{t-1}(N) \leq \binom{m-1-(t-1)+2}{2} = c + 1$. Let P be a prime ideal of height $h = \text{ht } P \leq c$. We claim that $P \not\supseteq I_{t-1}(N)$. Clearly $P \not\supseteq I_t(M)$, since $\text{ht } I_t(M) = c + 1 > h$. Then the localization M_P of M at P has the form (up to a change of coordinates)

$$M_P = \begin{bmatrix} I_t & 0 \\ 0 & C \end{bmatrix},$$

where I_t is an identity matrix of size $t \times t$, 0 represents a matrix of zeroes, and C is a symmetric matrix of size $(m - t) \times (m - t)$. N is obtained from M by deleting a row and a column, possibly after a generic invertible linear transformation. Then the localization N_P of N at P has the form (up to a change of coordinates)

$$N_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & C \end{bmatrix} \quad \text{or} \quad N_P = \begin{bmatrix} I_t & 0 \\ 0 & C' \end{bmatrix}.$$

In fact, the morphism φ_P defined by M_P is an isomorphism on a free direct summand of rank at least t , hence the morphism φ'_P defined by N_P is an isomorphism on a free direct summand of rank at least $t - 1$. But then

$$I_{t-1}(N_P) = I_{t-1}(N)_P = R_P,$$

so $P \not\supseteq I_{t-1}(N)$. Since this holds for any prime P of R of height $h \leq c$, $\text{ht } I_{t-1}(N) = c + 1$. Then $I_{t-1}(N)$ defines an arithmetically Cohen–Macaulay, symmetric determinantal scheme X' of codimension $c + 1$, and $I_{t-1}(N) = I_{X'}$.

By Theorem 1.22,

$$\text{ht } I_t(M)/I_t(O) \leq 1,$$

then

$$\text{ht } I_t(O) \geq \text{ht } I_t(M) - 1 = c.$$

It follows that $\text{ht } I_t(O) = c$ and $I_t(O)$ is Cohen–Macaulay by Theorem 1.15. Therefore $I_Y = H_*^0(I_t(O)) = I_t(O)$, and the scheme Y is arithmetically Cohen–Macaulay of codimension c . Moreover,

$$\text{ht } I_{t-1}(O) \geq \text{ht } I_t(M) = c + 1 > c = \text{ht } I_t(O),$$

so Y is generically complete intersection by Theorem 1.18. \square

Remark 2.2. If X is a symmetric determinantal scheme of codimension $c + 1$ with defining matrix M , $I_X = I_t(M)$, we prove that:

- $\text{ht } I_{t-1}(O) \geq c + 1$, where O is almost-symmetric and obtained from M by deleting the last row.
- $\text{ht } I_{t-1}(N) = c + 1$, where N is symmetric and obtained from M by deleting the last row and column.

Notice that by Theorem 1.18 this implies that the scheme Y defined by $I_t(O)$ is generically complete intersection. However, it does not imply the same result for X , nor for the scheme X' defined by $I_{t-1}(N)$.

We are now ready to prove the main result of this paper.

Theorem 2.3. Any symmetric determinantal scheme in \mathbb{P}^n can be obtained from a linear variety by a finite sequence of ascending elementary G -biliaisons.

Proof. Let $X \subset \mathbb{P}^n$ be a symmetric determinantal scheme. We follow the notation of Definition 1.3. Let $M = (F_{ij})$ be the matrix whose minors of size $t \times t$ define X . From the definition $F_{ij} = F_{ji}$ for all i, j , and the matrix is t -homogeneous. Let $c + 1$ be the codimension of X , $c := \binom{m-t+2}{2} - 1$. If $t = 1$ or $t = m$, then X is a complete intersection,

therefore we can perform a finite sequence of descending elementary CI-biliaisons to a linear variety. Therefore, we concentrate on the case when $2 \leq t < m$.

Let O be the matrix obtained from M by deleting the last row, after performing generic invertible row and column operations that preserve the symmetry of M . O is a t -homogeneous matrix of size $(m-1) \times m$. Let Y be the scheme whose saturated ideal is generated by the $t \times t$ minors of O . By Proposition 2.1, Y is an arithmetically Cohen–Macaulay scheme of codimension c . Notice that Y is standard determinantal exactly when $t = m-1$. For our purpose, it is important to observe that Y is generically complete intersection. In particular, it satisfies the property G_0 . Therefore, a biliaison on Y is a G-biliaison, hence also an even G-liaison. This was proved in [16] for Y satisfying property G_1 and extended in [14] to Y satisfying property G_0 .

Let N be the matrix obtained from M by deleting the last row and column. N is a t -homogeneous symmetric matrix of size $(m-1) \times (m-1)$. Let X' be the scheme cut out by the $(t-1) \times (t-1)$ minors of N . Both X and X' are contained in Y . We denote by H a hyperplane section divisor on Y . We are going to show that

$$X \sim X' + aH \quad \text{for some } a > 0,$$

where \sim denotes linear equivalence of divisors. This will prove that X is obtained by an elementary biliaison from X' . Continuing in this manner, after $t-1$ biliaisons we reduce to the case $t=1$, when the scheme X is a complete intersection. Then we can perform descending CI-biliaisons to a linear variety.

Let $\mathcal{I}_{X|Y}, \mathcal{I}_{X'|Y}$ be the ideal sheafs on Y of X and X' . We then need to show that

$$\mathcal{I}_{X|Y} \cong \mathcal{I}_{X'|Y}(-a) \quad \text{for some } a > 0. \quad (2)$$

A system of generators of $I_{X|Y} = H_*^0(\mathcal{I}_{X|Y}) = I_t(M)/I_Y$ is given by the images in the coordinate ring of Y of the $t \times t$ minors of M

$$I_{X|Y} = (M_{i_1, \dots, i_t; j_1, \dots, j_t} \mid 1 \leq i_1 < i_2 < \dots < i_t \leq m, 1 \leq j_1 < j_2 < \dots < j_t \leq m).$$

Here $M_{i_1, \dots, i_t; j_1, \dots, j_t}$ denotes the image of the determinant of the submatrix of M consisting of rows i_1, \dots, i_t and columns j_1, \dots, j_t in the coordinate ring of Y . The saturated ideal of Y is minimally generated by the minors of size $t \times t$ of M that do not involve the last row. Notice that this minimal system of generators of I_Y can be completed to a minimal system of generators of I_X by adding all the minors of size $t \times t$ of M that involve both the last row and the last column. Therefore, a minimal system of generators of $I_{X|Y}$ is given by

$$I_{X|Y} = (M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \mid 1 \leq i_1 < \dots < i_{t-1} \leq m-1, 1 \leq j_1 < \dots < j_{t-1} \leq m-1).$$

A minimal system of generators of $I_{X'|Y} = H_*^0(\mathcal{I}_{X'|Y}) = I_{t-1}(N)/I_Y$ is given by the images in the coordinate ring of Y of the $(t-1) \times (t-1)$ minors of N

$$I_{X'|Y} = (M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}} \mid 1 \leq i_1 < \dots < i_{t-1} \leq m-1, 1 \leq j_1 < \dots < j_{t-1} \leq m-1).$$

Here $M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}}$ denotes the image of the determinant of the submatrix of M consisting of rows i_1, \dots, i_{t-1} and columns j_1, \dots, j_{t-1} in the coordinate ring of Y .

In order to prove the isomorphism (2), it suffices to check that the quotients

$$\frac{M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m}}{M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}}} \quad (3)$$

are all equal as elements of $H^0(\mathcal{K}_Y(a))$, where \mathcal{K}_Y is the sheaf of total quotient rings of Y . This also gives us an easy way to compute the value of a as the difference $\deg(M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m}) - \deg(M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}}) = \deg(F_{m, m})$.

Equality (3) is readily verified, once we show that

$$M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \cdot M_{k_1, \dots, k_{t-1}; l_1, \dots, l_{t-1}} - M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \cdot M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}} \in I_Y.$$

The proof is then completed by the following lemmas. \square

Since we could not find an adequate reference in the literature, we need to prove the following two lemmas about the minors of a matrix.

Lemma 2.4. *Let M be a matrix of size $m \times m$ and let $M_{i_1, \dots, i_a; j_1, \dots, j_a}$ denote the minor of the submatrix of M consisting of rows i_1, \dots, i_a and columns j_1, \dots, j_a . Let I be the ideal generated by the minors of M of size $(a+1) \times (a+1)$. Then*

$$M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_a; l_1, \dots, l_a} - M_{k_1, \dots, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \in I.$$

Proof. We start by proving the thesis when $i_b = k_b$ for $b = 1, \dots, a-1$. So we want to show that

$$M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_a} - M_{i_1, \dots, i_{a-1}, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \in I. \quad (4)$$

This is essentially an application of Sylvester's identity:

$$\begin{aligned} & M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; j_1, \dots, j_{a-1}, l_a} - M_{i_1, \dots, i_{a-1}, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; j_1, \dots, j_{a-1}, l_a} \\ &= M_{i_1, \dots, i_{a-1}; j_1, \dots, j_{a-1}} \cdot M_{i_1, \dots, i_a, k_a; j_1, \dots, j_a, l_a} \end{aligned}$$

For our purpose, we only need that the difference belongs to I . See [20, p. 33], for a general statement and a proof of Sylvester's identity.

$$\begin{aligned} & M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_a} - M_{i_1, \dots, i_{a-1}, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \\ &= M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_a} \\ &\quad - \sum_{b=1}^{a-1} M_{i_1, \dots, i_a; j_1, \dots, j_b, l_{b+1}, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_b, j_{b+1}, \dots, j_a} \end{aligned}$$

$$\begin{aligned}
& + \sum_{b=1}^{a-1} M_{i_1, \dots, i_a; j_1, \dots, j_b, l_{b+1}, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_b, j_{b+1}, \dots, j_a} \\
& - M_{i_1, \dots, i_{a-1}, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \\
& = \sum_{b=1}^a (M_{i_1, \dots, i_a; j_1, \dots, j_b, l_{b+1}, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_b, j_{b+1}, \dots, j_a} \\
& - M_{i_1, \dots, i_a; j_1, \dots, j_{b-1}, l_b, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_{b-1}, j_b, \dots, j_a}).
\end{aligned}$$

Each summand is of the form

$$\begin{aligned}
& M_{i_1, \dots, i_a; j_1, \dots, j_b, l_{b+1}, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_b, j_{b+1}, \dots, j_a} \\
& - M_{i_1, \dots, i_a; j_1, \dots, j_{b-1}, l_b, \dots, l_a} \cdot M_{i_1, \dots, i_{a-1}, k_a; l_1, \dots, l_{b-1}, j_b, \dots, j_a},
\end{aligned}$$

for $b = 1, \dots, a$. In particular, all the minors in the expression have all the rows and the columns in common, except possibly for one. So Sylvester's identity applies, and the thesis follows.

Let us now prove the thesis in full generality. We are going to use (4), and we will proceed in an analogous manner to the proof above. We want to show that

$$M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_a; l_1, \dots, l_a} - M_{k_1, \dots, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \in I.$$

Rewrite the difference as

$$\begin{aligned}
& M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_a; l_1, \dots, l_a} - M_{k_1, \dots, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \\
& = M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_a; l_1, \dots, l_a} \\
& - \sum_{b=1}^{a-1} M_{i_1, \dots, i_b, k_{b+1}, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_b, i_{b+1}, \dots, i_a; l_1, \dots, l_a} \\
& + \sum_{b=1}^{a-1} M_{i_1, \dots, i_b, k_{b+1}, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_b, i_{b+1}, \dots, i_a; l_1, \dots, l_a} \\
& - M_{k_1, \dots, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a} \\
& = \sum_{b=1}^a M_{i_1, \dots, i_b, k_{b+1}, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_b, i_{b+1}, \dots, i_a; l_1, \dots, l_a} \\
& - M_{i_1, \dots, i_{b-1}, k_b, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_{b-1}, i_b, \dots, i_a; l_1, \dots, l_a}.
\end{aligned}$$

Each summand is of the form

$$\begin{aligned}
& M_{i_1, \dots, i_b, k_{b+1}, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_b, i_{b+1}, \dots, i_a; l_1, \dots, l_a} \\
& - M_{i_1, \dots, i_{b-1}, k_b, \dots, k_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_{b-1}, i_b, \dots, i_a; l_1, \dots, l_a},
\end{aligned}$$

hence by (4) it belongs to I . This concludes the proof. \square

Remark 2.5. Notice that following the steps of the proof, one can write down explicitly $M_{i_1, \dots, i_a; j_1, \dots, j_a} \cdot M_{k_1, \dots, k_a; l_1, \dots, l_a} - M_{k_1, \dots, k_a; j_1, \dots, j_a} \cdot M_{i_1, \dots, i_a; l_1, \dots, l_a}$ as a combination of the minors of size $(a + 1) \times (a + 1)$ of the matrix M . This is not relevant for our purposes.

The following lemma concludes the proof of Theorem 2.3.

Lemma 2.6. Let M be a matrix of size $m \times m$ and let $M_{i_1, \dots, i_t; j_1, \dots, j_t}$ denote the minor of the submatrix of M consisting of rows i_1, \dots, i_t and columns j_1, \dots, j_t . Let I_Y be the ideal generated by the determinants of the submatrices of M of size $t \times t$ of M , that do not contain the last row. Then

$$M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \cdot M_{k_1, \dots, k_{t-1}; l_1, \dots, l_{t-1}} \\ - M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \cdot M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}} \in I_Y.$$

Proof. It is enough to prove that the statement holds for a matrix of indeterminates $M = (x_{ij})$, $1 \leq i \leq j \leq m$. From Lemma 2.4, we have that

$$M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \cdot M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \\ - M_{k_1, \dots, k_{t-1}, m; j_1, \dots, j_{t-1}, m} \cdot M_{i_1, \dots, i_{t-1}, m; l_1, \dots, l_{t-1}, m} \in I \subseteq I_Y. \quad (5)$$

Here I is the ideal generated by the minors of M of size $t + 1$. Clearly, $I \subseteq I_Y$.

Expanding the determinant $M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m}$ about column m we obtain

$$M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} = \sum_{h=1}^t (-1)^h x_{i_h, m} M_{i_1, \dots, i_{h-1}, i_{h+1}, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}}.$$

We adopt the convention that $i_t = j_t = k_t = l_t = m$.

Substituting this expression in Eq. (5), we get

$$M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \cdot M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \\ - M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \cdot M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \\ = \sum_{h=1}^t (-1)^h x_{i_h, m} M_{i_1, \dots, i_{h-1}, i_{h+1}, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}} M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \\ - \sum_{h=1}^t (-1)^h x_{k_h, m} M_{k_1, \dots, k_{h-1}, k_{h+1}, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}} M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \in I_S.$$

The coefficient of $x_{m, m}$ in (5) is then

$$(-1)^t M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}} M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \\ - (-1)^t M_{k_1, \dots, k_{t-1}; l_1, \dots, l_{t-1}} M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m}.$$

Since (5) is an equation of the form $\alpha x_{m,m} + \beta = 0 \pmod{I_Y}$, and since this equality has to hold for any $x_{m,m}$, we deduce that $\alpha = 0 \pmod{I_Y}$. Equivalently,

$$M_{i_1, \dots, i_{t-1}; j_1, \dots, j_{t-1}} M_{k_1, \dots, k_{t-1}, m; l_1, \dots, l_{t-1}, m} \\ - M_{k_1, \dots, k_{t-1}; l_1, \dots, l_{t-1}} M_{i_1, \dots, i_{t-1}, m; j_1, \dots, j_{t-1}, m} \in I_Y,$$

that is what we wanted to prove. \square

We want to emphasize a consequence of the proof of Theorem 2.3.

Corollary 2.7. *Every symmetric determinantal scheme X can be G -bilinked in $t - 1$ steps to a complete intersection, where t is the size of the minors defining X . In particular, every symmetric determinantal scheme is glicci.*

We end with an example that shows that the proof of Theorem 2.3 does not extend to a field K of characteristic 2. The matrix M below was brought to our attention by W. Bruns as an example of a matrix such that $\text{ht } I_2(M) = 3$ is maximal, but $I_3(M) = 0$ if the field K has characteristic 2.

Example 2.8. Let K be an algebraically closed field of characteristic 2, let $X \subseteq \mathbb{P}^3$ be the fat point with $I_X = (x, y, z)^2 \subseteq R = K[x, y, z, w]$. X is symmetric determinantal, since $I_X = I_2(M)$ where

$$M = \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$$

and $\text{ht } I_2(M) = 3 = \binom{3-2+2}{2}$. Any invertible row and column operations that preserve the symmetry and homogeneity of M yield a matrix M' of the form

$$M' = \begin{bmatrix} 0 & L_1 & L_2 \\ L_1 & 0 & L_3 \\ L_2 & L_3 & 0 \end{bmatrix}$$

where L_1, L_2, L_3 are linearly independent linear forms. So the matrix O has the form

$$O = \begin{bmatrix} 0 & L_1 & L_2 \\ L_1 & 0 & L_3 \end{bmatrix}$$

and $\text{ht } I_2(O) = 1$. Moreover, deleting the last column of O we obtain

$$N = \begin{bmatrix} 0 & L_1 \\ L_1 & 0 \end{bmatrix}$$

and $\text{ht } I_2(N) = 1$. Therefore, for X we cannot produce a G-biliaison as discussed in Theorem 2.3. Notice however that X is standard determinantal, corresponding to the matrix

$$\begin{bmatrix} x & y & z & 0 \\ 0 & x & y & z \end{bmatrix}.$$

Hence X can be obtained from the point $[0 : 0 : 0 : 1]$ via a G-biliaison.

Acknowledgments

I express my gratitude to Juan Migliore and Claudia Polini for their kind invitation and hospitality during my visit at the University of Notre Dame. I also thank them for useful comments and discussions about the material of this paper. I am grateful to Aldo Conca for useful comments on the first draft of this paper, and for finding a better proof of Lemma 1.16.

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