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Group graded algebras and almost polynomial growth

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ABSTRACT

Let F be a field of characteristic 0, G a finite abelian group and A a G-graded algebra. We prove that A generates a variety of G-graded algebras of almost polynomial growth if and only if A has the same graded identities as one of the following algebras:

(1) FC_p, the group algebra of a cyclic group of order p, where p is a prime number and $p \mid |G|$;
(2) $UT_2^G(F)$, the algebra of 2×2 upper triangular matrices over F endowed with an elementary G-grading;
(3) E, the infinite dimensional Grassmann algebra with trivial G-grading;
(4) in case $2 \mid |G|$, $E^{\mathbb{Z}_2}$, the Grassmann algebra with canonical $\mathbb{Z}_2$-grading.

1. Introduction

Let A be an associative algebra over a field F of characteristic zero graded by a finite abelian group G. Graded polynomials and graded polynomial identities are defined in a natural way and there is a copious literature on the subject (see for instance [3,5,14]). An effective way of measuring the graded polynomial identities satisfied by A is through the study of the asymptotic behavior of its sequence of G-codimensions $c_n^G(A)$, $n = 1, 2, \ldots$.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set, $F\langle X, G\rangle$ the free G-graded algebra on X and $\mathrm{Id}^G(A)$ the ideal of G-graded identities of A. One considers the relatively free G-graded algebra $F\langle X, G\rangle/\mathrm{Id}^G(A)$ and denotes by $c_n^G(A)$ the dimension of the subspace of multilinear elements in the homogeneous components of n free generators. The sequence $c_n^G(A)$, $n = 1, 2, \ldots$, is called the sequence of G-codimensions of A and one can compare $c_n^G(A)$ and $c_n(A)$, $n = 1, 2, \ldots$, the ordinary codimension

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sequence of A. It turns out that $c_n(A) \leqslant c_n^G(G)$, and in case A is a PI-algebra, i.e. satisfies an ordinary (non-graded) identity, $c_n^G(A) \leqslant |G|^n c_n(A)$ [11].

Throughout we shall assume that A is a PI-algebra. In this case, by a result of Regev [17], the sequence $c_n(A)$ is exponentially bounded. Hence by the above, also $c_n^G(A)$ is exponentially bounded. Moreover in [12] and [13] it was proved that if A is any PI-algebra, $\lim_{n\to\infty} \sqrt[n]{c_n(A)} = \exp(A)$ exists and is an integer called the PI-exponent of A. Recently in [2] and [8] the authors computed the exponential rate of growth of the sequence of G-graded codimensions for any G-graded PI-algebra A and it turns out to be still an integer.

From this result in particular it follows that no intermediate growth (between polynomial and exponential) is allowed. This fact resembles the ordinary (non-graded) case. In fact by a result of Kemer [15], (see also [14]), it was known that the sequence of codimensions of a PI-algebra is either polynomially bounded or grows exponentially. This phenomenon still holds for Lie PI-algebras [16] but fails in general for non-associative PI-algebras. In fact in [10] it was shown that for any real number $0 < \alpha < 1$ there exists an algebra A such that $c_n(A)$ grows as n^{n^α}.

At the light of the above results we shall search for G-graded PI-algebras, or varieties of G-graded algebras, of minimal exponential growth. Now, recall that if $\mathcal{V} = \mathrm{var}^G(A)$ is the variety generated by a G-graded algebra A, we write $\mathrm{Id}^G(\mathcal{V}) = \mathrm{Id}^G(A)$, $c_n^G(\mathcal{V}) = c_n^G(A)$, and the growth of $\mathcal{V}$ is the growth of the sequence $c_n^G(\mathcal{V})$, $n = 1, 2, \ldots$. Also we say that $\mathcal{V}$ has polynomial growth if $c_n^G(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^G(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

In this paper we classify the varieties of G-graded algebras having almost polynomial growth. We shall prove that such a variety is generated by one of the following algebras: 1) UT_2^G, the algebra of 2×2 upper triangular matrices over F endowed with an elementary G-grading induced by a pair of elements of G, 2) FC_p, the group algebra over F of a cyclic group of prime order p, $p \mid |G|$, endowed with the natural grading induced by $C_p = \langle g \rangle$, where $g \in G$, $o(g) = p$, 3) E, the infinite dimensional Grassmann algebra with trivial grading, 4) $E^{\mathbb{Z}2}$, the Grassmann algebra with its natural $\mathbb{Z}_2$-grading. This last case occurs only when $2 \mid |G|$.

As a consequence we shall get that if A is a G-graded PI-algebra, then $\mathcal{V} = \mathrm{var}^G(A)$ has polynomial growth if and only if E, UT_2^G, FC_p do not belong to $\mathcal{V}$, where UT_2^G is endowed with any G-grading and p runs over all primes dividing $|G|$. In case $2 \mid |G|$, one must add $E^{\mathbb{Z}_2}$, to the above list.

Our main tool in proving this result will be the explicit description of the G-exponent given in [2] and [8]. We shall also make use of a basic result recently proved independently by Aljadeff–Belov [1] and Sviridova [19] allowing us to reduce our problem to the study of the Grassmann envelope of a finite dimensional $G \times \mathbb{Z}_2$-graded algebra.

2. Graded identities and graded codimensions

Throughout the paper F will denote a field of characteristic zero, G a finite abelian group, $|G| = k$, and A a G-graded algebra over F. Since codimensions do not change upon extension of the base field, we shall assume throughout that F is algebraic closed. Let $G = \{g_1 = e, g_2, \ldots, g_k\}$ and let $A = \bigoplus_{i=1}^k A_{g_i}$, where $A_{g_i} A_{g_j} \subseteq A_{g_i g_j}$, $1 \leqslant i, j \leqslant k$, and the A_{g_i}'s are the homogeneous components of A.

Let $F\langle X, G\rangle$ be the free associative G-graded algebra on a countable set X over F. Write X as

$$X = \bigcup_{i=1}^{k} X_{g_i},$$

where $X_{g_i} = \{x_{1,g_i}, x_{2,g_i}, \ldots\}$ are disjoint sets, and the elements of X_{g_i} have homogeneous degree g_i. In general, given a monomial $x_{i_1,g_{j_1}} \cdots x_{i_t,g_{j_t}}$ its homogeneous degree is $g_{j_1} \cdots g_{j_t}$. If $\mathcal{F}_{g_i}$ is the subspace of the free algebra $F\langle X, G\rangle$ generated by all monomials in the variables of X having homogeneous degree g_i, then $F\langle X, G\rangle = \bigoplus_i \mathcal{F}_{g_i}$ is the natural G-grading of $F\langle X, G\rangle$.

Recall that an element $f = f(x_{1,g_1}, \dots, x_{t_1,g_1}, \dots, x_{1,g_k}, \dots, x_{t_k,g_k})$ of $F\langle X, G\rangle$ is called a graded polynomial. Also, f is a graded polynomial identity of A, and we write $f \equiv 0$, if f vanishes under all evaluations $x_{i,g_j} \to a_{g_j} \in A_{g_j}$, $1 \leqslant j \leqslant k$.

For $n \geqslant 1$ we define

$$P_n^G = \operatorname{span}_F\{x_{\sigma(1),g_{i_{\sigma(1)}}} \cdots x_{\sigma(n),g_{i_{\sigma(n)}}} \mid \sigma \in S_n,\ g_{i_1}, \dots, g_{i_n} \in G\},$$

the space of multilinear G-graded polynomials in the variables $x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}, g_{i_j} \in G$.

The graded identities of A form an ideal

$$\mathrm{Id}^G(A) = \{f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$$

which is invariant under all graded endomorphisms of $F\langle X, G\rangle$ (we say that $\mathrm{Id}^G(A)$ is a T_G-ideal). Since $\operatorname{char} F = 0$, $\mathrm{Id}^G(A)$ is determined by its multilinear polynomials i.e., by the sequence of subspaces $P_n^G \cap \mathrm{Id}^G(A)$, $n = 1, 2, \dots$. Then one defines

$$P_n^G(A) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)},$$

and

$$c_n^G(A) = \dim_F P_n^G(A), \quad n \geqslant 1,$$

is called the nth G-graded codimension of A.

We denote by P_n the space of multilinear polynomials in the variables $x_1, \dots, x_n$ and by $\mathrm{Id}(A) = \{f \in F\langle Y\rangle \mid f \equiv 0 \text{ on } A\}$ the T-ideal of ordinary polynomial identities of A. Then $c_n(A) = \dim_F P_n/(P_n \cap \mathrm{Id}(A))$ is the n-th ordinary codimension of A.

It is well known that for a general algebra A satisfying an ordinary polynomial identity, the sequence of codimensions is exponentially bounded [17]. The same conclusion about the G-graded codimensions can be drawn in case A satisfies an ordinary polynomial identity. In fact the following inequalities hold (see [11])

$$c_n(A) \leqslant c_n^G(A) \leqslant |G|^n c_n(A), \quad n \geqslant 1, \tag{1}$$

and the sequence of G-graded codimensions is exponentially bounded.

In [12] and [13] it was proved that if A is any algebra satisfying an ordinary polynomial identity, then the limit $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is an integer called the exponent of the algebra A. Recently in [2] and [8] an analogous results was proved for graded algebras satisfying an ordinary polynomial identity. In fact the authors proved that if A is a G-graded PI-algebra, then the limit $G\text{-}\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ exists and is an integer called the G-exponent of A.

We point out that these results were accomplished by using methods of representation theory of the group $G \wr S_n$, the wreath product of G and S_n.

Let $E = \langle e_1, e_2, \dots \mid e_i e_j = -e_j e_i\rangle$ be the infinite dimensional Grassmann algebra over F. E has a natural $\mathbb{Z}_2$-grading, $E = E_0 \oplus E_1$ where E_0 and E_1 are the subspaces of E spanned by the monomials in the $e_i's$ of even and odd length respectively.

A basic theorem that we shall need in the sequel is the following result proved independently by Aljadeff–Belov in [1] and Sviridova in [19]. We should point out that this theorem was proved in [1] for non-necessarily abelian groups. We first recall that if A is a $G \times \mathbb{Z}_2$-graded algebra, then the Grassmann envelope of A is

$$E(A) = \bigoplus_{g\in G} (E_0 \otimes A_{(g,0)} \oplus E_1 \otimes A_{(g,1)})$$

where $A = \bigoplus_{(g,i)\in G\times\mathbb{Z}_2} A_{(g,i)}$ is the decomposition of A into its homogeneous components.

Theorem 1. *(See [1,19].) Let A be a G-graded algebra satisfying an ordinary polynomial identity. Then there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that* $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$.

Next we recall how to compute the G-exponent of a PI G-graded algebra A. According to the above theorem there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$.

By the Wedderburn–Malcev theorem [7], we write $B = B' + J$ where B' is a maximal semisimple subalgebra of B and $J = J(B)$ is the Jacobson radical of B. It is well known that J is a graded ideal (see [6]), moreover we may assume that B' is a $G \times \mathbb{Z}_2$-graded subalgebra of B (see [18]). Hence we can write $B' = B_1 \oplus \cdots \oplus B_k$ where $B_1, \ldots, B_k$ are $G \times \mathbb{Z}_2$-graded simple algebras. Now, in [8] it is proved that

$$G\text{-}\exp(A) = G\text{-}\exp\bigl(E(B)\bigr) = \max \dim(C_1 \oplus \cdots \oplus C_h)$$

where $C_1, \ldots, C_h \in \{B_1, \ldots, B_k\}$ are distinct and $C_1 J C_2 J \cdots J C_h \neq 0$.

Another theorem that we shall need in the sequel is the following

Theorem 2. *(See [9].) Let $A = \bigoplus_{g \in G} A_g$ be an F-algebra graded by a finite group G. Then the sequence $c_n^G(A)$, $n = 1, 2, \ldots,$ is polynomially bounded if and only if for some positive integers m and r the algebra A satisfies the following graded identities:*

1) $[x_{1,1_G}, x_{2,1_G}] \cdots [x_{2m-1,1_G}, x_{2m,1_G}] \equiv 0$;

2)
$$x_{1,1_G} x_{3,1_G} \cdots x_{2r-1,1_G} x_{1,g} x_{2,1_G} x_{4,1_G} \cdots x_{2r,1_G} - \sum_{1 \neq \tau \in H} a_\tau x_{\tau(1),1_G} x_{\tau(3),1_G} \cdots x_{\tau(2r-1),1_G} x_{1,g} x_{\tau(2),1_G} x_{\tau(4),1_G} \cdots x_{\tau(2r),1_G} \equiv 0$$

for all $g \in G$, where H is the subgroup of S_{2r} generated by the transpositions $(12), (34), \ldots, (2r-1\ 2r)$, *and $a_\tau \in F$ depends on $g \in G$;*

3) *the ideal of A generated by $\bigoplus_{g \neq 1} A_g$ is nilpotent.*

3. The algebra FC_p

Let $C_p = \langle g \rangle$ be a cyclic group of order p and $A = FC_p$ the group algebra of C_p over F. Clearly A has a natural C_p-grading, $A = \bigoplus_{i=0}^{p-1} A_{g^i}$, where $A_{g^i} = Fg_i$, $0 \leqslant i \leqslant p-1$. It is clear that if $g \in G$, then A can be regarded as a G-graded algebra by setting $A_h = 0$ for all $h \notin \langle g \rangle$. We shall assume this point of view throughout this paper.

Next we study the C_p-graded algebra $A = FC_p$ and we prove that it generates a variety of G-graded algebras of almost polynomial growth, for any finite group G such that $p \mid |G|$.

Since FC_p is commutative, the graded identities of FC_p are easily computed and we have

Remark 3. $\mathrm{Id}^G(FC_p)$ is generated as a T_G-ideal by the polynomials $[x_{1,g^i}, x_{1,g^j}]$, $0 \leqslant i \leqslant j \leqslant p-1$.

Proposition 4. *Let G be an abelian group and p a prime such that $p \mid |G|$. Then* $\mathrm{var}^G(FC_p)$ *has almost polynomial growth and* $G\text{-}\exp(FC_p) = p$.

Proof. Since FC_p is commutative, it is easily checked that $c_n^G(FC_p) = p^n$. Hence $G\text{-}\exp(FC_p) = p$. This is also seen by [2], since FC_p is graded simple.

Let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^G(FC_p)$. We need to show that $c_n^G(\mathcal{U})$ is polynomially bounded. We shall do so by showing that if B is a generating algebra of $\mathcal{U}$, then B satisfies the conditions of Theorem 2.

Since $\mathrm{Id}^G(B) \nsubseteq \mathrm{Id}^G(FC_p)$ there exists a polynomial $f \in \mathrm{Id}^G(B)$, $f \notin \mathrm{Id}^G(FC_p)$. Moreover, by the standard multilinearization process we may assume that f is multilinear. Since f is not an identity of

FC_p, we can reduce f modulo $\mathrm{Id}^G(FC_p)$ and get a non-trivial identity of B that we shall still call f. Hence, recalling that the variables of homogeneous degree $h \in G \setminus C_p$ lie in $\mathrm{Id}^G(FC_p)$ and the variables of homogeneous degree g^i commute modulo the identities of FC_p, we may assume that f is of the form

$$f = x_{1,g^{i_1}} x_{2,g^{i_2}} \cdots x_{k,g^{i_k}}, \tag{2}$$

for some $1 \leqslant i_1, \ldots, i_k \leqslant p-1$.

Now fix t, $1 \leqslant t \leqslant p-1$. Since C_p is a cyclic group of order p, for all j, $1 \leqslant j \leqslant k$, we can write $g^{i_j} = g^{tm_j}$. Hence $x_{j,g^{i_j}}$ has the same homogeneous degree as a product of m_j variables of homogeneous degree g^t.

By applying a suitable endomorphism of the free G-graded algebra to (2), it follows that we can find $m \geqslant k$ such that

$$x_{1,g^t} x_{2,g^t} \cdots x_{m,g^t} \tag{3}$$

is a graded identity of B, for all t, $1 \leqslant t \leqslant p-1$.

If we now take any monomial of length $m(p-1)$ in variables of homogeneous degree different from 1, it is clear that it is a consequence of one of the polynomials in (3), modulo $\mathrm{Id}^G(FC_p)$, i.e., it is a graded identity of B. Also, if we insert variables of homogeneous degree 1 into any such monomial, we still get an identity of B. The outcome of the above discussion is that

$$\bigoplus_{\substack{h \in G \\ h \neq 1}} B_h$$

generates a nilpotent ideal of B. We now apply Theorem 2 to B and conclude that $\mathcal{U} = \mathrm{var}(B)$ has polynomial growth. $\square$

4. Classifying varieties of almost polynomial growth

Let UT_2 be the algebra of 2×2 upper triangular matrices over the field F and denote by UT_2^G the algebra UT_2 endowed with a G-grading. In [20] we classified the G-gradings on UT_2. It turns out that if G is any (non-necessarily abelian) group, then any G-grading on UT_2 is elementary and can be induced by a pair $(1, g)$, for some $g \in G$. Let e_{ij} be the usual matrix units. We recall that in this grading e_{11} and e_{22} have homogeneous degree 1 and e_{12} has homogeneous degree $g \in G$.

In [20] we computed the $\mathbb{Z}_2$-cocharacter of $UT_2^{\mathbb{Z}_2}$ and showed that $UT_2^{\mathbb{Z}_2}$ generates a variety of $\mathbb{Z}_2$-graded algebras of almost polynomial growth. We point out that, by looking at the decomposition into homogeneous spaces, it is readily seen that [20] actually shows the following:

Proposition 5. *If G is any group, UT_2^G generates a variety of G-graded algebras of almost polynomial growth.*

Recall that the infinite dimensional Grassmann algebra E over F has a natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$. Let us denote by E, the Grassmann algebra with trivial grading and by $E^{\mathbb{Z}_2}$ the algebra E with the above $\mathbb{Z}_2$-grading; notice that if $2 \mid |G|$, we can regard $E^{\mathbb{Z}_2}$ with induced G-grading.

Now by [15], E generate a variety of algebras of almost polynomial growth and, by [21], $E^{\mathbb{Z}_2}$ generates a variety of $\mathbb{Z}_2$-graded of almost polynomial growth. Reading this result in terms of G-gradings we get

Proposition 6. *E generates a variety of G-graded algebras of almost polynomial growth. Moreover if $2 \mid |G|$, also $E^{\mathbb{Z}_2}$ generates such a variety of G-graded algebras.*

In what follows we shall need the description of finite dimensional G-graded simple algebras given in [4]. It should be mentioned that this result holds for arbitrary groups (and not only for finite abelian groups). We have

Theorem 7. *(See [4].) Let A be a G-graded simple algebra. Then there exists a subgroup H of G, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer m and an m-tuple $(g_1 = 1, g_2, \dots, g_m) \in G^m$ such that A is G-graded isomorphic to $R = F^\alpha H \otimes M_m(F)$ where $R_g = \operatorname{span}_F\{b_h \otimes e_{ij} \mid g = g_i^{-1} h g_j\}$. Here $b_h \in F^\alpha H$ is a representative of $h \in H$.*

Next we prove the following technical lemma.

Lemma 8. *Let A be a G-graded algebra and $t \geqslant 1$. If every monomial of P_n^G containing at least t variables of homogeneous degree different from 1 is an identity of A, then $c_n^G(A) \leqslant \beta n^{t-1} c_n(A)$, for some constant β.*

Proof. Let $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)$ be a basis of $P_n \pmod{\operatorname{Id}(A)}$. Then, for any $\sigma \in S_n$,

$$x_{\sigma(1)} \cdots x_{\sigma(n)} = \sum_i \alpha_{\sigma,i} f_i \quad \bigl(\operatorname{mod} \operatorname{Id}(A)\bigr).$$

Now every variables x_i can be written as $x_i = x_{i,g_1} + \cdots + x_{i,g_k}$ where $G = \{g_1 = 1, \dots, g_k\}$. This says, by eventually putting equal to zero some of the x_{i,g_j}, that for any fixed n-tuple $(h_1, \dots, h_n) \in G^n$,

$$x_{\sigma(1),h_{\sigma(1)}} \cdots x_{\sigma(n),h_{\sigma(n)}} = \sum_i \alpha_{\sigma,i} f_i(x_{1,h_1}, \dots, x_{n,h_n}) \quad \bigl(\operatorname{mod} \operatorname{Id}(A)\bigr).$$

Since $\operatorname{Id}(A) \subseteq \operatorname{Id}^G(A)$ we get that

$$\bigl\{ f_i(x_{1,h_1}, \dots, x_{n,h_n}) \bigm| (h_1, \dots, h_n) \in G^n,\ 1 \leqslant i \leqslant m \bigr\}$$

spans $P_n^G \pmod{\operatorname{Id}^G(A)}$.

Moreover by the hypothesis we have only to consider n-tuples $(h_1, \dots, h_n)$ such that $h_i = 1$ except for at most $t-1$ elements. By counting we get $c_n^G(A) \leqslant \beta n^{t-1} c_n(A)$, for some constant β. $\square$

Now we are able to prove the following

Theorem 9. *Let G be a finite abelian group and A a G-graded algebra satisfying an ordinary polynomial identity. Then $c_n^G(A) \leqslant an^t$, for some constants a and t, if and only if either $|G|$ is odd and $UT_2^G, FC_p, E \notin \operatorname{var}^G(A)$, for all primes p, such that $p \mid |G|$ and for all G-gradings on UT_2 or $|G|$ is even and $UT_2^G, FC_p, E, E^{\mathbb{Z}_2} \notin \operatorname{var}^G(A)$.*

Proof. Since the algebras $UT_2^G, FC_p, E, E^{\mathbb{Z}_2}$ have exponential growth of the G-codimensions, it is clear that if $\operatorname{var}^G(A)$ has polynomial growth, $UT_2^G, FC_p, E \notin \operatorname{var}^G(A)$ (we add $E^{\mathbb{Z}_2}$ if $2 \mid |G|$).

Conversely, suppose that $UT_2^G, FC_p, E \notin \operatorname{var}^G(A)$, and in case $2 \mid |G|$, also $E^{\mathbb{Z}_2} \notin \operatorname{var}^G(A)$. By Theorem 1, there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $\operatorname{var}^G(A) = \operatorname{var}^G(E(B))$. We shall study the algebra $E(B)$. Write $B = B' + J$ where $J = J(B)$ is the Jacobson radical of B and B' is a maximal semisimple $G \times \mathbb{Z}_2$-graded subalgebra.

Let $B' = B_1 \oplus \cdots \oplus B_t$ be the decomposition of B' into its G-graded simple components. Then, according to Theorem 7, we have that, for $1 \leqslant i \leqslant t$, $B_i \cong M_{n_i}(F^{\alpha_i} H_i)$, where H_i is a subgroup of $G \times \mathbb{Z}_2$ and α_i is a corresponding 2-cocycle.

Suppose first that for some i, we have that $B_i \cong M_n(F^\alpha H)$, and $n > 1$. Then $C = Fe_{11} \oplus Fe_{12} \oplus Fe_{22}$ is a $G \times \mathbb{Z}_2$-graded subalgebra of B_i. Here the e_{ij}'s are the usual matrix units, e_{11} and e_{22} have homogeneous degree $(e, 0)$, where $e = 1_G$, and e_{12} has homogeneous degree $(g, 0)$ or $(g, 1)$, for some $g \in G$.

But then, either

$$E(C) = \begin{pmatrix} E_0 & E_0 \\ 0 & E_0 \end{pmatrix} \quad \text{or} \quad E(C) = \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}.$$

It is easily checked that in either case $\mathrm{Id}^G(E(C)) = \mathrm{Id}^G(UT_2^G)$ where UT_2^G has elementary grading induced by the pair (e, g). Thus $E(C) \subseteq E(B_i) \in \mathrm{var}^G(E(B))$, and this contradicts the fact that $UT_2^G \notin \mathrm{var}^G(E(B))$. Thus $B' = F^{\alpha_1}H \oplus \cdots \oplus F^{\alpha_t}H$.

Let $F^{\alpha}H$ be one of the $G \times \mathbb{Z}_2$-graded simple components. Suppose first that $(g, 0) \in H$, for some $g \in G$ of order a prime p. Then being $C_p = \langle (g, 0)\rangle$ cyclic, we may assume that α is trivial on C_p. Thus $E(FC_p) = E_0 \otimes FC_p$ has the same G-identities of FC_p. It follows that $FC_p \in \mathrm{var}(E(B))$, contrary to the assumption.

Suppose now that $(g, 1) \in H$ with $g \in G$ of order a prime p. It is clear that we get into the previous case unless $p = 2$ and, so, G is of even order. In this last case we have

$$E(FH) \supseteq E_0 \otimes F(e, 0) \oplus E_1 \otimes F(g, 1) \cong E_0 \oplus E_1 = E^{\mathbb{Z}_2} \in \mathrm{var}^G\big(E(B)\big),$$

a contradiction.

The only cases left are $H = \{e\} \times \mathbb{Z}_2$ or H trivial. In the first case $E(FH) \cong E$ with trivial grading and $E \in \mathrm{var}^G(E(B))$ is not allowed. Thus H is the trivial subgroup and $B' = F \oplus \cdots \oplus F$ is a direct sum of copies of F with trivial grading.

By taking the Grassmann envelope we get $E(B) = E(B') + E(J)$ and since $E(B') \subseteq E(B)_e$, we obtain that $\bigoplus_{g\in G,\, g\neq 1} E(B)_g \subseteq E(J)$.

Now, the algebra B is finite dimensional, hence $J = J(B)$ is a nilpotent ideal. It follows that $\bigoplus_{g\in G,\, g\neq 1} E(B)_g$ generates a nilpotent ideal of $E(B)$, of index of nilpotence, say, t. This implies that every monomial of P_n^G containing at least t variables of homogeneous degree different from 1 is an identity of $E(B)$. Then, by Lemma 8, $c_n^G(E(B)) \leqslant \beta n^{t-1} c_n(E(B))$, for some constant β.

Now, let $\mathrm{var}(E(B))$ be the ordinary variety of algebras generated by $E(B)$, i.e., we consider only algebras with trivial grading. Then if UT_2 is the algebra UT_2^G with trivial grading, by hypothesis, $E, UT_2 \notin \mathrm{var}(E(B))$. But then, by a result of Kemer [15], $\mathrm{var}(E(B))$ has polynomial growth. It follows that $c_n(E(B)) \leqslant \gamma n^q$, for some γ, q, and, so $c_n^G(A) = c_n^G(E(B)) \leqslant \beta n^{t-1} c_n(E(B)) \leqslant \beta\gamma n^{t-1} n^q$ and the proof is complete. $\square$

We immediately get the following consequence.

Corollary 10. *Let $\mathcal{V}$ be a variety of G-graded algebras. Then $\mathcal{V}$ has almost polynomial growth if and only if $\mathcal{V} = \mathrm{var}^G(A)$ where either $A \cong UT_2^G$, for some G grading on UT_2, or $A \cong FC_p$, for some prime p such that $p \mid |G|$, or $A \cong E$, the Grassmann algebra with trivial grading or, in case $|G|$ is even, $A \cong E^{\mathbb{Z}_2}$.*

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