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On semiabelian p-groups

Danny Neftin

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

The family of semiabelian p-groups is the minimal family that contains $\{1\}$ and is closed under quotients and semidirect products with finite abelian p-groups. Kisilevsky and Sonn have solved the minimal ramification problem for a certain subfamily $\mathcal{G}_p$ of the family of semiabelian p-groups. We show that $\mathcal{G}_p$ is in fact the entire family of semiabelian p-groups and by this complete their solution to all semiabelian p-groups.

1. Introduction

This paper is motivated by the minimal ramification problem for p-groups. Given a p-group G it is an open problem to find the minimal number of primes ramified in a G-extension of $\mathbb{Q}$ (see [8]). As a consequence of Minkowski's Theorem this number is greater or equal to $d(G)$, the minimal number of generators of G. In [5], Kisilevsky and Sonn proved this number is exactly $d(G)$ for a family of p-groups denoted by $\mathcal{G}_p$ and defined as follows:

Definition 1.1. Let $\mathcal{G}_p$ be the minimal family that satisfies:

(1) any abelian p-group is in $\mathcal{G}_p$,
(2) if $H, G \in \mathcal{G}_p$ then the standard wreath product $H \wr G$ is also in $\mathcal{G}_p$,
(3) if $G \in \mathcal{G}_p$ and $G \to \Gamma$ is a rank preserving epimorphism, i.e. with $d(G) = d(\Gamma)$, then $\Gamma \in \mathcal{G}_p$.

This family is contained in the family of semiabelian groups (see [5]):

Definition 1.2. The family of *semiabelian* groups $\mathcal{S}$ is the minimal family of groups that satisfies:

E-mail address: dartin@gmail.com.

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(1) $\{1\} \in \mathcal{S}$,
(2) if A is a finite abelian group and $H \in \mathcal{S}$ acts on A then the semidirect product $A \rtimes H$ is in $\mathcal{S}$,
(3) if $G \in \mathcal{S}$ and $G \to \Gamma$ is an epimorphism then $\Gamma \in \mathcal{S}$.

We shall prove that $\mathcal{G}_p$ is precisely the family of semiabelian p-groups. In fact this is an immediate corollary of the following theorem:

Theorem 1.3. *Let G be a semiabelian p-group. Then there are abelian p-groups $A_1, A_2, \ldots, A_r$ for which there is a rank preserving epimorphism $A_1 \wr (A_2 \wr \cdots \wr A_r) \to G$.*

By this we complete the solution of the minimal ramification problem for semiabelian p-groups.

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2. Properties of decompositions

The family of semiabelian groups has appeared in many forms in problems that arise from field theory (e.g., geometric realizations [3,7,9], generic extensions [9] and the minimal ramification problem [5]). The following notion of a decomposition is used in [3] to characterize semiabelian groups:

Definition 2.1. Let G be a non-trivial group. A *decomposition* of G is an abelian normal subgroup $A \lhd G$ and a proper subgroup $H < G$ such that $G = AH$.

Dentzer [3] showed that a non-trivial group G is semiabelian if and only if there is a decomposition $G = AH$ where H is semiabelian.

In the following lemma we summarize several properties of such decompositions which we shall use repeatedly. Let $\Phi(G)$ denote the Frattini subgroup of a group G. Recall that if G is a p-group then $\Phi(G) = G^p[G,G]$.

Lemma 2.2. *Let G be a p-group with decomposition $G = AH$. Let $\overline{A} = A/A^p[A,H]$, $\pi : A \to \overline{A}$ be the natural map and let M be a minimal subgroup of A for which $\pi(M) = \overline{A}$. Then:*

(1) *$[A,H]$ is a subgroup of A that is normal in G,*
(2) *$\Phi(G) = A^p[A,H]\Phi(H)$,*
(3) *$\overline{A}$ is a non-trivial elementary abelian p-group and $d(\overline{A}) = d(M)$.*

If we assume in addition that

$$A \cap H \subseteq A^p[A,H] \cap \Phi(H), \tag{2.1}$$

then:

(4) *there is an isomorphism:*

$$\psi : G/\Phi(G) \cong \overline{A} \times \big(H/\Phi(H)\big),$$

that is given explicitly for all $a \in A$, $h \in H$ by:

$$\psi\big(ah\Phi(G)\big) = \big(aA^p[A,H], h\Phi(H)\big).$$

(5) *$d(G) = d(\overline{A}) + d(H)$.*

Proof. (1) For $a \in A$, $h \in H$, we have $[a,h] = a^{-1}h^{-1}ah = a^{-1}a^h$ where $a^h := h^{-1}ah \in A$ and hence $[A,H] \leqslant A$.

Since $[A,H]$ is a subgroup of A it is centralized by A. For $h, h' \in H$ and $a \in A$ we have $[a,h]^{h'} = [a^{h'}, h^{h'}] \in [A,H]$ and hence $[A,H]^{h'} \subseteq [A,H]$. It follows that $[A,H]$ is normalized by A and H and hence is a normal subgroup of G.

(2) Clearly

$$A^p[A,H]\Phi(H) = A^p[A,H]H^p[H,H] \subseteq G^p[G,G] = \Phi(G).$$

To show the converse we prove that for $g_1 = a_1h_1$, $g_2 = a_2h_2 \in G$, where $a_1, a_2 \in A$ and $h_1, h_2 \in H$, the commutator $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ and g_1^p are elements of $A^p[A,H]\Phi(H)$. We use the following identities:

$$[x, yz] = [x,z][x,y]^z,$$

$$[xy, z] = [x,z]^y[y,z],$$

where $x^y = y^{-1}xy$. It follows that:

$$[g_1, g_2] = [a_1h_1, g_2] = [a_1, g_2]^{h_1}[h_1, g_2].$$

Expanding the commutators on the right hand side we have:

$$[a_1, g_2] = [a_1, a_2h_2] = [a_1, h_2][a_1, a_2]^{h_2} = [a_1, h_2] \in [A,H],$$

$$[h_1, g_2] = [h_1, a_2h_2] = [h_1, h_2][h_1, a_2]^{h_2} \in [H,H][A,H]^{h_2}.$$

Since by part (1), $[A,H] \lhd G$ we have:

$$[g_1, g_2] = [a_1, h_2]^{h_1}[h_1, h_2][h_1, a_2]^{h_2} \in [A,H][H,H][A,H] = [A,H][H,H].$$

We therefore have $[G,G] \subseteq [A,H][H,H]$. To prove that $g_1^p \in A^p[A,H]\Phi(H)$ we use the equality:

$$g_1^p = a_1^p h_1^p \pmod{[G,G]}.$$

It follows that $g_1^p \in A^pH^p[G,G] \subseteq A^p[A,H]\Phi(H)$.

(3) Since all p-powers of A are in $A^p[A,H]$, $\overline{A}$ is an elementary abelian p-group. Assume on the contrary $\overline{A} = \{1\}$. Then A is contained in $\Phi(G)$ and hence the equality $G = AH$ implies by [6, Corollary 10.3.3] that $G = H$. This contradicts the assumption that H is a proper subgroup of G as part of a decomposition.

Let us show that $d(\overline{A}) = d(M)$. Let $\overline{A} = \langle \bar{a}_1, \ldots, \bar{a}_d \rangle$ where $d := d(\overline{A})$. Since $\pi(M) = \overline{A}$, $d(M) \geqslant d(\overline{A})$. Furthermore, M contains elements $a_1, \ldots, a_d$ such that $\pi(a_i) = \bar{a}_i$ for $i = 1, \ldots, d$. The subgroup $M' := \langle a_1, \ldots, a_d \rangle \leqslant M$ maps under π onto $\overline{A}$ and hence by the minimality of M, $M = M'$. It follows that $d(M) = d$.

(4) Let us first prove $G/\Phi(G) \cong \overline{A} \times H/\Phi(H)$ under the assumption $A \cap H = \{1\}$. In such case $G = A \rtimes H$. Since by part (1) $A^p[A,H] \leqslant A$, we have $A^p[A,H] \cap \Phi(H) = \{1\}$ and hence part (2) shows

$$\Phi(G) = A^p[A,H] \rtimes \Phi(H).$$

We therefore have:

$$G/\Phi(G) = (A \rtimes H)\big/\big(A^p[A,H] \rtimes \Phi(H)\big). \tag{2.2}$$

As A^p is characteristic in A we have $A^p \lhd G$. Together with part (1) this implies that $A^p[A,H]$ is a normal subgroup of G. Therefore the right hand side of (2.2) is isomorphic to:

$$\bigl(A \rtimes H/A^p[A,H] \rtimes \{1\}\bigr)\big/\bigl(A^p[A,H] \rtimes \Phi(H)/A^p[A,H] \rtimes \{1\}\bigr)$$

and hence to:

$$\bigl(\bigl(A/A^p[A,H]\bigr) \rtimes H\bigr)\big/\bigl(\{1\} \rtimes \Phi(H)\bigr) = (\overline{A} \rtimes H)\big/\bigl(1 \rtimes \Phi(H)\bigr). \tag{2.3}$$

Since $[A,H] = \{a^{-1}a^h \mid a \in A,\ h \in H\}$, H acts trivially on $A/[A,H]$, and hence the actions in the semidirect products in (2.3) are trivial. We therefore have an isomorphism:

$$G/\Phi(G) \cong (\overline{A} \times H)\big/\bigl(\{1\} \times \Phi(H)\bigr) \cong \overline{A} \times \bigl(H/\Phi(H)\bigr).$$

Let us prove the assertion without assuming further that $A \cap H = \{1\}$. Note that $A \cap H$ is a normal subgroup of G and let $G_0 = G/A \cap H$, $A_0 = A/A \cap H$, $H_0 = H/A \cap H$. Since $A_0 \cap H_0 = \{1\}$ we have $G_0 = A_0 \rtimes H_0$. In particular, the assertion holds for the decomposition $G_0 = A_0 H_0$. Thus,

$$G_0/\Phi(G_0) \cong \overline{A}_0 \times H_0/\Phi(H_0), \tag{2.4}$$

where $\overline{A}_0 = A_0/A_0^p[A_0, H_0]$. Since,

$$\begin{aligned}\Phi(H_0) = H_0^p[H_0,H_0] = H^p[H,H]/A \cap H = \Phi(H)/A \cap H,\\ A_0^p[A_0,H_0] = A^p[A,H]/A \cap H,\end{aligned} \tag{2.5}$$

we have:

$$\begin{aligned}\overline{A}_0 = (A/A \cap H)\big/\bigl(A^p[A,H]/A \cap H\bigr) \cong A/A^p[A,H] = \overline{A},\\ H_0/\Phi(H_0) = (H/A \cap H)\big/\bigl(\Phi(H)/A \cap H\bigr) \cong H/\Phi(H).\end{aligned}$$

By part (2) we also have:

$$\Phi(G_0) = A_0^p[A_0,H_0]\Phi(H_0) = A^p[A,H]\Phi(H)/A \cap H = \Phi(G)/A \cap H.$$

We therefore have:

$$\begin{aligned}G/\Phi(G) &\cong (G/A \cap H)\big/\bigl(\Phi(G)/A \cap H\bigr) = G_0/\Phi(G_0)\\ &\cong \overline{A}_0 \times \bigl(H_0/\Phi(H_0)\bigr) \cong \overline{A} \times H/\Phi(H).\end{aligned}$$

Since in all of the above isomorphisms the coset of $a \in A$ (resp. $h \in H$) passes to the coset of a (resp. h) in the image, the resulting isomorphism

$$\psi : G/\phi(G) \to \overline{A} \times H/\Phi(H)$$

is given for all $a \in A$, $h \in H$ by:

$$\psi\bigl(ah\Phi(G)\bigr) = \bigl(aA^p[A,H], h\Phi(H)\bigr).$$

(5) By part (4) we have $d(G/\Phi(G)) = d(\overline{A} \times (H/\Phi(H)))$. Since $\overline{A}$ and $H/\Phi(H)$ are p-groups one has

$$d\big(\overline{A} \times \big(H/\Phi(H)\big)\big) = d(\overline{A}) + d\big(H/\Phi(H)\big).$$

Recall that $d(H) = d(H/\Phi(H))$ (see e.g., the Basis Theorem in [1, §5.4]). We therefore have $d(G) = d(\overline{A}) + d(H)$. $\square$

By iterating Lemma 2.2(4) we have:

Corollary 2.3. *Let H_0 be a p-group. Let $H_1 \geqslant H_2 \geqslant \cdots \geqslant H_k$ and $A_1, \ldots, A_k$ be subgroups of H_0 such that for each $i = 1, \ldots, k$, $H_{i-1} = A_i H_i$ is a decomposition and $A_i \cap H_i \subseteq A_i^p[A_i, H_i] \cap \Phi(H_i)$. Let $\overline{A}_i := A_i/A_i^p[A_i, H_i]$. Then there is an isomorphism*

$$\psi : H_0/\Phi(H_0) \cong \left(\prod_{i=1}^{k} \overline{A}_i\right) \times H_k/\Phi(H_k)$$

such that for all $a_i \in A_i$, $i = 1, \ldots, k$, and $h \in H_k$:

$$\psi\big(a_1 \ldots a_k h\Phi(H_0)\big) = \big(a_1 A_1^p[A_1, H_1], \ldots, a_k A_k^p[A_k, H_k], h\Phi(H_k)\big). \tag{2.6}$$

Note that since $H_0 = A_1 \ldots A_k H_k$, every element $x \in H_0$ can be written as $a_1 \ldots a_k h$, for some $a_i \in A_i$, $i = 1, \ldots, k$, $h \in H_k$ and hence (2.6) provides a description of ψ for all elements of $H_0/\Phi(H_0)$.

3. Minimal decompositions

A key ingredient in our proof of Theorem 1.3 is to find a decomposition $G = AH$ such that $d(G) = d(\overline{A}) + d(H)$, where $\overline{A} = A/A^p[A, H]$. We shall prove this is the case for minimal decompositions.

Definition 3.1. Let G be a semiabelian group. A *minimal decomposition* of G consists of the following data.

(1) a minimal normal abelian subgroup $A \lhd G$ for which there is a semiabelian proper subgroup H' of G satisfying $G = AH'$,

(2) a minimal semiabelian subgroup $H \leqslant G$ for which $G = AH$ (for the same A given in (1)).

By Dentzer's result [3] any non-trivial semiabelian group G has a decomposition and therefore also a minimal one.

In order to apply Lemma 2.2(5), we prove that minimal decompositions satisfy (2.1).

Proposition 3.2. *Let G be a semiabelian p-group with a minimal decomposition $G = AH$. Then $A \cap H \subseteq A^p[A, H] \cap \Phi(H)$.*

Proof. We divide the proof into two parts: (1) $A \cap H \subseteq A^p[A, H]$, (2) $A \cap H \subseteq \Phi(H)$.

(1) Let $\pi_A : A \to \overline{A}$ where $\overline{A} := A/A^p[A, H]$. Assume, on the contrary, that there is an $a \in A \cap H$ with non-trivial image $\bar{a} := \pi_A(a)$. By Lemma 2.2(3), $\overline{A}$ is an elementary abelian p-group, and hence can be viewed as an $\mathbb{F}_p$-vector space. We can choose an $\mathbb{F}_p$-subspace $B_1 \subseteq \overline{A}$ such that $\overline{A} = \langle \bar{a} \rangle \oplus B_1$.

The group $A_1 := \pi_A^{-1}(B_1)$ is a proper subgroup of A. By definition of $\overline{A}$, H acts trivially by conjugation on $\overline{A}$. Thus, as a preimage of an H-invariant group, A_1 is H-invariant and hence a normal subgroup of G.

We also have $\pi_A(A_1)\pi_A(A \cap H) = \overline{A}$ and therefore

$$A_1(A \cap H)A^p[A,H] = A.$$

Since $A_1 \supseteq A^p[A,H]$ this implies that $A_1(A \cap H) = A$ and therefore that

$$A_1 H = A_1(A \cap H)H = AH = G.$$

Thus, A_1 is a proper subgroup of A that is normal in G such that $G = A_1 H$, contradicting the minimality of A.

(2) Let us show that $A \cap H \subseteq \Phi(H)$ by induction on $|G|$. Assume that for any semiabelian group G_0 with $|G_0| < |G|$ and any minimal decomposition $G_0 = BK$, we have $B \cap K \subseteq \Phi(K)$.

Let $\pi_H : H \to H/\Phi(H)$ and assume on the contrary there is an $a \in A \cap H$ for which $\hat{a} := \pi_H(a)$ is non-trivial. Let $H_1 = H$ and $H_i = A_{i+1}H_{i+1}$, $i = 1, 2, \ldots, k-1$, be a sequence of minimal decompositions such that H_k is the first for which $\hat{a} \notin \pi_H(H_k)$.

Let $\overline{A}_i = A_i / A_i^p[A_i, H_i]$ and $\pi_i : A_i \to \overline{A}_i$. By the induction hypothesis and part (1), $A_i \cap H_i \subseteq A_i^p[A_i, H_i] \cap \phi(H_i)$, for $i = 2, \ldots, k$. Thus, we can apply Corollary 2.3 and obtain an isomorphism

$$\psi : H/\Phi(H) \cong \left(\prod_{i=2}^{k} \overline{A}_i \right) \times H_k/\Phi(H_k)$$

such that for all $a_i \in A_i$, $i = 2, \ldots, k$, $h \in H_k$:

$$\psi\big(\pi_H(a_2 \ldots a_k h)\big) = \big(\pi_2(a_2), \ldots, \pi_k(a_k), \pi_H(h)\big). \tag{3.1}$$

As $H_{k-1} = A_k H_k$, (3.1) implies:

$$\begin{aligned} \psi\big(\pi_H(H_{k-1})\big) &= \{1\}^{k-2} \times \overline{A}_k \times H_k/\Phi(H_k), \\ \psi\big(\pi_H(H_k)\big) &= \{1\}^{k-1} \times H_k/\Phi(H_k). \end{aligned} \tag{3.2}$$

Write $a = a_2 a_3 \ldots a_k h$ for $a_i \in A_i$ and $h \in H_k$, $i = 2, \ldots, k$. Since $\hat{a} \in \pi_H(H_{k-1}) \setminus \pi_H(H_k)$, (3.2) implies:

$$\psi(\hat{a}) \in \big(\{1\}^{k-2} \times \overline{A}_k \times H_k/\Phi(H_k)\big) \big\backslash \big(\{1\}^{k-1} \times H_k/\Phi(H_k)\big),$$

and hence by (3.1), $\pi_k(a_k) \neq 1$.

Let $\pi_k(a_k), x_1, \ldots, x_r$ be a basis of $\overline{A}_k$ and let $A'_k := \pi_k^{-1}(\langle x_1, \ldots, x_r \rangle)$. Then A'_k is a proper subgroup of A_k which is normal in H_{k-1}. Since $\langle \pi_k(a_k) \rangle \pi_k(A'_k) = \overline{A}_k$ and as $A'_k \supseteq A_k^p[A_k, H_k]$, we have $\langle a_k \rangle A'_k = A_k$.

The group $U_{k-1} := A'_k H_k$ is a semiabelian subgroup of H_{k-1}. Since A'_k is a proper subgroup of A_k and A_k is minimal we deduce that U_{k-1} is a proper subgroup of H_{k-1}. Iteratively, define a semiabelian subgroup $U_i := A_{i+1} U_{i+1}$ of H_i for $i = 1, \ldots, k-2$. The decompositions $H_k = A_{i+1} H_{i+1}$ are minimal and hence each U_i is a proper subgroup of H_i for $i = 1, \ldots, k-1$.

We now claim that $AU_1 = G$. We have:

$$AU_1 = AA_2 \ldots A_{k-1} A'_k H_k = AA_2 \ldots A_{k-1} H_k A'_k H_k. \tag{3.3}$$

Since

$$a_k = a_{k-1}^{-1} \ldots a_2^{-1} a h^{-1} \in A_{k-1} \ldots A_2 A H_k = AA_2 \ldots A_{k-1} H_k,$$

the right hand side of (3.3) contains:

$$AA_2 \dots A_{k-1}\langle a_k\rangle A'_k H_k = A_1 \dots A_k H_k = G.$$

It follows that $G = AU_1$ which contradicts the minimality of $H_1 = H$. $\square$

Combining Lemma 2.2 and Proposition 3.2 we have:

Theorem 3.3. *Let G be a semiabelian p-group and $G = AH$ a minimal decomposition. Then $d(G) = d(\bar{A}) + d(H)$. In particular $d(H) < d(G)$.*

Proof. By Proposition 3.2, $A \cap H \subseteq A^p[A, H] \cap \Phi(H)$. By Lemma 2.2(5) this implies $d(G) = d(\bar{A}) + d(H)$. By Lemma 2.2(3), $\bar{A}$ is non-trivial and hence $d(H) < d(G)$. $\square$

4. Equality of families

Let G be a semiabelian p-group. We shall prove Theorem 1.3 in three steps. In Step I, we use a minimal decomposition $G = AH$ to construct a rank preserving epimorphism $\psi_1 : A \rtimes H \to G$. In Step II, we construct a subgroup $M \leqslant A$ and a rank preserving epimorphism $\psi_2 : M \wr H \to A \rtimes H$. In Step III, we prove Theorem 1.3 by iterating Steps I and II.

Step I. At first, we use Theorem 3.3 to prove the following corollary:

Corollary 4.1. *Let G be a semiabelian p-group with a minimal decomposition $G = AH$. Let $G_0 := A \rtimes H$ with respect to the action induced by conjugation in G. Then there is a rank preserving epimorphism $\psi_1 : G_0 \to G$.*

Proof. Let $\psi_1 : G_0 \to G$ be defined by $\psi_1(a, h) = ah$. It is a homomorphism since for all $a_i \in A$, $h_i \in H$, $i = 1, 2$,

$$\psi_1(a_1, h_1)\psi_1(a_2, h_2) = a_1h_1a_2h_2 = a_1a_2^{h_1^{-1}} h_1h_2 = \psi_1\big((a_1, h_1)(a_2, h_2)\big).$$

Since A and H are in $\mathrm{Im}(\psi)$, ψ is surjective. Let $A_0 = A \rtimes 1$, $H_0 = 1 \rtimes H \leqslant G_0$ and $\bar{A}_0 = A_0/A_0^p[A_0, H_0]$. Since

$$A_0 \cap H_0 = \{1\} \subseteq {A_0}^p[A_0, H_0] \cap \Phi(H_0),$$

it follows from Lemma 2.2(5) that $d(G_0) = d(\bar{A}_0) + d(H_0)$. We have $H_0 \cong H$ and since

$$[A_0, H_0] = \big\langle a^{-1}a^h \bigm| a \in A_0,\ h \in H_0\big\rangle \cong [A, H],$$

we also have $\bar{A}_0 \cong \bar{A}$. In particular, $d(G_0) = d(\bar{A}) + d(H)$. By Theorem 3.3 $d(G) = d(\bar{A}) + d(H)$ and hence $d(G_0) = d(G)$. $\square$

Step II. We show that the group G_0 in Corollary 4.1 is a rank preserving epimorphic image of a corresponding wreath product. We first recall the following definition:

Definition 4.2. Let H be a group and A an abelian group.

(1) Let $M_1^H(A)$ be the induced H-module, i.e. the abelian group of all functions $f : H \to A$ with the H-action $f^h(x) = f(xh^{-1})$ for all $x, h \in H$, $f \in M_1^H(A)$.
(2) For every $a \in A$, let $a_* \in M_1^H(A)$ be defined by $a_*(1) = a$ and $a_*(h) = 1$ for $h \neq 1$.

(3) The wreath product $A \wr H$ is the semidirect product $M_1^H(A) \rtimes H$ with respect to the above H-action.

Note that for $a \in A$, $h \in H$, one has $a_*^h(x) = a$ if $x = h$ and $a_*^h(x) = 1$ otherwise.

To compute the rank of wreath products we shall use the following well-known lemma (see e.g., [10] or [2]).

Lemma 4.3. *Let A and H be p-groups and assume A is abelian. Then*

$$d(A \wr H) = d(A) + d(H).$$

Proposition 4.4. *Let $G = A \rtimes H$, where A and H are p-groups and A is abelian. Let $\overline{A} = A/A^p[A, H]$ and $\pi : A \to \overline{A}$ be the natural map. Let M be a minimal subgroup of A for which $\pi(M) = \overline{A}$. Then there is a rank preserving epimorphism*

$$\psi_2 : M \wr H \to G.$$

Proof. Let ψ_2 be the epimorphism from $A \wr H$ to G (see [4, Lemma 16.4.3]) defined for all $f \in M_1^H(A)$ and $h \in H$ by[1]:

$$\psi_2(f, h) = \left(\prod_{\sigma \in H} f(\sigma)^\sigma, h\right).$$

We claim that the restriction of ψ_2 to $M \wr H$ remains surjective. As $\psi_2(1, h) = h$ for all $h \in H$, we have $H \subseteq \operatorname{Im}(\psi_2)$. Since for every $a \in M$, $\psi_2(a_*, 1) = a$, we have $M \subseteq \operatorname{Im}(\psi_2)$. As $\pi(M) = \overline{A}$ we have $MA^p[A, H] = A$ and hence $A^p[A, H]\langle M, H\rangle = G$. Since $A^p[A, H]$ is contained in $\Phi(G)$ this implies $\langle M, H\rangle = G$. It follows that $\operatorname{Im}(\psi_2) \supseteq G$ which proves the claim. It remains to show that $d(M \wr H) = d(G)$. By Lemma 4.3, $d(M \wr H) = d(M) + d(H)$. By Lemma 2.2(3), $d(M) = d(\overline{A})$. By Theorem 3.3, $d(G) = d(\overline{A}) + d(H)$. It follows that:

$$d(M \wr H) = d(M) + d(H) = d(\overline{A}) + d(H) = d(G). \qquad \square$$

Step III. Let G be a semiabelian p-group. The composition $\psi_2 \circ \psi_1$ gives a rank preserving epimorphism from $M \wr H \to G$ where $M \leqslant G$ is abelian and $H < G$ is semiabelian. To prove Theorem 1.3 we iterate this process using the following well-known lemma. For the sake of completeness we include a proof of this lemma.

Lemma 4.5. *Let A be a finite abelian group and $\psi : G \to \Gamma$ an epimorphism of finite groups. Then there is an epimorphism $\tilde{\psi} : A \wr G \to A \wr \Gamma$.*

Proof. We shall treat the Γ-module $M_1^\Gamma(A)$ as a G-module via the map ψ. By the Frobenius Reciprocity Theorem

$$\operatorname{Hom}\bigl(A, M_1^\Gamma(A)\bigr) \cong \operatorname{Hom}_G\bigl(M_1^G(A), M_1^\Gamma(A)\bigr), \tag{4.1}$$

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[1] In [4], ψ_2 was defined by $\psi_2(f, h) = (\prod_{\sigma \in H} f(\sigma)^{\sigma^{-1}}, h)$. The source of the difference is in the definition of f^τ. In [4], $f^\tau(\sigma) = f(\tau\sigma)$.

where Hom_G denotes the group of G-homomorphisms. The isomorphism (4.1) associates to a homomorphism $i : A \to M_1^\Gamma(A)$, a homomorphism of G-modules $i^* : M_1^G(A) \to M_1^\Gamma(A)$ that is given by:

$$i^*(f) = \prod_{g \in G} i\bigl(f(g)\bigr)^{\psi(g)}.$$

Let $i : A \to M_1^\Gamma(A)$ be the homomorphism $i(a) = a_*$. Then

$$i^*(f)(\gamma) = \left(\prod_{g \in G} f(g)_*^{\psi(g)}\right)(\gamma) = \prod_{\{g \in G:\ \psi(g)=\gamma\}} f(g), \tag{4.2}$$

is the function that sums over the values of $f \in M_1^G(A)$ on the fiber $\psi^{-1}(\gamma)$.

We claim that i^* is surjective. Let $f \in M_1^\Gamma(A)$. For every $\gamma \in \Gamma$, fix an element $g_\gamma \in G$ for which $\psi(g_\gamma) = \gamma$ and define $\tilde{f} = \prod_{\gamma \in \Gamma} f(\gamma)_*^{g_\gamma} \in M_1^G(A)$. In particular, $\tilde{f}(g) = 1$ for $g \notin \{g_\gamma \mid \gamma \in \Gamma\}$ and $\tilde{f}(g_\gamma) = f(\gamma)$. By (4.2), $i^*(\tilde{f})(\gamma) = \tilde{f}(g_\gamma) = f(\gamma)$ for all $\gamma \in \Gamma$. It follows that for every $f \in M_1^\Gamma(A)$, $i^*(\tilde{f}) = f$, proving the claim.

Since $i^* : M_1^G(A) \to M_1^\Gamma(A)$ is an epimorphism of G-modules, i^* induces an epimorphism $\psi_2' : M_1^G(A) \rtimes G \to M_1^\Gamma(A) \rtimes G$. Since $\ker(\psi) \lhd G$ acts trivially on $M_1^\Gamma(A)$, $\ker(\psi)$ is a normal subgroup of $M_1^\Gamma(A) \rtimes G$. In particular, ψ_2' induces an epimorphism

$$\psi_2 : A \wr G = M_1^G(A) \rtimes G \to \bigl(M_1^\Gamma(A) \rtimes G\bigr) / \ker(\psi) \cong M_1^\Gamma(A) \rtimes \Gamma = A \wr \Gamma. \qquad \square$$

Proof of Theorem 1.3. We argue by induction on $|G|$. Let $G = AH$ be a minimal decomposition. By Corollary 4.1, there is a rank preserving epimorphism $\psi_1 : A \rtimes H \to G$. Let $\bar{A} = A/A^p[A, H]$, $\pi : A \to \bar{A}$ be the natural map and let A_1 be a minimal subgroup of A for which $\pi(A_1) = \bar{A}$. By Proposition 4.4 there is a rank preserving epimorphism $\psi_2 : A_1 \wr H \to A \rtimes H$. Thus $\psi = \psi_2 \circ \psi_1 : A_1 \wr H \to G$ is a rank preserving epimorphism.

By the induction hypothesis there are abelian p-groups $A_2, \ldots, A_r$ and a rank preserving epimorphism $\phi_0 : A_2 \wr (A_3 \wr \cdots \wr A_r) \to H$. By Lemma 4.5, ϕ_0 induces an epimorphism $\phi : A_1 \wr (A_2 \wr \cdots \wr A_r) \to A_1 \wr H$. Using the equality $d(A_2 \wr \cdots \wr A_r) = d(H)$ and Lemma 4.3 we have:

$$d(A_1 \wr H) = d(A_1) + d(H) = d(A_1) + d(A_2 \wr \cdots \wr A_r) = d(A_1 \wr \cdots \wr A_r)$$

and hence ϕ is rank preserving. It follows that $\psi \circ \phi : A_1 \wr (A_2 \wr \cdots \wr A_r) \to G$ is a rank preserving epimorphism. $\square$

As a corollary we have:

Corollary 4.6. *The families $\mathcal{G}_p$ and $\mathcal{S}_p$ are equal.*

Proof. The inclusion $\mathcal{G}_p \subseteq \mathcal{S}_p$ follows from [5]. For abelian p-groups $A_1, \ldots, A_r$ the iterated wreath product $A_1 \wr (A_2 \wr \cdots \wr A_r)$ is in $\mathcal{G}_p$. By Theorem 1.3, every semiabelian group is a rank preserving epimorphism of such an iterated wreath product and hence in $\mathcal{G}_p$. Thus, $\mathcal{G}_p = \mathcal{S}_p$. $\square$

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