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On related varieties to the commuting variety of a semisimple Lie algebra

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ABSTRACT

Let \mathfrak{g} be a semisimple Lie algebra of finite dimension. The nullcone \mathcal{N} of \mathfrak{g} is the set of (x, y) in $\mathfrak{g} \times \mathfrak{g}$ such that x and y are nilpotents and are in the same Borel subalgebra. The main result of this paper is that \mathcal{N} is a closed and irreducible subvariety of $\mathfrak{g} \times \mathfrak{g}$ and its normalization morphism is bijective.

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1. Introduction

The basic field \mathbb{k} is an algebraic closed field of characteristic zero. Let \mathfrak{g} be a semisimple Lie algebra of finite dimension, let G be its adjoint group and let \mathfrak{g}^* be its dual. We identify \mathfrak{g} and \mathfrak{g}^* by the Killing form. We denote by \mathfrak{b} a Borel subalgebra of \mathfrak{g} , by \mathfrak{h} a Cartan subalgebra of \mathfrak{g} contained in \mathfrak{b} , and by $b_{\mathfrak{g}}$ and $\mathrm{rk} \mathfrak{g}$ the dimensions of \mathfrak{b} and \mathfrak{h} , respectively. Let W be the Weyl group of \mathfrak{g} with respect to \mathfrak{h} . The symmetric algebra of \mathfrak{g} is denoted by $S(\mathfrak{g})$ and the subalgebra of its G -invariant elements is denoted by $S(\mathfrak{g})^G$. Let $\mathcal{B}_{\mathfrak{g}}$ be the set of (x, y) in $\mathfrak{g} \times \mathfrak{g}$ such that x and y are in the same Borel subalgebra. In Section 4, we show the following theorem.

Theorem 1.1.

- (i) *The variety $\mathcal{B}_{\mathfrak{g}}$ is closed in $\mathfrak{g} \times \mathfrak{g}$ and irreducible of dimension $3b_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}$, but it is not normal.*
- (ii) *The algebra of W -invariant regular functions on $\mathfrak{h} \times \mathfrak{h}$ is isomorphic to the algebra of G -invariant regular functions on $\mathcal{B}_{\mathfrak{g}}$.*

The variety $\mathcal{B}_{\mathfrak{g}}$ contains two interesting varieties, the nullcone and the commuting variety. The nullcone of \mathfrak{g} , denoted by \mathcal{N} , is defined by the set of zeros of the G -invariant polynomial functions f on $\mathfrak{g} \times \mathfrak{g}$ such that $f(0) = 0$. According to Mumford [13], it can be equivalently defined as the variety of pairs (x, y) in $\mathcal{B}_{\mathfrak{g}}$ such that x and y are nilpotents. It is well known that the nullcone plays a fundamental role in the theory of invariants and its applications. This cone was introduced and studied by Hilbert in his famous paper “Ueber die vollen Invariantensystem” [5]. Kraft and Wallach studied the geometry of the nullcone [10]. They showed that the nullcone of any number of copies of the adjoint representation of G on \mathfrak{g} is irreducible and they gave a resolution of singularities of this variety. They also studied the case in which the polarizations of a set of invariant functions defining the nullcone of a representation V define the nullcone of a direct sum of several copies of V [11]. This question was previously studied by Losik, Michor and Popov [12], who analyzed the relationship between $\mathbb{k}[V^{\oplus n}]^G$, the G -invariant algebra of n -summands of a finite dimensional algebraic G -module V , and $\mathrm{pol}_n \mathbb{k}[V]^G$, the subalgebra of $\mathbb{k}[V^{\oplus n}]^G$ generated by the polarizations of all elements in $\mathbb{k}[V]^G$. Popov provided a general algorithm for determining the irreducible components of maximal dimension of the nullcone using the weights of the representation and their multiplicities [15]. In Section 5 of this paper, we prove the following theorem.

Theorem 1.2.

- (i) *The nullcone \mathcal{N} is closed in $\mathfrak{g} \times \mathfrak{g}$ and irreducible of dimension $3(b_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$.*
- (ii) *The codimension of the set of its singular points is at least two and the normalization morphism of \mathcal{N} is bijective.*

The other interesting variety of $\mathcal{B}_{\mathfrak{g}}$ is the commuting variety. We recall that the commuting variety $\mathcal{C}_{\mathfrak{g}}$ of \mathfrak{g} is the set of (x, y) in $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. We use a result reported by Joseph on $\mathcal{C}_{\mathfrak{g}}$ [8], a result that has been generalized by Tevelev [17], to prove that $\mathbb{k}[\mathcal{B}_{\mathfrak{g}}]^G$ (see Notation) and $S(\mathfrak{h} \times \mathfrak{h})^W$ are isomorphic.

In this paper, we also prove some other results for $\mathcal{B}_{\mathfrak{g}}$. We use the homogeneous generators $p_1, \dots, p_{\mathrm{rk} \mathfrak{g}}$ of $S(\mathfrak{g})^G$, chosen so that the sequence of their degrees $d_1, \dots, d_{\mathrm{rk} \mathfrak{g}}$ is increasing. According to Bourbaki, $d_1 + \dots + d_{\mathrm{rk} \mathfrak{g}} = b_{\mathrm{rk} \mathfrak{g}}$ [2]. Let $X := \sigma(\mathfrak{h} \times \mathfrak{h})$ when σ is the morphism from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{k}^{b_{\mathfrak{g}} + \mathrm{rk} \mathfrak{g}}$ defined by

$$\sigma(x, y) = (p_1^{(0)}(x, y), \dots, p_1^{(d_1)}(x, y), \dots, p_{\mathrm{rk} \mathfrak{g}}^{(0)}(x, y), \dots, p_{\mathrm{rk} \mathfrak{g}}^{(d_{\mathrm{rk} \mathfrak{g}})}(x, y)),$$

where $p_i^{(n)}$ are the second-order polarizations of p_i of bidegree $(d_i - n, n)$ (see Notation). We show in Section 3 that there is an isomorphism between $(\mathfrak{h} \times \mathfrak{h})/W$ and the normalization of X , the normalization morphism of X is bijective, and the codimension of the complement of the set of regular

points of X is at least two. Conversely, the variety $\sigma(\mathcal{B}_{\mathfrak{g}})$ is equal to X and $\mathcal{B}_{\mathfrak{g}}$ is an irreducible component of $\sigma^{-1}(X)$.

2. Notation

We consider the diagonal action of G on $\mathfrak{g} \times \mathfrak{g}$ and the diagonal action of W on $\mathfrak{h} \times \mathfrak{h}$. Let \mathfrak{u} be the set of the nilpotent elements in \mathfrak{b} . Let \mathcal{R} be the root system of \mathfrak{h} in \mathfrak{g} , let \mathcal{R}_+ be the positive root system defined by \mathfrak{b} and let Π be the set of simple roots of \mathcal{R}_+ . We denote by \mathbf{B} the normalizer of \mathfrak{b} in G , by \mathbf{U} its unipotent radical, and by \mathbf{H} and $N_G(\mathfrak{h})$ the centralizer and normalizer, respectively, of \mathfrak{h} in G . We use the following notations:

- If $G \times A \rightarrow A$ is an action of G on the algebra A , we denote by A^G the subalgebra of the G -invariant elements of A .
- For $i = 1, \dots, \text{rk } \mathfrak{g}$, the second-order polarizations of p_i of bidegree $(d_i - n, n)$, denoted by $p_i^{(n)}$, are the unique elements in $(S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}))^G$ satisfying the relation

$$p_i(ax + by) = \sum_{n=0}^{d_i} a^{d_i-n} b^n p_i^{(n)}(x, y)$$

for all $a, b \in \mathbb{C}$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

- $\langle \cdot, \cdot \rangle$ is the Killing form of \mathfrak{g} .
- For $i = 1, \dots, \text{rk } \mathfrak{g}$, ε_i is the element of $S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ defined by

$$\langle \varepsilon_i(x), v \rangle = p'_i(x)(v) \quad \forall x, v \in \mathfrak{g}$$

when $p'_i(x)$ is the differential of p_i at x for $i = 1, \dots, \text{rk } \mathfrak{g}$.

- For $i = 1, \dots, \text{rk } \mathfrak{g}$, the 2-polarizations of ε_i of bidegree $(d_i - m - 1, m)$, denoted by $\varepsilon_i^{(m)}$, are the unique elements in $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ satisfying the relation

$$\varepsilon_i(ax + by) = \sum_{n=0}^{d_i-1} a^{d_i-n-1} b^n \varepsilon_i^{(n)}(x, y)$$

for all $a, b \in \mathbb{C}$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

- \mathfrak{g}' is the set of regular elements of \mathfrak{g} .
- $\mathfrak{h}' := \mathfrak{g}' \cap \mathfrak{h}$.
- For x, y in \mathfrak{g} , $P_{x,y}$ is the linear span of the set $\{x, y\}$ over \mathbb{k} .
- π is the morphism from \mathfrak{h} to $\mathbb{k}^{\text{rk } \mathfrak{g}}$ defined by $\pi(x) := (p_1(x), \dots, p_{\text{rk } \mathfrak{g}}(x))$.
- For an algebraic variety X and a point $x \in X$, $T_x X$ is the tangent space of X at x .
- For an algebraic variety X and a point $x \in X$, $\mathcal{O}_{x,X}$ is the local ring of X at x , \mathfrak{M}_x is its maximal ideal and $\hat{\mathcal{O}}_{x,X}$ is the completion of $\mathcal{O}_{x,X}$ for the \mathfrak{M}_x -adic topology.
- For an algebraic \mathbf{B} -variety X , $G \times_{\mathbf{B}} X$ is the quotient of $G \times X$ under the action of \mathbf{B} given by $b \cdot (g, x) := (gb^{-1}, b(x))$.
- For an algebraic \mathbf{B} -variety X and ϕ' a morphism on $G \times X$, we define the morphism ϕ from $G \times_{\mathbf{B}} X$ to $G \cdot X$ through the quotient by ϕ' , i.e. $\phi(y) = \phi'(g, x)$, where (g, x) is a representative element of y in $G \times_{\mathbf{B}} X$.
- For $x \in \mathfrak{b}$, $x = x_0 + x_+$ with $x_0 \in \mathfrak{h}$ and $x_+ \in \mathfrak{u}$.
- h is the element of \mathfrak{h} such that $\beta(h) = 1$ for all β in Π , and $h(t)$ is the one-parameter subgroup of G generated by adh . Then for all x in \mathfrak{b} ,

$$\lim_{t \rightarrow 0} h(t)(x) = x_0.$$

- According to the identification of \mathfrak{g} and \mathfrak{g}^* , $S(\mathfrak{g})$ identifies with the algebra of polynomial functions on \mathfrak{g} and $S(\mathfrak{h})$ identifies with the algebra of polynomial functions on \mathfrak{h} .

3. On the variety $\sigma(\mathfrak{h} \times \mathfrak{h})$

We denote by $(x, y) \mapsto \overline{(x, y)}$ the canonical map from $\mathfrak{h} \times \mathfrak{h}$ to $(\mathfrak{h} \times \mathfrak{h})/W$.

Proposition 3.1. *The set $X := \sigma(\mathfrak{h} \times \mathfrak{h})$ is closed in $\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$.*

Proof. For $m := \sup\{d_i, i \in \{1, \dots, \text{rk } \mathfrak{g}\}\}$, let $(t_i)_{i=1, \dots, m+1}$ be in \mathbb{k} , pairwise different, and let α be the morphism from $\mathfrak{h} \times \mathfrak{h}$ to \mathfrak{h}^{m+1} defined by

$$\alpha(x, y) := (x + t_1 y, \dots, x + t_{m+1} y) \quad (1)$$

and let δ be the morphism from \mathfrak{h}^{m+1} to $\mathbb{k}^{(m+1) \text{rk } \mathfrak{g}}$ defined by

$$\delta(x_1, \dots, x_{m+1}) := (\pi(x_1), \dots, \pi(x_{m+1})). \quad (2)$$

Let β be the morphism from $\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$ to $\mathbb{k}^{(m+1) \text{rk } \mathfrak{g}}$ defined by

$$\beta((z_i^{(j)})_{1 \leq i \leq \text{rk } \mathfrak{g}, 0 \leq j \leq d_i}) := \left(\sum_{j=0}^{d_1} t_1^j z_1^{(j)}, \sum_{j=0}^{d_2} t_1^j z_2^{(j)}, \dots, \sum_{j=0}^{d_{\text{rk } \mathfrak{g}}} t_{m+1}^j z_{\text{rk } \mathfrak{g}}^{(j)} \right).$$

Since $(t_i)_{i=1, \dots, m+1}$ are pairwise different, β is an isomorphism from $\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$ to $\beta(\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}})$. Since $S(\mathfrak{h})$ is an integral extension of $\mathbb{k}[p_1, \dots, p_{\text{rk } \mathfrak{g}}]$, π is a finite morphism and δ is too. Hence $\delta \circ \alpha(\mathfrak{h} \times \mathfrak{h})$ is closed in $\mathbb{k}^{(m+1) \text{rk } \mathfrak{g}}$. Furthermore, $\delta \circ \alpha(\mathfrak{h} \times \mathfrak{h})$ is contained in the image of β . Then $\sigma(\mathfrak{h} \times \mathfrak{h})$ is closed since $\beta \circ \sigma(\mathfrak{h} \times \mathfrak{h}) = \delta \circ \alpha(\mathfrak{h} \times \mathfrak{h})$. \square

Proposition 3.2. *Let (X_n, μ) be the normalization of $X = \sigma(\mathfrak{h} \times \mathfrak{h})$.*

- There exists a bijection $\bar{\sigma}$ from $(\mathfrak{h} \times \mathfrak{h})/W$ to X such that $\bar{\sigma}(\overline{(x, y)}) = \sigma(x, y)$ for all (x, y) in $\mathfrak{h} \times \mathfrak{h}$.
- There exists a unique morphism σ_n from $(\mathfrak{h} \times \mathfrak{h})/W$ to X_n such that $\mu \circ \sigma_n = \bar{\sigma}$.
- The morphism σ_n is an isomorphism.
- The morphism μ is bijective.

Proof. (i) Since $\sigma(w(x, y)) = \sigma(x, y)$, for all w in W and (x, y) in $\mathfrak{h} \times \mathfrak{h}$, $\bar{\sigma}$ is well defined and surjective. Let $(x, y), (x', y')$ be in $\mathfrak{h} \times \mathfrak{h}$ such that $\bar{\sigma}(\overline{(x, y)}) = \bar{\sigma}(\overline{(x', y')})$. Then:

$$\begin{aligned} \bar{\sigma}(\overline{(x, y)}) = \bar{\sigma}(\overline{(x', y')}) &\Rightarrow \sigma(x, y) = \sigma(x', y') \\ &\Rightarrow p_i(x + ty) = p_i(x' + ty') \quad \forall t \in \mathbb{k} \text{ and } \forall i \in \{1, \dots, \text{rk } \mathfrak{g}\}. \end{aligned}$$

Since W is finite, there exist t_1 and t_2 , two different elements of \mathbb{k} and w in W such that we have the following system:

$$\begin{cases} x' + t_1 y' = w(x) + t_1 w(y), \\ x' + t_2 y' = w(x) + t_2 w(y) \end{cases}$$

and then $x' = w(x)$ and $y' = w(y)$; hence, $\overline{(x', y')} = \overline{(x, y)}$ and thus $\bar{\sigma}$ is injective and therefore it is bijective.

(ii) The variety $(\mathfrak{h} \times \mathfrak{h})/W$ is a quotient of a smooth variety by a finite group, so it is normal. Since $\bar{\sigma}$ is bijective, it is a dominant morphism. Since $(\mathfrak{h} \times \mathfrak{h})/W$ is normal and since $\bar{\sigma}$ is a dominant morphism, there exists a unique morphism σ_n from $(\mathfrak{h} \times \mathfrak{h})/W$ to X_n such that $\mu \circ \sigma_n = \bar{\sigma}$ [4, Chapter II, Example 3.8]. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{h} \times \mathfrak{h}/W & \xrightarrow{\sigma_n} & X_n \\ & \searrow \bar{\sigma} & \downarrow \mu \\ & & X. \end{array}$$

(iii) Let Ψ be the comorphism of σ_n from $\mathbb{k}[X_n]$ to $S(\mathfrak{h} \times \mathfrak{h})^W$.

We show that Ψ admits an inverse. Let Φ be the map from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $\mathbb{k}[X_n]$ defined by $\Phi(Q) = P$, where $P(z) = Q(x, y)$ when (x, y) is in $\mathfrak{h} \times \mathfrak{h}$ and such that $\sigma_n(x, y) = z$. The map Φ is well defined. In fact, let (x, y) and (x', y') be in $\mathfrak{h} \times \mathfrak{h}$ such that $\sigma_n(x, y) = \sigma_n(x', y')$. We have to prove

$$Q(x, y) = Q(x', y').$$

We have:

$$\sigma_n(x, y) = \sigma_n(x', y') \Rightarrow \mu \circ \sigma_n(x, y) = \mu \circ \sigma_n(x', y') \Rightarrow \sigma(x, y) = \sigma(x', y').$$

It has been proved that there exists w in W such that $x' = w(x)$ and $y' = w(y)$ and then

$$Q(x', y') = Q(w(x), w(y)) = Q(x, y).$$

It is clear that the map Φ is an algebra homomorphism. We now prove that P belongs to $\mathbb{k}[X_n]$. Let Γ be the graph of Q and let $\tilde{\Gamma}$ be the image of Γ under $\sigma_n \times 1_{\mathbb{k}}$. Then $\tilde{\Gamma}$ is the graph of P and it is closed since $\sigma_n \times 1_{\mathbb{k}}$ is a finite morphism and Γ is closed. Let χ be the projection of $\tilde{\Gamma}$ onto X_n . Then χ is a bijection. According to Zariski's main theorem [14], χ is an isomorphism since X_n is normal. Hence P is a regular function on X_n .

We now show that $\Phi = \Psi^{-1}$. Let P be in $\mathbb{k}[X_n]$ and let Q be in $S(\mathfrak{h} \times \mathfrak{h})^W$. We have

$$\Phi \circ \Psi(P) = \Phi(P \circ \sigma_n) = P \quad \text{and} \quad \Psi \circ \Phi(Q) = \Phi(Q) \circ \sigma_n.$$

We now calculate $(\Phi(Q) \circ \sigma_n)(x, y)$ for (x, y) in $\mathfrak{h} \times \mathfrak{h}$:

$$(\Phi(Q) \circ \sigma_n)(x, y) = \Phi(Q)(\sigma_n(x, y)) = Q(x, y),$$

and then $\Psi \circ \Phi(Q) = Q$. Hence Ψ is an isomorphism from $\mathbb{k}[X_n]$ to $S(\mathfrak{h} \times \mathfrak{h})^W$, which proves the assertion.

(iv) Since $\mu \circ \sigma_n = \bar{\sigma}$, σ_n is an isomorphism and $\bar{\sigma}$ is bijective, then μ is bijective. \square

Remark 3.1. According to Hunziker [7], $\bar{\sigma}$ is an isomorphism if and only if the algebra $S(\mathfrak{h} \times \mathfrak{h})^W$ is generated by the 2-polarizations of elements of $S(\mathfrak{h})^W$, that is, for g of type A_n , B_n and C_n . For g of type D_n , $S(\mathfrak{h} \times \mathfrak{h})^W$ is not always generated by the polarizations of elements of $S(\mathfrak{h})^W$. In particular, for n even, all the polarizations of elements of $S(\mathfrak{h})^W$ have even total degree and $S(\mathfrak{h} \times \mathfrak{h})^W$ contains elements of odd total degree. Therefore, $\mathbb{k}[X_n]$ strictly contains $\mathbb{k}[X]$.

Proposition 3.3. Let $\Gamma := \{(x, y) \in \mathfrak{h} \times \mathfrak{h} \mid P_{x,y} \cap \mathfrak{h}' \neq \emptyset\}$. We then have the following:

- (i) $\text{codim}_X X \setminus \sigma(\Gamma) \geq 2$.
- (ii) For all z in $\sigma(\Gamma)$, z is a regular point of X .

Proof. (i) Let (x, y) be in $(\mathfrak{h} \times \mathfrak{h}) \setminus \Gamma$. We have

$$(x, y) \in (\mathfrak{h} \times \mathfrak{h}) \setminus \Gamma \Rightarrow P_{x,y} \cap \mathfrak{h}' = \emptyset \Rightarrow x, y \in \mathfrak{h} \setminus \mathfrak{h}'.$$

Then $(\mathfrak{h} \times \mathfrak{h}) \setminus \Gamma$ is contained in $\mathfrak{h} \setminus \mathfrak{h}' \times \mathfrak{h} \setminus \mathfrak{h}'$. Hence, $\text{codim}_{\mathfrak{h} \times \mathfrak{h}} (\mathfrak{h} \times \mathfrak{h}) \setminus \Gamma \geq 2$, from which the assertion follows since σ is a finite morphism.

(ii) Let (x, y) be an element in Γ and let $(t_i)_{i=1, \dots, m+1}$ be in \mathbb{k} , pairwise different, chosen such that $x + t_i y$ is regular for all i , with $m = \sup\{d_i, i \in \{1, \dots, \text{rk } \mathfrak{g}\}\}$. Let β be the morphism from $\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$ to $\mathbb{k}^{(m+1)\text{rk } \mathfrak{g}}$ defined by

$$\beta((z_i^{(j)})_{1 \leq i \leq \text{rk } \mathfrak{g}, 0 \leq j \leq d_i}) = \left(\sum_{j=0}^{d_1} t_1^j z_1^{(j)}, \sum_{j=0}^{d_2} t_1^j z_2^{(j)}, \dots, \sum_{j=0}^{d_{\text{rk } \mathfrak{g}}} t_{m+1}^j z_{\text{rk } \mathfrak{g}}^{(j)} \right).$$

Since $(t_i)_{i=1, \dots, m+1}$ are pairwise different, then β is an isomorphism from $\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$ to $\beta(\mathbb{k}^{b_{\mathfrak{g}} + \text{rk } \mathfrak{g}})$. Moreover, $\beta(\sigma(\mathfrak{h} \times \mathfrak{h}))$ is equal to $\kappa(\mathfrak{h} \times \mathfrak{h})$, where κ is the morphism from $\mathfrak{h} \times \mathfrak{h}$ to $\mathbb{k}^{(m+1)\text{rk } \mathfrak{g}}$ defined by

$$\kappa(x, y) := (\pi(x + t_1 y), \dots, \pi(x + t_{m+1} y)).$$

Let $\kappa_{1,2}$ be the morphism from $\mathfrak{h} \times \mathfrak{h}$ to $\mathbb{k}^{2\text{rk } \mathfrak{g}}$ defined by

$$\kappa_{1,2}(x, y) := (\pi(x + t_1 y), \pi(x + t_2 y))$$

and let φ be the projection from $\kappa(\mathfrak{h} \times \mathfrak{h})$ to $\kappa_{1,2}(\mathfrak{h} \times \mathfrak{h})$ defined by

$$\varphi(\pi(x + t_1 y), \dots, \pi(x + t_{m+1} y)) := (\pi(x + t_1 y), \pi(x + t_2 y)).$$

Let $(\kappa_{1,2})_*$ and φ_* be the comorphisms of $\kappa_{1,2}$ and φ , respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{\kappa(x,y), \tau(\mathfrak{h} \times \mathfrak{h})} & \xrightarrow{\kappa_*} & \hat{\mathcal{O}}_{(x,y), \mathfrak{h} \times \mathfrak{h}} \\ & \searrow \varphi_* & \downarrow (\kappa_{1,2})_*^{-1} \\ & & \hat{\mathcal{O}}_{\kappa_{1,2}(x,y), \kappa_{1,2}(\mathfrak{h} \times \mathfrak{h})} \end{array}$$

Indeed, since $x + t_i y$ is a regular point, π is an étale morphism at $x + t_i y$ for $i = 1, 2$, and hence $(\kappa_{1,2})_*$ is an isomorphism. Recall that α and δ are defined by (1) and (2) in the proof of Proposition 3.1. Then we obtain the following commutative diagram:

$$\begin{array}{ccc}
 \mathfrak{h} \times \mathfrak{h} & \xrightarrow{\alpha} & \alpha(\mathfrak{h} \times \mathfrak{h}) \\
 \downarrow \kappa_{1,2} & & \downarrow \delta \\
 \kappa_{1,2}(\mathfrak{h} \times \mathfrak{h}) & \xleftarrow{\varphi} & \kappa(\mathfrak{h} \times \mathfrak{h}).
 \end{array}$$

Let α_* and δ_* be the comorphisms of α and δ , respectively. Since α is an isomorphism, $(\alpha_*)^{-1} \circ (\kappa_{1,2})_*$ is an isomorphism. Hence the coordinates on $\alpha(\mathfrak{h} \times \mathfrak{h})$ are in $\hat{\mathcal{O}}_{\kappa_{1,2}(x,y), \kappa_{1,2}(\mathfrak{h} \times \mathfrak{h})}$. Then $p_i(x + t_j y)$ are in the image of φ_* . Since $p_i(x + t_j y)$ generate $\hat{\mathcal{O}}_{\kappa(x,y), \kappa(\mathfrak{h} \times \mathfrak{h})}$, φ_* is surjective. Since $\hat{\mathcal{O}}_{\kappa_{1,2}(x,y), \kappa_{1,2}(\mathfrak{h} \times \mathfrak{h})}$ and $\hat{\mathcal{O}}_{\kappa(x,y), \kappa(\mathfrak{h} \times \mathfrak{h})}$ have the same Krull dimension $2 \operatorname{rk} \mathfrak{g}$, $\ker \varphi_* = \{0\}$. Then φ_* is an isomorphism. Hence $\hat{\mathcal{O}}_{\kappa(x,y), \kappa(\mathfrak{h} \times \mathfrak{h})}$ is a regular local ring, so $\kappa(x, y)$ is a smooth point of $\kappa(\mathfrak{h} \times \mathfrak{h})$. Hence $\sigma(x, y)$ is a smooth point of $\sigma(\mathfrak{h} \times \mathfrak{h})$ since β is an isomorphism from $\sigma(\mathfrak{h} \times \mathfrak{h})$ to $\kappa(\mathfrak{h} \times \mathfrak{h})$. Therefore for (x, y) in Γ , $\sigma(x, y)$ is regular in $\sigma(\mathfrak{h} \times \mathfrak{h})$. \square

We recall that the commuting variety $\mathcal{C}_{\mathfrak{g}}$ of \mathfrak{g} is the set of (x, y) in $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$.

Proposition 3.4. *The variety X is the image of the commuting variety $\mathcal{C}_{\mathfrak{g}}$ of \mathfrak{g} by σ .*

Proof. Let (x, y) be in $\mathcal{C}_{\mathfrak{g}}$ and let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions of x and y . Since $[x, y] = 0$, $[x_s, y_s] = 0$. Then x_s and y_s belong to the same Cartan subalgebra of \mathfrak{g} . Since the Cartan subalgebras are conjugate under G , there exists g in G such that $g(x_s)$ and $g(y_s)$ are in \mathfrak{h} . Since

$$p_i(x + ty) = p_i(x_s + ty_s) \quad \forall i = 1, \dots, \operatorname{rk} \mathfrak{g}$$

and σ is G -invariant,

$$\sigma(x, y) = \sigma(g(x_s), g(y_s)).$$

Hence $\sigma(\mathcal{C}_{\mathfrak{g}})$ is contained in X . Since $\mathcal{C}_{\mathfrak{g}}$ contains $\mathfrak{h} \times \mathfrak{h}$, X is equal to $\sigma(\mathcal{C}_{\mathfrak{g}})$. \square

4. On the subvariety $\mathcal{B}_{\mathfrak{g}}$ of $\mathfrak{g} \times \mathfrak{g}$

We consider the action of \mathbf{B} on $G \times \mathfrak{b} \times \mathfrak{b}$ given by $b.(g, x, y) = (gb^{-1}, b(x), b(y))$. Let γ' be the morphism from $G \times \mathfrak{b} \times \mathfrak{b}$ to $\mathfrak{g} \times \mathfrak{g}$ defined by

$$\gamma'(g, x, y) = (g(x), g(y))$$

and let γ be the morphism from $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ to $\mathfrak{g} \times \mathfrak{g}$ defined through the quotient by γ' . Let $\mathcal{B}_{\mathfrak{g}}$ be the set of (x, y) in $\mathfrak{g} \times \mathfrak{g}$ such that x and y are in the same Borel subalgebra. Then $\mathcal{B}_{\mathfrak{g}}$ is the image of $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ by γ .

Proposition 4.1. *The morphism γ is a desingularization of $\mathcal{B}_{\mathfrak{g}}$. The subvariety $\mathcal{B}_{\mathfrak{g}}$ is closed in $\mathfrak{g} \times \mathfrak{g}$ and irreducible of dimension $3b_{\mathfrak{g}} - \operatorname{rk} \mathfrak{g}$. Moreover, it is not normal.*

Proof. Since $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ is a vector bundle on the smooth variety G/\mathbf{B} then it is a smooth variety. We denote by \mathcal{B} the subvariety of the Borel subalgebras of \mathfrak{g} and by $\tilde{\mathcal{B}}$ the subvariety of elements (u, x, y) of $\mathcal{B} \times \mathfrak{g} \times \mathfrak{g}$ such that u contains x and y . The map from $G \times \mathfrak{g} \times \mathfrak{g}$ to $G/\mathbf{B} \times \mathfrak{g} \times \mathfrak{g}$ defined by $(g, x, y) \mapsto (\bar{g}, g(x), g(y))$, where \bar{g} is the image of g in G/\mathbf{B} , defines through the quotient an

isomorphism ν from $G \times_{\mathbf{B}} \mathfrak{g} \times \mathfrak{g}$ to $G/\mathbf{B} \times \mathfrak{g} \times \mathfrak{g}$. Since $\mathcal{B} \times \mathfrak{g} \times \mathfrak{g}$ and $G/\mathbf{B} \times \mathfrak{g} \times \mathfrak{g}$ are isomorphic, we obtain the following commutative diagram:

$$\begin{array}{ccc} G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b} & \longrightarrow & \mathcal{B} \times \mathfrak{g} \times \mathfrak{g} \\ & \searrow \gamma & \downarrow \\ & & \mathcal{B}_{\mathfrak{g}}. \end{array}$$

Hence γ is a proper morphism since G/\mathbf{B} is a projective variety. Since $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ is a closed subset in $G \times_{\mathbf{B}} \mathfrak{g} \times \mathfrak{g}$, $\mathcal{B}_{\mathfrak{g}}$ is closed in $\mathfrak{g} \times \mathfrak{g}$ and it is irreducible as the image of an irreducible variety.

It remains to show that γ is a birational morphism. Let

$$\Omega_{\mathfrak{g}} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \setminus \{0\} \subset \mathfrak{g}', \dim P_{x,y} = 2\}.$$

The subset $\Omega_{\mathfrak{g}}$ is an open subset of $\mathfrak{g} \times \mathfrak{g}$ and $\Omega_{\mathfrak{g}} \cap \mathcal{B}_{\mathfrak{g}}$ is a nonempty open subset of $\mathcal{B}_{\mathfrak{g}}$. Let (x, y) be an element in $\Omega_{\mathfrak{g}} \cap \mathcal{B}_{\mathfrak{g}}$. According to [1, Corollary 2] and [9, Theorem 9], the subspace $V_{x,y}$ generated by $\{\varepsilon_i^{(m)}(x, y), i = 1, \dots, \text{rk } \mathfrak{g}, m = 0, \dots, d_i\}$ is the unique Borel subalgebra that contains x and y ; hence $\nu^{-1}(V_{x,y}, x, y)$ is the unique point in $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ above (x, y) . Therefore, γ is birational.

Since γ is birational,

$$\dim \mathcal{B}_{\mathfrak{g}} = \dim(G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}) = 3b_{\mathfrak{g}} - \text{rk } \mathfrak{g}.$$

Since γ is a birational morphism, for \mathfrak{g} nontrivial, if $\mathcal{B}_{\mathfrak{g}}$ were normal, by Zariski's main theorem a finite fiber of γ would have cardinality of 1. However, for x in \mathfrak{h}' , a Borel subalgebra contains x if and only if it contains \mathfrak{h} . Hence the set of Borel subalgebras of \mathfrak{g} containing x is finite of cardinality $|W|$ and then $|\gamma^{-1}(x, 0)| = |W|$. Therefore, $\mathcal{B}_{\mathfrak{g}}$ is not normal. \square

Let $(\mathcal{B}_{\mathfrak{g},n}, \eta)$ be the normalization of $\mathcal{B}_{\mathfrak{g}}$ and let ι be the canonical injection of $\mathfrak{h} \times \mathfrak{h}$ in $\mathcal{B}_{\mathfrak{g}}$.

Lemma 4.1. *There exists a unique closed immersion ι_n from $\mathfrak{h} \times \mathfrak{h}$ to $\mathcal{B}_{\mathfrak{g},n}$ such that $\eta \circ \iota_n = \iota$.*

Proof. Since γ is a dominant morphism from $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ to $\mathcal{B}_{\mathfrak{g}}$ and since $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ is normal, there exists a unique morphism γ_n from $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ to $\mathcal{B}_{\mathfrak{g},n}$ such that $\eta \circ \gamma_n = \gamma$ [4, Chapter II, Example 3.8]. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b} & \xrightarrow{\gamma_n} & \mathcal{B}_{\mathfrak{g},n} \\ & \searrow \gamma & \downarrow \eta \\ & & \mathcal{B}_{\mathfrak{g}}. \end{array}$$

Let i be the canonical injection of $\mathfrak{h} \times \mathfrak{h}$ into $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$. Hence, taking $\iota_n = \gamma_n \circ i$, $\eta \circ \iota_n = \iota$ since $\iota = \gamma \circ i$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{h} \times \mathfrak{h} & \xrightarrow{\iota_n} & \iota_n(\mathfrak{h} \times \mathfrak{h}) \\ & \searrow \iota & \downarrow \eta \\ & & \mathcal{B}_{\mathfrak{g}}. \end{array}$$

Since γ_n and i are closed, ι_n is closed. \square

Proposition 4.2. *We have the following properties:*

- (i) $\sigma(\mathcal{B}_{\mathfrak{g}})$ is equal to X .
- (ii) $\mathcal{B}_{\mathfrak{g}}$ is an irreducible component of $\sigma^{-1}(X)$.

Proof. (i) Let (x, y) be in $\mathcal{B}_{\mathfrak{g}}$. Since σ is G -invariant, we can assume that x and y are in \mathfrak{b} . We have

$$\begin{aligned} p_i(x + ty) &= p_i(x_0 + ty_0) \quad \forall i = 1, \dots, \operatorname{rk} \mathfrak{g} \\ \Rightarrow \quad \sigma(x, y) &= \sigma(x_0, y_0), \end{aligned}$$

and hence $\sigma(\mathcal{B}_{\mathfrak{g}})$ is contained in X , from which the equality follows, since $\mathfrak{h} \times \mathfrak{h}$ is contained in $\mathcal{B}_{\mathfrak{g}}$.

(ii) For z in $\mathbb{k}^{b_{\mathfrak{g}} + \operatorname{rk} \mathfrak{g}}$, according to Charbonnel and Moreau [3], the dimension of $\sigma^{-1}(z)$ is $3b_{\mathfrak{g}} - 3\operatorname{rk} \mathfrak{g}$, and hence

$$\dim \sigma^{-1}(X) = 3b_{\mathfrak{g}} - \operatorname{rk} \mathfrak{g} = \dim \mathcal{B}_{\mathfrak{g}}.$$

Moreover, $\mathcal{B}_{\mathfrak{g}}$ is irreducible by Proposition 4.1. Hence $\mathcal{B}_{\mathfrak{g}}$ is an irreducible component of $\sigma^{-1}(X)$. \square

Let τ' be the morphism from $G \times \mathfrak{b} \times \mathfrak{b}$ to $\mathfrak{h} \times \mathfrak{h}$ defined by

$$\tau'(g, x, y) = (x_0, y_0).$$

For b in \mathbf{B} ,

$$\tau'(b.(g, x, y)) = \tau'(gb^{-1}, b(x), b(y)) = (b(x)_0, b(y)_0) = (x_0, y_0).$$

Since $\mathbf{B} = \mathbf{UH}$, there exists u in \mathbf{U} and h in \mathbf{H} such that $b = uh$. Then $b(x) = b(x_0 + x_+) = u(x_0) + b(x_+)$, $u(x_0)$ is in $x_0 + u$ and $b(x_+)$ is in u , and hence $b(x)$ is in $x_0 + u$. Therefore, τ' is constant on the \mathbf{B} -orbits. Hence there exists a morphism τ from $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ to $\mathfrak{h} \times \mathfrak{h}$ defined by

$$\tau(\overline{(g, x, y)}) = (x_0, y_0).$$

Lemma 4.2. *Let z and z' be in $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ such that $\gamma(z') = \gamma(z)$. Then there exists w in W such that $\tau(z') = w.\tau(z)$.*

Proof. Let (g, x, y) and (g', x', y') in $G \times \mathfrak{b} \times \mathfrak{b}$ be two representatives of z and z' respectively. We have

$$\begin{aligned} \gamma(z) = \gamma(z') &\Rightarrow (g'(x'), g'(y')) = (g(x), g(y)) \\ &\Rightarrow (x', y') = g'^{-1}g.(x, y). \end{aligned}$$

Since $G = \bigcup_{w \in W} \mathbf{U}w\mathbf{B}$, there exist u in \mathbf{U} , w in W , and g_w in $\mathbf{N}_G(\mathfrak{h})$ a representative of w and b in \mathbf{B} such that $g'^{-1}g = ug_wb$. Let $x = x_0 + x_+$, $x' = x'_0 + x'_+$, $y = y_0 + y_+$ and $y' = y'_0 + y'_+$. We have

$$(x', y') = (ug_wb(x), ug_wb(y)) \Rightarrow (u^{-1}(x'), u^{-1}(y')) = (g_wb(x), g_wb(y)).$$

Since $b(x) \in x_0 + u$, $b(y) \in y_0 + u$, $u^{-1}(x') \in x'_0 + u$ and $u^{-1}(y') \in y'_0 + u$, there exist $v_x, v_y, v_{x'}$ and $v_{y'}$ in u such that

$$\begin{aligned} b(x) &= x_0 + v_x, & b(y) &= y_0 + v_y, \\ u^{-1}(x') &= x'_0 + v_{x'} & \text{and} & \quad u^{-1}(y') = y'_0 + v_{y'}. \end{aligned}$$

Hence

$$x'_0 + v_{x'} = g_w(x_0) + g_w(v_x) \quad \text{and} \quad y'_0 + v_{y'} = g_w(y_0) + g_w(v_y).$$

Since $v_x \in \sum_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha$, $g_w(v_x) \in \sum_{\alpha \in \mathcal{R}} \mathfrak{g}^{g_w \cdot \alpha} = \sum_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha$. Therefore, $g_w(v_x) - v_{x'}$ is in $\sum_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha$ and $g_w(x_0) = x'_0$; similarly, $g_w(y_0) = y'_0$. Hence $\tau(z') = g_w \cdot \tau(z)$. \square

Proposition 4.3. *There exists a uniquely defined homomorphism Φ from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $\mathbb{k}[\mathcal{B}_{\mathfrak{g},n}]^G$ such that*

$$\Phi(P) \circ \iota_n = P, \quad \forall P \in S(\mathfrak{h} \times \mathfrak{h})^W.$$

Proof. Let P be in $S(\mathfrak{h} \times \mathfrak{h})^W$, let Γ be the graph of $P \circ \tau$ and let Γ' be the image of Γ by $\gamma_n \times id_{\mathbb{k}}$. We want to prove that Γ' is a graph of an element Q in $\mathbb{k}[\mathcal{B}_{\mathfrak{g},n}]^G$. Let (x, t) and (x, t') be in Γ' . Let z and z' be in $G \times_{\mathbf{B}} \mathfrak{b} \times \mathfrak{b}$ such that

$$(z, t), (z', t') \in \Gamma \quad \text{and} \quad \gamma_n(z) = \gamma_n(z') = x.$$

By Lemma 4.2, there exists w in W such that $\tau(z') = w \cdot \tau(z)$ and hence

$$P \circ \tau(z') = P \circ \tau(z) \quad \text{and} \quad t = t'.$$

Then Γ' is a graph of a function Q on $\mathcal{B}_{\mathfrak{g},n}$. Since Γ and γ_n are closed, Γ' is closed too. Let θ be the projection of Γ' on $\mathcal{B}_{\mathfrak{g},n}$. Since θ is bijective and $\mathcal{B}_{\mathfrak{g},n}$ is normal, then by Zariski's main theorem θ is an isomorphism and hence Q is a regular function on $\mathcal{B}_{\mathfrak{g},n}$. Since γ_n and Γ are G -invariant, and Γ' is G -invariant and Q is G -invariant, and then Q is in $\mathbb{k}[\mathcal{B}_{\mathfrak{g},n}]^G$. Hence we have a homomorphism Φ from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $\mathbb{k}[\mathcal{B}_{\mathfrak{g},n}]^G$ such that

$$\Phi(P) \circ \gamma_n = P \circ \tau.$$

Then, for P in $S(\mathfrak{h} \times \mathfrak{h})^W$,

$$\Phi(P) \circ \iota_n = \Phi(P) \circ \gamma_n \circ i = P \circ \tau \circ i = P,$$

since $\tau \circ i = id_{\mathfrak{h} \times \mathfrak{h}}$. \square

Proposition 4.4. *The homomorphism Φ is an isomorphism from $S(\mathfrak{h} \times \mathfrak{h})^W$ to $\mathbb{k}[\mathcal{B}_{\mathfrak{g}}]^G$.*

Proof. There are canonical homomorphisms

$$\begin{aligned} \mathbb{k}[\mathcal{B}_{\mathfrak{g}}]^G &\rightarrow \mathbb{k}[\mathcal{C}_{\mathfrak{g}}]^G, & \mathbb{k}[\mathcal{C}_{\mathfrak{g}}]^G &\rightarrow S(\mathfrak{h} \times \mathfrak{h})^W, \\ S(\mathfrak{g} \times \mathfrak{g})^G &\rightarrow \mathbb{k}[\mathcal{B}_{\mathfrak{g}}]^G & \text{and} & \quad S(\mathfrak{g} \times \mathfrak{g})^G \rightarrow \mathbb{k}[\mathcal{C}_{\mathfrak{g}}]^G \end{aligned}$$

given by restrictions. Since G is a reductive group, the two last arrows are surjective. Therefore the first one is surjective and the second one is an isomorphism [8, Theorem 2.9]. For all (x, y) in $\mathfrak{b} \times \mathfrak{b}$, the intersection of the closure of $\mathbf{B} \cdot (x, y)$ and $\mathfrak{h} \times \mathfrak{h}$ is nonempty. According to Section 2,

$$\lim_{t \rightarrow 0} h(t)(x, y) = (x_0, y_0).$$

Therefore, any G -orbit in \mathcal{B} has a nonempty intersection with $\mathcal{C}_{\mathfrak{g}}$. As a consequence, the first arrow is an isomorphism since $G \cdot (\mathfrak{h} \times \mathfrak{h})$ is dense in $\mathcal{C}_{\mathfrak{g}}$ [16]. Hence the composition of the two first arrows is an isomorphism whose inverse is Φ . \square

5. On the nullcone

Let $\mathfrak{N}_{\mathfrak{g}}$ be the nilpotent cone of \mathfrak{g} and let $\mathfrak{N}'_{\mathfrak{g}}$ be the set of its regular points. We denote by \mathcal{N} the set of (x, y) in $\mathcal{B}_{\mathfrak{g}}$ such that x and y are nilpotents and by \mathcal{N}' the set of its smooth points. Let ϑ be the restriction of γ to $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$. Then \mathcal{N} is the image of $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ by ϑ .

Proposition 5.1. *The morphism ϑ is a desingularization of \mathcal{N} . The variety \mathcal{N} is closed in $\mathfrak{g} \times \mathfrak{g}$ and irreducible of dimension $3(b_{\mathfrak{g}} - \text{rk } \mathfrak{g})$.*

Proof. Since $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ is a vector bundle on the smooth variety G/\mathbf{B} , it is a smooth variety. Recall that \mathcal{B} is the subvariety of the Borel subalgebras of \mathfrak{g} and $\tilde{\mathcal{B}}$ is the subvariety of elements (u, x, y) of $\mathcal{B} \times \mathfrak{g} \times \mathfrak{g}$ such that u contains x and y . Hence there exists an embedding ϵ from $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ to $\tilde{\mathcal{B}}$ such that ϑ is the composition of ϵ and the canonical projection of $\tilde{\mathcal{B}}$ on $\mathfrak{g} \times \mathfrak{g}$. Therefore, ϑ is a proper morphism since G/\mathbf{B} is a projective variety.

The set $\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}} \cap \mathcal{N}$ is an open subset of \mathcal{N} . Let (x, y) be in $\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}} \cap \mathcal{N}$. Since x is regular, there exists a unique Borel subalgebra containing x , so $\vartheta^{-1}(x, y)$ is a single point. Hence ϑ is birational.

Since \mathcal{N} is the image of $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ and since G/\mathbf{B} is projective variety, \mathcal{N} is closed. Since $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ is irreducible, \mathcal{N} is irreducible. Since ϑ is birational,

$$\dim \mathcal{N} = \dim(G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}) = 3(b_{\mathfrak{g}} - \text{rk } \mathfrak{g}). \quad \square$$

Lemma 5.1. *Let (x, y) be an element of \mathcal{N} and let (v, w) be in $T_{(x,y)}\mathcal{N}$. Then $v + tw$ is contained in $T_{x+ty}\mathfrak{N}$ for all t in \mathbb{k} .*

Proof. Let p'_i be the differential of p_i for $i = 1, \dots, \text{rk } \mathfrak{g}$. By [9, Proposition 0.1], the ideal of \mathfrak{N} is generated by the set $\{p_1, \dots, p_{\text{rk } \mathfrak{g}}\}$. Then it suffices to show that $p'_i(x + ty)(v + tw) = 0$ for all $i = 1, \dots, \text{rk } \mathfrak{g}$. We have

$$\begin{aligned} p'_i(x + ty)(v + tw) &= \frac{d}{da} p_i(x + ty + a(v + tw))|_{a=0} \\ &= \frac{d}{da} p_i(x + av + t(y + aw))|_{a=0} \\ &= \sum_{m=0}^{d_i} t^m \frac{d}{da} p_i^{(m)}(x + av, y + aw)|_{a=0}. \end{aligned}$$

Since (v, w) is in $T_{(x,y)}\mathcal{N}$ and since $\mathcal{N} \subset \sigma^{-1}(\{0\})$,

$$\frac{d}{da} p_i^{(m)}(x + av, y + aw)|_{a=0} = 0 \quad \forall m = 0, \dots, d_i - 1 \text{ and } i = 1, \dots, \text{rk } \mathfrak{g}.$$

Then

$$p'_i(x + ty)(v + tw) = 0 \quad \forall i = 1, \dots, \text{rk } \mathfrak{g},$$

and the claim follows. \square

Let t be in \mathbb{k} . Consider the morphisms

$$\begin{aligned} \theta' : G \times \mathfrak{u} &\rightarrow \mathfrak{N} \\ (g, u) &\mapsto g(u), \\ \rho : G \times \mathfrak{u} \times \mathfrak{u} &\rightarrow G \times \mathfrak{u} \\ (g, u, u') &\mapsto (g, u + tu'), \\ \tau : \mathcal{N} &\rightarrow \mathfrak{N} \\ \tau(x, y) &\mapsto x + ty. \end{aligned}$$

Let ϑ' be the morphism from $G \times_{\mathbf{B}} \mathfrak{u}$ to \mathfrak{N} defined through the quotient by θ' and let τ' be the morphism from $G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u}$ to $G \times_{\mathbf{B}} \mathfrak{u}$ defined through the quotient by ρ . Then we have the following commutative diagram:

$$\begin{array}{ccc} G \times_{\mathbf{B}} \mathfrak{u} \times \mathfrak{u} & \xrightarrow{\vartheta} & \mathcal{N} \\ \tau' \downarrow & & \downarrow \tau \\ G \times_{\mathbf{B}} \mathfrak{u} & \xrightarrow{\vartheta'} & \mathfrak{N}. \end{array}$$

Lemma 5.2. *Let (x, y) be in \mathcal{N} . If the intersection of $P_{x,y}$ and \mathfrak{g}' is not empty, then (x, y) is regular in \mathcal{N} .*

Proof. It suffices to prove the lemma for x regular. Let (x, y) be in \mathcal{N} such that x is regular and let (v, w) be in $T_{(x,y)}\mathcal{N}$. By Lemma 5.1, $v + tw$ is in $T_{x+ty}\mathfrak{N}$ for all t in \mathbb{k} . Let $t \neq 0$ be in \mathbb{k} such that $x + ty$ is regular. Since x is regular, there exists (ξ, ω_1) in $T_{(1,x)}G \times_{\mathbf{B}} \mathfrak{u}$ such that $[\xi, x] + \omega_1 = v$ and since $x + ty$ is regular, there exists ω_2 in \mathfrak{u} such that $[\xi, x + ty] + \omega_2 = v + tw$. Then, for $\omega'_2 = \frac{1}{t}(\omega_2 - \omega_1)$, $[\xi, y] + \omega'_2 = w$. Hence,

$$\begin{aligned} T_{(x,y)}\mathcal{N} \subset \{ (v, w) \in \mathfrak{g} \times \mathfrak{g} \mid \exists \xi \in \mathfrak{g} \text{ and } \omega_1, \omega_2 \in \mathfrak{u}, \text{ verifying} \\ [\xi, x] + \omega_1 = v \text{ and } [\xi, y] + \omega_2 = w \}. \end{aligned}$$

Let μ be the morphism from $\mathfrak{g} \times \mathfrak{u} \times \mathfrak{u}$ to $\mathfrak{g} \times \mathfrak{g}$ defined by

$$\mu(\xi, \omega_1, \omega_2) = ([\xi, x] + \omega_1, [\xi, y] + \omega_2).$$

Then $T_{(x,y)}\mathcal{N}$ is contained in the image of μ . The set $\mu^{-1}(\{0\})$ is equal to the set of elements $(\xi, \omega_1, \omega_2)$ such that $[\xi, x]$ and $[\xi, y]$ are in \mathfrak{u} . Let $(\xi, \omega_1, \omega_2)$ be in $\mu^{-1}(\{0\})$. Since x is regular, $[\xi, x]$ is in \mathfrak{u} if and only if ξ is in \mathfrak{b} . Hence $\dim \mu^{-1}(\{0\}) = b_{\mathfrak{g}}$, and it follows that

$$\dim T_{(x,y)}\mathcal{N} \leq \dim \mathfrak{g} \times \mathfrak{u} \times \mathfrak{u} - \dim \mu^{-1}(\{0\}) = 3(b_{\mathfrak{g}} - \text{rk } \mathfrak{g}) = \dim \mathcal{N}.$$

As a result, (x, y) is regular in \mathcal{N} . \square

Corollary 5.1. *The codimension of the set of singular points of \mathcal{N} is at least two.*

Proof. By Lemma 5.2 we have

$$(\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}_{\mathfrak{g}} \cup \mathfrak{N}_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}}) \cap \mathcal{N} \subset \mathcal{N}',$$

and then

$$\mathcal{N} \setminus \mathcal{N}' \subset \mathcal{N} \setminus (\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}_{\mathfrak{g}} \cup \mathfrak{N}_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}}).$$

Since $\mathcal{N} \setminus (\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}_{\mathfrak{g}} \cup \mathfrak{N}_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}})$ is the image of $G \times_{\mathbf{B}} u \setminus u' \times u \setminus u'$,

$$\begin{aligned} \dim \mathcal{N} \setminus \mathcal{N}' &\leq \dim \mathcal{N} \setminus (\mathfrak{N}'_{\mathfrak{g}} \times \mathfrak{N}_{\mathfrak{g}} \cup \mathfrak{N}_{\mathfrak{g}} \times \mathfrak{N}'_{\mathfrak{g}}) \\ &\leq \dim G \times_{\mathbf{B}} u \setminus u' \times u \setminus u' \\ &= 3(b_{\mathfrak{g}} - \operatorname{rk} \mathfrak{g}) - 2. \quad \square \end{aligned}$$

For an α simple root, let $\mathfrak{p}_{\alpha} = \mathfrak{g}^{-\alpha} + \mathfrak{b}$ be the minimal parabolic subalgebra corresponding to α and let L be the set of all Borel subalgebras contained in \mathfrak{p}_{α} . Then L is a projective line. We call such a line a projective line of type α . For (x, y) in \mathcal{N} , we denote by $\mathcal{B}_{x,y}$ the set of Borel subalgebras containing x and y .

Lemma 5.3. *Let (x, y) be in \mathcal{N} and let \mathfrak{b} and \mathfrak{b}' be two elements in $\mathcal{B}_{x,y}$. There exist a sequence of projective lines $(L_i)_{1 \leq i \leq q}$ and a sequence $(b_i)_{0 \leq i \leq q}$ in $\mathcal{B}_{x,y}$ such that $b_0 = \mathfrak{b}$, $b_q = \mathfrak{b}'$, and b_{i-1} and b_i for all $i = 1, \dots, q$ are in L_i .*

Proof. The proof is inspired by [6, (6.5)]. There exists g in G such that $\mathfrak{b}' = g(\mathfrak{b})$ and by the Bruhat decomposition there exist b, b' in \mathbf{B} , w in W and n_w in $N_G(\mathfrak{h})$ representing w such that $g = bn_w b'$. Since b' is in \mathbf{B} , we can assume that $g = bn_w$. We prove the lemma by induction on the length $l(w)$ of w .

Suppose that $l(w) = 1$; then $w = s_{\alpha}$ for a simple root α . Let $u = b^{-1}(x)$ and $v = b^{-1}(y)$. Therefore, u and v belong to $u \cap s_{\alpha}(u)$, the nilpotent radical of the minimal parabolic subalgebra \mathfrak{p}_{α} . Then x and y belong too, and hence x and y are in all Borel subalgebras of \mathfrak{p}_{α} , so it follows that

$$L_{\alpha} \subset \mathcal{B}_{x,y} \quad \text{and} \quad \mathfrak{b}, \mathfrak{b}' \in L_{\alpha}.$$

Suppose the lemma is true for $l(w) \leq q - 1$.

Let $w = s_1 s_2 \dots s_q$ be a reduced expression, where each s_i is the reflection associated with the simple root α_i . Let $u = b^{-1}(x)$ and $v = b^{-1}(y)$, so u and v are in $u \cap w(u)$. We have $u = \bigoplus_{\gamma \in \mathcal{R}^+} \mathfrak{g}^{\gamma}$, and then $w(u) = \bigoplus_{\gamma \in \mathcal{R}^+, w(\gamma) > 0} \mathfrak{g}^{\gamma}$. Hence $u \cap w(u) = \bigoplus_{\gamma \in \mathcal{R}^+, w(\gamma) > 0} \mathfrak{g}^{\gamma}$. Since $w(\alpha_q) < 0$, $u \cap w(u) \subset u \cap s_{\alpha_q}(u)$, the nilpotent radical of \mathfrak{p}_{α_q} , and then u and v are in $u \cap s_{\alpha_q}(u)$, and hence x and y are too. In particular, x and y are in $s_q(\mathfrak{b})$. As a result, \mathfrak{b} and $s_q(\mathfrak{b})$ are in L_{α_q} and $s_q(\mathfrak{b})$, and $bn_{w'}(s_q(\mathfrak{b}))$ are in $\mathcal{B}_{x,y}$, where $w' = s_1 \dots s_{q-1}$ and $n_{w'} \in N_G(\mathfrak{h})$ is a representative of w' . Since $l(w') = q - 1$, by induction, there exist a sequence of projective lines $(L_i)_{1 \leq i \leq q-1}$ and a sequence $(b_i)_{0 \leq i \leq q-1}$ in $\mathcal{B}_{x,y}$ such that

$$b_0 = s_q(\mathfrak{b}), \quad b_{q-1} = bn_{w'}(s_q(\mathfrak{b})) \quad \text{and} \quad b_{i-1}, b_i \in L_i, \quad \forall 1 \leq i \leq q - 1.$$

We have

$$b, s_q(b) \in L_{\alpha_q} = L_0,$$

and then the sequence of projective lines $(L_i)_{0 \leq i \leq q-1}$ and the sequence b, b_0, \dots, b_{q-1} verify

$$b_{q-1} = b', \quad b, b_0 \in L_0 \quad \text{and} \quad b_{i-1}, b_i \in L_i \quad \forall 1 \leq i \leq q-1.$$

This completes the proof. \square

Proposition 5.2. *The set $\mathcal{B}_{x,y}$ is connected.*

Proof. Let b and b' be two elements in $\mathcal{B}_{x,y}$ and let X be the connected component of $\mathcal{B}_{x,y}$ containing b . By Lemma 5.3, there exist a sequence of projective lines $(L_i)_{1 \leq i \leq q}$ and a sequence $(b_i)_{0 \leq i \leq q}$ in $\mathcal{B}_{x,y}$ such that

$$b_0 = b, \quad b_q = b' \quad \text{and} \quad b_{i-1}, b_i \in L_i \quad \forall i = 1, \dots, q.$$

Since b and b_1 are in L_1 and since L_1 is connected, X contains L_1 and b_1 . By induction on q , suppose that X contains b_{q-1} . Since b_{q-1} and b_q are in L_q and since L_q is connected, X contains L_q and $b_q = b'$. Hence $\mathcal{B}_{x,y}$ is connected. \square

Let (\mathcal{N}_n, τ) be the normalization of \mathcal{N} .

Proposition 5.3. *The morphism τ is bijective.*

Proof. Since $G \times_{\mathbf{B}} u \times u$ is normal and since ϑ is dominant, there exists a unique morphism ϑ_n from $G \times_{\mathbf{B}} u \times u$ to \mathcal{N}_n such that $\vartheta = \tau \circ \vartheta_n$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} G \times_{\mathbf{B}} u \times u & \xrightarrow{\vartheta_n} & \mathcal{N}_n \\ & \searrow \vartheta & \downarrow \tau \\ & & \mathcal{N}. \end{array}$$

Let (x, y) be in \mathcal{N} . Then $\vartheta^{-1}(x, y) = \vartheta_n^{-1}(\tau^{-1}(x, y))$. Since ϑ is a birational proper morphism, ϑ_n is too. Then $\vartheta^{-1}(x, y)$ is the disjoint union of the fibers of ϑ_n at the elements of $\tau^{-1}(x, y)$. By Zariski's main theorem, $\vartheta_n^{-1}(z)$ is connected for all z in \mathcal{N}_n . Since $\mathcal{B}_{x,y}$ is connected, $\vartheta^{-1}(x, y)$ is too. Therefore, $|\tau^{-1}(x, y)| = 1$. As a result, τ is bijective since τ is surjective. \square

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