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On the involution module of $\mathrm{GL}_n(2^f)$

Lars Pforte

Department of Mathematics, National University of Ireland, Maynooth, Co. Kildare, Ireland

A R T I C L E I N F O

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For any group G the set of involutions $\mathcal{J}$ in G, that is, the set of group elements that have order two, forms a G-set under conjugation. The corresponding kG-permutation module $k\mathcal{J}$ is the involution module of G. Here k is an algebraically closed field of characteristic two. In this paper we discuss aspects of the involution module of the general linear group $\mathrm{GL}_n(2^f)$. We determine almost all components of this module. Furthermore we present a vertex and the Green correspondent of each component.

1. Introduction

The goal of this paper is to investigate the involution module of the general linear group $\mathrm{GL}_n(2^f)$, where $f \geqslant 1$ is an integer. We start with an introduction to the idea of the involution module. Also in this section we develop the necessary notation and state some important results which we employ in our work.

In Section 2 we present a partial decomposition of the involution module of $\mathrm{GL}_n(2^f)$. In Theorem 3.1 the possible vertices of a component of our involution module are given. By component we mean an indecomposable summand. Sections 4–8 focus on each of those possible candidates. Finally we summarize our results in Theorem 9.1, followed by some further observations.

1.1. The involution module

Let G be a finite group and let k be an algebraically closed field of characteristic 2. By $\mathcal{J}$ we denote the set of involutions in G, that is, the set of elements in G of order two. Then G acts on $\mathcal{J}$ by conjugation. In particular we obtain the kG-permutation module $k\mathcal{J}$. This module is called the *involution module* of G. In the paper [15], G.R. Robinson investigated the projective components of

E-mail address: larspforte@yahoo.ie.

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this module using the Frobenius–Schur indicator. Later J. Murray studied the involution module in general in [8–11]. He, too, worked with the Frobenius–Schur indicator, but also used block-theoretical methods, such as the defect class of a block. Furthermore P. Collings studied parts of the involution module of the symmetric group in his PhD thesis [4], focusing on the fixed point free involutions in the symmetric group $\mathrm{Sym}(n)$. Finally the author studied the involution module of the special linear group $\mathrm{SL}_2(2^f)$ in [12].

The main motivation of this paper is to study the involution module of the general linear group $\mathrm{GL}_n(2^f)$. We are able to determine the number of components and describe the vertex and Green correspondent of each component. However as many of our calculations are still valid for any prime number p, we present most results for $\mathrm{GL}_n(p^f)$ and k of characteristic p.

1.2. Notation

Throughout this paper let k be an algebraically closed field of prime characteristic p. Also let $f, n \geqslant 1$ be integers and set $q := p^f$. By $\mathbb{F}_q$ we mean the finite field with q elements. Our main group of interest is the general linear group $\mathrm{GL}_n(q)$, that is, the group of all invertible $n \times n$-matrices with entries in $\mathbb{F}_q$. For convenience we denote this group by GL_n in the following. Note that both the upper-triangular matrices in GL_n and the upper-triangular matrices in GL_n with ones on the main diagonal form subgroups of GL_n. We denote them by B_n and U_n, respectively. Furthermore U_n is a Sylow-p-subgroup of both B_n and GL_n. As is standard I_n denotes the identity matrix in GL_n. For integers $1 \leqslant k < l \leqslant n$ let $E_{k,l}(\alpha)$ be the $n \times n$-matrix with zeros everywhere, except for the (k,l)-entry which is $\alpha \in \mathbb{F}_q$. Then $F_{k,l} := \{I_n + E_{k,l}(\alpha)\colon\ \alpha \in \mathbb{F}_q\}$ is the subgroup of U_n of matrices where all entries off the main diagonal are zero, except for the (k,l)-entry which can be anything in $\mathbb{F}_q$. Finally for any two integers $r, s \geqslant 0$ we define $\mathrm{GL}_{r,s} := \mathrm{GL}_r \times \mathrm{GL}_s$, where GL_0 is the trivial group.

Next let $\lambda_1, \ldots, \lambda_t \geqslant 0$ such that $\lambda_1 + \cdots + \lambda_t = n$. Then if $A_r \in \mathrm{GL}_{\lambda_r}$, for $r = 1, \ldots, t$, we define $D_n(A_1, A_2, \ldots, A_t)$ as that matrix in GL_n that has the matrices $A_1, \ldots, A_t$ on its main diagonal and zeros everywhere else. Likewise $D_n(A_1\bullet, A_2\bullet, \ldots, A_t)$ denotes a matrix with the matrices $A_1, \ldots, A_t$ on its main diagonal, zeros below and arbitrary elements in $\mathbb{F}_q$ above. Note that $D_n(A_1\bullet, A_2\bullet, \ldots, A_t)$ is not unique, but in our considerations it does not matter what the specific entries above the diagonal of matrices $A_1, \ldots, A_t$ are. In the same sense we define the groups $D_n(H_1, H_2, \ldots, H_t)$ and $D_n(H_1\bullet, H_2\bullet, \ldots, H_t)$, for $H_r \leqslant \mathrm{GL}_{\lambda_r}$. Finally set $\mathrm{GP}_{\lambda_1,\ldots,\lambda_r} := D_n(\mathrm{GL}_{\lambda_1}\bullet, \mathrm{GL}_{\lambda_2}\bullet, \ldots, \mathrm{GL}_{\lambda_r})$. Note that $\mathrm{GP}_{\lambda_1,\ldots,\lambda_r}$ is known as a parabolic subgroup of GL_n.

Still let $n \geqslant 1$. Then $\mathcal{W}_n$ denotes the group of permutation matrices in GL_n. Note that we can identify a permutation matrix with a unique permutation in the symmetric group $\mathrm{Sym}(n)$. In fact $\omega \in \mathrm{Sym}(n)$ corresponds to the permutation matrix $(\delta_{k,\omega(l)})_{k,l}$, where $\delta_{-,-}$ denotes the Kronecker-symbol.

Finally we discuss some block theory of GL_n. Refer to [6] and [16] for definitions and more details. A block of GL_n has either full defect or is of defect zero. There are exactly $q-1$ of each type. Also the module $k_{B_n}{\uparrow}^{\mathrm{GL}_n}$ has a unique irreducible projective component St_n, which is called the *Steinberg module*. This module is self-dual and has dimension $q^{\binom{n}{2}}$. Furthermore the center $\mathrm{Z}(\mathrm{GL}_n)$ of GL_n acts trivially on $k_{B_n}{\uparrow}^{\mathrm{GL}_n}$. This follows since $\mathrm{Z}(\mathrm{GL}_n) = \{\alpha I_n\colon\ \alpha \in \mathbb{F}_q^*\}$ is a normal subgroup of B_n. In particular $\mathrm{Z}(\mathrm{GL}_n)$ acts trivially on St_n.

Let $\mathcal{S}$ denote the GL_n-representation corresponding to St_n. For $A \in \mathrm{GL}_n$ and $j = 0, 1, \ldots, q-2$, we define $\mathcal{S}^j(A) := (\det(A))^j \cdot \mathcal{S}(A)$. Then $\mathcal{S}^j$ is a projective irreducible GL_n-representation. We denote the corresponding GL_n-module by St_n^j. If $(\mathrm{St}_n^j)^*$ denotes the dual of St_n^j, then $(\mathrm{St}_n^j)^* = \mathrm{St}_n^{q-1-j}$. The modules $\mathrm{St}_n, \mathrm{St}_n^1, \ldots, \mathrm{St}_n^{q-2}$ are all the projective irreducible GL_n-modules. As every block of defect zero contains a unique irreducible projective module we let B_j^z denote the block that contains St_n^j, for $j = 0, 1, \ldots, q-2$. Thus

$$B_j^z = \mathrm{St}_n^j \otimes \mathrm{St}_n^{q-1-j}, \quad \text{as } \mathrm{GL}_{n,n}\text{-modules}. \tag{1}$$

In particular B_j^z is projective and irreducible as a $\mathrm{GL}_{n,n}$-module.

1.3. Broué's Theorem

In this section we present a method to study vertices of the components of permutation modules. The following approach is due to M. Broué [2].

Let G be a finite group. As usual for a subgroup $H \leqslant G$, we denote the normalizer of H in G by $\mathrm{N}_G(H)$. Suppose that $V \leqslant G$ is a p-group and X is a G-set. By $\mathrm{Fix}_X(V)$ we mean the set of elements in X that stay fixed under the action of V. Then $\mathrm{Fix}_X(V)$ is an $\mathrm{N}_G(V)$-set, and $k\,\mathrm{Fix}_X(V)$ is an $\mathrm{N}_G(V)$-module.

Theorem 1.1 *(Broué). Let $V \leqslant G$ be a p-group and X a G-set. Then kX, as kG-module, and $k\,\mathrm{Fix}_X(V)$, as $k\mathrm{N}_G(V)$-module, have the same number of components with vertex V. Furthermore the components of $k\,\mathrm{Fix}_X(V)$ with vertex V are the Green correspondents with respect to $(G, V, \mathrm{N}_G(V))$ of the components of kX with vertex V.*

For a subgroup H of G, we have $k_H{\uparrow}^G \cong k(G/H)$. Here G/H is the set of left cosets of H in G, regarded as a G-set under translation. Hence $k_H{\uparrow}^G$ and $k\,\mathrm{Fix}_{G/H}(V)$ have the same number of components with vertex V, for any p-subgroup V of G. The next statement follows from Mackey's lemma and the fact that $k\,\mathrm{Fix}_{G/H}(V)$ is a direct summand of $(k_H{\uparrow}^G){\downarrow}_{\mathrm{N}_G(V)}$.

Lemma 1.2. *Let H and V be subgroups of G, with V a p-group. Then*

$$k\,\mathrm{Fix}_{G/H}(V) \cong \bigoplus_{\substack{g \in H\backslash G/\mathrm{N}_G(V),\\ V \leqslant H^g}} k_{H^g \cap \mathrm{N}_G(V)}{\uparrow}^{\mathrm{N}_G(V)}, \quad \text{as } k\mathrm{N}_G(V)\text{-modules}.$$

1.4. Inflation and deflation

Let $\varphi : G \to R$ be a group homomorphism. Set $\overline{H} := \varphi(H)$, for $H \leqslant G$. Next let L be a group such that $\overline{G} \leqslant L \leqslant R$. Then every kL-module N can be considered as a module for G using inflation. We denote the resulting kG-module by N^φ. If $\varphi|_H$ denotes the restriction of φ to some subgroup H of G, then

$$N^\varphi{\downarrow}_H \cong N^{\varphi|_H}, \quad \text{as } kH\text{-modules}. \tag{2}$$

Next let M be a kG-module with $\ker\varphi$ acting trivially on M. Then M can be considered as a $k\overline{G}$-module via deflation. We denote the arising $k\overline{G}$-module by $\varphi(M)$. If $H \leqslant G$, then $\varphi(M){\downarrow}_{\overline{H}} \cong \varphi|_H(M{\downarrow}_H)$, as $k\overline{H}$-modules. Furthermore $(\varphi(M))^\varphi \cong M$ and $\varphi(N^\varphi) \cong N{\downarrow}_{\overline{G}}$, as kG-modules and $k\overline{G}$-modules, respectively. Also inflation and deflation commute with taking the dual of a module.

Lemma 1.3. *Let $\varphi : G \to R$ be a homomorphism of groups. Also let $H \leqslant G$ so that $\ker\varphi \leqslant H$. Then for every $k\overline{H}$-module N, $(N{\uparrow}^{\overline{G}})^\varphi \cong (N^{\varphi|_H}){\uparrow}^G$, as kG-modules.*

Proof. The isomorphism is given by $\sum_{j=1}^t x_j \otimes n_j \mapsto \sum_{j=1}^t \varphi(x_j) \otimes n_j$, where $\{x_1, \dots, x_t\}$ is a left transversal for G/H. $\square$

Next for $K \leqslant R$ we define $\varphi^{-1}(K) := \{g \in G\colon\ \varphi(g) \in K\}$. Observe that $\varphi(\varphi^{-1}(K)) = K \cap \varphi(G)$.

Lemma 1.4. *Let $\varphi : G \to R$ be a homomorphism of groups. Furthermore suppose that N is an indecomposable $k\overline{G}$-module. Assume that $\overline{W} \leqslant \overline{G}$ is a vertex of N and $V \leqslant G$ is a vertex of N^φ. Then $\overline{W}$ is $\overline{G}$-conjugate to $\overline{V}$ and V is G-conjugate to a Sylow-p-subgroup of $\varphi^{-1}(\overline{W})$.*

Proof. Set $H := \varphi^{-1}(\overline{W})$. Then $\overline{H} = \overline{W}$ and $\ker\varphi \leqslant H$. Thus N^φ is a component of $(N_{\overline{H}}\uparrow^{\overline{G}})^\varphi$. Also $(N_{\overline{H}}\uparrow^{\overline{G}})^\varphi \cong (N_{\overline{H}})^{\varphi|_H}\uparrow^G$, by Lemma 1.3, and so N^φ is relatively H-projective. Hence w.l.o.g. $V \leqslant S$, for some $S \in \mathrm{Syl}_p(H)$.

As $V \leqslant \varphi^{-1}(\overline{V})$, we get that N^φ is a component of $(N^\varphi\downarrow_{\varphi^{-1}(\overline{V})})\uparrow^G$. Hence N is a component of $\varphi((N^\varphi\downarrow_{\varphi^{-1}(\overline{V})})\uparrow^G)$. Since $N^\varphi\downarrow_{\varphi^{-1}(\overline{V})} \cong (N\downarrow_{\overline{V}})^{\varphi|_{\varphi^{-1}(\overline{V})}}$, it follows with Lemma 1.3 that $\varphi((N^\varphi\downarrow_{\varphi^{-1}(\overline{V})})\uparrow^G) \cong (N\downarrow_{\overline{V}})\uparrow^{\overline{G}}$. Thus, $\overline{W} \leqslant_{\overline{G}} \overline{V}$. As $\overline{V} \leqslant_{\overline{G}} \overline{S} \leqslant \overline{W}$ we conclude that $\overline{W} =_{\overline{G}} \overline{V}$, that is, $\overline{V}$ is $\overline{G}$-conjugate to $\overline{W}$.

Since $\ker\varphi$ acts trivially on N^φ, V contains a Sylow-p-subgroup of $\ker\varphi$. But then $V \cap \ker\varphi = S \cap \ker\varphi$, and since $\overline{V} = \overline{S}$, it follows that $V = S$. □

In the following let $\mathrm{p}^\sharp(M)$ denote the number of projective components of a module M, and let $P(V)$ be the projective cover of V.

Lemma 1.5. *Let $\varphi : G \to \overline{G}$ be an epimorphism of groups, let $H \leqslant G$ and suppose that $\ker\varphi$ is a p-group, with $H \cap \ker\varphi = \langle 1\rangle$. Then $\mathrm{p}^\sharp(k_H\uparrow^G) = \mathrm{p}^\sharp(k_{\overline{H}}\uparrow^{\overline{G}})$.*

Proof. Since $\ker\varphi$ is a normal p-subgroup of G, Clifford theory shows that the irreducible kG-modules are in bijection with the irreducible $k\overline{G}$-modules by deflation/inflation with respect to φ. Hence for every irreducible kG-module V there exists an irreducible $k\overline{G}$-module $\overline{V}$ such that $V \cong \overline{V}^\varphi$. The lemma follows if the multiplicity of $P(V)$ as a summand of $k_H\uparrow^G$ equals the multiplicity of $P(\overline{V})$ as a summand of $k_{\overline{H}}\uparrow^{\overline{G}}$.

Assume $P(V)$ appears exactly d times in $k_H\uparrow^G$. Then Theorem 3 in [14] shows that $P(k_H)$ appears exactly d times in $V\downarrow_H$. Since $\ker\varphi|_H = \langle 1\rangle$, every kH-module M can be considered as a $k\overline{H}$-module by deflation with respect to $\varphi|_H$. Then $\varphi|_H(V\downarrow_H) \cong \varphi|_H(\overline{V}^\varphi\downarrow_H) \cong \overline{V}\downarrow_{\overline{H}}$, by (2). As certainly $\varphi|_H(P(k_H)) \cong P(k_{\overline{H}})$ and $P(k_H) \cong (P(k_{\overline{H}}))^{\varphi|_H}$, it follows that $P(k_{\overline{H}})$ appears exactly d times in $\overline{V}\downarrow_{\overline{H}}$. Again by Theorem 3 in [14], we conclude that $P(\overline{V})$ appears exactly d times in $k_{\overline{H}}\uparrow^{\overline{G}}$. □

2. The involution module of $\mathrm{GL}_n(2^f)$

In this section let $p = 2$ and let $G := \mathrm{GL}_n(2^f)$, for some integer $f \geqslant 1$. We aim to study the involution module of G. First we determine all involutions in G. As is standard, $\lfloor r\rfloor$ denotes the greatest integer less or equal to $r \in \mathbb{R}$.

Lemma 2.1. *There are exactly $\lfloor\frac{n}{2}\rfloor$ conjugacy classes of involutions in G.*

Proof. It is a straightforward exercise to show that a Jordan form of order 2 does only contain 1-by-1 and 2-by-2 Jordan blocks, with all its eigenvalues equal to one. In particular there are exactly $\lfloor\frac{n}{2}\rfloor$ such Jordan forms. It furthermore follows that each involution $X \in G$ is G-conjugate to its Jordan form. □

Our goal in this section is to partially decompose the involution module into direct summands that are easier to handle. Hence we determine the G-conjugacy classes of the involutions in G and calculate the centralizer of a representative for each of these classes. Let $i = 1, \ldots, \lfloor\frac{n}{2}\rfloor$. We define

$$T_i := \begin{pmatrix} I_i & & I_i \\ & I_{n-2i} & \\ & & I_i \end{pmatrix}. \tag{3}$$

This notation means that T_i has 1's on the main diagonal and a 1 in each entry $(j, n-i+j)$, for $j = 1, \ldots, i$. All other entries are zero. Note that T_i is an involution. We denote the centralizer of T_i

in G by $\mathrm{C}_G(T_i)$. We define

$$C_i := \{D_n(A\bullet, B\bullet, A)\colon\ A \in \mathrm{GL}_i,\ B \in \mathrm{GL}_{n-2i}\}. \tag{4}$$

In the following let $m := \lfloor \frac{n}{2} \rfloor$.

Lemma 2.2. *Let $i = 1, \ldots, m$. Then $\mathrm{C}_G(T_i) = C_i$. Moreover $T_1, \ldots, T_m$ represent the m different conjugacy classes of elements of order p in G. In particular they represent the m different conjugacy classes of involutions in G.*

Proof. An easy matrix calculation shows that $\mathrm{C}_G(T_i) = C_i$. Also observe that the rank of $T_i - I_n$ equals i. Hence different T_i lie in different conjugacy classes. The last part of the statement follows from Lemma 2.1. $\square$

This provides a first partial decomposition of the involution module $k\mathfrak{I}$ as now:

$$k\mathfrak{I} \cong \bigoplus_{i=1}^{m} k_{\mathrm{C}_G(T_i)}\uparrow^G = \bigoplus_{i=1}^{m} k_{C_i}\uparrow^G. \tag{5}$$

3. A more general framework

For the remainder of this paper we work in the following more general framework. Let k be an algebraically closed field of prime characteristic p. Also let $G := \mathrm{GL}_n(q)$, where $q = p^f$, for some $f \geqslant 1$. Furthermore set $m := \lfloor \frac{n}{2} \rfloor$. For $i = 1, \ldots, m$ consider the element T_i and the group C_i as defined in (3) and (4), respectively. Note that still C_i is the centralizer of the element T_i, which is an element of order p.

Our aim is to study the module $\bigoplus_{i=1}^{m} k_{C_i}\uparrow^G$, which includes the case of the involution module of $\mathrm{GL}_n(2^f)$, that is, when $p = 2$. In the following we fix $i \in \{1, \ldots, m\}$ and focus on the direct summand $k_{C_i}\uparrow^G$. In addition to C_i we define the following groups:

$$\begin{aligned}
S_i &:= C_i \cap U_n = \{D_n(A\bullet, B\bullet, A) \in G\colon\ A \in U_i,\ B \in U_{n-2i}\},\\
N_i &:= \{D_n(A\bullet, B\bullet, \alpha A)\colon\ A \in \mathrm{GL}_i,\ B \in \mathrm{GL}_{n-2i},\ \alpha \in \mathbb{F}_q^\times\},\\
R_i &:= D_n(\mathrm{GL}_i\bullet, \mathrm{GL}_{n-2i}\bullet, \mathrm{GL}_i).
\end{aligned} \tag{6}$$

Then $S_i \leqslant C_i \leqslant N_i \leqslant R_i \leqslant G$. Note that $|S_i| = q^{\binom{n}{2}-\binom{i}{2}} = |C_i|_p = |N_i|_p$. So S_i is a Sylow-p-subgroup of both C_i and N_i.

Furthermore for $r, s \geqslant 1$ such that $r + s \leqslant n$ we set

$$V_{r,s} := D_n(I_r\bullet, U_{n-r-s}\bullet, I_s). \tag{7}$$

Then, if defined for i and n, we have $V_{i+1,i+1} \leqslant V_{i,i+1}, V_{i+1,i} \leqslant V_{i,i} \leqslant S_i$. In [13] the author has shown the following:

Theorem 3.1. *Suppose M is a component of $k_{C_i}\uparrow^G$. Then, one of the following groups is a vertex of M:*

$$S_i, \quad V_{i,i}, \quad V_{i+1,i}, \quad V_{i,i+1}, \quad V_{i+1,i+1}, \quad \langle I_n \rangle.$$

Furthermore if M has a trivial vertex, then $i = \lfloor \frac{n}{2} \rfloor$.

Observe that these six groups are not conjugate in G as they all have different size apart from $V_{i,i+1}$ and $V_{i+1,i}$, which however have different numbers of fixed points on the natural GL_n-module $(\mathbb{F}_q)^n$.

Next we state some observations. It is easy to see that C_i is normal in N_i. In fact $N_i = \mathrm{N}_G(C_i)$, but we do not require this stronger fact and thus omit a proof. Also observe that $N_i/C_i \cong \mathbb{F}_q^\times$ is the cyclic group of order $q-1$. Hence

$$k_{C_i}\!\uparrow^{N_i} \cong \bigoplus_{r=0}^{q-2} W_r, \tag{8}$$

where W_r is a one-dimensional N_i-module, and $D_n(A\bullet, B\bullet, \alpha A) \in N_i$ acts on W_r by multiplication by α^r. In particular, $W_0 \cong k_{N_i}$. Furthermore observe that $W_{q-1-r} \otimes W_r \cong k_{N_i}$. If W_r^* denotes the dual of W_r, then

$$W_r^* \cong W_{q-1-r}, \quad \text{for all } r = 0, \dots, q-2. \tag{9}$$

Since C_i acts trivially on W_r, for all $r = 0, \dots, q-2$, it follows that S_i is contained in a vertex of W_r. Thus W_r has vertex S_i, as $S_i \in \mathrm{Syl}_p(N_i)$.

Next consider the epimorphism

$$\mathcal{R} : R_i \to \mathrm{GL}_{i,i}: \quad D_n(A\bullet, B\bullet, C) \mapsto (A, C). \tag{10}$$

Note that $\mathcal{R}(C_i) = \Delta\,\mathrm{GL}_i$, where $\Delta H := \{(h,h):\ h \in H\}$, for any group H. Moreover $\ker \mathcal{R} \leqslant C_i$. Then $k_{C_i}\!\uparrow^{R_i} \cong (k_{\Delta\,\mathrm{GL}_i}\!\uparrow^{\mathrm{GL}_{i,i}})^{\mathcal{R}}$, by Lemma 1.3. But $k_{\Delta\,\mathrm{GL}_i}\!\uparrow^{\mathrm{GL}_{i,i}}$ is the direct sum of all p-blocks of GL_i, regarded as $\mathrm{GL}_{i,i}$-modules. In particular

$$k_{C_i}\!\uparrow^{R_i} \cong \bigoplus_{\substack{B \text{ a } p\text{-block} \\ \text{of } \mathrm{GL}_i}} B^{\mathcal{R}}. \tag{11}$$

As each B is a component of $k_{\Delta\,\mathrm{GL}_i}\!\uparrow^{\mathrm{GL}_{i,i}}$, we now have a decomposition of $k_{C_i}\!\uparrow^{R_i}$ into a direct sum of indecomposable R_i-modules.

Theorem 3.2. (a)

$$k_{C_i}\!\uparrow^{G} \cong \bigoplus_{r=0}^{q-2} W_r\!\uparrow^{G}. \tag{12}$$

Furthermore $W_r\!\uparrow^G$ has at least one component with vertex S_i. In particular, $k_{C_i}\!\uparrow^G$ has at least $q-1$ such components.

(b)

$$k_{C_i}\!\uparrow^{G} \cong \bigoplus_{\substack{B \text{ a } p\text{-block} \\ \text{of } \mathrm{GL}_i}} B^{\mathcal{R}}\!\uparrow^{G}. \tag{13}$$

Furthermore if B has full defect then $B^{\mathcal{R}}\!\uparrow^G$ has at least one component with vertex S_i. If B has defect zero, then $B^{\mathcal{R}}\!\uparrow^G$ has at least one component with vertex $V_{i,i}$. In particular, $k_{C_i}\!\uparrow^G$ has at least $q-1$ components with vertex $V_{i,i}$.

(c) *For all* $r = 0, 1, \ldots, q-2$ *there is a unique block* B *of* GL_i *of full defect such that* $B^{\mathcal{R}}\uparrow^G$ *is a direct summand of* $W_r\uparrow^G$. *We denote this block by* B_r. *Then for* $s := \gcd(q-1, i)$ *and* $q-1 = s \cdot t$, *we have*

$$W_r\uparrow^G \cong \begin{cases} B_r^{\mathcal{R}}\uparrow^G, & \text{if } i = 1 \text{ or } s \nmid r, \\ B_r^{\mathcal{R}}\uparrow^G \oplus \bigoplus_{u=0}^{s-1} (B_{l+ut}^z)^{\mathcal{R}}\uparrow^G, & \text{if } i \geqslant 2 \text{ and } s \mid r, \end{cases}$$

where $l \in \{0, 1, \ldots, t-1\}$ *such that* $il \equiv -r \bmod (q-1)$. *Such an* l *is uniquely determined in the case* $s \mid r$.

Proof. Part (a) follows from (8) and transitivity of induction. Furthermore, as W_r has vertex S_i, the same must be true for at least one component of $W_r\uparrow^G$.

The decomposition (13) in part (b) follows from (11). Next recall that a block B of GL_i is either of defect zero or full defect. In the first case B is projective as a $\mathrm{GL}_{i,i}$-module. Since $V_{i,i} \in \mathrm{Syl}_p(\ker \mathcal{R})$ we conclude from Lemma 1.4 that $B^{\mathcal{R}}$ has vertex $V_{i,i}$. In the second case B has vertex ΔU_i. Since $\Delta U_i \leqslant \mathcal{R}(R_i)$ and $S_i \in \mathrm{Syl}_p(\mathcal{R}^{-1}(\Delta U_i))$, Lemma 1.4 implies that $B^{\mathcal{R}}$ has vertex S_i.

It remains to prove part (c). Note that $k_{C_i}\uparrow^{R_i}$ has exactly $q-1$ components with vertex S_i, which arise from the blocks of full defect. Since W_r has vertex S_i, the direct summand $W_r\uparrow^{R_i}$ of $k_{C_i}\uparrow^{R_i}$ must have a component of vertex S_i. Hence this must be some $B^{\mathcal{R}}$ for some p-block B of GL_i with full defect.

Next if $i = 1$, then the blocks of full defect coincide with the blocks of defect zero and thus we have $W_r\uparrow^{R_i} \cong B_r^{\mathcal{R}}$. Hence in the following let $i \geqslant 2$. Recall from (1) that $B_j^z \cong \mathrm{St}_i^j \otimes \mathrm{St}_i^{q-1-j}$ is an irreducible $\mathrm{GL}_{i,i}$-module. Then $(B_j^z)^{\mathcal{R}}$ is an irreducible R_i-module. First we show that $(B_j^z)^{\mathcal{R}}$ is a component of $W_r\uparrow^{R_i}$ if and only if $(B_j^z)^{\mathcal{R}}$ appears in the head of $W_r\uparrow^{R_i}$. We only need to verify the sufficiency of this claim. So let $(B_j^z)^{\mathcal{R}}$ appear in the head of $W_r\uparrow^{R_i}$. Then B_j^z appears in the head of $\mathcal{R}(W_r\uparrow^{R_i})$. But B_j^z is projective, and thus must be a component of $\mathcal{R}(W_r\uparrow^{R_i})$. Therefore $(B_j^z)^{\mathcal{R}}$ is a component of $W_r\uparrow^{R_i}$.

Next we claim W_{-ij} appears in the socle of $(B_j^z)^{\mathcal{R}}\downarrow_{N_i}$. As $(B_j^z)^{\mathcal{R}}$ is a component of $k_{C_i}\uparrow^{R_i}$ there exists some $\lambda = \lambda_1 \otimes \lambda_2 \in \mathrm{St}_i^j \otimes \mathrm{St}_i^{q-1-j}$, such that C_i acts trivially on λ. The claim follows once we have shown that $kN_i \cdot \lambda \cong W_{-ij}$. So let $X \in N_i$. Then $X = D_n(A\bullet, B\bullet, \alpha A) \in D_n(I_{n-i}, \alpha I_i) \cdot C_i$. Thus

$$X \cdot \lambda = D_n(I_{n-i}, \alpha I_i) \cdot \lambda = \lambda_1 \otimes \bigl((\alpha I_i) \cdot \lambda_2\bigr).$$

But $\alpha I_i \in Z(\mathrm{GL}_i)$ acts trivially on St_i, and therefore $(\alpha I_i) \cdot \lambda_2 = \alpha^{-ij} \cdot \lambda_2$. In particular $D_n(A\bullet, B\bullet, \alpha A) \cdot \lambda = \alpha^{-ij} \cdot \lambda$, and thus $kN_i \cdot \lambda \cong W_{-ij}$.

By Frobenius reciprocity it now follows that $(B_j^z)^{\mathcal{R}}$ appears in the head of $W_{-ij}\uparrow^{R_i}$. Overall the previous two paragraphs show that $(B_j^z)^{\mathcal{R}}$ is a component of $W_r\uparrow^{R_i}$ if and only if $ij \equiv -r \bmod (q-1)$.

Next suppose $s \nmid r$. Since s divides both $q-1$ and i, there is no $l \geqslant 0$, such that $il \equiv -r \bmod (q-1)$. In particular, there are no blocks of defect zero contributing towards $W_r\uparrow^{R_i}$, and so $W_r\uparrow^{R_i} \cong B_r$.

If $s \mid r$, then there exists $l \in \{0, 1, \ldots, t-1\}$ such that $il \equiv -r \bmod (q-1)$, as $\gcd(i/s, t) = 1$. Now $i(l+ut) \equiv -r \bmod (q-1)$, for all $u = 0, 1, \ldots, s-1$. So $B_l^z, B_{l+t}^z, \ldots, B_{l+(s-1)t}^z$ are components of $W_r\uparrow^{R_i}$. Since t different integers in $\{0, 1, \ldots, q-2\}$ are divisible by s, we have accounted for all blocks of defect zero. In particular this completes the proof. $\square$

Remark 3.3. We claim that the block B_0 as given by Theorem 3.2(c) is the principal block of GL_i. Since $W_0 \cong k_{N_i}$, we know that k_{R_i} appears in the socle of $W_0\uparrow^{R_i}$. Hence $\mathcal{R}(W_0\uparrow^{R_i})$ has $k_{\mathrm{GL}_{i,i}}$ in its socle. Consequently, considered as a GL_i-module it contains the trivial GL_i-module. In particular the principal block must appear as a component of $\mathcal{R}(W_0\uparrow^{R_i})$, and thus equals B_0.

We conclude this section with a useful duality on GL_n. Let $\eta \in \mathcal{W}_n$ be the permutation matrix that corresponds to $(1,n)(2,n-1)\ldots \in \mathrm{Sym}(n)$. We define

$$\varphi : G \to G: \quad g \mapsto \left(\left(g^{-1}\right)^T\right)^{\eta}.$$

Then φ is an automorphism. Furthermore it is a straightforward exercise to show that $\varphi(C_i) = C_i$ and $\varphi(V_{i,i+1}) = V_{i+1,i}$. Consequently we obtain

Lemma 3.4. *The module $k_{C_i}\uparrow^G$ has the same number of components with vertex $V_{i,i+1}$ and $V_{i+1,i}$.*

4. The group S_i as a vertex

Let $i \in \{1,\ldots,\lfloor \frac{n}{2} \rfloor\}$. In this and the following sections we consider each of the six groups found in Theorem 3.1 and investigate how many components of $k_{C_i}\uparrow^G$ have each group as a vertex. If V is the p-group in question then by Theorem 1.1 it is enough to determine the number of components of $k\,\mathrm{Fix}_{G/C_i}(V)$, considered as a $k\mathrm{N}_G(V)$-module. Furthermore for each component of $k_{C_i}\uparrow^G$ with vertex V we determine its Green correspondent with respect to $(G, V, \mathrm{N}_G(V))$.

We start with the Sylow-p-subgroup S_i of C_i. Since any G-conjugate of S_i contained in C_i must be a C_i-conjugate of S_i we obtain by Lemma 1.2 that

$$k\,\mathrm{Fix}_{G/C_i}(S_i) \cong k_{\mathrm{N}_{C_i}(S_i)}\uparrow^{\mathrm{N}_G(S_i)}, \quad \text{as } k\mathrm{N}_G(S_i)\text{-modules}.$$

At first we describe the normalizer $\mathrm{N}_G(S_i)$ of S_i in G. Recall the definition of the groups $F_{k,l}$, as given in subsection 1.2.

Lemma 4.1.

$$\mathrm{N}_G(S_i) = \left\{D_n(A\bullet, B\bullet, AZ)\colon\ A \in B_i,\ B \in B_{n-2i},\ Z \in \mathrm{C}_{\mathrm{GL}_i}(U_i)\right\} = \mathrm{N}_{N_i}(S_i)\cdot F_{n-i+1,n}.$$

Proof. Let H denote the described set in the middle and let $X \in \mathrm{N}_G(S_i)$. Then ${S_i}^X \leqslant R_i$. By Bruhat decomposition (see [1]) we have $X = g_1\mu g_2$, for $g_1, g_2 \in B_n$ and $\mu \in \mathcal{W}_n$. As $B_n \leqslant R_i$ we get $({S_i}^{g_1})^{\mu} \leqslant R_i$. Note that for every $k \leqslant l$, there is some $h \in {S_i}^{g_1}$ with $h_{k,l} \neq 0$. Hence μ acts on the sets $\{1,\ldots,i\}$, $\{i+1,\ldots,n-i\}$ and $\{n-i+1,\ldots,n\}$. That means, $\mu \in R_i$, and so $X \in R_i$.

Write $X = D_n(A\bullet, B\bullet, C)$, for $A, C \in \mathrm{GL}_i$ and $B \in \mathrm{GL}_{n-2i}$. Since $S_i = {S_i}^X$ we have $A, C \in \mathrm{N}_{\mathrm{GL}_i}(U_i) = B_i$, $B \in \mathrm{N}_{\mathrm{GL}_{n-2i}}(U_{n-2i}) = B_{n-2i}$ and $A^{-1}C \in \mathrm{C}_{\mathrm{GL}_i}(U_i)$. As now $C = AZ$, where $Z \in \mathrm{C}_{\mathrm{GL}_i}(U_i)$, we conclude that $\mathrm{N}_G(S_i) \subseteq H$.

Next let $X = D_n(A\bullet, B\bullet, AZ) \in H$. Note that $Z = \alpha \cdot I_{1,i}(\beta)$, for some $\alpha \in \mathbb{F}_q^\times$ and $\beta \in \mathbb{F}_q$. Then $X = D_n(A\bullet, B\bullet, \alpha A)\cdot D_n(I_{n-i}, I_{1,i}(\beta)) \in N_i \cdot F_{n-i+1}$. The fact that $F_{1,i}$ centralizes U_i, shows that $F_{n-i+1,n}$ normalizes S_i. In particular we have $X \in \mathrm{N}_{N_i}(S_i)\cdot F_{n-i+1,n}$, that is, $H \subseteq \mathrm{N}_{N_i}(S_i)\cdot F_{n-i+1,n}$.

Finally $\mathrm{N}_{N_i}(S_i)\cdot F_{n-i+1,n} \leqslant \mathrm{N}_G(S_i)$ is obvious as $F_{n-i+1,n} \leqslant \mathrm{N}_G(S_i)$. $\square$

Lemma 4.2. *For all $r = 0,1,\ldots,q-2$, let W_r be the one-dimensional kN_i-module given in (8). Then*

$$k_{\mathrm{N}_{C_i}(S_i)}\uparrow^{\mathrm{N}_G(S_i)} \cong \bigoplus_{r=0}^{q-2}(W_r{\downarrow_{\mathrm{N}_{N_i}(S_i)}})\uparrow^{\mathrm{N}_G(S_i)}.$$

Furthermore $(W_r{\downarrow_{\mathrm{N}_{N_i}(S_i)}})\uparrow^{\mathrm{N}_G(S_i)}$, for $r = 0,1,\ldots,q-2$, are pairwise non-isomorphic indecomposable $\mathrm{N}_G(S_i)$-modules with vertex S_i.

Proof. Note that $N_i = C_i \cdot \mathrm{N}_{N_i}(S_i)$. So $k_{C_i}{\uparrow}^{N_i}{\downarrow}_{\mathrm{N}_{N_i}(S_i)} \cong k_{\mathrm{N}_{C_i}(S_i)}{\uparrow}^{\mathrm{N}_{N_i}(S_i)}$, by Mackey's lemma. Hence the decomposition follows from (8).

Since $\mathrm{N}_{N_i}(S_i) \cap F_{n+1-i,n} = \langle I_n \rangle$, we get by Lemma 4.1 and Mackey's lemma that

$$\left(W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}{\uparrow}^{\mathrm{N}_G(S_i)}\right){\downarrow}_{F_{n+1-i,n}} \cong k_{\langle I_n \rangle}{\uparrow}^{F_{n+1-i,n}}.$$

As $F_{n+1-i,n}$ is a p-group, $k_{\langle I_n \rangle}{\uparrow}^{F_{n+1-i,n}}$ is indecomposable. Thus so is $W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}{\uparrow}^{\mathrm{N}_G(S_i)}$. Its vertex is S_i, as W_r has vertex S_i.

Next let $T := D_n(I_{n-i}, \mathrm{Z}(\mathrm{GL}_i)) \leqslant \mathrm{N}_G(S_i)$. Then $\mathrm{N}_G(S_i) = \mathrm{N}_{N_i}(S_i) \cdot F_{n-i+1,n} \cdot T$, by Lemma 4.1. Clearly $F_{n-i+1,n} \leqslant \mathrm{C}_G(T)$, and thus $T^g \cap \mathrm{N}_{N_i}(S_i) = T$, for all $g \in F_{n-i+1,n}$. Since $|\mathrm{N}_G(S_i) : \mathrm{N}_{N_i}(S_i)| = q$, it follows from Mackey's lemma that $((W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}){\uparrow}^{\mathrm{N}_G(S_i)}){\downarrow}_T$ is the direct sum of q copies of $W_r{\downarrow}_T$. As the various $W_r{\downarrow}_T$ are non-isomorphic, the same is true for the various $(W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}){\uparrow}^{\mathrm{N}_G(S_i)}$. □

We summarize our results on the components that have vertex S_i. Recall the homomorphism $\mathcal{R} : R_i \to \mathrm{GL}_{i,i}$ given by (10).

Theorem 4.3. *Let $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Then $k_{C_i}{\uparrow}^G$ has exactly $q-1$ components with vertex S_i. In decomposition (12) each summand $W_r{\uparrow}^G$ has exactly one component M_r with vertex S_i. In decomposition (13), M_r is a component of $B_r^{\mathcal{R}}{\uparrow}^G$. Furthermore with respect to $(G, S_i, \mathrm{N}_G(S_i))$ the Green correspondent of M_r is $(W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}){\uparrow}^{\mathrm{N}_G(S_i)}$. Finally, the modules $M_0, M_1, \dots, M_{q-2}$ are pairwise non-isomorphic.*

Proof. By Lemma 4.2 we see that $k_{C_i}{\uparrow}^G$ has exactly $q-1$ components with vertex S_i. Then by Theorem 3.2(a) each of the $q-1$ modules $W_r{\uparrow}^G$ in the decomposition (12) has exactly one such component, which we denote by M_r. By Theorem 3.2(b) + (c) it follows that M_r is a component of $B_r^{\mathcal{R}}{\uparrow}^G$.

As W_r is a component of $W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}{\uparrow}^{N_i}$, we get that M_r is a component of $(W_r{\downarrow}_{\mathrm{N}_{N_i}(S_i)}{\uparrow}^{\mathrm{N}_G(S_i)}){\uparrow}^G$. Now the rest of statement follows by Lemma 4.2. □

5. On the groups $V_{r,s}$

Let $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Recall the groups $V_{r,s}$, as defined in (7), for integers $r, s \geqslant 1$, such that $r+s \leqslant n$. Below we study $k\,\mathrm{Fix}_{G/C_i}(V_{r,s})$, as an $\mathrm{N}_G(V_{r,s})$-module, for certain pairs (r,s). In particular we need to understand $\mathrm{N}_G(V_{r,s})$.

Lemma 5.1. *Let $r, s \geqslant 1$ be positive integers, such that $r+s \leqslant n$. Then*

$$\mathrm{N}_{\mathrm{GL}_n}(V_{r,s}) = D_n(\mathrm{GL}_r \bullet, B_{n-r-s} \bullet, \mathrm{GL}_s).$$

Proof. Clearly $D_n(\mathrm{GL}_r \bullet, B_{n-r-s}\bullet, \mathrm{GL}_s)$ normalizes $V_{r,s}$. Now the Bruhat decomposition and the observation that if $\mu \in \mathcal{W}_n$ normalizes $V_{r,s}$, then $\mu = D_n(\mu_1, I_{n-r-s}, \mu_2)$, for some $\mu_1 \in \mathcal{W}_r$ and $\mu_2 \in \mathcal{W}_s$, imply the rest. □

Lemma 5.2. *Let $r, s \geqslant i$ so that $r+s \leqslant n$, and let $V = V_{r,s}$. Then*

$$k\,\mathrm{Fix}_{G/C_i}(V) \cong k_{\mathrm{N}_{C_i}(V)}{\uparrow}^{\mathrm{N}_G(V)}, \quad \textit{as } \mathrm{N}_G(V)\textit{-modules}.$$

Proof. By Lemma 1.2 it is enough to show that if $V^X \leqslant C_i$, for some $X \in G$, then $X \in \mathrm{N}_G(V) \cdot C_i$. However observe that $R_i = \mathrm{N}_G(V_{i,i}) \cdot C_i \leqslant \mathrm{N}_G(V) \cdot R_i$, and hence $\mathrm{N}_G(V) \cdot C_i = \mathrm{N}_G(V) \cdot R_i$. Since $B_n \leqslant \mathrm{N}_G(V) \cap R_i$ and $C_i \leqslant R_i$ it follows from the Bruhat decomposition that it is sufficient to show that if $V^\mu \leqslant R_i$, for some $\mu \in \mathcal{W}_n$, then $\mu \in \mathrm{N}_G(V) \cdot R_i$.

So let $V^{\mu} \leqslant R_i$, for some $\mu \in \mathcal{W}_n$. Observe that there is some $\lambda \in \mathcal{W}_n \cap R_i$ such that for $\mu' := \mu\lambda$ we have $\mu'(i+1) < \cdots < \mu'(n-i)$. For more details see the proof of Lemma 3.4 in [13].

It is a straightforward exercise to show that $\omega(\{1,\dots,i\}) \subseteq \{1,\dots,r\}$ and $\omega(\{n-i+1,\dots,n\}) \subseteq \{n-s+1,\dots,n\}$, for all $\omega \in \mathcal{W}_n$ such that $V^{\omega} \leqslant R_i$. Hence this is true for μ' and we conclude from Lemma 5.1 that $\mu' \in \mathrm{N}_G(V)$. This concludes the proof. $\square$

Recall the homomorphism $\mathcal{R}: R_i \to \mathrm{GL}_{i,i}$ given by (10).

Lemma 5.3. *Under the above assumptions let M be a component of the module $(B^{\mathcal{R}}{\downarrow_{\mathrm{N}_{R_i}(V)}}){\uparrow^{\mathrm{N}_G(V)}}$ with vertex V. Then the Green correspondent of M with respect to $(G, V, \mathrm{N}_G(V))$ is a component of $B^{\mathcal{R}}{\uparrow^G}$.*

Proof. It follows from Mackey's lemma that $(B^{\mathcal{R}}{\downarrow_{\mathrm{N}_{R_i}(V)}}){\uparrow^{\mathrm{N}_G(V)}}$ is a direct summand of $(B^{\mathcal{R}}{\uparrow^G}){\downarrow_{\mathrm{N}_G(V)}}$. Hence there exists a component N of $B^{\mathcal{R}}{\uparrow^G}$ such that M is a component of $N{\downarrow_{\mathrm{N}_G(V)}}$. Now a result by Burry and Carlson [3] implies that N is the Green correspondent of M with respect to $(G, V, \mathrm{N}_G(V))$. $\square$

6. The group $V_{i,i}$ as a vertex

Let $i \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$. In this section we determine the number of components of $k_{C_i}{\uparrow^G}$ that have vertex $V_{i,i}$. As $V_{1,1} = S_1$, let $i \geqslant 2$. Furthermore set $V := V_{i,i}$. By Theorem 1.1 and Lemma 5.2 it is enough to determine the components of $k_{\mathrm{N}_{C_i}(V)}{\uparrow^{\mathrm{N}_G(V)}}$ with vertex V. Note that $\mathrm{N}_G(V) = D_n(\mathrm{GL}_i\,\bullet, B_{n-2i}\bullet, \mathrm{GL}_i) \leqslant R_i$, by Lemma 5.1. Let $\mathcal{R}$ be the epimorphism given by (10), and let $\mathcal{R}' := \mathcal{R}|_{\mathrm{N}_G(V)}$ be its restriction to $\mathrm{N}_G(V)$. Recall that $B_0^z, B_1^z, \dots, B_{q-2}^z$ denote the p-blocks of GL_i of defect zero.

Lemma 6.1. *The module $k_{\mathrm{N}_{C_i}(V)}{\uparrow^{\mathrm{N}_G(V)}}$ has exactly $q-1$ components with vertex V. They are given by $(B_j^z)^{\mathcal{R}|_{\mathrm{N}_G(V)}}$, for $j = 0, 1, \dots, q-2$. In particular these components are pairwise non-isomorphic.*

Proof. Note that $\mathcal{R}'(\mathrm{N}_G(V)) = \mathrm{GL}_{i,i}$, $\mathcal{R}'(\mathrm{N}_{C_i}(V)) = \Delta\,\mathrm{GL}_i$ and $\ker\mathcal{R}' = D_n(I_i\bullet, B_{n-2i}\bullet, I_i) \leqslant \mathrm{N}_{C_i}(V)$. Hence $k_{\mathrm{N}_{C_i}(V)}{\uparrow^{\mathrm{N}_G(V)}} \cong (k_{\Delta\,\mathrm{GL}_i}{\uparrow^{\mathrm{GL}_{i,i}}})^{\mathcal{R}'}$, by Lemma 1.3.

Clearly, $V \in \mathrm{Syl}_p(\ker\mathcal{R}')$. Hence Lemma 1.4 implies that the components of $k_{\mathrm{N}_{C_i}(V)}{\uparrow^{\mathrm{N}_G(V)}}$ that have vertex V are exactly the $\mathcal{R}'$-inflations of the projective components of $k_{\Delta\,\mathrm{GL}_i}{\uparrow^{\mathrm{GL}_{i,i}}}$. These are the p-blocks $B_0^z, B_1^z, \dots, B_{q-2}^z$ of GL_i of defect zero. As they are pairwise non-isomorphic the proof is complete. $\square$

Now we can summarize our results on components with vertex $V_{i,i}$.

Theorem 6.2. *Let $i \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$. Then $k_{C_i}{\uparrow^G}$ has exactly $q-1$ components with vertex $V_{i,i}$. Next let $s := \gcd(q-1, i)$. In decomposition (12) the module $W_r{\uparrow^G}$ has exactly s components with vertex $V_{i,i}$ if and only if s divides r. Otherwise it has no such components. In decomposition (13) each $(B_r^z)^{\mathcal{R}}{\uparrow^G}$ admits exactly one component M_r with vertex $V_{i,i}$. With respect to $(G, V_{i,i}, \mathrm{N}_G(V_{i,i}))$ the Green correspondent of M_r is $(B_r^z)^{\mathcal{R}|_{\mathrm{N}_G(V_{i,i})}}$. Finally the components $M_0, M_1, \dots, M_{q-2}$ are pairwise non-isomorphic.*

Proof. By Lemma 6.1 we know that $k_{C_i}{\uparrow^G}$ has exactly $q-1$ components with vertex V. By Theorem 3.2(c) it follows that $(B_r^z)^{\mathcal{R}}{\uparrow^G}$ in the decomposition (13) admits a unique component M_r with vertex V. The statement on $W_r{\uparrow^G}$ follows from Theorem 3.2(b).

By Lemma 6.1 we know that $(B_r^z)^{\mathcal{R}|_{\mathrm{N}_G(V)}}$ has vertex V. Since $\mathrm{N}_G(V) = \mathrm{N}_{R_i}(V)$, Lemma 5.3 now implies that the Green correspondent of $(B_r^z)^{\mathcal{R}|_{\mathrm{N}_G(V)}}$ is M_r. Finally $M_0, M_1, \dots, M_{q-2}$ are pairwise non-isomorphic, as their Green correspondents are pairwise non-isomorphic, by Lemma 6.1. $\square$

7. The groups $V_{i,i+1}$, $V_{i+1,i}$ and $V_{i+1,i+1}$ as a vertex

Let $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and let $(r,s) \in \{(i,i+1),(i+1,i),(i+1,i+1)\}$ provided that $r+s \leqslant n$. By Theorem 1.1 and Lemma 5.2 we know that $k_{C_i}\uparrow^G$ and $k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})}$ have the same number of components with vertex $V_{r,s}$. We have $\mathrm{N}_G(V_{r,s}) = D_n(\mathrm{GL}_r\bullet, B_{n-r-s}\bullet, \mathrm{GL}_s)$, by Lemma 5.1. Also observe that $\mathrm{N}_{C_i}(V_{r,s}) = \{D_n(A\bullet, B\bullet, A)\colon\ A \in \mathrm{GL}_i,\ B \in B_{n-2i}\}$. Next we define the epimorphism

$$\mathcal{P}_{r,s} : \mathrm{N}_G(V_{r,s}) \to \mathrm{GL}_{r,s}\colon \quad D_n(A\bullet, B\bullet, C) \mapsto (A, C). \tag{14}$$

Then $\ker \mathcal{P}_{r,s} = D_n(I_r\bullet, B_{n-r-s}\bullet, I_s)$, and $V_{r,s}$ is a Sylow-p-subgroup of $\ker \mathcal{P}_{r,s}$. For every $0 \leqslant t \leqslant i$ we define the group

$$Q^t_{r,s} := \bigl\{\bigl(D_{t+r-i}(A\bullet, X), D_s(Y\bullet, A)\bigr)\colon\ A \in \mathrm{GL}_t,\ X \in B_{r-i},\ Y \in B_{s-t}\bigr\}. \tag{15}$$

Then $Q^t_{r,s} \leqslant \mathrm{GP}_{(t,r-i)} \times \mathrm{GP}_{(s-t,s)} \leqslant \mathrm{GL}_{t+r-i,s}$. Also $\mathcal{P}_{r,s}(\mathrm{N}_{C_i}(V_{r,s})) = Q^i_{r,s}$. Recall the definition of the groups $\mathrm{GP}_{(t,r-i)}$ and $\mathrm{GP}_{(s-t,s)}$ from Section 1.2.

Lemma 7.1. *We have*

$$k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})} \cong \bigl(k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}\bigr)^{\mathcal{P}_{r,s}}.$$

Also the number of components of $k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})}$ that have vertex $V_{r,s}$ is given by $\mathrm{p}^\sharp(k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}})$.

Proof. Since $\ker \mathcal{P}_{r,s} \leqslant \mathrm{N}_{C_i}(V_{r,s})$, the first assertion follows from Lemma 1.3. As $V_{r,s} \in \mathrm{Syl}_p(\ker \mathcal{P}_{r,s})$, Lemma 1.4 implies the second assertion. $\square$

Lemma 7.2.

$$\text{(a)}\quad \mathrm{p}^\sharp\bigl(k_{Q^i_{i+1,i+1}}\uparrow^{\mathrm{GL}_{i+1,i+1}}\bigr) \leqslant \mathrm{p}^\sharp\bigl(k_{Q^i_{i,i+1}}\uparrow^{\mathrm{GL}_{i,i+1}}\bigr),$$

$$\text{(b)}\quad \mathrm{p}^\sharp\bigl(k_{Q^t_{i,i+1}}\uparrow^{\mathrm{GL}_{t,i+1}}\bigr) \leqslant \mathrm{p}^\sharp\bigl(k_{Q^{t-1}_{i,i+1}}\uparrow^{\mathrm{GL}_{t-1,i+1}}\bigr), \quad \textit{for } 1 \leqslant t \leqslant i,$$

$$\text{(c)}\quad \mathrm{p}^\sharp\bigl(k_{Q^0_{i,i+1}}\uparrow^{\mathrm{GL}_{0,i+1}}\bigr) \leqslant 1.$$

Proof. For every integer t such that $0 \leqslant t \leqslant i$ we define the subgroup

$$K_t := \bigl\{(X,Y) \in \mathrm{GP}_{(1,t)} \times \mathrm{GL}_{i+1}\colon\ \det(X)\cdot\det(Y) = 1\bigr\},$$

of $\mathrm{GL}_{t+1,i+1}$. Furthermore consider the epimorphism

$$\varphi_t : K_t \to \mathrm{GL}_{t,i+1}\colon \quad \bigl(D_{t+1}(\alpha\bullet, A), B\bigr) \mapsto (A, B).$$

Then $\ker \varphi_t = \{(D_{t+1}(1\bullet, I_t), I_{i+1})\}$ is a normal p-subgroup of K_t.

(a) Let $\omega' \in \mathcal{W}_{i+1}$ be the permutation matrix that corresponds to the permutation $(1,2,\ldots,i+1) \in \mathrm{Sym}(i+1)$. Then the $(\mathrm{GP}_{(1,i)}, \mathrm{GP}_{(i,1)})$-double cosets of GL_{i+1} are represented by $\{I_{i+1}, \omega'\}$. This follows from the Bruhat decomposition and the observation that for $\eta \in \mathcal{W}_{i+1}$ we have $\eta \in \mathrm{GP}_{(1,i)} \cdot \mathrm{GP}_{(i,1)}$ if and only if $\eta(i+1) \neq 1$, and $\eta \in \mathrm{GP}_{(1,i)} \cdot \omega' \cdot \mathrm{GP}_{(i,1)}$ if and only if $\eta(i+1) = 1$.

Set $\omega := (\omega', I_{i+1}) \in \mathrm{GL}_{i+1,i+1}$. Then the $(K_i, Q^i_{i+1,i+1})$-double cosets of $\mathrm{GL}_{i+1,i+1}$ are represented by $\{(I_{i+1}, I_{i+1}), \omega\}$. Now by Mackey's lemma we have

$$\big(k_{Q^i_{i+1,i+1}}\uparrow^{\mathrm{GL}_{i+1,i+1}}\big)\downarrow_{K_i} \cong k_{Q^i_{i+1,i+1}\cap K_i}\uparrow^{K_i} \oplus k_T\uparrow^{K_i},$$

where $T := (Q^i_{i+1,i+1})^{\omega} \cap K_i$. One checks easily that

$$T = \big\{\big(D_{i+1}(\alpha, A), D_{i+1}(\beta\bullet, A)\big) \in K_i\colon\ A \in \mathrm{GL}_i,\ \alpha, \beta \in \mathbb{F}_q^{\times}\big\}.$$

Observe that $(I_{1,i+1}(1), I_{i+1}) \in \ker\varphi_t \cap Q^i_{i+1,i+1} \cap K_i$. As $\ker\varphi_t$ is a normal p-group, $k_{Q^i_{i+1,i+1}\cap K_i}\uparrow^{K_i}$ is projective-free. Hence $\mathrm{p}^\sharp(k_{Q^i_{i+1,i+1}}\uparrow^{\mathrm{GL}_{i+1,i+1}}) \leqslant \mathrm{p}^\sharp(k_T\uparrow^{K_i})$. Since $\ker\varphi_t \cap T$ is trivial and $\varphi_t(T) = Q^i_{i,i+1}$, Lemma 1.5 implies that $\mathrm{p}^\sharp(k_T\uparrow^{K_i}) = \mathrm{p}^\sharp(k_{Q^i_{i,i+1}}\uparrow^{\mathrm{GL}_{i,i+1}})$. This concludes part (a).

(b) Let $1 \leqslant t \leqslant i$. First note that $\mathrm{GL}_{t,i+1} = K_{t-1} \cdot Q^t_{i,i+1}$. This follows as for all $(A, B) \in \mathrm{GL}_{t,i+1}$ and $\alpha := \det(B) \cdot \det(A^{-1})$ we have

$$(A, B) = \big(I_t, B \cdot D_{i+1}\big(\alpha^{-1}, I_{i-t}, A^{-1}\big)\big) \cdot \big(A, D_{i+1}(\alpha, I_{i-t}, A)\big).$$

Now $(k_{Q^t_{i,i+1}}\uparrow^{\mathrm{GL}_{t,i+1}})\downarrow_{K_{t-1}} \cong k_{Q^t_{i,i+1}\cap K_{t-1}}\uparrow^{K_{t-1}}$. Observe that $\ker(\varphi_{t-1}) \cap Q^t_{i,i+1} \cap K_{t-1}$ is trivial and $\varphi_{t-1}(Q^t_{i,i+1} \cap K_{t-1}) = Q^{t-1}_{i,i+1}$. So $\mathrm{p}^\sharp(k_{Q^t_{i,i+1}\cap K_{t-1}}\uparrow^{K_{t-1}}) = \mathrm{p}^\sharp(k_{Q^{t-1}_{i,i+1}}\uparrow^{\mathrm{GL}_{t-1,i+1}})$, by Lemma 1.5. Thus

$$\mathrm{p}^\sharp\big(k_{Q^t_{i,i+1}}\uparrow^{\mathrm{GL}_{t,i+1}}\big) \leqslant \mathrm{p}^\sharp\big(k_{Q^t_{i,i+1}\cap K_{t-1}}\uparrow^{K_{t-1}}\big) = \mathrm{p}^\sharp\big(k_{Q^{t-1}_{i,i+1}}\uparrow^{\mathrm{GL}_{t-1,i+1}}\big).$$

(c) Note that $Q^0_{i,i+1} \cong B_{i+1}$ and $\mathrm{GL}_{0,i+1} \cong \mathrm{GL}_{i+1}$. As $\mathrm{GL}_{i+1} = \mathrm{SL}_{i+1} \cdot B_{i+1}$ and $\mathrm{SL}_{i+1} \cap B_{i+1} = SB_{i+1}$ it follows from Mackey's lemma that

$$\mathrm{p}^\sharp\big(k_{Q^0_{i,i+1}}\uparrow^{\mathrm{GL}_{0,i+1}}\big) = \mathrm{p}^\sharp\big(k_{B_{i+1}}\uparrow^{\mathrm{GL}_{i+1}}\big) \leqslant \mathrm{p}^\sharp\big(k_{SB_{i+1}}\uparrow^{\mathrm{SL}_{i+1}}\big) = 1. \qquad \square$$

Next we define the two homomorphisms

$$\begin{aligned} \tau_r : \mathrm{GP}_{(i,r-i)} &\to \mathrm{GL}_i, \\ D_r(A\bullet, X) &\mapsto A, \end{aligned} \quad \text{and} \quad \begin{aligned} \mu_s : \mathrm{GP}_{(s-i,i)} &\to \mathrm{GL}_i, \\ D_s(X\bullet, B) &\mapsto B, \end{aligned}$$

Note that since $r, s \in \{i, i+1\}$, the variable X either represents a scalar or nothing, that is, these homomorphism might just be the identity. Also recall that St_r denotes the Steinberg module in GL_r.

Lemma 7.3. *The module St_r is a component of $(\mathrm{St}_i)^{\tau_r}\uparrow^{\mathrm{GL}_r}$, and the module St_s is a component of $(\mathrm{St}_i)^{\mu_s}\uparrow^{\mathrm{GL}_s}$.*

Proof. We only prove the first assertion of the statement, as the second one is shown similarly. As the statement is trivial if $r = i$, let $r = i + 1$. Since St_{i+1} is a projective irreducible module, by Frobenius reciprocity it is enough to show that $\mathrm{Hom}_{kGP_{(i,1)}}((\mathrm{St}_i)^{\tau_r}, \mathrm{St}_r\downarrow_{\mathrm{GP}_{(i,1)}}) \neq 0$. We claim that $\mathrm{St}_r\downarrow_{\mathrm{GP}_{(i,1)}} \cong ((\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}})\uparrow^{\mathrm{GP}_{(i,1)}}$, which then completes the proof.

Observe that $(\mathrm{St}_i)^{\tau_r|_{\mathrm{GL}_{i,1}}} \cong (\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}}$, by (2), and since $\ker(\tau_r|_{\mathrm{GL}_{i,1}})$ is a p'-group, $(\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}}$ is projective, by Lemma 1.4. Hence $(k_{B_i}\uparrow^{\mathrm{GL}_i})^{\tau_r|_{\mathrm{GL}_{i,1}}} \cong k_{D_{i+1}(B_i,\mathbb{F}_q^{\times})}\uparrow^{\mathrm{GL}_{i,1}}$, by Lemma 1.3, and thus $(\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}}\uparrow^{\mathrm{GP}_{(i,1)}}$ is a projective summand of $k_{D_{i+1}(B_i,\mathbb{F}_q^{\times})}\uparrow^{\mathrm{GP}_{(i,1)}}$. But $D_{i+1}(B_i, \mathbb{F}_q^{\times}) = \mathrm{GP}_{(i,1)} \cap (B_{i+1})^{\omega}$, where $\omega \in \mathcal{W}_{i+1}$ corresponds to $(1, 2, \ldots, i+1) \in \mathrm{Sym}(i+1)$. So, by Mackey's lemma, $(\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}}\uparrow^{\mathrm{GP}_{(i,1)}}$ is a projective summand of $(k_{B_{i+1}}\uparrow^{\mathrm{GL}_{i+1}})\downarrow_{\mathrm{GP}_{(i,1)}}$.

As $\mathrm{p}^\sharp((k_{B_{i+1}}\uparrow^{\mathrm{GL}_{i+1}})\downarrow_{\mathrm{GP}_{(i,1)}}) \leqslant \mathrm{p}^\sharp((k_{B_{i+1}}\uparrow^{\mathrm{GL}_{i+1}})\downarrow_{U_{i+1}}) = 1$, we get that $\mathrm{St}_r\downarrow_{\mathrm{GP}_{(i,1)}}$ is the unique projective component of $(k_{B_{i+1}}\uparrow^{\mathrm{GL}_{i+1}})\downarrow_{\mathrm{GP}_{(i,1)}}$. Consequently $\mathrm{St}_r\downarrow_{\mathrm{GP}_{(i,1)}} \cong (\mathrm{St}_i)^{\tau_r}\downarrow_{\mathrm{GL}_{i,1}}\uparrow^{\mathrm{GP}_{(i,1)}}$. $\square$

Next we define $\xi_{r,s} := \tau_r \otimes \mu_s$, that is,

$$\xi_{r,s} : \mathrm{GP}_{(i,r-i)} \times \mathrm{GP}_{(s-i,s)} \to \mathrm{GL}_{i,i},$$
$$\big(D_r(A\bullet, X), D_s(Y\bullet, B)\big) \mapsto (A, B).$$

Furthermore let $\mathrm{St}_{r,s} := \mathrm{St}_r \otimes \mathrm{St}_s$, considered as a $\mathrm{GL}_{r,s}$-module. Then $\mathrm{St}_{r,s}$ is projective.

Lemma 7.4. *As $\mathrm{GL}_{r,s}$-modules, $\mathrm{St}_{r,s}$ is a direct summand of $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$ and $k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$. In particular, $\mathrm{p}^\sharp(k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}) \geqslant 1$.*

Proof. First we claim that $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$ is a direct summand of $k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$. Observe that $\xi_{r,s}(Q^i_{r,s}) = \Delta\,\mathrm{GL}_i$ and $\xi_{r,s}(\mathrm{GP}_{(i,r-i)} \times \mathrm{GP}_{(s-i,i)}) = \mathrm{GL}_{i,i}$. Since $\ker\xi_{r,s} \leqslant Q^i_{r,s}$, we obtain $k_{Q^i_{r,s}}\uparrow^{\mathrm{GP}_{(i,r-i)}\times\mathrm{GP}_{(s-i,i)}} \cong (k_{\Delta\,\mathrm{GL}_i}\uparrow^{\mathrm{GL}_{i,i}})^{\xi_{r,s}}$ from Lemma 1.3. Considering the block B_0^z of GL_i as a $\mathrm{GL}_{i,i}$-module, we see that $(B_0^z)^{\xi_{r,s}}$ is a direct summand of $k_{Q^i_{r,s}}\uparrow^{\mathrm{GP}_{(i,r-i)}\times\mathrm{GP}_{(s-i,i)}}$. Now the claim follows.

Next we show that $\mathrm{St}_{r,s}$ is a direct summand of $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$. Since $B_0^z \cong \mathrm{St}_{i,i}$, as $\mathrm{GL}_{i,i}$-modules, we get $(B_0^z)^{\xi_{r,s}} \cong (\mathrm{St}_i)^{\tau_r} \otimes (\mathrm{St}_i)^{\mu_s}$, as $\mathrm{GP}_{(i,1),(1,i)}$-modules. Then $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}} \cong (\mathrm{St}_i)^{\tau_r}\uparrow^{\mathrm{GL}_r} \otimes (\mathrm{St}_i)^{\mu_s}\uparrow^{\mathrm{GL}_s}$, as $\mathrm{GL}_{r,s}$-modules, follows by (10.17) in [5]. By Lemma 7.3 we know that St_r is a component of $(\mathrm{St}_i)^{\tau_r}\uparrow^{\mathrm{GL}_r}$ and St_s is a component of $(\mathrm{St}_i)^{\mu_s}\uparrow^{\mathrm{GL}_s}$. In particular the projective module $\mathrm{St}_{r,s}$ is a direct summand of $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$. $\square$

Lemma 7.5. *The $\mathrm{N}_G(V_{r,s})$-module $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is indecomposable with vertex $V_{r,s}$. It is the only such component of $k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})}$. Also $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is a component of $((B_0^z)^{\mathcal{R}}\downarrow_{\mathrm{N}_{R_i}(V_{r,s})})\uparrow^{\mathrm{N}_G(V_{r,s})}$.*

Proof. Lemmas 7.2 and 7.4 show that $\mathrm{p}^\sharp(k_{Q^i_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}) = 1$ for $(r,s) \in \{(i,i+1),(i+1,i+1)\}$. Hence $k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})}$ has a unique component with vertex $V_{r,s}$, by Lemma 7.1. The case $(r,s) = (i+1,i)$ now follows from Lemma 3.4.

From Lemmas 7.1 and 7.4 we get that $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is a direct summand of $k_{\mathrm{N}_{C_i}(V_{r,s})}\uparrow^{\mathrm{N}_G(V_{r,s})}$. As $V_{r,s} \in \mathrm{Syl}_p(\ker\mathcal{P}_{r,s})$, each component of $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ has vertex $V_{r,s}$, by Lemma 1.4. Hence $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is indecomposable with vertex $V_{r,s}$.

Lemma 7.4 shows that $\mathrm{St}_{r,s}$ is a component of $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}}$. So $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is a component of $((B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}})^{\mathcal{P}_{r,s}}$. Next observe that $\mathcal{P}_{r,s}(\mathrm{N}_{R_i}(V_{r,s})) = \mathrm{GP}_{(i,r-i)} \times \mathrm{GP}_{(s-i,s)}$. As $\ker\mathcal{P}_{r,s} \leqslant \mathrm{N}_{R_i}(V_{r,s})$, we conclude from Lemma 1.3 that $(B_0^z)^{\xi_{r,s}}\uparrow^{\mathrm{GL}_{r,s}} \cong (B_0^z)^{\gamma}\uparrow^{\mathrm{N}_G(V_{r,s})}$, where $\gamma := \xi_{r,s} \circ \mathcal{P}_{r,s}|_{\mathrm{N}_{R_i}(V_{r,s})}$. But $\xi_{r,s} \circ \mathcal{P}_{r,s}|_{\mathrm{N}_{R_i}(V_{r,s})} = \mathcal{R}|_{\mathrm{N}_{R_i}(V_{r,s})}$, and $(B_0^z)^{\mathcal{R}|_{\mathrm{N}_{R_i}(V)}} \cong (B_0^z)^{\mathcal{R}}\downarrow_{\mathrm{N}_{R_i}(V_{r,s})}$, by (2). $\square$

Theorem 7.6. *Let $i \in \{1,\ldots,\lfloor\frac{n}{2}\rfloor\}$ and $(r,s) \in \{(i,i+1),(i+1,i),(i+1,i+1)\}$, provided that $r+s \leqslant n$. Then $k_{C_i}\uparrow^G$ has exactly one component $M_{r,s}$ with vertex $V_{r,s}$. In decomposition (12), $M_{r,s}$ is a component of $W_0\uparrow^G \cong k_{N_i}\uparrow^G$. In decomposition (13), $M_{r,s}$ is a component of $(B_0^z)^{\mathcal{R}}\uparrow^G$. Finally with respect to $(G, V_{r,s}, \mathrm{N}_G(V_{r,s}))$ the Green correspondent of $M_{r,s}$ is $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$.*

Proof. By Theorem 1.1 and Lemma 7.5 we derive that $k_{C_i}\uparrow^G$ has a unique component $M_{r,s}$ with vertex $V_{r,s}$, whose Green correspondent with respect to $(G, V_{r,s}, \mathrm{N}_G(V_{r,s}))$ is $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$. Furthermore as $(\mathrm{St}_{r,s})^{\mathcal{P}_{r,s}}$ is a component of $((B_0^z)^{\mathcal{R}}\downarrow_{\mathrm{N}_{R_i}(V_{r,s})})\uparrow^{\mathrm{N}_G(V_{r,s})}$, by Lemma 7.5, it follows from Lemma 5.3 that $M_{r,s}$ is a component of $(B_0^z)^{\mathcal{R}}\uparrow^G$. In particular we have located $M_{r,s}$ in the decomposition (13). Now Theorem 3.2(c) shows that $M_{r,s}$ is a component of $W_0\uparrow^G$. $\square$

8. The trivial group as a vertex

The trivial group turns out to be the most exceptional vertex. By Theorem 3.1 we deduce that in the decomposition (5) the module $k_{C_m}{\uparrow}^G$, where $m := \lfloor \frac{n}{2} \rfloor$, is the only summand that could have a projective component.

Let us briefly only consider the case of the involution module, that is, when $p = 2$. Theorem 8 of [8] states that all projective components of $k\mathcal{I}$ are self-dual, irreducible and appear with multiplicity one. On the other hand all self-dual and irreducible projective kG-modules appear in $k\mathcal{I}$. But in the case $p = 2$, the Steinberg-module St_n is the only self-dual irreducible projective kG-module, and thus it is unique as a projective component of the involution module.

We have no equivalent statement to the above theorem in the case of odd characteristic. However below we verify that there is at least one projective component.

In the following let p again be arbitrary. First we describe the restriction of St_n^j to R_m, for all $j = 0, 1, \ldots, q-2$. We define

$$R_m^* := \{D_n(A, B, C)\colon\ A, C \in \mathrm{GL}_m,\ B \in \mathrm{GL}_{n-2m}\}.$$

Then $R_m^* \leqslant R_m$. Note that $n - 2m$ is either zero or one. Next we set $\mathcal{R}^* := \mathcal{R}|_{R_m^*}$, where $\mathcal{R} : R_m \to \mathrm{GL}_{m,m}$ is the homomorphism defined in (10). Finally we define the $\mathrm{GL}_{m,m}$-module $\mathrm{St}_{m,m} := \mathrm{St}_m \otimes \mathrm{St}_m$.

Lemma 8.1. $\mathrm{St}_n^j{\downarrow}_{R_m} \cong ((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j{\uparrow}^{R_m}$, *for all* $j = 0, 1, \ldots, q-2$.

Proof. Recall that St_m is a projective component of $k_{B_m}{\uparrow}^{\mathrm{GL}_m}$. So $\mathrm{St}_{m,m}$ is a projective component of $k_{B_m \times B_m}{\uparrow}^{\mathrm{GL}_{m,m}}$, by [7] (Proposition 1.2). Note that $\ker \mathcal{R}^*$ is a p'-group. Hence Lemma 1.4 implies that $(\mathrm{St}_{m,m})^{\mathcal{R}^*}$ is projective. Also $\mathcal{R}^*(B_n \cap R_m^*) = B_m \times B_m$ and $\ker \mathcal{R}^* \leqslant B_n$. So we conclude from Lemma 1.3 that $(k_{B_m \times B_m}{\uparrow}^{\mathrm{GL}_{m,m}})^{\mathcal{R}^*} \cong k_{B_n \cap R_m^*}{\uparrow}^{R_m^*}$. In particular $(\mathrm{St}_{m,m})^{\mathcal{R}^*}{\uparrow}^{R_m}$ is a projective direct summand of $k_{B_n \cap R_m^*}{\uparrow}^{R_m}$.

Next observe that $\mathrm{St}_n{\downarrow}_{R_m}$ is a direct summand of $(k_{B_n}{\uparrow}^G){\downarrow}_{R_m}$. As $U_n \leqslant R_m$, it follows that $\mathrm{p}^\sharp((k_{B_n}{\uparrow}^G){\downarrow}_{R_m}) \leqslant 1$. Hence $\mathrm{St}_n{\downarrow}_{R_m}$ is the unique projective component of $(k_{B_n}{\uparrow}^G){\downarrow}_{R_m}$. But Mackey's lemma implies that $k_{B_n \cap R_m^*}{\uparrow}^{R_m}$ is a direct summand of $(k_{B_n}{\uparrow}^G){\downarrow}_{R_m}$, since $R_m \cap B_n^g = B_n \cap R_m^*$, where

$$g := \begin{pmatrix} & & I_m \\ & I_{n-2m} & \\ I_m & & \end{pmatrix}.$$

Hence $\mathrm{St}_n{\downarrow}_{R_m} \cong (\mathrm{St}_{m,m})^{\mathcal{R}^*}{\uparrow}^{R_m}$. Our statement follows as $((\mathrm{St}_{m,m})^{\mathcal{R}^*}{\uparrow}^{R_m})^j \cong ((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j{\uparrow}^{R_m}$ and $\mathrm{St}_n^j{\downarrow}_{R_m} \cong (\mathrm{St}_n{\downarrow}_{R_m})^j$. $\square$

Lemma 8.2. *For every p-block B of G and $j = 0, 1, \ldots, q-2$ we have*

$$\mathrm{Hom}_{kG}(\mathrm{St}_n^j, B^{\mathcal{R}}{\uparrow}^G) \cong \mathrm{Hom}_{kR_m^*}\big(((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j, B^{\mathcal{R}^*}\big).$$

Proof. By Frobenius reciprocity and Lemma 8.1 we get

$$\begin{aligned}
\mathrm{Hom}_{kG}(\mathrm{St}_n^j, B^{\mathcal{R}}{\uparrow}^G) &\cong \mathrm{Hom}_{kR_m}(\mathrm{St}_n^j{\downarrow}_{R_m}, B^{\mathcal{R}}) \\
&\cong \mathrm{Hom}_{kR_m}\big(((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j{\uparrow}^{R_m}, B^{\mathcal{R}}\big) \\
&\cong \mathrm{Hom}_{kR_m^*}\big(((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j, B^{\mathcal{R}}{\downarrow}_{R_m^*}\big) \\
&\cong \mathrm{Hom}_{kR_m^*}\big(((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j, B^{\mathcal{R}^*}\big),
\end{aligned}$$

where the isomorphism $B^{\mathcal{R}}{\downarrow}_{R_m^*} \cong B^{\mathcal{R}^*}$ follows from (2). $\square$

Now we give the final statement on the projective components of $\bigoplus_{i=1}^m k_{C_i}\uparrow^G$. Recall that B_0^z denotes the block of GL_m that contains St_m. Then $B_0^z \cong \mathrm{St}_{m,m}$, as $\mathrm{GL}_{m,m}$-modules, by (1).

Theorem 8.3. *The Steinberg module St_n is a component of $k_{C_i}\uparrow^G$ if and only if $i = m$. In the decompositions (12) and (13) of $k_{C_m}\uparrow^G$ it appears as a component of $k_{N_m}\uparrow^G$, and $(B_0^z)^{\mathcal{R}}\uparrow^G$, respectively.*

Furthermore in the case of the involution module $k\mathcal{I}$, that is, when $p = 2$, the Steinberg module St_n is the unique projective component.

Proof. Since $(B_0^z)^{\mathcal{R}}$ is a direct summand of both $k_{N_m}\uparrow^G$ and $k_{C_m}\uparrow^G$, it is enough to show that St_n is a component of $(B_0^z)^{\mathcal{R}}$. Being projective and irreducible, St_n is a component of $B^{\mathcal{R}}\uparrow^G$ if and only if $\mathrm{Hom}_{kG}(\mathrm{St}_n, (B_0^z)^{\mathcal{R}}\uparrow^G)$ is non-trivial. But this is the case by Lemma 8.2 and the fact that $B_0^z \cong \mathrm{St}_{m,m}$, as $\mathrm{GL}_{m,m}$-modules. The rest of the statement has been established in the introduction to this section. $\square$

Lemma 8.4. *If $n = 2m$ and p is odd, then St_n^j is a projective component of $(B_j^z)^{\mathcal{R}}\uparrow^G$, where $j = \frac{q-1}{2}$.*

Proof. Since p is odd, j is an integer. As $n = 2m$ we have $((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j \cong ((\mathrm{St}_{m,m})^j)^{\mathcal{R}^*}$. Furthermore $(\mathrm{St}_{m,m})^j \cong \mathrm{St}_m^j \otimes \mathrm{St}_m^j \cong \mathrm{St}_m^j \otimes \mathrm{St}_m^{q-1-j} \cong B_j^z$. Now the statement follows from Lemma 8.2. $\square$

Note that in the above lemma the module St_n^j is irreducible, projective and self-dual. In particular this is not a counter-example to an odd-characteristic version of Theorem 8 in [8].

Lemma 8.5. *If $n = 2m + 1$ and p is odd, then St_n^j, where $j = \frac{q-1}{2}$, is not a component of $k_{C_m}\uparrow^G$, and thus not a component of $\bigoplus_{i=1}^m k_{C_i}\uparrow^G$.*

Proof. Suppose that St_n^j is a component of $k_{C_m}\uparrow^G$. Then there is some p-block B of G so that St_n^j is a component of $B^{\mathcal{R}}$. By Lemma 8.2, we have $\mathrm{Hom}_{kG}(\mathrm{St}_n^j, B^{\mathcal{R}}\uparrow^G) \cong \mathrm{Hom}_{kR_m^*}(((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j, B^{\mathcal{R}^*})$, which is then non-trivial. Thus there is a non-trivial homomorphism $\phi : ((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j \to B^{\mathcal{R}^*}$.

Next let $H := D_n(I_{m+1}, \mathrm{GL}_m)$. Then $((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j\downarrow_H \cong (k_{\mathrm{GL}_m} \otimes \mathrm{St}_m^j)^{\mathcal{R}^*}$. So ϕ gives rise to a $\langle I_m\rangle \times \mathrm{GL}_m$-homomorphism from $k_{\mathrm{GL}_m} \otimes (\mathrm{St}_m)^j$ to the block B. Hence St_m^j appears in the block B, and so $B \cong \mathrm{St}_m^j \otimes \mathrm{St}_m^{q-1-j} \cong B_j^z$. Now let $x \in ((\mathrm{St}_{m,m})^{\mathcal{R}^*})^j$ and set $y = \phi(x) \in (B_j^z)^{\mathcal{R}^*}$. Also let $\beta \in \mathbb{F}_q^*$, such that $\beta^j \neq 1$. Then

$$y = D_n(I_m, \beta, I_m) \cdot \phi(x) = \phi\big(D_n(I_m, \beta, I_m) \cdot x\big) = \phi\big(\beta^j \cdot x\big) = \beta^j \cdot \phi(x) = \beta^j \cdot y.$$

As this is a contradiction, ϕ does not exist. $\square$

9. The involution module of $\mathrm{GL}_n(2^f)$

We conclude this paper with a comprehensive description of the involution module $k\mathcal{I}$ of $\mathrm{GL}_n(2^f)$, which we introduced in Section 2. Let $G = \mathrm{GL}_n(2^f)$, for some integer $f \geqslant 1$ and let $m = \lfloor \frac{n}{2} \rfloor$. Then $k\mathcal{I} \cong \bigoplus_{i=1}^m k_{C_i}\uparrow^G$, by (5). Using Theorems 4.3, 6.2, 7.6 and 8.3 we obtain

Theorem 9.1. *Let $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then*

$$k_{C_i}\uparrow^G \cong \bigoplus_{B \text{ a } p\text{-block of } \mathrm{GL}_i} B^{\mathcal{R}}\uparrow^G.$$

Unless $B = B_0^z$, the summand $B^{\mathcal{R}}\uparrow^G$ is indecomposable, with the respective vertex S_i or $V_{i,i}$, depending on whether B is a block of full defect or of defect zero. The summand $(B_0^z)^{\mathcal{R}}\uparrow^G$ decomposes in the following way:

If $n = 2i$, then $(B_0^z)^{\mathcal{R}} \uparrow^G$ has two components, the Steinberg module, which is projective and a component that has vertex $V_{i,i}$.
If $n = 2i+1$, then $(B_0^z)^{\mathcal{R}} \uparrow^G$ has four components, the Steinberg module and a component with each of the respective groups $V_{i,i}$, $V_{i,i+1}$ and $V_{i+1,i}$ as a vertex.
If $n \geqslant 2i+2$, then $(B_0^z)^{\mathcal{R}} \uparrow^G$ has four components with each of the respective groups $V_{i,i}$, $V_{i,i+1}$, $V_{i+1,i}$ and $V_{i+1,i+1}$ as a vertex.

Finally all components of $k_{C_i}\uparrow^G$ are pairwise non-isomorphic.

Remark 9.2. Theorem 9.1 is also true for odd characteristic unless $i = \lfloor \frac{n}{2} \rfloor$. In this case the statement is true unless there are additional projective components. In fact the only unaccounted components would have to be projective. However only if $n = 2i$ are we aware of such an additional component, that is, St_n^j appearing as a component of $(B_j^z)^{\mathcal{R}} \uparrow^G$, where $j = \frac{q-1}{2}$. See Lemma 8.4.

We can now count the number of components of $k\mathcal{I}$.

Corollary 9.3. *The involution module $k\mathcal{I}$ of $\mathrm{GL}_n(2^f)$ decomposes into exactly $2^f(n-1) + (r-1)$ components, if $n = 2r$, and into $2^f(n-2) + (r+1)$ components, if $n = 2r+1$.*

Proof. Note that $S_1 = V_{1,1}$. First let $n = 2r$. We start with $r = 1$. Then $k\mathcal{I} \cong k_{C_1}\uparrow^G$. Hence there are $2^f - 1$ components with vertex S_1 and one projective component. So there are 2^f components, and thus the claim is true.

Now let $r \geqslant 2$. Then $k_{C_1}\uparrow^G$ has $2^f - 1$ components with vertex S_1 and three components with the respective vertices $V_{1,2}$, $V_{2,1}$ and $V_{2,2}$. For all $j \in \{2, \ldots, r-1\}$ we find that $k_{C_j}\uparrow^G$ has $2^f - 1$ components with vertex S_j, $2^f - 1$ components with vertex $V_{j,j}$, and another three components with the respective vertices $V_{j,j+1}$, $V_{j+1,j}$ and $V_{j+1,j+1}$. Finally $k_{C_r}\uparrow^G$ has $2^f - 1$ components with vertex S_r, $q-1$ components with vertex $V_{r,r}$ and one projective component. Altogether the claim follows for even n. Likewise one shows the claim for odd n. $\square$

Corollary 9.4. *The only self-dual components of $k_{C_i}\uparrow^G$ are the various components of $(B_0^z)^{\mathcal{R}} \uparrow^G$ and the component $B_0^{\mathcal{R}} \uparrow^G$.*

Proof. Let M be a self-dual component of $k_{C_i}\uparrow^G$. Then M is a component of $B^{\mathcal{R}}\uparrow^G$, for some 2-block B of G. But according to Theorem 9.1 no two components of $k_{C_i}\uparrow^G$ are isomorphic, and thus $B^{\mathcal{R}}\uparrow^G$ must be self-dual. So if B is of full defect, then (9) implies that $B^{\mathcal{R}}\uparrow^G$ is self-dual if and only if $B = B_0$. On the other hand if B is of defect zero, then using (1), we must have $B = B_0^z$. Considering their vertices we see that each component of $(B_0^z)^{\mathcal{R}}\uparrow^G$ is indeed self-dual. $\square$

Corollary 9.5. *For every $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the component $M_{i+1,i+1}$ of $k_{C_i}\uparrow^G$ that has vertex $V_{i+1,i+1}$ (see Theorem 7.6) is isomorphic to the component M_0 of $k_{C_{i+1}}\uparrow^G$ (see Theorem 6.2). Moreover this is the only way a component of the involution module can occur with multiplicity greater than one.*

Proof. Let M and M' be different components of the involution module that are isomorphic. Then there are integers i, j such that $1 \leqslant i < j \leqslant \lfloor \frac{n}{2} \rfloor$, and M is a component of $k_{C_i}\uparrow^G$ and M' is a component of $k_{C_j}\uparrow^G$. Now let V be a common vertex of M and M'. Since $i \neq \lfloor \frac{n}{2} \rfloor$, we know that V is non-trivial. Therefore V is G-conjugate to one of the groups $\{S_i, V_{i,i}, V_{i,i+1}, V_{i+1,i}, V_{i+1,i+1}\}$ and to one of groups $\{S_j, V_{j,j}, V_{j,j+1}, V_{j+1,j}, V_{j+1,j+1}\}$. A straightforward comparison of the size of the various groups shows that the only possibility is $V = V_{i+1,i+1}$ and $j = i+1$. Hence $M \cong M_{i+1,i+1}$, by Theorem 7.6. Furthermore with respect to $(G, V, \mathrm{N}_G(V))$ its Green correspondent is $(\mathrm{St}_{i+1,i+1})^{\mathcal{P}_{i+1,i+1}}$, which is isomorphic to $(\mathrm{St}_{j,j})^{\mathcal{R}|_{\mathrm{N}_G(V_{j,j})}}$. In particular $M \cong M_0$, where M_0 is described in Theorem 6.2. $\square$

In the following let $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Recall decomposition (8).

Lemma 9.6. *Set $s := \gcd(q-1, n)$ and let $r \in \{0, 1, \dots, q-2\}$. Then $W_r{\uparrow}^G$ is indecomposable if and only if $i = 1$ and $r \neq 0$ or $i \geqslant 2$ and $s \nmid r$.*

Proof. By Theorem 3.2(c) we see that $W_r{\uparrow}^G$ is surely not indecomposable if $i \geqslant 2$ and $s \mid r$. So suppose that $i = 1$. Then $W_r{\uparrow}^G \cong B_r^{\mathcal{R}}{\uparrow}^G$. Note that GL_1 is a $2'$-group. So the 2-blocks of full defect coincide with the blocks of defect zero. As the trivial module k_{GL_1} coincides with the Steinberg module, we conclude that $B_0 = B_0^z$. So it follows from Theorem 9.1 that $W_r{\uparrow}^G$ is indecomposable if and only if $r \neq 0$.

Next suppose that $i \geqslant 2$ and $s \nmid r$. Then Theorem 3.2(c) shows again that $W_r{\uparrow}^G \cong B_r^{\mathcal{R}}{\uparrow}^G$. As now $B_r \ncong B_0^z$, we conclude that $W_r{\uparrow}^G$ is indecomposable, by Theorem 9.1. That completes the proof. $\square$

We conclude this section by calculating the dimension of the modules $B^{\mathcal{R}}{\uparrow}^G$, for all 2-blocks B of GL_i. In particular we know the dimension of most of the components of the involution module $k\mathfrak{I}$ of $\mathrm{GL}_n(2^f)$. For any integer $m \geqslant 1$ we define

$$[m]_q := \prod_{j=1}^{m} (q^j - 1).$$

Lemma 9.7. *Let $r = 0, 1, \dots, q-2$ and set $s := \gcd(n, q-1)$. Then*

$$\dim_k \left(B_r^z\right)^{\mathcal{R}}{\uparrow}^G = q^{i(i-1)} \cdot \frac{[n]_q}{[i]_q^2 \cdot [n-2i]_q},$$

$$\dim_k B_r^{\mathcal{R}}{\uparrow}^G = \begin{cases} \frac{q^{\binom{i}{2}}}{q-1} \cdot \frac{[n]_q}{[i]_q \cdot [n-2i]_q}, & \text{if } i = 1 \text{ or } s \nmid r, \\ \frac{q^{\binom{i}{2}}}{q-1} \cdot \frac{[n]_q}{[i]_q^2 \cdot [n-2i]_q} \cdot \left([i]_q - s \cdot (q-1) \cdot q^{\binom{i}{2}}\right), & \text{otherwise}. \end{cases}$$

Proof. Since $\dim_k \mathrm{St}_i = q^{\binom{i}{2}}$, it follows from (1) that $\dim_k B_r^z = q^{i(i-1)}$. Hence $\dim_k (B_r^z)^{\mathcal{R}}{\uparrow}^G = q^{i(i-1)} \cdot |G : R_i|$, where $|G : R_i| = \frac{[n]_q}{[i]_q^2 \cdot [n-2i]_q}$.

Theorem 3.2(c) gives $B_r^{\mathcal{R}}{\uparrow}^G \cong W_r{\uparrow}^G$, if $i = 1$ or $s \nmid r$. Then $\dim_k W_r{\uparrow}^G = |G : N_i|$, where $|G : N_i| = \frac{q^{\binom{i}{2}}}{q-1} \cdot \frac{[n]_q}{[i]_q \cdot [n-2i]_q}$. Finally let $i \geqslant 2$ and $s \mid r$. Then $\dim_k B_r^{\mathcal{R}}{\uparrow}^G = |G : N_i| - s \cdot (q^{i(i-1)} \cdot |G : R_i|)$, by Theorem 3.2(c). $\square$

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