Journal of Algebra 398 (2014) 396–406

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Character correspondences in blocks with normal defect groups ☆

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ARTICLE INFO

Article history:
Received 11 April 2012
Available online 29 June 2012
Communicated by C.W. Eaton

Dedicated to G.R. Robinson on his 60th birthday

MSC:
20C15

Keywords:
Representation theory of finite groups
Character theory
Modular representation theory
Block theory of finite groups

ABSTRACT

In this paper we give an extension of the Glauberman correspondence to certain characters of blocks with normal defect groups.

1. Introduction

Let p be a fixed prime number. The Glauberman correspondence appears in many key places in the character theory of finite groups, specially in those connecting global and local representations. In what perhaps constitutes the most relevant case, the Glauberman correspondence asserts that if a finite p-group P acts as automorphisms on a finite group K of order not divisible by p, then there is a natural bijection between $\mathrm{Irr}_P(K)$, the P-invariant irreducible characters of K, and $\mathrm{Irr}(C)$, the irreducible characters of the fixed point subgroup $C = \mathbf{C}_K(P)$. (In fact, the Glauberman correspondence

☆ The research of the first author is partially supported by the Spanish Ministerio de Educación y Ciencia proyecto MTM2010-15296, and Prometeo/Generalitat Valenciana. The research of the second author is supported by the DFG priority program, SPP 1388.

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.06.007

is defined whenever P is a solvable group acting coprimely on a group G, see Chapter 13 of [Is]. Later on, M. Isaacs removed the hypothesis of P being solvable in [Is2].) The Glauberman correspondence (in the case where a p-group is acting) was noticed to be a consequence of the Brauer correspondence between blocks by J. Alperin in [A].

In 1980, E.C. Dade (using the Brauer correspondence) extended the Glauberman correspondence to a correspondence between certain defect zero characters of $\mathrm{Irr}_P(K)$ and $\mathrm{Irr}(C)$, whenever a group K is acted by a p-group P [D]. This was also discovered independently by H. Nagao (see Theorem (5.12.1) in [NT]), and we have called this the Dade–Glauberman–Nagao correspondence (DGN) in [NaTi].

While the Glauberman correspondence is a key ingredient in the reduction of the McKay conjecture to simple groups, and the Dade–Glauberman–Nagao correspondence plays an important role in a reduction of the Alperin Weight Conjecture [NaTi], it is somewhat remarkable that we need now an extension of the Dade–Glauberman–Nagao correspondence in order to carry out a reduction to simple groups of the unproven half of Brauer's Height Zero conjecture [NS].

To state our Theorem A below, we remind the reader that if G is a finite group, $N \lhd G$, then an irreducible character $\chi \in \mathrm{Irr}(G)$ has **relative p-defect zero** with respect to N (or that χ has **N-relative p-defect zero**) if

$$\bigl(\chi(1)/\theta(1)\bigr)_p = |G/N|_p,$$

where $\theta \in \mathrm{Irr}(N)$ is any irreducible constituent of the restriction χ_N. (Let us briefly mention here that the significance of relative p-defect zero characters, or, in their terminology, of *N-relatively projective characters*, was already pointed out by B. Külshammer and G.R. Robinson in the remarkable paper [KR], and that we shall be using here some techniques introduced by them.)

We need some new notation in order to state our main result. Let G be a finite group. If a p-subgroup P of G normalizes some subgroup $K \leqslant G$, then we denote by $\mathrm{Bl}(K|P)$ the set of p-blocks b of K such that the unique block of KP covering b has defect group P. (If P is contained in K, then $\mathrm{Bl}(K|P)$ becomes the set of blocks of K with defect group P.) If $\tau \in \mathrm{Irr}(G)$, then $\mathrm{bl}(\tau)$ is the p-block of G that contains τ. If $D \lhd G$ and $\mu \in \mathrm{Irr}(D)$, then G_μ is the stabilizer of μ in G, and if $\chi \in \mathrm{Irr}(G)$ lies over μ, then $\chi_\mu \in \mathrm{Irr}(G_\mu)$ is the Clifford correspondent of χ over μ.

Theorem A. *Let G be a finite group, and let p be a prime. Suppose that $K \lhd G$, P is a p-subgroup of $G = KP$ and $P \cap K = D \lhd G$. Let $C = \mathbf{N}_K(P)$. Then*:

(a) *There is a natural bijection $'\colon \mathrm{Bl}(K|P) \to \mathrm{Bl}(C|P)$.*
(b) *If $b \in \mathrm{Bl}(K|P)$, then there is a natural bijection*

$$'\colon \mathrm{Irr}_P(b) \to \mathrm{Irr}_P\bigl(b'\bigr),$$

where $\mathrm{Irr}_P(b)$ are the P-invariant irreducible characters of b.
(c) *If $\eta \in \mathrm{Irr}_P(b)$, then there is a unique C-conjugacy class of P-invariant irreducible constituents $\mu \in \mathrm{Irr}(D)$ of the restriction η_D such that $\mathrm{bl}(\eta_\mu) \in \mathrm{Bl}(K_\mu|P)$. Furthermore, η' is the unique irreducible D-relative p-defect zero constituent of η_C with multiplicity not divisible by p lying over μ.*
(d) *If $\eta \in \mathrm{Irr}_P(b)$, then*

$$\bigl[\eta_C, \eta'\bigr] \equiv \pm 1 \mod p.$$

Of course, when K is a p'-group, then the correspondence in Theorem A is the p-group case of the Glauberman correspondence, and when $D = 1$, this is the extension by Dade and Nagao. Also the case where $D \in \mathrm{Syl}_p(K)$ was recently obtained in [IN], in a totally different context.

2. Proofs

We fix a prime number p. We start with a few elementary results on p-blocks. Our notation follows [N], and all blocks below are p-blocks.

(2.1) Lemma. *Suppose that $K \lhd G$ has p-power index, and let b be a block of K. Let B be the unique block of G that covers b, and suppose that B has defect group P.*

(a) *Then b is G-invariant if and only if $KP = G$.*
(b) *Suppose that b is G-invariant. If B' is the Brauer First Main correspondent of B, then B' covers a unique block b' of $\mathbf{N}_K(P)$. Also, b and b' have defect group $P \cap K$.*

Proof. First of all, recall that b is covered by a unique block of G, by Corollary (9.6) of [N]. We start with (a). Suppose that $KP = G$. Let T be the stabilizer of b in G. If b_T is the Fong–Reynolds correspondent of B over b (Theorem (9.14) of [N]), then b_T has defect group P^g by the Fong–Reynolds correspondence. If $T < G$, then there is some proper normal subgroup $T \subseteq M \lhd G$. Then $P^g \subseteq M$ and this is not possible. Conversely, suppose that b is G-invariant. Then $KP = G$ by Fong's Theorem (9.17) of [N]. This completes the proof of (a).

For (b), we have by (a) that $G = KP$. Hence $\mathbf{N}_G(P) = \mathbf{N}_K(P)P$ and again by (a), we have that B' covers a unique block of $\mathbf{N}_K(P)$, which we call b'. By Fong's Theorem (9.17) of [N], we also have that $P \cap K$ is a defect group of b and that $P \cap K$ is a defect group of b'. $\square$

(2.2) Notation. Suppose that $K \subseteq G$, P is a p-subgroup of G normalizing K. As in the Introduction, we write $\mathrm{Bl}(K|P)$ for the blocks b of K such that the unique block B of KP covering b has defect group P. If P is contained in K, of course $\mathrm{Bl}(K|P)$ is the set of blocks of K with defect group P. If $b \in \mathrm{Bl}(K|P)$, notice that b is P-invariant by Lemma (2.1), and that we have uniquely defined a block b' of $\mathbf{N}_K(P)$, which is the unique block of $\mathbf{N}_K(P)$ covered by B', the Brauer First Main correspondent of B. We call b' the P-correspondent of b.

(2.3) Lemma. *If H is a subgroup of G, $\xi \in \mathrm{Irr}(H)$ belongs to b, and $\xi^G = \chi \in \mathrm{Irr}(G)$ belongs to B, then every defect group of b is contained in a defect group of B.*

Proof. This follows from Corollary (6.2) and Lemma (4.13) of [N]. $\square$

If $N \lhd G$ and $\delta \in \mathrm{Irr}(N)$, as we have said, we denote by G_δ the stabilizer of δ in G. Also $\mathrm{Irr}(G|\delta)$ is the set of the irreducible characters χ of G such that χ_N contains δ, and recall that the Clifford correspondence states that induction defines a bijection $\mathrm{Irr}(G_\delta|\delta) \to \mathrm{Irr}(G|\delta)$. If $\chi \in \mathrm{Irr}(G|\delta)$, then we denote by $\chi_\delta \in \mathrm{Irr}(G_\delta|\delta)$ the Clifford correspondent of χ over δ. Also, recall that

$$[\chi_N, \delta] = \big[(\chi_\delta)_N, \delta\big].$$

(For a proof, see Theorem (6.11) of [Is].) Also, if $\tau \in \mathrm{Irr}(H)$, then $\mathrm{bl}(\tau)$ is the unique p-block of H that contains τ.

(2.4) Lemma. *Suppose that G/K is a p-group and that $b \in \mathrm{Bl}(K)$ has defect group $D \lhd G$. Suppose that B, the only block of G covering b, has defect group P. If $\eta \in \mathrm{Irr}(b)$ is P-invariant, then η_D has a P-invariant constituent μ such that $\mathrm{bl}(\eta_\mu) \in \mathrm{Bl}(K_\mu|P)$. Also, any two of them are $\mathbf{N}_K(P)$-conjugate.*

Proof. Since η is P-invariant, then we have that b is P-invariant, and $G = KP$ by Lemma (2.1). Also, $K \cap P = D$. Let $\delta \in \mathrm{Irr}(D)$ be under η. Let $T = G_\delta$ be the stabilizer of δ in G. Since η is P-invariant (and then G-invariant), we have that $KT = G$ by the Frattini argument. Let $\xi \in \mathrm{Irr}(T \cap K)$ be the Clifford correspondent of η over δ, and let $\tau \in \mathrm{Irr}(T)$ be any character over ξ. By Mackey, $(\tau^G)_K =$

$(\tau_{K\cap T})^K$ contains η, and therefore $(\tau^G)_K = e\eta$. Hence $\tau_{T\cap K} = e\xi$. Now, the block $\mathrm{bl}(\tau)^G$ is defined and covers b, so necessarily $\mathrm{bl}(\tau)^G = B$. Now, let R be a defect group of $\mathrm{bl}(\tau)$. Recall that $D \subseteq R$ by Theorem (4.8) of [N]. We claim that R is a defect group of B. By Lemma (2.3), we have that R is contained in some G-conjugate of P. Since the block of ξ is T-invariant, by Theorem (9.26) of [N], we have that $R \cap T \cap K$ is a defect group of the block of $\mathrm{bl}(\xi)$. Now, $D \subseteq R \cap T \cap K$, and again by Lemma (2.3), we have that $R \cap T \cap K$ should be contained in D. Hence $R \cap T \cap K = D$. Since $\mathrm{bl}(\xi)$ is T-invariant, by Lemma (2.1), we have that $R(T \cap K) = T$. Therefore $RK = G$ and $R \cap K = R \cap T \cap K = D$. Since $P \cap K = D$, we conclude that $|R| = |P|$. Hence R and P are G-conjugate. Therefore P^g is a defect group of the block of G_δ that covers $\mathrm{bl}(\xi)$. Now, if $\mu = \delta^{g^{-1}}$ (and using that η is G-invariant), we have that μ is P-invariant, and that P is a defect group of the unique block that covers $\mathrm{bl}(\eta_\mu)$. Hence, $\mathrm{bl}(\eta_\mu) \in \mathrm{Bl}(K_\mu|P)$. Finally, if $\epsilon = \mu^g$ is P-invariant and P is also a defect group of the unique block B_1 of G_ϵ that covers η_ϵ, then we have that P^g is also a defect group of B_1. Thus $P^g = P^x$ for some $x \in G_\epsilon$. Hence $y = gx^{-1} \in \mathbf{N}_G(P)$ and $\mu^y = \epsilon$. Since $\mathbf{N}_G(P) = \mathbf{N}_K(P)P$, we may choose $y \in \mathbf{N}_K(P)$. □

(2.5) Lemma. *Suppose that $b \in \mathrm{Bl}(G|D)$, where $D \lhd G$. If $\eta \in \mathrm{Irr}(b)$, then η has D-relative p-defect zero.*

Proof. We have that b covers a block e of $E = D\mathbf{C}_G(D)$ with defect group D. If I is the stabilizer of e in G, and b' is the Fong–Reynolds correspondent of b over e, then we have that b' has defect group D, and that $\eta = (\eta')^G$ for some $\eta' \in \mathrm{Irr}(b')$. If $I < G$, then η' has D-relative p-defect zero by induction. If $\mu \in \mathrm{Irr}(D)$ lies under η', then $(\eta'(1)/\mu(1))_p = |I : D|_p$ and using that $|G : I|\eta'(1) = \eta(1)$, we see that η also has D-relative p-defect zero. So we may assume that $I = G$. Hence G/E is a p'-group by Theorem (9.22) of [N]. Now, let $\tau \in \mathrm{Irr}(E|\mu)$ below η. Thus $\eta(1)_p = \tau(1)_p$ by Corollary (11.29) of [Is]. Now using Theorem (9.12) of [N], and its notation, we have that $\tau = \theta_\mu$, where $\theta \in \mathrm{Irr}(E/D)$ has defect zero. Hence $(\tau(1)/\mu(1))_p = |E : D|_p$, and we deduce that $(\eta(1)/\mu(1))_p = |G : D|_p$, as desired. □

Now we restate the Dade–Glauberman–Nagao correspondence in a convenient way, while we add some new information that we shall need below and in [NS]. We use $\mathrm{dz}(H)$ to denote the p-defect zero irreducible characters of H.

(2.6) Theorem. *Suppose that $K \lhd G$, where G/K is a p-group. Suppose that $\theta \in \mathrm{Irr}(K)$ is G-invariant and has defect zero. Assume that the unique block B of G covering θ has defect group P, and let $C = \mathbf{N}_K(P)$.*

(a) *We have that*

$$\theta_C = e\theta' + p\Delta + \Xi,$$

where p does not divide e, $\theta' \in \mathrm{Irr}(C)$ has defect zero, all irreducible constituents of the character Δ have defect zero, and none of the irreducible constituents of the character Ξ has defect zero.

(b) *We have that*

$$\theta(1)_{p'} \equiv e|K : C|_{p'}\theta'(1)_{p'} \mod p$$

and

$$e \equiv \pm 1 \mod p.$$

(c) *If $x \in K$ with $P \in \mathrm{Syl}_p(\mathbf{C}_G(x))$, then*

$$\theta(x) \equiv e\theta'(x) \mod p.$$

(d) *If $\mathrm{dz}_G(K|P)$ is the set of G-invariant p-defect zero characters of K covered by a block of G with defect group P, then the map $\mathrm{dz}_G(K|P) \to \mathrm{dz}(C)$ given by $\theta \mapsto \theta'$ is a bijection.*

Proof. Parts (a) and (d) follow from Theorem (5.12.1) of [NT], and we freely use the proof of that theorem in what follows. Recall also that P is a complement of K in G and that $N = \mathbf{N}_G(P) = C \times P$, where $C = \mathbf{C}_K(P) = \mathbf{N}_K(P)$. Next we prove (b). By our hypotheses, we have that θ has an extension ψ to G. (See Problem (3.10) of [N] for an elementary proof.) It then follows that $\mathrm{Irr}(B) = \{\tau\psi \mid \tau \in \mathrm{Irr}(G/K)\}$, using Gallagher's Corollary (6.17) of [Is], and Theorem (9.2.b) of [N]. If B' is the Brauer First Main correspondent of B, we also have that $\mathrm{Irr}(B') = \{\tau\psi' \mid \tau \in \mathrm{Irr}(N/C)\}$, where $\psi' = \theta' \times 1_P$. By the definition of the dimension of blocks, we have

$$\dim(B) = \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)^2 = \theta(1)^2|P|.$$

In the same way,

$$\dim\bigl(B'\bigr) = \theta'(1)^2|P|.$$

Now, by using the formula at the third paragraph of Theorem (2.1) in [M], we have that

$$\dim(B)_{p'} = \bigl(|G:N|_{p'}\bigr)^2 \dim\bigl(B'\bigr)_{p'}.$$

We deduce that

$$\theta(1)_{p'} \equiv \pm|K:C|_{p'}\theta'(1)_{p'} \mod p, \tag{$*$}$$

as wanted.

Now, let $\rho = (\psi')^G$. This character can be decomposed as the sum of its $\tilde{B}$-components $\rho_{\tilde{B}}$ for $\tilde{B} \in \mathrm{Bl}(G)$. According to Corollary 5.3.2 of [NT] we have that

$$\rho_{\tilde{B}}(1)_p > \rho_B(1)_p = \rho(1)_p$$

for every $\tilde{B} \in \mathrm{Bl}(G) - \{B\}$. Now,

$$|G:N|\theta'(1) = \bigl(\psi'\bigr)^G(1) = \rho_B(1) + \sum_{\tilde{B} \in \mathrm{Bl}(G)-\{B\}} \rho_{\tilde{B}}(1)$$

and by dividing by $\rho(1)_p$, we obtain

$$|G:N|_{p'}\theta'(1)_{p'} \equiv \rho_B(1)_{p'} \mod p.$$

Write $(\rho_B)_K = e\theta$ for some integer e. Since $[(\rho_{\tilde{B}})_K, \theta] = 0$ for $\tilde{B} \in \mathrm{Bl}(G) - \{B\}$, we have that

$$e = \bigl[(\rho_{\tilde{B}})_K, \theta\bigr] = [\rho_K, \theta] = \bigl[\bigl(\theta'\bigr)^K, \theta\bigr] = \bigl[\theta_C, \theta'\bigr]$$

which we know is not divisible by p by Theorem (5.12.1) of [NT]. Hence $\rho_B(1)_{p'} = e\theta(1)_{p'}$ and $|K:C|_{p'}\theta'(1)_{p'} \equiv e\theta(1)_{p'}$. By using our previous congruence $(*)$, we deduce that

$$e \equiv \pm 1 \mod p,$$

as desired.

Finally we show (c). We have that $x \in K$ with $P \in \mathrm{Syl}_p(\mathbf{C}_G(x))$. Then $\mathrm{Cl}_N(x) = \mathrm{Cl}_G(x) \cap \mathbf{C}_G(P)$ by Lemma (4.16) of [N]. Now since $(B')^G = B$, we use that $(\lambda_{B'})^G = \lambda_B$ to obtain that

$$\left(\frac{|\mathrm{Cl}_G(x)|\psi(x)}{\psi(1)}\right)^* = \left(\frac{|\mathrm{Cl}_N(x)|\psi'(x)}{\psi'(1)}\right)^*.$$

Now

$$\frac{|G|\theta(x)}{|\mathbf{C}_G(x)|\theta(1)} = \frac{|K|\theta(x)}{|\mathbf{C}_K(x)|\theta(1)} = \frac{|K|_{p'}\theta(x)}{|\mathbf{C}_K(x)|\theta(1)_{p'}}$$

and in the same way,

$$\frac{|N|\theta'(x)}{|\mathbf{C}_N(x)|\theta'(1)} = \frac{|C|_{p'}\theta'(x)}{|\mathbf{C}_C(x)|\theta'(1)_{p'}}.$$

Now, $|\mathbf{C}_K(x) : \mathbf{C}_C(x)| = |\mathbf{C}_G(x) : \mathbf{C}_N(x)| = |\mathbf{C}_G(x) : \mathbf{N}_{\mathbf{C}_G(x)}(P)| \equiv 1 \mod p$ by Sylow theory. Since $1 = P \cap \mathbf{C}_K(x) \in \mathrm{Syl}_p(\mathbf{C}_K(x))$, we have that $\mathbf{C}_K(x)$ is not divisible by p. Using that $\psi(x) = \theta(x)$ and part (b), all this easily implies (d). $\square$

In the next results, we heavily use the following fact: if $D \lhd K$ and $\mu \in \mathrm{Irr}(D)$ is K-invariant, then there exists a natural bijection $\chi \mapsto {}_\mu\chi$ between the set $\mathrm{dz}(K/D)$ of the defect zero characters of K/D and the set $\mathrm{rdz}(K|\mu)$ of the D-relative p-defect zero characters of K over μ. (See Section 2 of [N1] for a proof of this result. In [N1] the character ${}_\mu\chi$ was denoted by χ_μ, but we have already used this notation in this paper.)

(2.7) Lemma. *Suppose that G/K is a p-group and that $b \in \mathrm{Bl}(K)$ has defect group $D \lhd G$. Suppose that B is the only block of G covering b, and assume that B has defect group P. Let $C = \mathbf{N}_K(P)$ and let b' be the P-correspondent of b. Let $\mu \in \mathrm{Irr}(D)$ be G-invariant, and suppose that $\chi \in \mathrm{dz}(K/D)$ is P-invariant.*

(a) *We have that ${}_\mu\chi \in \mathrm{Irr}(b)$ if and only if $\chi \in \mathrm{Irr}(b)$. In this case, the unique block of G/D covering $\{\chi\}$ has defect group P/D.*
(b) *Suppose that the block of G/D covering χ has defect group P/D, and let $\chi' \in \mathrm{Irr}(C/D)$ be the Dade–Glauberman–Nagao correspondent of χ. Then $\chi \in \mathrm{Irr}(b)$ if and only if $\chi' \in \mathrm{Irr}(b')$.*

Proof. Let $z \in \mathbf{C}_K(D)$ be p-regular. Then $D\langle z\rangle = D \times \langle z\rangle$ and $\hat{\mu}(z) = \mu(1)$, by the definition of the function $\hat{\mu}$ in [N1]. Therefore we have that

$$\left(({}_\mu\chi)^0\right)_{\mathbf{C}_K(D)} = \mu(1)\left(\chi^0\right)_{\mathbf{C}_K(D)}.$$

In particular, the Brauer characters χ^0 and $({}_\mu\chi)^0$ have irreducible constituents τ_1 and τ_2 in $\mathrm{IBr}(K)$, respectively, which lie over a common $\delta \in \mathrm{IBr}(\mathbf{C}_K(D))$. Let e be the block of δ. Suppose that ${}_\mu\chi \in \mathrm{Irr}(b)$. Since δ lies under $\tau_2 \in \mathrm{IBr}(b)$, then b covers e (Corollary (9.2) of [N]) and $e^K = b$ (since b is the only block covering e by Corollary (9.21) of [N]). Since the block of τ_1 also covers e, we have that $\tau_1 \in \mathrm{IBr}(b)$ and χ belongs to b. The same argument proves the converse.

Suppose now that $\chi \in \mathrm{Irr}(b)$, and consider $\tilde{\chi} \in \mathrm{Irr}(G/D)$ to be an extension of χ. Then $\tilde{\chi}$ belongs to B. If $L = \mathrm{Cl}_G(x)$ is a p-regular defect class for B, then, using that $\mathbf{C}_{G/D}(Dx) = \mathbf{C}_G(x)D/D$, we have that

$$\omega_{\tilde{\chi}}(\hat{L}) = \frac{|\mathrm{Cl}_G(x)|\chi(x)}{\chi(1)} = \left|\mathbf{C}_G(x)D : \mathbf{C}_G(x)\right|\left(\left|\mathrm{Cl}_{G/D}(Dx)\right|\frac{\tilde{\chi}(Dx)}{\tilde{\chi}(1)}\right),$$

where on the right hand side we view $\tilde{\chi}$ as a character of G/D. Now, by Theorem (4.4) of [N], we deduce that a defect group of the block of G/D containing $\tilde{\chi}$ is contained in P/D. Since this defect group has to complement K/D in G/D, the proof of (a) is now complete.

To prove (b), write $\chi = \bar{\chi} \in \mathrm{Irr}(K/D)$ viewed as a character of K/D and let $\tilde{B}$ be the block of G/D that covers $\{\bar{\chi}\}$. Recall that $\tilde{B}$ is contained in B if and only if the same happens with its Brauer correspondents $(\tilde{B})'$ and B'. (This follows easily from Lemma (3.2) of [N2], for instance.) By the construction of the DGN correspondence, we have that $(\tilde{B})'$ is the only block of $\mathbf{N}_G(P)/D$ covering $(\bar{\chi})' = \chi'$. Suppose that $\chi \in \mathrm{Irr}(b)$. Then $\tilde{B}$ is contained in B, and the same happens with Brauer First Main correspondents $(\tilde{B})'$ and B'. Then $\bar{\chi}'$ is covered by $(\tilde{B})'$ and thus χ' is covered by B'. Thus $\chi' \in \mathrm{Irr}(b')$. Suppose conversely that $\chi' \in \mathrm{Irr}(b')$, so B' contains $(\tilde{B})'$ and B contains $\tilde{B}$, so $\chi \in \mathrm{Irr}(b)$, since b is the only block of K covered by B. $\square$

(2.8) Theorem. *Suppose that G/K is a p-group, that $b \in \mathrm{Bl}(K)$ has defect group $D \lhd G$. Suppose that B, the only block of G covering b, has defect group P. Let $C = \mathbf{N}_K(P)$. Let $\mu \in \mathrm{Irr}(D)$ be G-invariant, and let $\mathrm{Irr}_P(b|\mu) = \mathrm{Irr}(K|\mu) \cap \mathrm{Irr}_P(b)$. Suppose that $\eta \in \mathrm{Irr}_P(b|\mu)$.*

(a) *We have that*

$$\eta_C = e\eta' + p\Delta + \Psi,$$

where $\eta' \in \mathrm{Irr}(C)$ has D-relative p-defect zero, p does not divide e, every irreducible constituent of Δ has D-relative p-defect zero and no irreducible constituent of Ψ has D-relative p-defect zero.

(b) *We have that*

$$\eta(1)_{p'} \equiv e|K : C|_{p'}\eta'(1)_{p'} \mod p$$

and

$$e \equiv \pm 1 \mod p.$$

(c) *If b' is the P-corresponding block of b, then the map*

$$\mathrm{Irr}_P(b|\mu) \to \mathrm{Irr}_P\left(b'\middle|\mu\right)$$

defined by $\eta \mapsto \eta'$ is a well-defined natural bijection.

(d) *If $x \in K$ is p-regular with $P \in \mathrm{Syl}_p(\mathbf{C}_G(x))$, we have that*

$$\frac{\eta(x)}{\mu(1)} \equiv e\frac{\eta'(x)}{\mu(1)} \mod p.$$

Proof. Again, we use Theorem (2.1) of [N1]. Hence, we have that the map $\mathrm{dz}(K/D) \to \mathrm{rdz}(K|\mu)$ given by $\chi \mapsto {}_{\mu}\chi$ is a canonical bijection. Also,

$${}_{\mu}\chi(k) = 0 = \chi(k)$$

for all $k \in K$ with $k_p \notin D$, and ${}_{\mu}\chi(k) = \hat{\mu}(k)\chi(k)$ if $k_p \in D$ (where $\hat{\mu}$ is the canonical extension of μ to $D\langle k\rangle$).

Suppose that $\gamma \in \mathrm{Irr}(C|\mu)$ and $\xi \in \mathrm{Irr}(K|\mu)$ have D-relative p-defect zero. Hence $\gamma = {}_{\mu}\theta$ and $\xi = {}_{\mu}\chi$ for some defect zero $\theta \in \mathrm{dz}(C/D)$ and $\chi \in \mathrm{dz}(K/D)$. Write $C = Dx_1 \cup \cdots \cup Dx_t$, as a disjoint union, and suppose that $Dx_1, \ldots, Dx_s$ are exactly the cosets Dx with $x_p \in D$. Notice that if $x_p \in D$

then $(dx)_p \in D$ for all $d \in D$ because $x_p \in D$ if and only if Dx has order not divisible by p. Using that D is contained in the kernel of θ and χ, it follows that

$$\begin{aligned}
[\xi_C, \gamma] = \bigl[({}_\mu\chi)_C, {}_\mu\theta\bigr] &= \frac{1}{|C|}\sum_{j=1}^{t}\Bigl(\sum_{d\in D} {}_\mu\chi(dx_j)\overline{{}_\mu\theta(dx_j)}\Bigr) \\
&= \frac{1}{|C|}\sum_{j=1}^{s}\Bigl(\sum_{d\in D} \chi(dx_j)\hat{\mu}(dx_j)\overline{\theta(dx_j)}\,\overline{\hat{\mu}(dx_j)}\Bigr) \\
&= \frac{1}{|C|}\sum_{j=1}^{s}\chi(x_j)\overline{\theta(x_j)}\Bigl(\sum_{d\in D}\hat{\mu}(dx_j)\overline{\hat{\mu}(dx_j)}\Bigr) \\
&= \frac{|D|}{|C|}\sum_{j=1}^{s}\chi(x_j)\overline{\theta(x_j)} \\
&= \frac{|D|}{|C|}\sum_{j=1}^{t}\chi(x_j)\overline{\theta(x_j)} = [\chi_{C/D}, \theta],
\end{aligned}$$

where we have used Lemma (8.14.c) of [Is].

Suppose now that $\eta \in \mathrm{Irr}_P(b|\mu)$; that is, $\eta \in \mathrm{Irr}(b)$ is P-invariant and lies over μ. In particular, we have that b is P-invariant and $G = KP$ and $D = K \cap P$, by Lemma (2.1). Then η has D-relative p-defect zero by Lemma (2.5), and therefore $\eta = {}_\mu\chi$ for a unique defect zero $\chi \in \mathrm{Irr}(K/D)$. By uniqueness, we see that χ is P-invariant. By Lemma (2.7), we have that the block of G/D that covers χ has defect group P/D. Then, by Theorem (2.6), we can write

$$\chi_C = e\chi' + p\Delta_0 + \Xi,$$

where p does not divide e, all irreducible constituents of Δ_0 have defect zero, and no irreducible constituent of Ξ has defect zero. Also, $e \equiv \pm 1 \mod p$ and

$$\chi(1)_{p'} \equiv e|K:C|_{p'}\chi'(1)_{p'} \mod p.$$

From this equality and the equation in the second paragraph, we deduce that

$$\eta_C = e\eta' + p\Delta + \Psi,$$

where $\eta' = {}_\mu(\chi') \in \mathrm{Irr}(b')$ (by Lemma (2.7)), and Δ and Ψ are as required in the statement. Also, $({}_\mu\chi(1))_{p'} = \chi(1)_{p'}$.

Now, using the same ideas as in the previous paragraphs, it is straightforward to prove that if b' is the P-corresponding block of b, then the map $\eta \mapsto \eta'$ defines a bijection $\mathrm{Irr}_P(b|\mu) \to \mathrm{Irr}_P(b'|\mu)$.

Finally, to prove (d), notice that $\mathbf{C}_{G/D}(Dx) = \mathbf{C}_G(x)/D$ since x is p-regular. Hence we can apply Theorem (2.6.d) to χ and use that $\hat{\mu}(x) = \mu(1)$. $\square$

In the notation of the previous theorem, notice that if χ is a defect zero character of K/D and $\eta = {}_\mu\chi$ for some G-invariant $\mu \in \mathrm{Irr}(D)$, then $\eta' = {}_\mu(\chi')$, where χ' is the Dade–Glauberman–Nagao correspondent of χ.

Proof of Theorem A. (a) We have that the map $' : \mathrm{Bl}(K|P) \to \mathrm{Bl}(C|P)$ given by $b \mapsto b'$ is a bijection by the Brauer First Main theorem and Lemma (2.1).

We prove the rest of the theorem simultaneously. Suppose that $b \in \mathrm{Bl}(K|P)$ and $\eta \in \mathrm{Irr}_P(b)$. By Lemma (2.4), η_D contains a P-invariant constituent μ such that, if $\tau = \eta_\mu$ is the Clifford correspondent of η over μ and $T = K_\mu$, then $\mathrm{bl}(\tau) \in \mathrm{Bl}(T|P)$. Also, any other choice of such a μ is C-conjugate to μ. Now, by the Clifford correspondence (see Lemma (2.5) of [S] for a proof), we have that

$$\eta_C = \left(\tau^K\right)_C = (\tau_{T\cap C})^C + \Xi,$$

where no irreducible constituent of Ξ lies over μ, and

$$\left[\eta_C, \rho^C\right] = [\tau_{T\cap C}, \rho]$$

for $\rho \in \mathrm{Irr}(T \cap C|\mu)$. Since induction of characters gives a bijection $\mathrm{Irr}(T \cap C|\mu) \to \mathrm{Irr}(C|\mu)$, which sends D-relative p-defect zero characters onto D-relative p-defect zero characters, part (c) of Theorem A follows now from Theorem (2.8) applied to τ and by defining $\eta' = (\tau')^C$. It is straightforward to check that our map is a bijection. $\square$

As happens with the Glauberman correspondence, the deeper applications of our correspondence will occur when we will have D, K and G as normal subgroups of some finite overgroup Γ and we wish to relate the characters in $\mathrm{Irr}(\Gamma|\eta)$ with the characters of $\mathrm{Irr}(\mathbf{N}_\Gamma(P)|\eta')$.

We finish this paper with an apparently elementary property of our correspondence that shall be used elsewhere, and that illustrates how we deal with this new correspondence. First, it is convenient to introduce the following definition.

(2.9) Definition. Suppose that K, P are subgroups of some finite group G, where P normalizes K and $P \cap K = D \lhd K$. Suppose that $\theta \in \mathrm{Irr}(K)$ is P-invariant. If the unique block of KP that covers the block of K that contains θ has defect group P, then we say that the P**-correspondent** of θ is **defined**. Also, the character $\theta' \in \mathrm{Irr}(\mathbf{N}_K(P))$ provided by Theorem A is called the P**-correspondent** of θ. (Notice that in this situation, we have that D is the defect group of the block of θ, using, for instance, Theorem (9.26) of [N].)

We shall use the following.

(2.10) Lemma. *Suppose that a p-group P acts on G, and suppose that $N \lhd G$ is P-invariant and that p does not divide $|N|$. If P acts trivially on G/N, then $G = \mathbf{C}_G(P)N$.*

Proof. This follows from Theorem (3.27) of [Is1]. $\square$

Suppose that $E \lhd G$, write $\bar{G} = G/E$ and use the bar-convention. Recall that if $K \subseteq G$, then $K/(E\cap K)$ is naturally isomorphic to the subgroup KE/E of G/E. Consequently, all the characters τ of K that contain $E \cap K$ in its kernel can be seen as characters $\bar{\tau}$ of $\bar{K} = KE/E$, where

$$\bar{\tau}(\bar{k}) = \bar{\tau}(Ek) = \tau(k)$$

for all $k \in K$.

(2.11) Theorem. *Suppose that K and P are subgroups of a finite group G, where P is a p-subgroup of G that normalizes K, with $K \cap P = D \lhd K$. Suppose that the P-correspondent $\theta' \in \mathrm{Irr}(\mathbf{N}_K(P))$ of $\theta \in \mathrm{Irr}(K)$ is defined, and suppose that $E \lhd G$ is such that $E \cap K \subseteq \ker(\theta)$. Let $\bar{G} = G/E$ and use the bar convention. Then $\overline{\mathbf{N}_K(P)} = \mathbf{N}_{\bar{K}}(\bar{P})$, the $\bar{P}$-correspondent of $\bar{\theta} \in \mathrm{Irr}(\bar{K})$ is defined and*

$$(\bar{\theta})' = \overline{\theta'}.$$

Proof. Let $M = KP$, let b be the block of θ, let B be the unique block of M that covers b, which we know has defect group P by hypothesis. Also, we know that b is P-invariant and has defect group $D = P \cap K \lhd M$. Let $\bar{b}$ be the block of $\bar{K}$ that contains $\bar{\theta} \in \mathrm{Irr}(\bar{K})$. We have that $\bar{b}$ is $\bar{M}$-invariant because $\bar{\theta}$ is $\bar{P}$-invariant. Now, if c is the block of $\theta \in \mathrm{Irr}(K/E \cap K)$ viewed as character of $K/E \cap K$ by Theorem (9.9.a) of [N], we have that $D(E \cap K)/(E \cap K)$ contains a defect group of c. However every defect group of c contains $\mathbf{O}_p(K/E \cap K)$ (Theorem (4.8) of [N]), and since D is normal in K, we conclude that $D(E \cap K)/(E \cap K)$ is the defect group of the block of $\theta \in \mathrm{Irr}(K/E \cap K)$. Now, since $K/E \cap K$ is naturally isomorphic to $KE/E = \bar{K}$, we conclude that the block of $\bar{\theta} \in \mathrm{Irr}(\bar{K})$ has defect group $DE/E = \bar{D} \lhd \bar{K}$.

Now $\bar{M}/\bar{K}$ is a p-group, so there is a unique block $\bar{B}$ of $\bar{M}$ that covers $\bar{b}$. We want to show next that $\bar{P}$ is a defect group of $\bar{B}$.

Let $\bar{\gamma} \in \mathrm{Irr}(\bar{M})$ be over $\bar{\theta}$, and let $\hat{\gamma} \in \mathrm{Irr}(ME)$ be the irreducible character of ME that contains E in its kernel and that corresponds to $\bar{\gamma}$. We have that $\bar{\gamma}$ belongs to $\bar{B}$. Also, we have that $\hat{\gamma}_M \in \mathrm{Irr}(M)$ lies over θ. Hence $\hat{\gamma}_M$ belongs to B. Also, notice that $\hat{\gamma}_M$ has $M \cap E$ in its kernel. In particular, B covers the principal block of $M \cap E$, and we conclude that $P \cap E$ is a Sylow p-subgroup of $M \cap E$ by Theorem (9.26) of [N]. By Theorem (9.9.a) of [N], we have that a defect group $P_0/E \cap M$ of $\hat{\gamma}_M \in \mathrm{Irr}(M/E \cap M)$ (viewed as a character of $M/E \cap M$) is contained in $P(E \cap M)/(E \cap M)$. Since $M/E \cap M$ and ME/E are naturally isomorphic, we deduce that the block $\bar{B}$ of $\bar{\gamma}$ has a defect group $P_0E/E = \bar{P}_0$ that is contained in $\bar{P}$. Also, since $\bar{b}$ is $\bar{M}$-invariant and has defect group $\bar{D} \lhd \bar{M}$, we know that $\bar{P}_0$ is a complement of $\bar{K}$ in $\bar{M}$. It is enough then to show that

$$|\bar{P}|/|\bar{D}| = |\bar{M}|/|\bar{K}|.$$

This easily reduces to checking

$$|M \cap E|/|E \cap P| = |K \cap E|/|D \cap E|,$$

which is true because $P \cap E \in \mathrm{Syl}_p(M \cap E)$ and $(M \cap E)/(K \cap E)$ is a p-group. We then conclude that $\bar{P}$ is a defect group of $\bar{B}$, and this proves that the $\bar{P}$-correspondent of $\bar{\theta}$ is defined.

Next we show that $\overline{\mathbf{N}_K(P)} = \mathbf{N}_{\bar{K}}(\bar{P})$. So if $J = \mathbf{N}_{KE}(PE)$, then we wish to show that $J = \mathbf{N}_K(P)E$. Let $I = J \cap K$. Hence $J = IE$. Since $P \cap E \in \mathrm{Syl}_p(M \cap E)$, we have that $D \cap E = P \cap K \cap E \in \mathrm{Syl}_p(K \cap E)$. Therefore $K \cap E/(D \cap E)$ is a normal p'-subgroup of $I/(D \cap E)$. Because KE/E is naturally isomorphic to $K/K \cap E$ and P acts trivially on J/DE (because PE/DE complements KE/DE in ME/DE), it follows that P acts trivially on $I/D(K \cap E)$. By Lemma (2.10), applied to the action of P on I/D and using that $D(E \cap K)/D$ is a normal p'-subgroup of I/D, we conclude that

$$I/D = D(K \cap E)/D\mathbf{C}_{I/D}(P) \subseteq D(K \cap E)/D\mathbf{C}_{K/D}(P) \subseteq D(E \cap K)\mathbf{N}_K(P)/D.$$

Hence $I \subseteq (K \cap E)\mathbf{N}_K(P)$, and therefore $J \subseteq E\mathbf{N}_K(P)$, which is trivially contained in $\mathbf{N}_{KE}(PE) = J$.

By Lemma (2.4), we know that the restriction θ_D has a unique $\mathbf{N}_K(P)$-conjugacy class of irreducible constituents $\mu \in \mathrm{Irr}(D)$ such that the block of $K_\mu P$ that covers the block of θ_μ has defect group P. Now, since $E \cap K$ is in the kernel of θ and $(\theta_\mu)^K = \theta$, then we have that $E \cap K \subseteq \ker(\theta_\mu)$ by elementary character theory. Now, by the first part of this proof we conclude that the block of $\overline{\theta_\mu} \in \mathrm{Irr}(K_\mu E/E)$ has defect group $\bar{D}$. Since $E \cap K$ is contained in the kernel of θ, we have that $E \cap D$ is contained in the kernel of μ, and therefore μ naturally corresponds to a character $\bar{\mu} \in \mathrm{Irr}(\bar{D})$. We also notice that the stabilizer $(\bar{K})_{\bar{\mu}} = K_\mu E/E$. Finally, we observe that $|N_K(P)E : DE|_p = |N_K(P) : D|_p$ using again that $D \cap E$ is a Sylow p-subgroup of $E \cap K$. Finally, we are able to use Theorem A(c) to easily check that

$$(\bar{\theta})' = \overline{\theta'},$$

as desired. $\square$

References

[A] J.L. Alperin, The main problem of block theory, in: Proceedings of the Conference on Finite Groups, Univ. of Utah, Park City, UT, 1975, Academic Press, New York, 1976, pp. 341–356.

[D] E.C. Dade, A correspondence of characters, in: Proc. Sympos. Pure Math., vol. 37, 1980, pp. 401–403.

[KR] B. Külshammer, G.R. Robinson, Characters of relatively projective modules II, J. Lond. Math. Soc. 36 (1987) 59–67.

[Is] M. Isaacs, Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, RI, 2006.

[Is1] M. Isaacs, Finite Group Theory, AMS, Providence, RI, 2008.

[Is2] I.M. Isaacs, Characters of solvable and symplectic groups, Amer. J. Math. 95 (1973) 594–635.

[IN] I.M. Isaacs, G. Navarro, Groups whose real irreducible characters have degree coprime to p, J. Algebra 356 (1) (2012) 195–206.

[M] G. Michler, Trace and defect of a block idempotent, J. Algebra 131 (1990) 496–501.

[N] G. Navarro, Characters and Blocks of Finite Groups, Cambridge University Press, 1998.

[N1] G. Navarro, Actions and characters in blocks, J. Algebra 275 (2004) 471–480.

[N2] G. Navarro, The McKay conjecture and Galois automorphisms, Ann. of Math. 160 (2004) 1129–1140.

[NS] G. Navarro, B. Späth, On Brauer's height zero conjecture, submitted for publication.

[NaTi] G. Navarro, Pham H. Tiep, A reduction theorem for the Alperin weight conjecture, Invent. Math. 184 (3) (2011) 529–565.

[NT] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Inc., Boston, MA, 1989.

[S] L. Sanus, Induction and character correspondences in groups of odd order, J. Algebra 249 (2002) 186–195.