Journal of Algebra 402 (2014) 174–177

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Trivial unit conjecture and homotopy theory

Shengkui Ye [a,b,*]

[a] Mathematics and Physics Centre, Xi'an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu 215123, China
[b] Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford, OX1 3LB, UK

ARTICLE INFO

Article history:
Received 17 April 2013
Available online 9 January 2014
Communicated by Michel Broué

Keywords:
Trivial unit conjecture
Group rings
Homotopy groups

ABSTRACT

A homotopy theoretic description is given for trivial unit conjecture in the group ring $\mathbb{Z}G$.

1. Introduction

Let G be a torsion-free group and $\mathbb{Z}G$ the integral group ring. The trivial unit conjecture for G says that any invertible element (unit) of $\mathbb{Z}G$ is of the form $\pm g$ for some $g \in G$ (cf. [6], Chapter 13). For solving such a conjecture, to the author's knowledge, almost all the approaches used are algebraic (cf. [1] and references therein). In this note, we give a homotopy theoretic description of such a conjecture.

Let X be a CW complex with fundamental group $\pi_1(X) = G$. For any integer $d \geqslant 2$ and map $f : S^d \to X \vee S^d$, we construct a CW complex $Y_f = (X \vee S^d) \cup_f e^{d+1}$. In this note, the following homotopy theoretic characterization is obtained:

* Correspondence to: Mathematics and Physics Centre, Xi'an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu 215123, China.
E-mail addresses: Shengkui.Ye@xjtlu.edu.cn, Shengkui.Ye@maths.ox.ac.uk.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.12.019

Theorem 1. *Let G be a torsion-free group. The trivial unit conjecture for G is true if and only if for an Eilenberg–Mac Lane space $X = BG$, the element $[f] \in \pi_d(\widetilde{X \vee S^d}, S^d)$ (the relative homotopy group of the universal covering space) vanishes for some lifting of S^d whenever the inclusion $i_f : X \to Y_f$ is a homotopy equivalence.*

All modules considered in this note are left modules. Let $\tilde{Y}_f$ be the universal covering space of Y_f and $C_i(\tilde{Y}_f)$ the i-th term of the cellular chain complex of $\tilde{Y}_f$. By definition, $C_i(\tilde{Y}_f)$ is a free $\mathbb{Z}G$-module spanned by the set of all i-cells. For the inclusion $i_f : X \to Y_f$, we have a cellular map $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ which lifts i_f. As the map i_f induces the identity homomorphism on fundamental groups of X and Y_f, we may assume that $\tilde{X}$ is a subspace of $\tilde{Y}_f$. The relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ of $(\tilde{Y}_f, \tilde{X})$ is of the following form

$$0 \to C_{d+1}(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \xrightarrow{\partial} C_d(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \to 0.$$

This is a chain complex whose terms are all vanishing except for the d-th term a free $\mathbb{Z}G$-module spanned by S^d and the $(d+1)$-th term a free $\mathbb{Z}G$-module spanned by e^{d+1}. Let $\gamma_f = \partial(1) \in \mathbb{Z}G$, the unique element determined by the boundary map ∂. We give a homotopy theoretic description of units in $\mathbb{Z}G$ as follows.

Lemma 2. *Let $\gamma_f \in \mathbb{Z}G$ be the element defined above. Then γ_f is an invertible element if and only if the inclusion $i_f : X \hookrightarrow Y_f$ is a homotopy equivalence.*

Proof. All the notations used in this proof are the same as defined before. Suppose that $\gamma_f = \partial(1)$ is an invertible element in $\mathbb{Z}G$. Then ∂ is both injective and surjective, which shows the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $\tilde{i}_f$ induces an isomorphism between the homology groups $H_i(\tilde{X})$ and $H_i(\tilde{Y}_f)$ for each $i \geqslant 0$. Since $\tilde{X}$ and $\tilde{Y}_f$ are both simply connected, $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence. Since i_f induces the identity homomorphism on fundamental groups, this shows that $i_f : X \to Y_f$ is a homotopy equivalence by the Whitehead theorem.

Conversely, suppose that $i_f : X \to Y_f$ is a homotopy equivalence. Then $\tilde{i}_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence, which implies that the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $\gamma_f = \partial(1)$ has a left inverse. It is a well-known fact that in the integral group ring of a torsion-free group, one-sided invertible element is also two-sided invertible (cf. Corollary 1.9 from [6, p. 38]). This finishes the proof. $\square$

Proof of Theorem 1. Let $X = BG$, the classifying space of G. Suppose that the trivial unit conjecture for G is true. For an integer $d \geqslant 2$ and a map $f : S^d \to X \vee S^d$, suppose that the CW complex $Y_f = (X \vee S^d) \cup_f e^{d+1}$ has its inclusion $i_f : X \to Y_f$ a homotopy equivalence. By Lemma 2, the element γ_f is a unit. Therefore, $\gamma_f = \pm g$ for some element $g \in G$. As the d-th and $(d+1)$-th terms of the relative chain complex are free $\mathbb{Z}G$-modules, we can view them as submodules of $C_i(\tilde{Y})$ $(i = d, d+1$ resp.). Since $\tilde{X}$ is a free G-CW complex and S^d is simply connected, the universal covering space $\widetilde{X \vee S^d}$ could be taken as the push out the following diagram

$$\begin{array}{ccc} G \times \mathrm{pt} & \to & \tilde{X} \\ \downarrow & & \downarrow \\ G \times S^d & \to & \tilde{X} \vee_G (G \times S^d). \end{array}$$

Since $X = BG$ is aspherical, $\tilde{X}$ is contractible. This implies that there is a homotopy equivalence $\tilde{X} \vee_G (G \times S^d) \simeq \vee_G S^d$, where $\vee_G S^d$ is the wedge of copies of S^d indexed by G. For any element $h \in G$, let $p_h : \widetilde{X \vee S^d} \to S^d$ be the projection onto the h-component of $\vee_G S^d$. Consider a lifting $\tilde{f}$ of f to the universal covering space as shown in the following diagram

$$\begin{array}{ccc} & & \widetilde{X \vee S^d} \\ & \tilde{f} \nearrow & \downarrow \\ S^d & \xrightarrow{f} & X \vee S^d. \end{array}$$

This $\tilde{f}$ actually determines the $(d+1)$-th boundary map in the chain complex of $\tilde{Y}_f$. By the definition of the boundary map ∂, the degree of the composition

$$S^d \xrightarrow{\tilde{f}} \widetilde{X \vee S^d} \xrightarrow{p_h} S^d$$

is zero when $h \neq g$ or ± 1 when $h = g$. Therefore, $\tilde{f}$ is homotopic to some map $\tilde{g}$ whose image occupies only the g-component S^d. This shows that $[f] := \tilde{f} \in \pi_d(\widetilde{X \vee S^d}, S^d)$ is vanishing, where S^d is viewed as the g-component S^d.

Conversely, suppose that γ is a nontrivial invertible element in $\mathbb{Z}G$. We will construct some map $f_\gamma : S^d \to X \vee S^d$ such that the inclusion $i_{f_\gamma} : X \to Y_f$ is a homotopy equivalence but $[f_\gamma] \in \pi_d(X \vee S^d, S^d)$ is not vanishing. Assume that $\gamma = \sum a_g g$ for $g \in G$ and $a_g \in \mathbb{Z}$. As in the first part of this proof, the universal covering space $\widetilde{X \vee S^d} = \tilde{X} \vee_G (G \times S^d)$ could be a free G-CW complex. Let $p_h : \tilde{X} \vee_G (G \times S^d) \simeq \vee_G S^d \to S^d$ be the projection onto the h-component. Define $\tilde{f}_\gamma : S^d \to \tilde{X} \vee_G (G \times S^d) \simeq \vee_G S^d$ as a cellular map such that the degree of the composition

$$S^d \xrightarrow{\tilde{f}_\gamma} \widetilde{X \vee S^d} \xrightarrow{p_h} S^d$$

is a_h for each $h \in G$. Denote by

$$\phi_\gamma : G \times S^d \to \tilde{X} \vee_G \left(G \times S^d\right)$$

the unique G-equivariant map determined by $\tilde{f}_\gamma$. Note that ϕ_γ is a G-equivariant between two free G-CW complexes. Passing to the quotient space, we get a map $f_\gamma : S^d \to \tilde{X} \vee_G (G \times S^d)/G = X \vee S^d$ such that the following diagram is commutative

$$\begin{array}{ccc} & & \widetilde{X \vee S^d} \\ & \tilde{f}_\gamma \nearrow & \downarrow \\ S^d & \xrightarrow{f_\gamma} & X \vee S^d. \end{array}$$

Construct a free G-CW complex $\tilde{Y}_\gamma = \widetilde{X \vee S^d} \cup_{\phi_\gamma} (G \times e^{d+1})$ as the push out of the following diagram

$$\begin{array}{ccc} G \times S^d & \xrightarrow{\phi_\gamma} & \tilde{X} \vee_G (G \times S^d) \\ \downarrow & & \downarrow \\ G \times e^{d+1} & \rightarrow & \tilde{Y}_f. \end{array}$$

This G-CW complex $\tilde{Y}_\gamma$ is actually the universal cover of $Y_\gamma := \tilde{Y}_\gamma / G$ (for more details on the construction, see the proof of Lemma 2.2 in [4] or p. 371 in [5]). According to Lemma 2, the inclusion $i_f : X \to Y_\gamma$ is a homotopy equivalence, since γ is a unit. Let $i_g : S^d \hookrightarrow \widetilde{X \vee S^d} = \tilde{X} \vee_G (G \times S^d)$ be the inclusion of S^d into the g-component. As γ is nontrivial, the map $\tilde{f}_\gamma$ is not homotopic to any map $S^d \to S^d \overset{i_g}{\hookrightarrow} \widetilde{X \vee S^d} = \tilde{X} \vee_G (G \times S^d)$ for any $g \in G$ by considering the degree of $p_h \tilde{f}_\gamma$ for each $h \in G$. This shows that $[f_\gamma] := \tilde{f}_\gamma \in \pi_d(\widetilde{X \vee S^d}, S^d)$ is not vanishing for any lifting of S^d. $\square$

Remark 3. For zero divisor conjecture in $\mathbb{Z}G$, some necessary conditions of homotopy descriptions are given in [2] and [3].

Acknowledgment

This note was finished when the author was a Phd student in National University of Singapore (NUS). He is grateful to his thesis advisor Professor A.J. Berrick for many discussions and constant encouragement.

References

[1] D.A. Craven, P. Pappas, On the unit conjecture for supersoluble group rings, I, arXiv:1010.1144.
[2] S.V. Ivanov, An asphericity conjecture and Kaplansky problem on zero divisors, J. Algebra 216 (1) (1999) 13–19.
[3] I.J. Leary, Asphericity and zero divisors in group algebras, J. Algebra 227 (2000) 362–364.
[4] W. Lück, L^2 invariants of regular coverings of compact manifolds and CW-complexes, in: R.J. Davermann, R.B. Sher (Eds.), Handbook of Geometric Topology, Elsevier, 2002.
[5] W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-Theory, Ergeb. Math. Grenzgeb., vol. 44, Springer, 2002.
[6] D.S. Passman, The Algebraic Structure of Group Rings, John Wiley & Sons, New York, London, Sydney, Toronto, 1977.