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# Groups having a faithful irreducible representation

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ABSTRACT

We address the problem of finding necessary and sufficient conditions for an arbitrary group, not necessarily finite, to admit a faithful irreducible representation over an arbitrary field.



## 1. Introduction

We are interested in the problem of finding necessary and sufficient conditions for a group to have a faithful irreducible linear representation. Various criteria have been found when the group in question is finite, as described in §2, so we will concentrate mainly on infinite groups.

Let $G$ be an arbitrary group. Recall that $\mathrm{Soc}(G)$ is the subgroup of $G$ generated by all minimal normal subgroups of $G$. It is perfectly possible for $\mathrm{Soc}(G)$ to be trivial. We denote by $\mathrm{S}(G)$, $\mathrm{T}(G)$ and $\mathrm{F}(G)$ the subgroups of $\mathrm{Soc}(G)$ generated by all minimal normal subgroups of $G$ that are non-abelian, torsion abelian and torsion-free abelian,

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respectively. For each prime $p$, let $\mathrm{T}(G)_p$ be the $p$-part of $\mathrm{T}(G)$. It is a vector space over $F_p$. Let $\Pi(G)$ be the set of all primes $p$ such that $\mathrm{T}(G)_p$ is non-trivial.

One readily verifies that $\mathrm{T}(G)\,\mathrm{F}(G)$ is abelian and throughout the paper we view $\mathrm{T}(G)\,\mathrm{F}(G)$ as a $\mathbb{Z}G$-module, with $G$ acting by conjugation. Thus, a minimal normal subgroup of $G$ contained in $\mathrm{T}(G)\,\mathrm{F}(G)$ is nothing but an irreducible $\mathbb{Z}G$-submodule of $\mathrm{T}(G)\,\mathrm{F}(G)$.

A normal subgroup $N$ of $G$ is said to be essential if every non-trivial normal subgroup of $G$ intersects $N$ non-trivially.

With this level of generality, it seems unavoidable that the desired necessary and sufficient conditions be stated separately. Our main results are as follows.

**Theorem 1.1.** *Let $G$ be a group and let $K$ be a field such that:*

(1) $\mathrm{Soc}(G)$ *is essential;*
(2) *If $K$ has prime characteristic p and $M$ is a minimal normal subgroup of $G$ as well as a non-abelian $p$-group that is not finitely generated, then $M$ admits a non-trivial irreducible representation over $K$.*
(3) $\mathrm{char}(K) \notin \Pi(G)$*;*
(4) $\mathrm{T}(G)$ *has a subgroup $S$ such that* $\mathrm{T}(G)/S$ *is locally cyclic and* $\mathrm{core}_G(S) = 1$.

*Then $G$ has a faithful irreducible representation over $K$.*

Although condition (1) plays a critical role for us, it is certainly not necessary, as illustrated by the free group $F\{x, y\}$ on 2 generators. It has no minimal normal subgroups and can be faithfully represented, e.g. via $x \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, as in irreducible subgroup of $\mathrm{GL}_2(\mathbb{C})$ (cf. [17, §2.1]).

In connection to condition (2), it is an open question whether a non-abelian simple $p$-group exists lacking non-trivial irreducible representations in prime characteristic $p$. The answer is known to be negative under the additional assumption that the group be finitely generated, as shown in [15, Theorem 6.3]. Regardless of the outcome of this question, let $M = M(\mathbb{Q}, \leq, F_p)$ be the McLain group [11], where $\leq$ is the usual order on the rational field $\mathbb{Q}$ and $p$ is a prime. Then $M$ is a minimal normal subgroup of $G = M \rtimes \mathrm{Aut}(M)$ (in fact, $M = \mathrm{Soc}(G)$) as well as a non-abelian $p$-group that is not finitely generated. Moreover, the only irreducible $F_pM$-module is the trivial one (since $M$ is a locally finite $p$-group). Groups like $G$ lie beyond the scope of Theorem 1.1. Perhaps surprisingly, $M \rtimes \mathrm{Aut}(M)$ does have a faithful irreducible representation over $F_p$ (see [20] for details).

In regards to the necessity of conditions (3) and (4), we have the following result.

**Theorem 1.2.** *Let $G$ be a group such that $[G : C_G(N)] < \infty$ for every minimal normal subgroup $N$ of $G$ contained in $T(G)$. Let $K$ be a field and suppose that $G$ admits a faithful* *irreducible representation over $K$. Then* $\mathrm{char}(K) \notin \Pi(G)$ *and* $\mathrm{T}(G)$ *has a subgroup $S$ such that* $\mathrm{T}(G)/S$ *is locally cyclic and* $\mathrm{core}_G(S) = 1$.

The most general contributions to this problem have been made by Tushev. Theorems 1.1 and 1.2 extend [23, Theorem 1], which requires that every minimal normal subgroup of $G$ be finite. Under this assumption, $[G : C_G(N)]$ above is finite, $\mathrm{F}(G)$ is trivial, every irreducible $\mathbb{Z}G$-submodule of $\mathrm{T}(G)$ is finite, and every non-abelian minimal normal subgroup $M$ of $G$ is a direct power of a finite non-abelian simple group. In particular, $M$ is not a $p$-group for any prime $p$, and it therefore has non-trivial irreducible representations over any field (cf. Lemma 5.3).

The condition $[G : C_G(N)] < \infty$ cannot be removed with impunity from Theorem 1.2 as the examples from [20] show.

When $G$ itself is finite [23, Theorem 1] or, alternatively, Theorems 1.1 and 1.2 yield a full criterion (no additional hypotheses are required).

In prior work, Tushev [22] found necessary and sufficient conditions for a locally polycyclic, solvable group of finite Prüfer rank to have a faithful irreducible representation over an algebraic extension of a finite field. He recently [24] extended this line of research to solvable groups of finite Prüfer rank over an arbitrary field.

Also recently, Bekka and de la Harpe [2] found a criterion for a countable group to have faithful irreducible unitary complex representation, although their sense of irreducibility differs from the purely algebraic meaning given in this paper.

## 2. A review of the finite case

The development of our problem for finite groups makes an interesting story which is often reported with a certain degree of inaccuracy in the literature. Moreover, all known criteria, obviously equivalent to each other, are easily obtained from one another by means of a straightforward argument that does not involve groups, their representations or whether they are faithful or not. This prompted us to review the history of this problem for finite groups in more detail than usual and to indicate how a direct translation between the various criteria can be carried out.

For the sake of our historical review we adopt the following conventions: $G$ stands for an arbitrary finite group and a representation of $G$ means a complex representation, unless otherwise stated.

At the dawn of the twentieth century, a well known and established necessary condition was that the center of $G$ be cyclic. A partial converse was shown by Fite [6] as early as 1906. He proved that if $G$ has prime power order or, more generally, if $G$ is the direct product of such groups (that is, if $G$ is nilpotent), then the above condition is also sufficient.

The first example of a finite centerless group that admits no faithful irreducible representation was given by Burnside [1, Note F] in 1911. It was the semidirect product

$$(C_3 \times C_3) \rtimes C_2, \tag{2.1}$$

where $C_2$ acts on $C_3 \times C_3$ without non-trivial fixed points. (Another well-known example of a similar kind, due to Isaacs [9, Exercise 2.19] is

$$(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3, \tag{2.2}$$

where, again, $C_3$ acts on $C_2 \times C_2 \times C_2 \times C_2$ without non-trivial fixed points.)

In [1, Note F] Burnside found a sufficient condition as well. He showed that if $G$ does not contain two distinct minimal normal subgroups whose orders are powers of the same prime, then $G$ has a faithful irreducible representation. This condition is not necessary, a fact recognized by Burnside. To illustrate this phenomenon, let

$$G = V \rtimes T,$$

where $V$ is a vector space of finite dimension $d > 1$ over a finite field $F_q$ with $q > 2$ elements, prime characteristic $p$, and $T$ is the diagonal subgroup of $\mathrm{GL}(V)$ with respect to some basis of $V$. Clearly, $G$ is the direct product of $d$ copies of

$$F_q^+ \rtimes F_q^*,$$

which admits a faithful irreducible representation of dimension $q - 1$. When $q > 2$ this group has trivial center, so the corresponding tensor power representation of $G$ is not only irreducible but also faithful.

Observe that while $G$ has $d > 1$ distinct minimal normal subgroups of order $q$, the normal subgroup they generate, namely $V$, is clearly a cyclic $F_pG$-module. This is precisely the condition that Burnside missed.

Since the center of a nilpotent group intersects every non-trivial normal subgroup non-trivially, Fite's result follows from Burnside's. The groups $F_q^+ \rtimes F_q^*$ used above show that the converse fails. Because of his work in [1, Note F] we will refer to the problem at hand as Burnside's problem or Burnside's question.

The relevance of $\mathrm{Soc}(G)$ to his problem was already apparent to Burnside in 1911, although the nature of $\mathrm{Soc}(G)$ was not sufficiently understood at the time for him to produce a solution. The structure of $\mathrm{Soc}(G)$ was elucidated by Remak [16] in 1930, and his description has been implicitly or explicitly used by every author who eventually wrote about Burnside's question.

The first paper addressing Burnside's problem was written by Shoda [18] in 1930. An error in [18] was quickly pointed by Akizuki in a private letter to Shoda. This letter included the first correct solution to Burnside's question. Akizuki's criterion and a sketch of his proof appeared in a second paper by Shoda [19] in 1931, which also included an independent proof of Akizuki's criterion by Shoda.

Let us outline Akizuki's criterion as it appears in [19]. Let $T$ be one of the factors appearing in a decomposition of $\mathrm{T}(G)$ as a direct product of abelian minimal normal subgroups of $G$, and let $s$ be the total number of these factors isomorphic to $T$ via an isomorphism that commutes with the inner automorphisms of $G$. The group $T$ is isomorphic to the direct product of, say $r$, copies of $\mathbb{Z}_p$ for some prime $p$. The total number of endomorphisms of $T$ that commute with all inner automorphisms of $G$ is of the form $p^g$ for some positive integer $g$. Akizuki's criterion is that $G$ admits a faithful irreducible representation if and only if $sg \leq r$, and this holds for all factors $T$ indicated above.

The reader will likely be as baffled by this criterion as subsequent writers on this subject were. Akizuki's condition seemed difficult to verify in practice, to say the least, and alternative criteria were sought.

A second criterion was produced by Weisner [25] in 1939. According to Weisner, $G$ has a faithful irreducible representation if and only if for every $p \in \Pi(G)$ there exists a maximal subgroup of $\mathrm{T}(G)_p$ that contains no normal subgroup of $G$ other than the trivial subgroup.

Shortly after Weisner's paper, Tazawa [21] extended Burnside's problem and asked under what conditions $G$ would admit a faithful representation with $k$ irreducible constituents. His answer was given along the same lines as Akizuki's criterion.

The next paper on the subject was written by Nakayama [12] in 1947. He was the first to consider Burnside's problem over fields of prime characteristic, obtaining a full criterion which is a slight modification of Weisner's criterion as stated over $\mathbb{C}$. Nakayama seems to have been unaware of Weisner's prior work. A great deal of [12] is devoted to solve several generalizations of Burnside's problem.

The following paper on Burnside's question was written Kochendörffer [10] in 1948. He gave another proof of Akizuki's criterion and, unaware of Weisner's paper, stated and proved Weisner's criterion. He wished to produce a condition that was easily verifiable in practice, and proved that if all Sylow subgroups of $G$ have cyclic center then $G$ has a faithful irreducible representation (cf. [9, Exercise 5.25]). This sufficient condition is an immediate consequence of the one proved by Burnside in 1911. In a second part of his paper Kochendörffer addressed and solved Burnside's question for fields of prime characteristic, unaware of Nakayama's prior solution.

The next solution to Burnside's problem was given by Gaschütz [7] in 1954. According to Gaschütz, $G$ has a faithful irreducible representation if and only if $\mathrm{T}(G)$ is a cyclic $\mathbb{Z}G$-module.

With the benefit of hindsight, let us try to explain *directly* why the criteria of Akizuki, Weisner and Gaschütz are equivalent to each other. For this purpose, let us slightly translate each of the above criteria into a more favorable language.

Since $\mathrm{T}(G)$ is a completely reducible $\mathbb{Z}G$-module, it is readily seen that $\mathrm{T}(G)$ is a cyclic $\mathbb{Z}G$-module if and only if $\mathrm{T}(G)_p$ is a cyclic $F_pG$-module for each $p \in \Pi(G)$.

Given $p \in \Pi(G)$, a maximal subgroup, say $M$, of $\mathrm{T}(G)_p$ is the kernel a non-trivial linear functional $\lambda : \mathrm{T}(G)_p \to F_p$, and the largest normal subgroup of $G$ contained in $M$ is therefore

$$\bigcap_{g\in G} gMg^{-1} = \bigcap_{g\in G} \ker({}^g\lambda),$$

where ${}^g\lambda : \mathrm{T}(G)_p \to F_p$ is the linear character defined by

$$({}^g\lambda)(v) = \lambda(g^{-1}vg), \quad v \in \mathrm{T}(G)_p.$$

Since a proper subspace of the dual space $\mathrm{T}(G)_p^*$ annihilates a non-zero subspace of $\mathrm{T}(G)_p$, we see that Weisner's condition is that $\mathrm{T}(G)_p^*$ be a cyclic $F_pG$-module for every $p \in \Pi(G)$.

It is easily seen (cf. §3) that if $K$ is a field, $A$ is a $K$-algebra with involution, and $V$ is a finite dimensional completely reducible $A$-module then $V$ is cyclic if and only if so is its dual $V^*$. This shows directly that the criteria of Weisner and Gaschütz are equivalent.

Let $T$ be an abelian minimal normal subgroup of $G$, that is, an irreducible $F_pG$-module of $\mathrm{T}(G)_p$ for some $p \in \Pi(G)$. The quantities $s$, $g$ and $r$ of Akizuki's criterion are respectively equal to the multiplicity of $T$ in $\mathrm{T}(G)_p$, the dimension of the field $F = \mathrm{End}_{F_pG}(T)$ over $F_p$, and the dimension of $T$ itself over $F_p$. Now $T$ is an $F$-vector space of dimension $r/g$, so Akizuki's criterion is that the multiplicity of $T$ in $\mathrm{T}(G)_p$ be less than or equal to $\dim_F(T)$. The above translation of Akizuki's criterion was already known to Pálfy [13] in 1979, who gave another proof of Akizuki's criterion, this time over a splitting field for $G$ of characteristic not dividing $\mathrm{T}(G)$. (Pálfy credited Kochendörffer for this result, unaware of Nakayama's prior work.)

It is easily seen (cf. §3) that if $R$ is a left artinian ring and $V$ is a completely reducible $R$-module then $V$ is cyclic if and only if for every irreducible submodule $W$ of $V$ the multiplicity of $W$ in $V$ does not exceed $\dim_D W$, where $D = \mathrm{End}_R W$. This shows directly that the criteria of Akizuki, Weisner and Gaschütz are all equivalent to each other, with generic arguments that involve no groups at all.

Burnside's question and its various solutions have also appeared in book form. See the books by Huppert [8], Doerk and Hawkes [5], and Berkovich and Zhmud' [3], for instance. Incidentally, Zhmud' [26] found alternative necessary and sufficient conditions for $G$ to admit a faithful representation with exactly $k$ irreducible representations, a problem previously considered by Tazawa [21].

We close this section with an example. Given a finite field $F_q$ with $q$ elements and a vector space $V$ of finite dimension $d$ over $F_q$ we consider the subgroup

$$G(d,q) = V \rtimes F_q^*$$

of the affine group $V \rtimes \mathrm{GL}(V)$ and set

$$G(q) = F_q^+ \rtimes F_q^*.$$

For any non-trivial linear character $\lambda : F_q^+ \to \mathbb{C}^*$ the induced character

$$\chi = \mathrm{ind}_{F_q^+}^{G(q)} \lambda$$

is readily seen to be a faithful and irreducible of degree $q-1$. Any hyperplane $P$ of $V$ gives rise a group epimorphism

$$G(d,q) \to G(q)$$

with kernel $P$. This produces $(q^d-1)/(q-1)$ different irreducible characters of $G(d,q)$ of degree $q-1$, each of them having a hyperplane of $V$ as kernel. On the other hand, $G(d,q)$ has $q-1$ different linear characters with $V$ in the kernel. Since

$$\frac{q^d-1}{q-1} \times (q-1)^2 + (q-1) \times 1^2 = q^d(q-1) = |G(d,q)|,$$

these are all the irreducible characters of $G(d,q)$. In particular, $G(d,q)$ has no faithful irreducible characters if $d>1$. On the other hand, if $q>2$ the center of $G(d,q)$ is trivial. Thus, to each finite field $F_q$, with $q>2$, there corresponds an infinite family of finite centerless groups, namely $G(d,q)$ with $d>1$, having no faithful irreducible characters for virtually tautological reasons. Moreover, every non-linear irreducible character of $G(d,q)$ is uniquely determined by its (non-trivial) kernel. Observe that the groups (2.1) and (2.2) are isomorphic to $G(2,3)$ and $G(2,4)$, respectively.

## 3. Cyclic modules and their duals

All modules appearing in this paper are assumed to be left modules.

**Lemma 3.1.** *Let $R$ be a left artinian ring and let $V$ be a completely reducible $R$-module. Then $V$ is cyclic if and only if for each irreducible component $W$ of $V$ the multiplicity of $W$ in $V$ is less than or equal to $\dim_D(W)$, where $D = \mathrm{End}_R(W)$.*

**Proof.** Since there are finitely many isomorphism types of irreducible $R$-modules, we may reduce to the case when all irreducible submodules of $V$ are isomorphic. Let $W$ be one of them. Replacing $R$ by its image in $\mathrm{End}(V)$ we may also assume that $J(R)=0$. Thus we may restrict to the case $R = M_n(D)$ and $W = D^n$. A cyclic $R$-module is nothing but a quotient of $R$ by a left ideal. This is just a left ideal of $R$, which is isomorphic to the direct sum of at most $n$ copies of $W$. $\square$

**Corollary 3.2.** *Let $K$ be a field and let $A$ be a $K$-algebra with involution. Suppose that $V$ is a completely reducible finite dimensional $A$-module. Then $V$ is cyclic if and only if so is $V^*$.*

**Proof.** Since every submodule of $V$ is isomorphic to its double dual, we may reduce to the case when all irreducible submodules of $V$ are isomorphic. Let $W$ be one of them. Then the transpose map

$$\mathrm{End}_K(W) \to \mathrm{End}_K(W^*)$$

sends $D = \mathrm{End}_A(W)$ onto $D^0 = \mathrm{End}_A(W^*)$, the division algebra opposite to $D$. Since $\dim_K(W) = \dim_K W^*$, it follows that $\dim_D(W) = \dim_{D^0}(W^*)$. Now apply Lemma 3.1. $\square$

**Corollary 3.3.** *Let $K$ be a field and let $A$ be a $K$-algebra with involution. Suppose that $V$ is a finite dimensional $A$-module. Then $V$ is cyclic if and only if the irreducible components of $V/\mathrm{Rad}(V)$ satisfy the conditions of Lemma 3.1, and $V^*$ is cyclic if and only if the irreducible components of $\mathrm{Soc}(V)$ satisfy the conditions of Lemma 3.1.*

**Proof.** Clearly $V$ is cyclic if and only if so is $V/\mathrm{Rad}(V)$. Moreover,

$$V^*/\mathrm{Rad}(V^*) \cong \mathrm{Soc}(V)^*,$$

and by Lemma 3.1 we know that $\mathrm{Soc}(V)^*$ is cyclic if and only if so is $\mathrm{Soc}(V)$. $\square$

In particular, if $V/\mathrm{Rad}(V)$ (resp. $\mathrm{Soc}(V)$) is irreducible then $V$ (resp. $V^*$) is cyclic. In the extreme case when $V$ is uniserial then so is $V^*$ and, moreover, $V$ and $V^*$ are automatically cyclic.

## 4. Modules lying over others

**Lemma 4.1.** *Let $K$ be a field. If $U$ and $V$ are $K$-vector spaces with proper subspaces $X$ and $Y$, respectively, then the subspace $X \otimes V + U \otimes Y$ of $U \otimes V$ is proper.*

**Proof.** There are non-zero linear functionals $\alpha : U \to K$ and $\beta : V \to K$ such that $\alpha(X) = 0$ and $\beta(Y) = 0$. Let $\gamma : U \otimes V \to K$ be the unique linear functional such that $\gamma(u \otimes v) = \alpha(u)\beta(v)$ for all $u \in U$ and $v \in V$. Then $\gamma \neq 0$ and $\gamma(X \otimes V + U \otimes Y) = 0$, as required. $\square$

**Lemma 4.2.** *Let $K$ be a field, let $G_1, \dots, G_n$ be groups and set $G = G_1 \times \cdots \times G_n$. Let $J_1, \dots, J_n$ be proper left ideals of $KG_1, \dots, KG_n$, respectively. Then the left ideal of $KG$ generated by $J_1, \dots, J_n$ is proper.*

**Proof.** Arguing by induction, we are reduced to the case $n = 2$, which follows from Lemma 4.1 by means of the isomorphism of $K$-algebras $KG \cong KG_1 \otimes KG_2$. $\square$

Given a ring $R$, a subring $S$, an $R$-module $V$, and an $S$-module $U$, we say that $V$ lies over $U$ if $U$ is isomorphic to an $S$-submodule of $V$.

**Lemma 4.3.** *Let $(G_i)_{i \in I}$ be a family of groups and let $G$ be the direct product of them. Let $K$ be a field and for each $i \in I$ let $V_i$ be an irreducible $KG_i$-module. Then there is an irreducible $KG$-module $V$ lying over all $V_i$.*

**Proof.** For each $i \in I$ we have $V_i \cong KG_i/J_i$, where $J_i$ is a maximal left ideal of $KG_i$. Suppose the result is false. Then

$$1 = x_1 y_1 + \cdots + x_n y_n, \tag{4.1}$$

where $x_i \in KG$ and each $y_i$ is in some $J_k$. There is a finite subset $L$ of $I$ such that for $H = \prod_{\ell \in L} G_\ell$ all $J_k$ and $x_i$ above are inside of $KH$, against Lemma 4.2. □

**Lemma 4.4.** *Let $H$ be a subgroup of a group $G$ and let $K$ be a field. Let $V$ be an irreducible $KH$-module. Then there exists an irreducible $KG$-module lying over $V$.*

**Proof.** As shown in [14, Lemma 6.1.2], a proper left ideal of $KH$ generates a proper left ideal of $KG$. The result is an immediate consequence of this observation. □

## 5. Restricting scalars

**Lemma 5.1.** *Let $B$ be a ring with an irreducible $B$-module $V$. Suppose there is a subring $A$ of $B$ and a family $(u_i)_{i\in I}$ of units of $B$ such that: $B$ is the sum of all subgroups $Au_i$; $u_i A = Au_i$ for all $i \in I$; given any $i, j \in I$ there is $k \in I$ such that $u_i u_j A = u_k A$. Then*

(a) *$V$ has maximal $A$-submodule if and only if $V$ has a minimal $A$-submodule, in which case $V$ is a completely reducible $A$-module (homogeneous if all $u_i$ commute elementwise with $A$).*
(b) *If $I$ is finite then $V$ is a completely reducible $A$-module of length $\le |I|$.*

**Proof.** (a) Suppose first that $V$ has maximal $A$-submodule $M$. Since $u_i A = Au_i$ for all $i$, it follows that every $u_i M$ is an $A$-submodule of $V$. But each $u_i$ is a unit, so $u_i M$ is a maximal $A$-submodule of $V$. Let $N$ be in the intersection of all $u_i M$. Since $u_i u_j A$ is equal to some $u_k A$, we see that $N$ is invariant under all $u_i$. But $N$ is closed under addition and $B$ is the sum of all $Au_i$, so $N$ is $B$-invariant. The irreducibility of $V$ implies that $N = 0$. Thus $V$ is isomorphic, as $A$-module, to a submodule of a completely reducible $A$-module, namely the direct sum of all $V/u_i M$. In particular, $V$ has an irreducible $A$-submodule.

Suppose next that $V$ has an irreducible $A$-submodule $W$. Then every $u_i W$ is also $A$-irreducible and $V$ is the sum of all of them. Thus $V$ is the direct sum of sum of them. Removing one summand produces a maximal $A$-submodule.

(b) In this case $V$ is a finitely generated $A$-module, so it has a maximal $A$-submodule by Zorn's Lemma. By above, $V$ is an $A$-submodule of a completely reducible $A$-module of length $\le |I|$. □

**Corollary 5.2.** *Let $G$ be a group and let $L/K$ be a finite radical field extension. Let $V$ be an irreducible $LG$-module. Then $V$ is a completely reducible homogeneous $KG$-module. In particular, $V$ has an irreducible $KG$-submodule.*

**Proof.** Arguing by induction, we may assume that $L = K[x]$, where $x^n \in K$ for some $n$, so Lemma 5.1 applies with $B = LG$, $A = KG$ and $u_i = x^i$ for $1 \le i \le n$. $\square$

**Lemma 5.3.** *Let $G$ be a non-trivial group and let $K$ be a field. If $K$ has prime characteristic $p$ we assume that $G$ is not a $p$-group. Then $G$ has a non-trivial irreducible representation over $K$.*

**Proof.** Our hypothesis ensures the existence of $x \in G$ and a non-trivial linear character $\lambda : \langle x \rangle \to K[\zeta]$, where $\zeta$ is a root of unity. By Lemma 4.4 there is an irreducible $G$-module $V$ over $K[\zeta]$ lying over $\lambda$. By Corollary 5.2 there is an irreducible $KG$-submodule $U$ of $V$. Since $V = K[\zeta]U$ is not trivial, neither is $U$. $\square$

**Note 5.4.** Lemma 5.1 is a generalization of [14, Theorem 7.2.16], which states that an irreducible $KG$-module $V$ has an irreducible $KN$-submodule, for a normal $N$ subgroup of $G$, provided $[G : N]$ is finite. Our statement of Lemma 5.1 was designed to accommodate Corollary 5.2 as well.

Observe that Corollary 5.2 fails if $L/K$ is an arbitrary field extension, even if $\dim_L(V) = 1$. Indeed, let $K$ be an arbitrary field, $L = K((t))$, $G = U(K[[t]])$ and $V = L$. Then $V$ is an irreducible $LG$-module of dimension one. However, when viewed as a $KG$-module, the submodules of $V$ form the following doubly descending/ascending infinite chain, where $R = K[[t]]$:

$$V \supset \cdots \supset Rt^{-2} \supset Rt^{-1} \supset R \supset Rt \supset Rt^2 \supset \cdots \supset 0.$$

Given a group $G$, a representation of a subgroup of $G$ is said to be $G$-faithful if its kernel contains no non-trivial normal subgroups of $G$.

**Lemma 5.5.** *Let $G$ be a group and let $K$ be a field. Let $p$ be a prime different from* $\mathrm{char}(K)$. *Suppose $N$ is a normal subgroup of $G$ contained in $\mathrm{T}(G)_p$ with a maximal subgroup containing no non-trivial normal subgroup of $G$. Then $N$ has a $G$-faithful irreducible representation over $K$.*

**Proof.** By assumption there exists a one-dimensional $G$-faithful module $V$ of $N$ over a $K[\zeta]$, where $\zeta^p = 1$. By Corollary 5.2, there is an irreducible $KN$-submodule $W$ of $V$. As $K[\zeta]W = V$, it is clear that $W$ is $G$-faithful. $\square$

## 6. The torsion-free abelian part of Soc($G$)

Recall that a torsion-free abelian characteristically simple group is divisible and hence a vector space over $\mathbb{Q}$.

**Lemma 6.1.** *Let $A$ be a non-trivial torsion-free divisible abelian group, let $K$ be a field and let $p$ be a prime different from* $\mathrm{char}(K)$*. Then there exists an irreducible representation $R : A \to \mathrm{GL}(V)$ over $K$ such that $R(A) \cong \mathbb{Z}_{p^\infty}$.*

**Proof.** There is a subgroup $S$ of $A$ such that $B = A/S \cong \mathbb{Z}_{p^\infty}$. Let $L$ be a splitting field over $K$ for the family of polynomials $t^{p^i} - 1$ for all $i \geq 1$. Then $G = L^*$ acts $K$-linearly on the $K$-vector space $V = L$ by multiplication. We have an injective linear character $\lambda : B \to G$ whose image $\lambda(B) = C$ consists of all roots of $t^{p^i} - 1$, $i \geq 1$, in $L$. Since $L/K$ is algebraic, $K[C] = K(C) = L$. Thus, the representation $A \to B \to G \to \mathrm{GL}(V)$ satisfies our requirements. $\square$

**Lemma 6.2.** *Let $G$ be a group and let $K$ be a field. Suppose $A = \prod_{i \in I} A_i$, where each $A_i$ is an irreducible $\mathbb{Z}G$-submodule of* $\mathrm{F}(G)$ *and $|I| \leq \aleph_0$. Then $A$ has a $G$-faithful irreducible module over $K$.*

**Proof.** There is an injection $i \mapsto p_i$, where each $p_i$ is a prime different from $\mathrm{char}(K)$. Lemma 6.1 ensures the existence, for each $i \in I$, of an irreducible representation $R_i : A_i \to \mathrm{GL}(V_i)$ over $K$ such that $R(A_i) \cong \mathbb{Z}_{p_i^\infty}$. By Lemma 4.3, there is an irreducible representation $R : A \to \mathrm{GL}(V)$ lying over all $R_i$. For a fixed $i \in I$, we have $V = \sum_{a \in A} aV_i = \bigoplus_{a \in J} aV_i$ for some subset $J$ of $A$, so $R(A_i) \cong \mathbb{Z}_{p_i^\infty}$.

Suppose, if possible, that $A$ contains a non-trivial normal $N$ subgroup of $G$ which acts trivially on $V$, and let $1 \neq x \in N$. Then $x = x_{i_1} \cdots x_{i_n}$, where $1 \neq x_{i_k} \in A_{i_k}$. Since $A_{i_1}$ is an irreducible $\mathbb{Z}G$-module, there is $r \in \mathbb{Z}G$ such that $r \cdot x_{i_1}$ does not act trivially on $V_{i_1}$. Let $y_{i_k} = r \cdot x_{i_k} \in A_{i_k}$ for $1 \leq k \leq n$. Then $r \cdot x = y_{i_1} \cdots y_{i_n} \in N$ and $T(y_{i_1}) \cdots T(y_{i_n}) = 1_V$, which is impossible since these factors commute pairwise, the order of $T(y_{i_1})$ is a positive power of $p_{i_1}$, and the order of every $T(y_{i_k})$, $k > 1$, is a power of $p_{i_k}$. $\square$

**Lemma 6.3.** *Let $G$ be a group and let $K$ be a field. Suppose $A = \prod_{i \in I} A_i$, where each $A_i$ is an irreducible $\mathbb{Z}G$-submodule of* $\mathrm{F}(G)$ *and $|I|$ is infinite. Then $A$ has a $G$-faithful irreducible module over $K$.*

**Proof.** Let $L$ be an extension of $K$ that is algebraically closed and satisfies $|L| = |I|$. Since $L^*$ is a divisible group, we have $L^* = T \times R$, where $T$ is the torsion subgroup of $L^*$ and $R$ and is torsion-free. Moreover, $R = \prod_{i \in I} R_i$, where $R_i \cong \mathbb{Q}$. Relabeling the factors of $A$, we have $A = B \times C$, where $B = \prod_{n \geq 1} B_n$ and $C = \prod_{i \in I} C_i$. Let $p_1, p_2, \ldots$ be the list of all primes different from $\mathrm{char}(K)$. For each $p_k$, let $T_k$ be the $p_k$-part of $T$.

There is a family of linear characters $\lambda_i : C_i \to L^*$ such that $\lambda_i(C_i) = R_i$ for every $i \in I$ and a family of linear characters $\mu_k : B_k \to L^*$ such that $\mu_i(B_k) = T_k$. Let $\alpha : A \to L^*$ be the linear character determined by the $\lambda_i$ and $\mu_k$, so that $\alpha(A) = L^*$.

Every $\lambda_i$ and $\mu_k$ are non-trivial, each $A_i$ is an irreducible $\mathbb{Z}G$-module, and the $R_i$ and $T_k$ are independent. We infer that $\alpha$ is $G$-faithful. The group $G = L^*$ acts via $K$-linear automorphisms on the $K$-vector space $V = L$ by multiplication. The composition $A \to L^* \to \mathrm{GL}(V)$ is a linear representation of $A$ on $V$ over $K$. As such, it is irreducible since $\alpha(A) = L^*$, and it is $G$-faithful as so is $\alpha$. $\square$

## 7. Groups with a faithful irreducible representation

**Theorem 7.1.** *Let $G$ be a group and let $K$ be a field such that:*

(1) $\mathrm{Soc}(G)$ *is essential;*
(2) *If $K$ has prime characteristic $p$ and $M$ is a minimal normal subgroup of $G$ as well as a non-abelian $p$-group that is not finitely generated, then $M$ admits a non-trivial irreducible representation over $K$.*
(3) $\mathrm{char}(K) \notin \Pi(G)$*;*
(4) $\mathrm{T}(G)$ *has a subgroup $S$ such that $\mathrm{T}(G)/S$ is locally cyclic and* $\mathrm{core}_G(S) = 1$*.*

*Then $G$ has a faithful irreducible representation over $K$.*

**Proof.** We first show that $\mathrm{S}(G)$ has a $G$-faithful irreducible module over $K$. By [17, 3.3.11], we have $\mathrm{S}(G) = \prod_{i \in I} S_i$, where each $S_i$ is a non-abelian minimal normal subgroup of $G$. Let $i \in I$. If it is not the case that $K$ has prime characteristic $p$ and $S_i$ is a $p$-group, then Lemma 5.3 ensures that $S_i$ has a non-trivial irreducible representation over $K$. If $S_i$ is finitely generated then, being perfect, it has a non-trivial irreducible representation over $K$ by [15, Theorem 6.3]. If $K$ has prime characteristic $p$ and $S_i$ is a $p$-group that is not finitely generated, then $S_i$ has a non-trivial irreducible representation over $K$ by condition (2). Thus, in any case, there is a non-trivial irreducible $KS_i$-module, say $V_i$. By Lemma 4.3, there is an irreducible $K\,\mathrm{S}(G)$-module $V$ lying over all $V_i$. It then follows from [17, 3.3.12] that $V$ is $G$-faithful.

Next note that, by [17, 3.3.11], we have $\mathrm{F}(G) = \prod_{i \in I} A_i$, where each $A_i$ is a torsion-free abelian minimal normal subgroup of $G$. It then follows from Lemmas 6.2 and 6.3 that $\mathrm{F}(G)$ has a $G$-faithful irreducible module $W$ over $K$.

Finally we show that $\mathrm{T}(G)$ has a $G$-faithful irreducible module over $K$. Indeed, let $S$ be the subgroup of $\mathrm{T}(G)$ ensured by condition (4). Then

$$S = \bigoplus_{p \in \Pi(G)} S \cap \mathrm{T}(G)_p.$$

Since $\mathrm{T}(G)/S$ is locally cyclic and $\mathrm{core}_G(S) = 1$, we have

$$[\mathrm{T}(G)_p : S \cap \mathrm{T}(G)_p] = p, \quad p \in \Pi(G).$$

Obviously, we still have $\mathrm{core}_G(S \cap \mathrm{T}(G)_p) = 1$. On the other hand, condition (3) ensures $\mathrm{char}(K) \notin \Pi(G)$. It thus follows from Lemma 5.5 that, for every $p \in \Pi(G)$, there is a $G$-faithful irreducible module $X_p$ of $\mathrm{T}(G)_p$ over $K$. By Lemma 4.3, there is an irreducible $K\,\mathrm{T}(G)$-module $X$ lying over all $X_p$. It is clear that $X$ must be $G$-faithful.

Since $\mathrm{Soc}(G) = \mathrm{S}(G) \times \mathrm{F}(G) \times T(G)$, Lemma 4.3 ensures the existence of an irreducible $\mathrm{Soc}(G)$-module $Y$ over $K$ lying over $V$, $W$ and $X$. It follows from [17, 3.3.11 and 3.3.12] that $Y$ is $G$-faithful.

By Lemma 4.4, there is an irreducible $KG$-module $Z$ lying over $Y$. By condition (1), $\mathrm{Soc}(G)$ is essential, so $Z$ must be faithful. □

**Lemma 7.2.** *Let $N \trianglelefteq G$ be groups, where $N$ is abelian of prime exponent $p$. Let $K$ be a field and suppose that $V$ admits a faithful irreducible $KG$-module. Let $H$ be any subgroup of $C_G(N)$. Assume that $V$ has an irreducible $KH$-submodule $W$. Then* $\mathrm{char}(K) \neq p$ *and there is a maximal subgroup $M$ of $N$ such that* $\mathrm{core}_G(M) = 1$.

**Proof.** Since $W$ is $KH$-irreducible, $D = \mathrm{End}_{KH}(W)$ is a division ring. Let $L$ be a maximal subfield of $D$ containing $K$. Then $W$ is an irreducible $LH$-module and $\mathrm{End}_{LH}(W) = L$. Since $N$ is a central subgroup of $H$, it follows that $N$ acts on $W$ via scalar operators given a linear character, say $\lambda : N \to L^*$.

Suppose, if possible, that $\lambda$ is trivial. Since $N \trianglelefteq G$ and $V$ is irreducible, we see that $N$ acts trivially on $V$, against the fact that $G$ acts faithfully on it. Thus, $\lambda$ is non-trivial, so $L$ has a root of unity of order $p$, whence $p \neq \mathrm{char}(L) = \mathrm{char}(K)$. Let $M = \ker(\lambda)$. Then $[N : M] = p$, so $M$ is a maximal subgroup of $N$. Moreover, since $V$ is $KG$-irreducible, we have $V = \sum_{g \in G} gW$, so $1 = \ker_G(V) \supseteq \bigcap_{g \in G} gMg^{-1}$. □

**Theorem 7.3.** *Let $G$ be a group such that $[G : C_G(N)] < \infty$ for every minimal normal subgroup $N$ of $G$ contained in $T(G)$. Let $K$ be a field and suppose that $G$ admits a faithful irreducible module $V$ over $K$. Then* $\mathrm{char}(K) \notin \Pi(G)$ *and* $\mathrm{T}(G)$ *has a subgroup $S$ such that* $\mathrm{T}(G)/S$ *is locally cyclic and* $\mathrm{core}_G(S) = 1$.

**Proof.** Given $p \in \Pi(G)$, let $N$ be an irreducible $F_pG$-submodule of $T(G)_p$ with $F_pG$-homogeneous component $M$. Since $[G : C_G(N)] < \infty$, [14, Theorem 7.2.16] ensures that $V$ has an irreducible $KH$-submodule, where $H = C_G(N) = C_G(M)$. It follows from Lemma 7.2 that $\mathrm{char}(K) \neq p$ and there is linear character $\lambda_M : M \to K[\zeta]$, $\zeta^p = 1$, such that

$$\mathrm{core}_G(\ker \lambda_M) = 1. \tag{7.1}$$

Since $T(G)_p$ is the direct product of its $F_pG$-homogeneous components there is a unique linear character $\lambda_p : T(G)_p \to K[\zeta]$ extending all $\lambda_M$. If we set $S_p = \ker \lambda_p$, then $[T(G)_p : S_p] = p$. A non-trivial normal subgroup of $G$ contained in $T(G)_p$ must intersect at least one $F_pG$-homogeneous component of $T(G)_p$ non-trivially. This and (7.1) yield $\mathrm{core}_G(S_p) = 1$. Since $T(G)$ is the direct product of all $T(G)_p$, it is now clear that the direct product $S$ of all $S_p$, satisfies the stated requirements. □

**Example 7.4.** Here we extend the example given at the end of §2. Let $F$ be a field of prime characteristic $p$, let $V$ be an $F$-vector space of dimension $> 1$ (not necessarily finite), and set $G = V \rtimes F^*$. We claim that, provided $F$ is finite, $G$ has no faithful irreducible representation over any field. Indeed, if $|F| < \infty$ then $[G : V] < \infty$, so in view of [14, Theorem 7.2.16] and Lemma 7.2, we are reduced to show that every $F_p$-hyperplane of $V$ contains a non-zero $F$-subspace of $V$. This verification can be further reduced to the case $\dim_F(V) = 2$, in which case a routine counting argument yields the desired result (which fails, in general, for $F$ infinite).

For reference elsewhere, we next show that McLain's group (which has no minimal normal subgroups) admits faithful irreducible representations in every possible characteristic.

**Example 7.5.** Let $M$ be the McLain group [11] defined over a division ring $D$ and let $K$ be any field. If $\mathrm{char}(D) = \mathrm{char}(K) = p$ is prime the only irreducible representation of $M$ over $K$ is the trivial one. In every other case, $M$ has a faithful irreducible representation over $K$.

The first assertion follows from the fact that when $\mathrm{char}(D) = p$ is prime, $M$ is a locally finite $p$-group. For each $\alpha < \beta \in \mathbb{Q}$ consider the subgroup $N_{\alpha,\beta} = \{1 + ae_{\alpha,\beta} \mid a \in D\}$ of $M$. Let $I$ be the set of all pairs $(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q}$ such that $\alpha < 0$ and $1 < \beta$ and let $N$ be the subgroup of $M$ generated by all $N_{\alpha,\beta}$ with $(\alpha, \beta) \in I$. Then $N$ is an essential normal abelian subgroup of $M$, equal to the direct product of all $N_{\alpha,\beta}$ with $(\alpha, \beta) \in I$.

Suppose first $D$ has prime characteristic $p \neq \mathrm{char}(K)$. For each $(\alpha, \beta) \in I$ let $\lambda_{\alpha,\beta} : N_{\alpha,\beta} \to K[\zeta]^*$, $\zeta^p = 1$, be a non-trivial linear character and let $\lambda : N \to K[\zeta]^*$ be the unique extension of the $\lambda_{\alpha,\beta}$ to $N$. Then $\lambda$ is $M$-faithful, for every non-trivial normal subgroup of $M$ contains at least one (in fact, infinitely many) $N_{\alpha,\beta}$ with $(\alpha, \beta) \in I$. It follows from Lemma 4.4 that $M$ has a faithful irreducible module $U$ over $K[\zeta]$. By Corollary 5.2, there is an irreducible $KM$-submodule $V$ of $U$. Since $K[\zeta]V = U$, it is clear that $V$ is faithful.

Suppose next that $\mathrm{char}(D) = 0$. Choose any $0 \neq x \in D^+$ and any prime $p \neq \mathrm{char}(K)$. Then there is a non-trivial linear character $\mu : \langle x \rangle \to K[\zeta]^*$, where $\zeta^p = 1$. By Lemma 4.4, there is an irreducible $D^+$-module $U$ over $K[\zeta]$ lying over $\mu$. For each $(\alpha, \beta) \in I$ let $U_{\alpha,\beta}$ be the irreducible $N_{\alpha,\beta}$-module over $K[\zeta]$ obtained from $U$ via the isomorphisms $D^+ \to N_{\alpha,\beta}$ given by $a \mapsto 1 + ae_{\alpha,\beta}$. By Lemma 4.3, there is an irreducible $N$-module $V$ over $K[\zeta]$ lying over all $U_{\alpha,\beta}$. This implies, as before, that $M$ has a faithful irreducible module over $K$.

## 8. Necessary and sufficient conditions for nilpotent groups

**Proposition 8.1.** *Let $G$ be a nilpotent group. Let $Z$ be the center of $G$ and let $T$ be the torsion subgroup of $Z$. Then the following conditions are equivalent:*

(a) *$G$ admits a faithful irreducible representation.*
(b) *$Z$ is isomorphic to a subgroup of the multiplicative group of a field.*
(c) *$Z$ admits a faithful irreducible representation.*
(d) *$T$ is locally cyclic.*
(e) *$T$ is a subgroup of $\mathbb{Q}/\mathbb{Z}$.*
(f) *For each prime $p$, the $p$-part of $T$ is a subgroup of $\mathbb{Z}_{p^\infty}$.*

**Proof.** Suppose first that $V$ is a faithful irreducible $G$-module over a field $K$ and let $D = \mathrm{End}_{KG}(V)$. Let $L$ be a maximal subfield $L$ of $D$ containing $K$. Then $L = \mathrm{End}_{LG}(V)$, which yields an injective group homomorphism $Z \to L^*$. This shows that (a) implies (b), which obviously implies (c).

Suppose next that $V$ is a faithful irreducible $Z$-module over a field $K$. By Lemma 4.4 there is an irreducible $KG$-module $U$ lying over $V$. Suppose, if possible, that $N$ is a non-trivial normal subgroup of $G$ acting trivially on $V$. Since $G$ is nilpotent, $N \cap Z(G)$ is a non-trivial normal subgroup of $G$ acting trivially on $U$ and hence on $V$, a contradiction. Thus (c) implies (a).

Clearly (b) implies (d). The converse was proven by Cohn [4]. It is well-known that (d), (e) and (f) are equivalent. $\square$

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