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An alternating matrix and a vector, with application to Aluffi algebras

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ABSTRACT

Let $\mathbf{X}$ be a generic alternating matrix, $\mathbf{t}$ be a generic row vector, and J be the ideal $\mathrm{Pf}_4(\mathbf{X}) + I_1(\mathbf{tX})$. We prove that J is a perfect Gorenstein ideal of grade equal to the grade of $\mathrm{Pf}_4(\mathbf{X})$ plus two. This result is used by Ramos and Simis in their calculation of the Aluffi algebra of the module of derivations of the homogeneous coordinate ring of a smooth projective hypersurface. We also prove that J defines a domain, or a normal ring, or a unique factorization domain if and only if the base ring has the same property. The main object of study in the present paper is the module $\mathcal{N}$ which is equal to the column space of $\mathbf{X}$, calculated mod $\mathrm{Pf}_4(\mathbf{X})$. The module $\mathcal{N}$ is a self-dual maximal Cohen–Macaulay module of rank two; furthermore, J is a Bourbaki ideal for $\mathcal{N}$. The ideals which define the homogeneous coordinate rings of the Plücker embeddings of the Schubert subvarieties of the Grassmannian of planes are used in the study of the module $\mathcal{N}$.

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1. Introduction

Aluffi [1] introduced a class of algebras which are intermediate between the symmetric algebra and the Rees algebra of an ideal in order to define the characteristic cycle of a hypersurface parallel to the conormal cycle in intersection theory. These algebras were investigated by Nejad and Simis [28], who called them Aluffi algebras. At the end of his paper, Aluffi observed that it would be computationally desirable to up-grade his methods to more general schemes. A first step in the direction of Aluffi's proposed up-grade is a good notion of the Rees algebra of a module, such as the one described by Simis, Ulrich, and Vasconcelos in [30]. A second step in the direction of Aluffi's proposed up-grade is a good notion of the Aluffi algebra of a module, such as the one introduced by Ramos and Simis in [29].

Ramos and Simis compute the Aluffi algebra of the module of derivations of the homogeneous coordinate ring of a smooth projective hypersurface. In other contexts this module is also called the module of tangent vector fields or the differential idealizer or the module of logarithmic derivations. As part of the Ramos–Simis program, it is necessary to understand the homological nature of the ideal $J = \mathrm{Pf}_4(\mathbf{X}) + I_1(\mathbf{tX})$, where $\mathbf{X}$ is a generic alternating matrix and $\mathbf{t}$ is a generic row vector. Simis told us that he and Ramos conjectured that J is a Gorenstein ideal of height two more than the height of $\mathrm{Pf}_4(\mathbf{X})$. The purpose of this paper is to prove the Ramos–Simis conjecture.

Let R_0 be an arbitrary commutative Noetherian ring, $\mathfrak{f}$ be an integer with $4 \le \mathfrak{f}$,

$$\mathcal{R} = R_0[\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}\} \cup \{t_i \mid 1 \le i \le \mathfrak{f}\}]$$

be a polynomial ring in $\binom{\mathfrak{f}}{2} + \mathfrak{f}$ indeterminates, $\mathbf{X}$ be the $\mathfrak{f} \times \mathfrak{f}$ alternating matrix with $x_{i,j}$ in position (row i, column j) for $i < j$, $\mathbf{t}$ be the $1 \times \mathfrak{f}$ matrix with t_j in column j, I be the ideal $\mathrm{Pf}_4(\mathbf{X})$ which is generated by the set of Pfaffians of the principal 4×4 submatrices of $\mathbf{X}$, K be the ideal $I_1(\mathbf{tX})$, which is generated by the entries of the product of $\mathbf{t}$ times $\mathbf{X}$, and J be the ideal $I + K$ of $\mathcal{R}$. The main result in the paper, Theorem 4.8, is that J is a perfect Gorenstein ideal in $\mathcal{R}$ of grade $\binom{\mathfrak{f}-2}{2} + 2$. In particular, if R_0 is a Gorenstein ring, then $\mathcal{R}/J$ is a Gorenstein ring. Some consequences of the main result are contained in Corollary 5.3 where it is shown that $\mathcal{R}/J$ is a domain, or a normal ring, or a unique factorization domain if and only if the base ring R_0 has the same property.

The main ingredient in the proof of Theorem 4.8 takes place over the polynomial ring

$$R = R_0[\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}\}].$$

We prove in Lemma 3.1 that "the column space of $\mathbf{X}$, calculated mod I", which is equal to the submodule

$$C = \{\mathbf{X}\theta \in (\tfrac{R}{I})^{\mathfrak{f}} \mid \theta \in (\tfrac{R}{I})^{\mathfrak{f}}\} \quad \text{of } (\tfrac{R}{I})^{\mathfrak{f}}, \tag{1.0.1}$$

is a perfect R-module of projective dimension $\binom{\mathfrak{f}-2}{2}$. If R_0 is a Cohen–Macaulay domain, then we prove, in Observation 3.14.d, that the module of (1.0.1) is a self-dual maximal Cohen–Macaulay R/I-module of rank two; and we prove, see Remark 4.4.b, that the ideal $J(\mathcal{R}/I)$, which is the central object in this paper, is a Bourbaki ideal for $\mathcal{R} \otimes_R C$; and therefore, homological properties of C are inherited by $\mathcal{R}/J$. For more discussion about Bourbaki ideals, see, for example, [2,26,4,30].

The ideal I is the ideal of "quadratic relations" which define the homogeneous coordinate ring of the image of the Plücker embedding of the Grassmannian $\mathrm{Gr}(2,\mathfrak{f})$ into projective space $\mathbb{P}^{\binom{\mathfrak{f}}{2}-1}$. Properties of the Schubert subvarieties of $\mathrm{Gr}(2,\mathfrak{f})$ play a crucial role in our proof of the properties of the module (1.0.1).

The ideal $I_1(\mathbf{tX})$ has already been studied. If $\mathfrak{f}$ is odd, then $I_1(\mathbf{tX})$ is a type two almost complete intersection ideal introduced by Huneke and Ulrich in [16] and further studied in [23]. If $\mathfrak{f}$ is even, then $I_1(\mathbf{tX})$ is a mixed ideal; its unmixed part is $I_1(\mathbf{tX}) + \mathrm{Pf}_{\mathfrak{f}}(\mathbf{X})$; see, for example, [24]. This ideal is a deviation two, grade $\mathfrak{f}-1$ Gorenstein ideal also introduced in [16] and further studied in [21,31,22].

2. Notation, conventions, and preliminary results

2.1. Let M and N be modules over a commutative Noetherian ring R. Whenever the meaning is unambiguous, we write M^*, $M \otimes N$, $\mathrm{Hom}(M,N)$, and $\bigwedge^i M$ in place of $\mathrm{Hom}_R(M,R)$, $M \otimes_R N$, $\mathrm{Hom}_R(M,N)$, and $\bigwedge_R^i M$, respectively.

2.2. An element x of a ring R is *regular* on the R-module M if x is a non-zero-divisor on M. In other words, if $xm = 0$ for some element $m \in M$, then $m = 0$.

2.3. If x is a non-nilpotent element of a commutative Noetherian ring R, then *the localization of R at x*, denoted R_x, is the ring $S^{-1}R$ where S is the set $\{1, x, x^2, x^3, \dots\}$. If x is a regular element of R, then we use the notation R_x and $R[x^{-1}]$ interchangeably.

2.4. We denote the ring of integers by $\mathbb{Z}$.

2.1. Perfection

2.5. Let R be a Noetherian ring, I be a proper ideal of R, and M be a non-zero finitely generated R-module.

(a) The *grade* of I is the length of a maximal regular sequence on R which is contained in I. (If R is Cohen–Macaulay, then the grade of I is equal to the height of I.)
(b) The R-module M is called *perfect* if the grade of the annihilator of M (denoted $\operatorname{ann}_R M$) is equal to the projective dimension of M (denoted $\operatorname{pd}_R M$). The inequality

$$\operatorname{grade}(\operatorname{ann}_R M) \le \operatorname{pd}_R M \tag{2.5.1}$$

holds automatically if $M \neq 0$.
(c) If M is a perfect R-module, then

$$\operatorname{pd}_{R_P} M_P = \operatorname{grade}\operatorname{ann}_{R_P} M_P = \operatorname{grade}\operatorname{ann}_R M$$

for all prime ideals P in the support of M. (See, for example, [6, Prop. 16.17].)
(d) If R is a polynomial ring over a field or over the ring of integers and M is a finitely generated graded R-module, then M is a perfect R-module if and only if M is a Cohen–Macaulay R-module. (This is not the full story. For more information, see, for example, [6, Prop. 16.19] or [5, Thm. 2.1.5].)
(e) The ideal I in R is called a *perfect ideal* if R/I is a perfect R-module. A perfect ideal I of grade g is a *Gorenstein ideal* if $\operatorname{Ext}^g_R(R/I, R)$ is a cyclic R-module.

The concept of perfection is particularly useful because of the "Persistence of Perfection Principle", which is also known as the "transfer of perfection"; see [14, Prop. 6.14] or [6, Thm. 3.5].

Theorem 2.6. *Let $R \to S$ be a homomorphism of Noetherian rings, M be a perfect R-module, and $\mathbb{P}$ be a resolution of M by projective R-modules. If $S \otimes_R M \neq 0$ and*

$$\operatorname{grade}(\operatorname{ann} M) \le \operatorname{grade}(\operatorname{ann}(S \otimes_R M)),$$

then $S \otimes_R M$ is a perfect S-module with $\operatorname{pd}_S(S \otimes_R M) = \operatorname{pd}_R M$ and $S \otimes_R \mathbb{P}$ is a resolution of $S \otimes_R M$ by projective S-modules.

2.2. Multilinear algebra

2.7. Many of our calculations are made in a coordinate-free manner. If the calculation is coordinate free, then the signs take care of themselves. In particular, when working with Pfaffians, we prefer to use elements of an exterior algebra rather than to define and keep track of sign conventions which mimic operations that take place in an exterior algebra.

Let R be a commutative Noetherian ring and F be a free module of finite rank $\mathfrak{f}$ over R. We make much use of the exterior algebras $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$, the fact that $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$ are modules over one another, and the fact that the even part of an exterior algebra comes equipped with a divided power structure. The rules for a divided power algebra are recorded in [12, section 7] or [10, Appendix 2]. (In practice these rules say that $w^{(a)}$ behaves like $w^a/(a!)$ would behave if $a!$ were a unit in R.)

2.8. We recall some of the properties of the divided power structure on the subalgebra $\bigwedge^{2\bullet} F$ of the exterior algebra $\bigwedge^{\bullet} F$. Suppose that $e_1, \dots, e_{\mathfrak{f}}$ is a basis for the free R-module F and

$$f_2 = \sum_{1 \le i_1 < i_2 \le \mathfrak{f}} a_{i_1,i_2}\, e_{i_1} \wedge e_{i_2}$$

is an element of $\bigwedge^2 F$, for some a_{i_1,i_2} in R. Let A be the $\mathfrak{f} \times \mathfrak{f}$ alternating matrix with

$$A_{i,j} = \begin{cases} a_{i,j}, & \text{if } i < j, \\ 0, & \text{if } i = j, \text{ and} \\ -a_{i,j}, & \text{if } j < i. \end{cases}$$

For each positive integer ℓ, the ℓ-th divided power of f_2 is

$$f_2^{(\ell)} = \sum_I A_I e_I \in \bigwedge^{2\ell} F,$$

where the 2ℓ-tuple $I = (i_1, \dots, i_{2\ell})$ roams over all increasing sequences of integers with $1 \le i_1$ and $i_{2\ell} \le \mathfrak{f}$, $e_I = e_{i_1} \wedge \dots \wedge e_{i_{2\ell}}$, and A_I is the Pfaffian of the submatrix of A which consists of rows and columns $\{i_1, \dots, i_{2\ell}\}$, in the given order. Furthermore, $\bigwedge^{2\bullet} F$ is a DGΓ-module over $\bigwedge^{\bullet} F^*$. In particular, if $\tau \in F^*$ and $v_1, \dots, v_s$ are homogeneous elements of $\bigwedge^{2\bullet} F$, then

$$\tau\left(v_1^{(\ell_1)} \wedge \dots \wedge v_s^{(\ell_s)}\right) = \sum_{j=1}^{s} \tau(v_j) \wedge v_1^{(\ell_1)} \wedge \dots \wedge v_j^{(\ell_j - 1)} \wedge \dots \wedge v_s^{(\ell_s)}. \tag{2.8.1}$$

For more details see, for example, [10, Appendix A2.4] or [8, Appendix and Sect. 2].

The following fact about the interaction of the module structures of $\bigwedge^{\bullet} F$ on $\bigwedge^{\bullet} F^*$ and $\bigwedge^{\bullet} F^*$ on $\bigwedge^{\bullet} F$ is well known; see [7, section 1] and [8, Appendix].

Proposition 2.9. *Let F be a free module of finite rank over a commutative Noetherian ring R. If $f_1 \in F$, $f_p \in \bigwedge^p F$, and $\phi_q \in \bigwedge^q(F^*)$, then*

$$(f_1(\phi_q))(f_p) = f_1 \wedge (\phi_q(f_p)) + (-1)^{1+q}\phi_q(f_1 \wedge f_p). \qquad \square$$

The following fact is important for our purposes. We prove it carefully in order to illustrate some of the ideas contained in 2.7 and 2.8.

Observation 2.10. *Let R be a commutative Noetherian ring, F be a free R-module of finite rank, and f_2 be an element of $\bigwedge^2 F$.*

(a) *If $\phi_3 \in \bigwedge^3 F^*$, then $[f_2(\phi_3)](f_2) = \phi_3(f_2^{(2)})$.*
(b) *If ϕ_1, ϕ_1', and ϕ_1'' are in F^*, then*

$$f_2(\phi_1 \wedge \phi_1' \wedge \phi_1'') = f_2(\phi_1 \wedge \phi_1') \cdot \phi_1'' - f_2(\phi_1 \wedge \phi_1'') \cdot \phi_1' + f_2(\phi_1' \wedge \phi_1'') \cdot \phi_1.$$

Proof. We prove (a) by showing that the two elements $[f_2(\phi_3)](f_2)$ and $\phi_3(f_2^{(2)})$ of F are equal by showing that $\phi_1\Big([f_2(\phi_3)](f_2)\Big) = \phi_1\Big(\phi_3(f_2^{(2)})\Big)$ for every element ϕ_1 of F^*. Observe that

$$\begin{aligned} \phi_1([f_2(\phi_3)](f_2)) &= -[f_2(\phi_3)][\phi_1(f_2)] &&= -[\phi_1(f_2)][f_2(\phi_3)] &&= -[\phi_1(f_2) \wedge f_2](\phi_3) \\ &= -[\phi_1(f_2^{(2)})](\phi_3) &&= -\phi_3[\phi_1(f_2^{(2)})] &&= \phi_1[\phi_3(f_2^{(2)})]. \end{aligned}$$

The first and last equalities hold because $\bigwedge^\bullet F$ is a module over the graded-commutative ring $\bigwedge^\bullet F^*$. The second and fifth equalities follow from the fact that the module actions of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$ and $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$ are compatible in the sense that

$$\phi_i(f_i) = f_i(\phi_i) \text{ for } \phi_i \in \textstyle\bigwedge^i F^* \text{ and } f_i \in \textstyle\bigwedge^i F. \tag{2.10.1}$$

The third equality is a consequence of the module action of $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$. The fourth equality is explained in (2.8.1).

The proof of (b) is similar. □

2.3. Mapping cone

The mapping cone of the map of complexes $c : F \to E$:

$$\begin{tikzcd} \cdots \arrow[r, "f_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "c_1"] & F_0 \arrow[d, "c_0"] \\ \cdots \arrow[r, "e_2"] & E_1 \arrow[r, "e_1"] & E_0 \end{tikzcd}$$

is the complex M:

$$\cdots \to \begin{matrix} E_3 \\ \oplus \\ F_2 \end{matrix} \xrightarrow{\begin{bmatrix} e_3 & c_2 \\ 0 & -f_2 \end{bmatrix}} \begin{matrix} E_2 \\ \oplus \\ F_1 \end{matrix} \xrightarrow{\begin{bmatrix} e_2 & c_1 \\ 0 & -f_1 \end{bmatrix}} \begin{matrix} E_1 \\ \oplus \\ F_0 \end{matrix} \xrightarrow{\begin{bmatrix} e_1 & c_0 \end{bmatrix}} E_0.$$

The three complexes are related by the short exact sequence

$$0 \to E \to M \to F[-1] \to 0;$$

and hence by the long exact sequence of homology

$$\cdots \to \mathrm{H}_1(M) \to \mathrm{H}_0(F) \to \mathrm{H}_0(E) \to \mathrm{H}_0(M) \to 0.$$

One can iterate the procedure. If F, E, and D are resolutions of $\mathrm{H}_0(F)$, $\mathrm{H}_0(E)$, and $\mathrm{H}_0(D)$, respectively,

$$0 \to \mathrm{H}_0(F) \xrightarrow{\gamma} \mathrm{H}_0(E) \xrightarrow{\beta} \mathrm{H}_0(D)$$

is an exact sequence of modules, and $c : F \to E$ is a map of complexes which covers γ, then the mapping cone M of c is a resolution of $\operatorname{coker}(\gamma) \cong \operatorname{im}(\beta)$. If $b : M \to D$ is a map of complexes which covers $\operatorname{im}\beta \subseteq \mathrm{H}_0(D)$, then the mapping cone of b is a resolution of $\operatorname{coker}\beta$.

2.4. The set up

2.11. We set up the data in a coordinate-free manner in 2.12 and 2.13; a version with coordinates is given in 2.14. The critical calculation, Lemma 3.1, involves "$x_{i,j}$'s", but not "t_i's"; the ambient ring for this calculation is called R. The information about R is given in 2.12.a, 2.13.a, and 2.14.a. The main result in the paper, Theorem 4.8, involves both "$x_{i,j}$'s" and "t_i's"; the ambient ring for this result is called $\mathcal{R}$. The ring $\mathcal{R}$ is an extension of R; the extra information about $\mathcal{R}$ is given in 2.12.b, 2.13.b, and 2.14.b.

Data 2.12. Let $\mathfrak{f}$ be a positive integer, R_0 a commutative Noetherian ring, and V be a free R_0-module of rank $\mathfrak{f}$.

(a) Let $R = \bigoplus_{i=0}^{\infty} R_i$ be the standard graded polynomial ring

$$R = \operatorname{Sym}_\bullet^{R_0}(\textstyle\bigwedge_{R_0}^2 V^*)$$

and F be the free R-module $F = R \otimes_{R_0} V$. Consider the R-module homomorphism

$$\xi \in \operatorname{Hom}_R(\textstyle\bigwedge_R^2 F^*, R) = \textstyle\bigwedge_R^2 F,$$

which is given as the composition

$$\xi : \textstyle\bigwedge_R^2 F^* = R \otimes_{R_0} \bigwedge_{R_0}^2 V^* = R \otimes R_1 \xrightarrow{\text{multiplication}} R.$$

(b) View $\bigwedge_{R_0}^2 V^* \oplus V$ as a bi-graded free R_0-module where each element of $\bigwedge_{R_0}^2 V^*$ has degree $(1, 0)$ and each element of V has degree $(0, 1)$. Let $\mathcal{R}$ be the bi-graded polynomial ring

$$\mathcal{R} = \operatorname{Sym}_\bullet^{R_0}(\textstyle\bigwedge_{R_0}^2 V^* \oplus V)$$

and $\mathcal{F}$ be the free $\mathcal{R}$-module $\mathcal{F} = \mathcal{R} \otimes_{R_0} V$. Consider the $\mathcal{R}$-module homomorphism

$$\tau \in \operatorname{Hom}_{\mathcal{R}}(\mathcal{F}, \mathcal{R}) = \mathcal{F}^*$$

which is given as the composition

$$\mathcal{F} = \mathcal{R} \otimes_{R_0} V = \mathcal{R} \otimes \mathcal{R}_{0,1} \xrightarrow{\text{multiplication}} \mathcal{R}.$$

(c) There is a natural inclusion map $R \hookrightarrow \mathcal{R}$ and a natural projection map $\mathcal{R} \twoheadrightarrow R$. The $\mathcal{R}$-module $\mathcal{F}$ of (b) is also equal to $\mathcal{F} = \mathcal{R} \otimes_R F$; furthermore, the element $\xi \in \bigwedge_R^2 F$ of (a) is also equal to the element

$$\xi = 1 \otimes \xi \text{ of } \textstyle\bigwedge_{\mathcal{R}}^2 \mathcal{F} = \mathcal{R} \otimes_R \bigwedge_R^2 F.$$

Notation 2.13. Adopt Data 2.12.

(a) Let
  (i) I be the ideal $I = \operatorname{im}(\xi^{(2)} : \bigwedge^4 F^* \to R)$, of R,
  (ii) A be the ring R/I,
  (iii) $\overline{}$ be the functor $A \otimes_R -$, and
  (iv) N be the cokernel of the map $\overline{d_1} : \bigwedge^3 \overline{F}^* \to \overline{F}^*$ where $d_1 : \bigwedge^3 F^* \to F^*$ is the map $d_1(\phi_3) = \xi(\phi_3)$, for $\phi_3 \in \bigwedge^3 F^*$.

(b) Let
  (i) K and J be the ideals

$$K = \operatorname{im}(\tau(\xi) : \mathcal{F}^* \to \mathcal{R}), \quad \text{and}$$
$$J = I\mathcal{R} + K$$

  of $\mathcal{R}$,
  (ii) $\mathcal{A}$ be the ring $\mathcal{R} \otimes_R A$, and
  (iii) $\mathcal{N}$ be the $\mathcal{R}$-module $\mathcal{R} \otimes_R N$.

Remark 2.14. Adopt Data 2.12 and Notation 2.13. If one picks dual bases $e_1, \dots, e_{\mathfrak{f}}$ for V and $e_1^*, \dots, e_{\mathfrak{f}}^*$ for V^*, let $x_{i,j}$ represent $e_j^* \wedge e_i^* \in \bigwedge_{R_0}^2 V^* = \mathcal{R}_{1,0}$, for $1 \le i < j \le \mathfrak{f}$, and let t_i represent $e_i \in V = \mathcal{R}_{0,1}$, for $1 \le i \le \mathfrak{f}$, then the following statements hold.

(a) The standard graded polynomial ring R is $R = R_0[\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}\}]$. Furthermore,
  (i) the element ξ of $\bigwedge^2 F$ is $\xi = \sum_{i<j} x_{i,j}\, e_i \wedge e_j$,

(ii) the matrix for the R-module homomorphism $d_0 : F^* \to F$, with $d_0(\phi_1) = \phi_1(\xi)$, with respect to the bases $\{e_j^*\}$ and $\{e_i\}$, is $-\mathbf{X}$, where $\mathbf{X}$ is the generic $\mathfrak{f} \times \mathfrak{f}$ alternating matrix whose entry in position (row i, column j) is

$$\begin{cases} x_{i,j} & \text{if } i < j \\ 0 & \text{if } i = j \\ -x_{j,i} & \text{if } j < i, \end{cases}$$

and

(iii) the ideal I of Notation 2.13 is equal to $\mathrm{Pf}_4(\mathbf{X})$, which is the ideal of R generated by the set of Pfaffians of the principal 4×4 submatrices of $\mathbf{X}$.

(b) The bi-graded polynomial ring $\mathcal{R}$ is $\mathcal{R} = R_0[\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}\} \cup \{t_i \mid 1 \le i \le \mathfrak{f}\}]$. Furthermore,

(i) the element τ of $\mathcal{F}^*$ is $\tau = \sum_i t_i e_i^*$,

(ii) the matrix for $\tau : \mathcal{F} \to \mathcal{R}$ with respect to the basis $\{e_i\}$ for $\mathcal{F}$ is the row vector

$$\mathbf{t} = [t_1, \dots, t_{\mathfrak{f}}],$$

(iii) the element $\tau(\xi)$ in $\mathcal{F}$ is an $\mathcal{R}$-module homomorphism $\mathcal{F}^* \to \mathcal{R}$ and the matrix for this homomorphism, with respect to the basis $\{e_i^*\}$ is the row vector $\mathbf{tX}$, and

(iv) the ideal K of Notation 2.13 is equal to $I_1(\mathbf{tX})$, which is the ideal of $\mathcal{R}$ generated by the entries of the product of $\mathbf{t}$ times $\mathbf{X}$, and

(v) the ideal J of Notation 2.13 is equal to $\mathrm{Pf}_4(\mathbf{X}) \cdot \mathcal{R} + I_1(\mathbf{tX})$.

Remark 2.15. Adopt the language of 2.12.a and 2.13.a. The following maps appear often in the paper:

$$\mathbb{M}: \quad \textstyle\bigwedge^3 F^* \xrightarrow{d_1} F^* \xrightarrow{d_0} F \xrightarrow{\delta_1} \bigwedge^3 F, \tag{2.15.1}$$

with $d_1(\phi_3) = \xi(\phi_3)$, $d_0(\phi_1) = \phi_1(\xi)$, and $\delta_1(f_1) = f_1 \wedge \xi$, for $\phi_3 \in \bigwedge^3 F^*$, $\phi_1 \in F^*$, and $f_1 \in F$. Use Observation 2.10.a and (2.8.1) to see that

$$(d_0 \circ d_1)(\phi_3) = [\xi(\phi_3)](\xi) = \phi_3(\xi^{(2)}) \quad \text{and} \quad (\delta_1 \circ d_0)(\phi_1) = [\phi_1(\xi)] \wedge \xi = \phi_1(\xi^{(2)});$$

so, in particular $A \otimes_R \mathbb{M}$ is a complex. In (3.11) we prove that a modification of $A \otimes_R \mathbb{M}$ is exact and in Observation 3.14 we prove that $A \otimes_R \mathbb{M}$ is exact.

If one uses the notation Remark 2.14.a, then the matrix for d_0 is $-\mathbf{X}$, the matrix for d_1 has $\mathfrak{f}$ rows and $\binom{\mathfrak{f}}{3}$ columns and the column corresponding to $e_k^* \wedge e_j^* \wedge e_i^*$, for $1 \le i < j < k \le \mathfrak{f}$, is

$$[0 \ \dots \ 0 \ \ x_{j,k} \ \ 0 \ \dots \ 0 \ \ -x_{i,k} \ \ 0 \ \dots \ 0 \ \ x_{i,j} \ \ 0 \ \dots \ 0]^{\mathrm{T}}, \tag{2.15.2}$$

where the non-zero entries appear in positions i, j, and k, respectively; see Observation 2.10.b and Remark 2.14.ai. (We use M^{T} to represent the transpose of the matrix M.) The matrix for δ_1 is the transpose of the matrix for d_1.

3. The main ingredient

In this section we prove the following result.

Lemma 3.1. *Adopt Data 2.12.a and Notation 2.13.a with $4 \le \mathfrak{f}$. If the base ring R_0 is an arbitrary commutative Noetherian ring, then the R-module N is perfect of projective dimension $\binom{\mathfrak{f}-2}{2}$.*

The proof of Lemma 3.1 is given in 3.12 at the end of the section. In Observation 3.14 we show that the module N of Lemma 3.1 is isomorphic to the module of (1.0.1).

Remark 3.2. The assertion of Lemma 3.1 does not hold for $\mathfrak{f} = 3$. Indeed, in the language of Remark 2.14, N, which is resolved by

$$0 \to R \xrightarrow{\begin{bmatrix} x_{2,3} & -x_{1,3} & x_{1,2} \end{bmatrix}^{\mathrm{T}}} R^3,$$

is not a perfect R-module and has projective dimension one, which is not equal to $\binom{\mathfrak{f}-2}{2}$. (We use M^{T} to represent the transpose of the matrix M.)

It is convenient to let

$$A' \text{ be the ring } A/(x_{1,2}\ ,\ x_{2,3}\ ,\ x_{1,3}),$$

in the language of Remark 2.14.a. The proof of Lemma 3.1 depends on Lemma 3.3 and on information about the rings A and A' which is contained in Lemma 3.7.

Lemma 3.3. *Adopt Data 2.12.a and Notation 2.13.a with $3 \le \mathfrak{f}$. If the base ring R_0 is a commutative Noetherian domain, then there is an exact sequence of A-modules:*

$$0 \to N \to A^3 \to A \to A' \to 0. \tag{3.3.1}$$

In particular, if (0) is the zero ideal of A, then $N_{(0)}$ is isomorphic to $A_{(0)} \oplus A_{(0)}$.

The proof of Lemma 3.3 is given in 3.11.

Remarks 3.4.

(a) A strengthened version of Lemma 3.3 may be found in Proposition 5.5.
(b) Lemma 3.3 does hold when $\mathfrak{f} = 3$; indeed, (3.3.1) becomes

$$0 \to \frac{R^3}{\left(\begin{bmatrix} x_{2,3} \\ -x_{1,3} \\ x_{1,2} \end{bmatrix}\right)} \xrightarrow{\begin{bmatrix} 0 & x_{1,2} & x_{1,3} \\ -x_{1,2} & 0 & x_{2,3} \\ -x_{1,3} & -x_{2,3} & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_{2,3} & -x_{1,3} & x_{1,2} \end{bmatrix}} R \to R_0 \to 0,$$

which is exact.

Definition 3.5. Adopt the language of 2.12.a, 2.13.a and 2.14.a. For each integer λ, between 1 and $\mathfrak{f}-1$, let I_λ be the ideal

$$I_\lambda = I + (\{x_{i,j} \mid 1 \le i < j \le \lambda\})$$

of R.

Example 3.6. Retain the notation of Definition 3.5. The ideal I_1 is equal to I (because the empty set generates the zero ideal) and the ideal $I_{\mathfrak{f}-1}$ is equal to $(\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}-1\})$ (because I is contained in the ideal $(\{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}-1\})$). In particular, $A = R/I_1$ and $A' = R/I_3$.

Lemma 3.7. *Adopt the language of* 2.12.a*,* 2.13.a*,* 2.14.a*, and* 3.5*. Let λ be an integer between* 1 *and* $\mathfrak{f}-1$*.*

(a) *If the base ring R_0 is an arbitrary commutative Noetherian ring, then I_λ is a perfect ideal in R of grade $\binom{\mathfrak{f}-2}{2} + \lambda - 1$. In particular, if $4 \le \mathfrak{f}$, then* $\operatorname{grade} I_3 = \operatorname{grade} I_1 + 2$.
(b) *If the base ring R_0 is a commutative Noetherian domain, then I_λ is a prime ideal.*
(c) *If the base ring R_0 is an arbitrary commutative Noetherian ring, then I is a Gorenstein ideal in the sense of* 2.5.e*. In particular, if R_0 is a Gorenstein ring, then R/I is a Gorenstein ring.*

Remark 3.8. The "in particular assertion" in (a) would be false if $\mathfrak{f}$ were equal to 3; because, in this case, I_1, which is equal to (0), has grade 0, and I_3, which is equal to $(x_{1,2}\ ,\ x_{1,3}\ ,\ x_{2,3})$, has grade 3. Of course, the parameter λ, which is assumed to be at most $\mathfrak{f}-1$, is not permitted to be 3, when $\mathfrak{f} = 3$.

Proof. (a, b) The ideal I_λ is equal to the ideal $\operatorname{Pf}(X; \lambda; \lambda)$ of [18]. The assertion follows from [18, Thm. 12]. The statement of [18, Thm. 12] only considers the case where R_0 is a domain; however, as soon as one knows that I_λ is a perfect ideal when R_0 is equal to the ring of integers and when R_0 is equal to a field, then I_λ built with $R_0 = \mathbb{Z}$ is a generically perfect ideal and consequently I_λ built over an arbitrary commutative Noetherian R_0 is a perfect ideal; see, for example [6, Prop. 3.2 and Thm. 3.3].

(c) A proof that R/I is a Gorenstein ring whenever R_0 is Gorenstein is given in [18, Thm. 17]. A more explicit statement and proof of this result is given in [3, Corollary]. In particular, when R_0 is equal to the ring of integers, then there exists a resolution $\mathbb{F}$ of R/I by free R-modules which has the property that the length of $\mathbb{F}$ is $\binom{\mathfrak{f}-2}{2}$ and the free module of $\mathbb{F}$ in position $\binom{\mathfrak{f}-2}{2}$ has rank one. Now let R_0 be an arbitrary commutative Noetherian ring. We explained in the proof of (a) and (b) that I is a perfect ideal in R. The "Persistence of Perfection Principle", Theorem 2.6, now guarantees that the back Betti number in a resolution of R/I by free R-modules is one; and therefore, I is a Gorenstein ideal in the sense of 2.5.e. $\square$

Remark 3.9. An alternate phrasing of the proof of Lemma 3.7, parts (a) and (b), (but really the same argument in a different form) involves the Grassmannian $\mathrm{Gr}(2,\mathfrak{f})$ of rank 2 free summands of the rank $\mathfrak{f}$ free R_0-module V. The ideal I is the ideal of "quadratic relations" which define the homogeneous coordinate ring of the image of the Plücker embedding of $\mathrm{Gr}(2,\mathfrak{f})$ into $\mathbb{P}(\bigwedge^2 V)$. The ideal I_λ defines the homogeneous coordinate ring of the Plücker embedding of the Schubert subvariety $\Omega(\mathfrak{f}-\lambda,\mathfrak{f})$ of $\mathrm{Gr}(2,\mathfrak{f})$. The Schubert subvariety $\Omega(\mathfrak{f}-\lambda,\mathfrak{f})$ consists of all W in $\mathrm{Gr}(2,\mathfrak{f})$ such that $i \le \mathrm{rank}(W \cap V_i)$ for the flag $V_1 \subsetneq V_2$ where V_1 is the summand of V with basis $e_{\lambda+1},\dots,e_{\mathfrak{f}}$ and $V_2 = V$. The original proofs that the homogeneous coordinate rings of the Schubert subvarieties of the Grassmannian are Cohen–Macaulay domains are [15, Thm. 3.1*, (3.10), Cor. 4.2], [25, Thm. 1], and [27, Thm. II.4.1 and Thm. III.4.1]. A version which contains many details is [6, Thm. 1.4, the bottom of page 52, Cor. 5.18, Thm. 6.3].

One consequence of Lemma 3.7 is that I is grade unmixed. This fact facilitates the identification of regular elements in A. Corollary 3.10 and its style of proof are used in the proof of Corollary 5.3.a.

Corollary 3.10. *Adopt the language of* 2.12.a, 2.13.a, *and* 2.14.a. *If the base ring R_0 is an arbitrary commutative Noetherian ring, then $x_{1,2}$, $x_{1,3}$ is a regular sequence on A.*

Proof. Every associated prime P of R/I in R has grade $PR_P = \binom{\mathfrak{f}-2}{2}$. Lemma 3.7.a assures that I_2, which equals $(I\ ,\ x_{1,2})$, is a perfect ideal of grade $\binom{\mathfrak{f}-2}{2}+1$ in R; hence $x_{1,2}$ is not in any associated prime of R/I (that is, $x_{1,2}$ is regular on R/I) and every associated prime P of $R/(I\ ,\ x_{1,2})$ in R has grade $PR_P = \binom{\mathfrak{f}-2}{2}+1$. We prove that $x_{1,3}$ is regular on $R/(I\ ,\ x_{1,2})$ by showing that $\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{grade} PR_P$ for all primes P of R which contain $(I\ ,\ x_{1,2}\ ,\ x_{1,3})$. Let P be such a prime. Consider the Pfaffian

$$x_{1,2}x_{3,j} - x_{1,3}x_{2,j} + x_{1,j}x_{2,3} \in I \subseteq P.$$

Thus, $x_{2,3}x_{1,j}$ is in P for $3 \le j \le \mathfrak{f}$. It follows that either I_3, which is $(I\ ,\ x_{1,2}\ ,\ x_{1,3}\ ,\ x_{2,3})$, is contained in P or $(I\ ,\ x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}}) \subseteq P$. Lemma 3.7.a ensures that I_3 has grade $\binom{\mathfrak{f}-2}{2}+2$. The ideal $(I\ ,\ x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}})$ is equal to $\mathrm{Pf}_4(\mathbf{X}')$ plus an ideal generated by $\mathfrak{f}-1$ indeterminates, where $\mathbf{X}'$ is $\mathbf{X}$ with row and column 1 deleted. Thus $(I\ ,\ x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}})$ has grade

$$\binom{\mathfrak{f}-3}{2}+\mathfrak{f}-1=\binom{\mathfrak{f}-2}{2}+2.$$

In either event, $\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{grade} P$ and the proof is complete. $\square$

3.11. Proof of Lemma 3.3. We prove that

$$\bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^* \xrightarrow{d_0'} A^3 \xrightarrow{\rho} A \to A' \to 0 \tag{3.11.1}$$

is an exact sequence of A-modules, where $d_1 : \bigwedge^3 F^* \to F^*$ is $d_1(\phi_3)=\xi(\phi_3)$, as given in Notation 2.13.aiv and Remark 2.15, d_0' is the composition

$$\overline{F}^* \xrightarrow{\overline{d_0}} \overline{F} = \bigoplus_{i=1}^{\mathfrak{f}} Ae_i \xrightarrow{\text{projection}} \bigoplus_{i=1}^{3} Ae_i,$$

where $d_0 : F^* \to F$ is $d_0(\phi_1)=\phi_1(\xi)$ as described in Remark 2.14.aii and Remark 2.15, and ρ is given by the matrix

$$\rho = \begin{bmatrix} x_{2,3} & -x_{1,3} & x_{1,2}\end{bmatrix}. \tag{3.11.2}$$

(The basis $e_1,\dots,e_{\mathfrak{f}}$ for F is introduced in Remark 2.14.) Once we show that (3.11.1) is an exact sequence, then the proof is complete. Indeed,

$$\begin{aligned} &N = \operatorname{coker}\overline{d_1} \cong \operatorname{im} d_0' = \ker\rho; \quad \text{hence,} \\ &0 \to N \to A^3 \xrightarrow{\rho} A \to A' \to 0 \end{aligned} \tag{3.11.3}$$

is exact, as claimed in (3.3.1).

We first show that (3.11.1) is a complex. To show that $d_0' \circ \overline{d_1}=0$ it suffices to show that the image of $d_0 \circ d_1$ is contained in $I \cdot F$ and this was done in Remark 2.15. The matrix for ρ is given in (3.11.2) and the matrix

$$d_0' = -\begin{bmatrix} 0 & x_{1,2} & x_{1,3} & x_{1,4} & \dots & x_{1,\mathfrak{f}} \\ -x_{1,2} & 0 & x_{2,3} & x_{2,4} & \dots & x_{2,\mathfrak{f}} \\ -x_{1,3} & -x_{2,3} & 0 & x_{3,4} & \dots & x_{3,\mathfrak{f}} \end{bmatrix} \tag{3.11.4}$$

for d_0' may be read from the discussion in Remark 2.15. It is clear that $\rho \circ d_0' = 0$ and that the complex (3.3.1) is exact at A and A'. We next show that (3.3.1) is exact at A^3. Suppose

$$\alpha = [a_1 \quad a_2 \quad a_3]^{\mathrm{T}}$$

is an element of A^3 with $\rho(\alpha) = 0$ in A. (We use M^{T} to represent the transpose of the matrix M.) In other words, $x_{2,3}a_1 - x_{1,3}a_2 + x_{1,2}a_3 = 0$ in A. In particular,

$$x_{2,3}a_1 \in (x_{1,2}\ ,\ x_{1,3})A \subseteq (x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}})A.$$

The ideal $(x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}})A$ of A is prime; indeed,

$$(x_{1,2}\ ,\ x_{1,3}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}}) + I = (x_{1,2}\ ,\ x_{1,3}\ ,\dots\ ,\ x_{1,\mathfrak{f}}) + \mathrm{Pf}_4(\mathbf{X}'),$$

where $\mathbf{X}'$ is the matrix $\mathbf{X}$ of Remark 2.14.aii with row one and column one deleted. The matrix $\mathbf{X}'$ is a generic alternating matrix which does not involve the variables $x_{1,2}\ ,\ \dots\ ,\ x_{1,\mathfrak{f}}$; so [18, Thm. 12] guarantees that $\mathrm{Pf}_4(\mathbf{X}')$ is prime; see, for example Lemma 3.7.

The product $x_{2,3}a_1$ is in the prime ideal $(x_{1,2},\ \dots,\ x_{1,\mathfrak{f}})A$ and $x_{2,3} \notin (x_{1,2},\ \dots,\ x_{1,\mathfrak{f}})A$; thus, $a_1 \in (x_{1,2},\ \dots,\ x_{1,\mathfrak{f}})A$ and a quick glance at (3.11.4) shows that there is an element $\overline{\phi_1}$ in $\overline{F}$ such that

$$\alpha - d_0'(\overline{\phi_1}) = [0 \quad a_2' \quad a_3']^{\mathrm{T}},$$

for some a_2' and a_3' in A. The equation $-x_{1,3}a_2' + x_{1,2}a_3' = 0$ in A shows that $x_{1,3}a_2'$ is an element of the prime ideal $(x_{1,2})A = I_2A$; see Lemma 3.7. Hence, a_2' is in $(x_{1,2})A$ and a further modification $\alpha - d_0'(\overline{\phi_1})$ by a boundary which only involves the first column of d_0' yields an element of the kernel of ρ of the form $[0 \quad 0 \quad a_3'']^{\mathrm{T}}$. The element a_3'' is zero because A is a domain; and therefore, $\alpha \in \operatorname{im} d_0'$.

The argument that (3.3.1) is exact at $\overline{F}^*$ is very similar to the preceding argument. Suppose $\alpha = [a_1\ ,\ \dots\ ,\ a_{\mathfrak{f}}]^{\mathrm{T}}$ is an element of $\ker d_0'$. The third row of the equation $d_0'\alpha = 0$ yields that $x_{3,\mathfrak{f}}a_f$ is an element of the prime ideal $I_{\mathfrak{f}-1}A$, in the language of Definition 3.5 and Lemma 3.7; but $x_{3,\mathfrak{f}} \notin I_{\mathfrak{f}-1}A$; so $a_{\mathfrak{f}} \in I_{\mathfrak{f}-1}$. On the other hand, for each $x_{i,j} \in I_{\mathfrak{f}-1}$,

$$\overline{d_1}(e_{\mathfrak{f}}^* \wedge e_j^* \wedge e_i^*) = x_{j,\mathfrak{f}}e_i^* - x_{i,\mathfrak{f}}e_j^* + x_{i,j}e_{\mathfrak{f}}^*;$$

hence there is an element $\overline{\phi_3} \in \bigwedge^3 \overline{F}^*$ so that

$$\alpha - \overline{d_1}(\overline{\phi_3}) = [a_1'\ ,\ \dots\ ,\ a_{\mathfrak{f}-1}'\ ,\ 0]^{\mathrm{T}}.$$

The third row of the equation $d_0'(\alpha - \overline{d_1}(\overline{\phi_3})) = 0$ yields that $x_{3,\mathfrak{f}-1}a_{\mathfrak{f}-1}' \in I_{\mathfrak{f}-2}A$. Use elements of the form $\overline{d_1}(e_{\mathfrak{f}-1}^* \wedge e_j^* \wedge e_i^*)$ to remove $a_{\mathfrak{f}-1}'$ (while keeping 0 in the bottom position). Continue in this manner to find $\overline{\phi_3}^{\dagger} \in \bigwedge^3 \overline{F}^*$ so that

$$\alpha - \overline{d_1}(\overline{\phi_3}^{\dagger}) = [a_1^{\dagger}\ ,\ a_2^{\dagger}\ ,\ a_3^{\dagger}\ ,\ 0\ ,\ \dots, 0]^{\mathrm{T}}.$$

The second equation of

$$\begin{bmatrix} 0 & x_{1,2} & x_{1,3} \\ -x_{1,2} & 0 & x_{2,3} \\ -x_{1,3} & -x_{2,3} & 0 \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \\ a_3^\dagger \end{bmatrix} = d_0'(\alpha - \overline{d_1}(\overline{\phi_3}^\dagger) = 0$$

yields $a_3^\dagger \in (x_{1,2})A$; hence there exists $\overline{\phi_3}^\ddagger \in \bigwedge^3 \overline{F}^*$, so that

$$\alpha - \overline{d_1}(\overline{\phi_3}^\ddagger) = [a_1^\ddagger\ ,\ a_2^\ddagger\ ,\ 0\ ,\ \dots\ ,\ 0]^{\mathrm{T}}.$$

Now one sees that $x_{1,2}a_1^\ddagger = x_{1,2}a_2^\ddagger = 0$ in the domain A; hence $a_1^\ddagger = a_2^\ddagger = 0$, α is a boundary, and (3.3.1) is exact.

The final assertion, that N has rank two as an A-module, is an immediate consequence of the exactness of (3.3.1). Indeed, A is a domain (see [18, Thm. 12] or Lemma 3.7) and $A'_{(0)} = 0$. $\square$

3.12. Proof of Lemma 3.1. The module N, built over an arbitrary ring R_0, is obtained from the module N, built over the ring of integers $\mathbb{Z}$, by way of the base change $R_0 \otimes_{\mathbb{Z}} -$. According to the theory of generic perfection (see, for example [6, Prop. 3.2 and Thm. 3.3]) in order to prove that N, built over an arbitrary ring R_0, is a perfect R-module, it suffices to prove that N is a perfect R-module when $R_0 = \mathbb{Z}$ and when R_0 is a field. Fix one of these choices for R_0 and consider the exact sequence of Lemma 3.3.

It was observed in Example 3.6 that $A = R/I_1$ and $A' = R/I_3$; consequently, Lemma 3.7.a guarantees that A and A' are perfect R-modules and $\operatorname{pd}_R A' = \operatorname{pd}_R A + 2$. (This is where the hypothesis $4 \le \mathfrak{f}$ is required; see Remark 3.8.) Let P be a prime ideal of R which is in the support of N. Lemma 3.3 shows that the module N embeds into a free A-module; hence, P is in the support of A and A_P is a Cohen–Macaulay ring. The localization A'_P is either zero or a Cohen–Macaulay ring with $\dim A'_P = \dim A_P - 2$. In either event, we apply the usual argument about the growth of depth in an exact sequence (see, for example, [5, Prop. 1.2.9]), to the localization of the exact sequence (3.3.1) at P in order to conclude that $\operatorname{depth} A_P \le \operatorname{depth} N_P$. At this point the inequalities

$$\operatorname{depth} N_P \le \dim N_P \le_* \dim A_P = \operatorname{depth} A_P \le \operatorname{depth} N_P \tag{3.12.1}$$

all hold; consequently, equality holds throughout. (The inequality labeled * holds because N_P is an A_P-module.) Thus, N_P is a Cohen–Macaulay R_P-module and

$$\operatorname{pd}_{R_P} N_P = \operatorname{pd}_{R_P} A_P = \operatorname{pd}_R A = \binom{\mathfrak{f}-2}{2}. \tag{3.12.2}$$

(The first equality is a consequence of the Auslander–Buchsbaum theorem; the second equality is explained in 2.5.c; and the third equality is a consequence of Lemma 3.7.) Thus, N is a perfect R-module of projective dimension $\binom{\mathfrak{f}-2}{2}$ (see 2.5.d, if necessary) and the proof is complete. $\square$

Section 4 is concerned with the ring $\mathcal{R}$ of Data 2.12 and Notation 2.13. The ring $\mathcal{R}$ is a polynomial ring over R and the $\mathcal{R}$-modules $\mathcal{N} = \mathcal{R} \otimes_R N$, $\mathcal{A} = \mathcal{R} \otimes_R A$, and $\mathcal{F} = \mathcal{R} \otimes_R F$ are obtained from the corresponding R-modules by way of a base change. It is convenient to record the results of the present section in the language of the future section.

Corollary 3.13. *Adopt Data 2.12 and Notation 2.13 with $4 \le \mathfrak{f}$. If the base ring R_0 is an arbitrary commutative Noetherian ring, then the $\mathcal{R}$-modules $\mathcal{N}$ and $\mathcal{A}$ are perfect of projective dimension $\binom{\mathfrak{f}-2}{2}$; furthermore $I\mathcal{R}$ is a Gorenstein ideal.*

Proof. Apply Lemmas 3.1 and 3.7. $\square$

We close this section by redeeming assorted promises. Assertion (a) was promised in Remark 2.15. Assertion (b) was promised in the introduction when we claimed that Section 3 is about the image of $\overline{d_0}$; however, until this point, it appears that Section 3 is about N, which is the cokernel of $\overline{d_1}$. The homological properties of N, which are listed in (c) and (d), were also promised in the introduction.

Observation 3.14. *Adopt the language of 2.12.a, 2.13.a, 2.14.a, and (2.15.1). Assume that R_0 is a domain.*

(a) *The complex $A \otimes_R \mathbb{M}$ is exact.*
(b) *The module N (of Lemma 3.1 and elsewhere) is isomorphic to the module of (1.0.1).*
(c) *The A-module N is self-dual.*
(d) *If R_0 is a Cohen–Macaulay domain, then N is a self-dual maximal Cohen–Macaulay A-module of rank two.*
(e) *If R_0 is a Gorenstein domain, and*

$$\mathbb{X}: \quad \cdots \xrightarrow{d_4} \mathbb{X}_3 \xrightarrow{d_3} \mathbb{X}_2 \xrightarrow{d_2} \bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^*$$

is a resolution of N by free A-modules, then

$$\mathbb{Y}: \quad \cdots \xrightarrow{d_4} \mathbb{X}_3 \xrightarrow{d_3} \mathbb{X}_2 \xrightarrow{d_2} \bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^* \xrightarrow{\overline{d_0}} \overline{F} \xrightarrow{\overline{\delta_1}} \bigwedge^3 \overline{F} \xrightarrow{d_2^*} \mathbb{X}_2^* \xrightarrow{d_3^*} \mathbb{X}_3^* \xrightarrow{d_4^*} \cdots$$

is a self-dual totally acyclic complex. (In other words, $\mathrm{H}_\bullet(\mathbb{Y}) = \mathrm{H}_\bullet(\mathbb{Y}^*) = 0$ and, after making the appropriate shift, $\mathbb{Y}^*$ is isomorphic to $\mathbb{Y}$.)

Proof. (a) We are supposed to prove that the complex

$$\bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^* \xrightarrow{\overline{d_0}} \overline{F} \xrightarrow{\overline{\delta_1}} \bigwedge^3 \overline{F} \tag{3.14.1}$$

is exact. (Recall from 2.13.aiii that $\overline{}$ is the functor $A \otimes_R -$.) We showed in (3.11.1) that

$$\bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^* \xrightarrow{\text{projection} \circ \overline{d_0}} A^3$$

is exact. It follows that

$$\operatorname{im} \overline{d_1} \subseteq \ker \overline{d_0} \subseteq \ker(\text{projection} \circ \overline{d_0}) = \operatorname{im} \overline{d_1}$$

and (3.14.1) is exact at $\overline{F}^*$.

We now prove that (3.14.1) is exact at $\overline{F}$. Let $f_1 = \sum_{i=1}^{\mathfrak{f}} a_i e_i$ be in $\ker \overline{\delta_1}$, with $a_i \in A_i$ and $e_1, \ldots, e_n$ a basis for $\overline{F}$. Use the coefficient of $e_1 \wedge e_i \wedge e_j$ in $0 = \overline{\delta_1}(f_1)$ in order to see that

$$x_{i,j} a_1 \in (x_{1,i} , x_{1,j}) \subseteq (x_{1,2} , x_{1,3} , \ldots , x_{1,\mathfrak{f}})$$

for all i and j with $2 \le i < j \le \mathfrak{f}$. The ideal $(x_{1,2} , x_{1,3} , \ldots , x_{1,\mathfrak{f}})$ of A is prime (indeed, $A/(x_{1,2} , x_{1,3} , \ldots , x_{1,\mathfrak{f}})$ is the domain defined by "Pf_4" of a smaller generic matrix) and $x_{i,j}$ is not in $(x_{1,2} , x_{1,3} , \ldots , x_{1,\mathfrak{f}})$. Therefore, $a_1 \in (x_{1,2} , x_{1,3} , \ldots , x_{1,\mathfrak{f}})$ and there is an element $\phi_1 \in \overline{F}^*$ with $f_1^\dagger = f_1 - \overline{d_0}(\phi_1) = \sum_{i=2}^{\mathfrak{f}} a_i^\dagger e_i$. (Recall that $-\mathbf{X}$ is the matrix for d_0.) The coefficient of $e_1 \wedge e_2 \wedge e_3$ in $0 = \overline{\delta_1}(f_1^\dagger)$ shows $a_2^\dagger x_{1,3}$ is in the prime ideal $(x_{1,2})$; hence, $a_2^\dagger \in (x_{1,2})$ and one may use the first column of $\mathbf{X}$ to remove $a_2^\dagger$ without damaging $a_1^\dagger = 0$. In other words, there exists $\phi_1^\ddagger \in \overline{F}^*$ with $f_1^\ddagger = f_1 - d_0(\phi_1^\ddagger) = \sum_{i=3}^{\mathfrak{f}} a_i e_i$. The coefficient of $e_1 \wedge e_2 \wedge e_j$ in $0 = \overline{\delta_1}(f_1^\ddagger)$ shows that $x_{1,2} a_j^\ddagger = 0$ for $3 \le j \le \mathfrak{f}$. Hence, $a_j^\ddagger = 0$ for $3 \le j \le \mathfrak{f}$, f_1 is a boundary in (3.14.1), and (3.14.1) is exact.

(b) Apply (a) to see that $N = \operatorname{coker} \overline{d_1} \cong \operatorname{im} \overline{d_0} =$ (1.0.1).

(c) The definition $N = \operatorname{coker} \overline{d_1}$ guarantees that $\bigwedge^3 \overline{F}^* \xrightarrow{\overline{d_1}} \overline{F}^* \to N \to 0$ is exact. Apply $\operatorname{Hom}_A(-, A)$ to learn that

$$0 \to N^* \to \overline{F}^{**} \xrightarrow{\overline{d_1}^*} \bigwedge^3 \overline{F}^{**}$$

is exact. It is easy to see that $\overline{F}^{**} \xrightarrow{\overline{d_1}^*} \bigwedge^3 \overline{F}^{**}$ is isomorphic to $\overline{F} \xrightarrow{\overline{\delta_1}} \bigwedge^3 \overline{F}$. Assertion (a) now gives that $N \cong \ker \overline{\delta_1} \cong \ker \overline{d_1}^* \cong N^*$.

(d) Lemma 3.1, especially (3.12.1), ensures that N is a maximal Cohen–Macaulay A-module. The rank of N is calculated in Lemma 3.3. The self-duality of N is established in (c).

(e) It follows from local duality (or the Auslander–Bridger formula, see, for example, [9, Thms. 1.4.8 and 1.4.9]) that the maximal Cohen–Macaulay module N over the Gorenstein ring A satisfies $\operatorname{Ext}_A^i(N, A) = 0$ for all positive i. So $\mathbb{X} \to N \to 0$ and $0 \to N^* \to \mathbb{X}^*$ are both acyclic. The complexes $\mathbb{X}$ and $\mathbb{X}^*$ may be patched together at $N \cong N^*$ to form the totally acyclic complex $\mathbb{Y}$. $\square$

4. The main result

The main result of the paper is Theorem 4.8 where we prove that J is a perfect Gorenstein ideal of grade $\binom{\mathfrak{f}-2}{2}+2$. We estimate the grade of J in Lemma 4.1 and we use the exact sequence (4.3.1) to estimate the projective dimension of $\mathcal{R}/J$.

Lemma 4.1. *Adopt the language of 2.12 and 2.13 with $3 \le \mathfrak{f}$. If the base ring R_0 is an arbitrary commutative Noetherian ring, then the height of the ideal J satisfies the inequality*

$$\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{ht} J.$$

Remark 4.2. The assertion of Lemma 4.1 is false when $\mathfrak{f}=2$ because in this case J equals $(t_1x_{1,2}\ ,\ t_2x_{1,2})$, which has height 1; see Remark 4.4.e for a continuation of this example. On the other hand, Lemma 4.1 does hold when $\mathfrak{f}=3$; indeed, in this case, J is the ideal generated by the maximal minors of the generic matrix

$$\begin{bmatrix} t_1 & t_2 & t_3 \\ x_{2,3} & -x_{1,3} & x_{1,2} \end{bmatrix};$$

see Remark 4.4.f for a continuation of this example.

Proof. It suffices to replace R_0 with R_0/p for some minimal prime ideal p in R_0 and to prove the result when R_0 is a domain. We use the language of Remark 2.14 and view J as the ideal $\operatorname{Pf}_4(\mathbf{X})+I_1(\mathbf{tX})$ in the ring $\mathcal{R}=R_0[\{x_{i,j}\},\{t_i\}]$. Let P be a prime ideal of $\mathcal{R}$ which contains J. We show

$$\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{ht} P.$$

If $t_1 \in P$, then $I'=\operatorname{Pf}_4(\mathbf{X})+(t_1)$ is a prime ideal of height $\binom{\mathfrak{f}-2}{2}+1$ which is contained in P; furthermore, the first entry of $\mathbf{tX}$ is a non-zero element of $P \setminus I'$. Thus, $\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{ht} P$.

If $t_1 \notin P$, then let $\mathbf{X}'$ be $\mathbf{X}$ with the first column removed, $\mathbf{X}''$ be $\mathbf{X}$ with the first row and first column removed, and I'' be the ideal $\operatorname{Pf}_4(\mathbf{X}'')$. Observe that I'' is a prime ideal of height $\binom{\mathfrak{f}-3}{2}$ (this is where we use the hypothesis that $3 \le \mathfrak{f}$); I'' is contained in P; and the entries of $\mathbf{tX}'$ form a regular sequence on $\mathcal{R}_{t_1}/I''\mathcal{R}_{t_1}$ in $P\mathcal{R}_{t_1}$. It follows that

$$\binom{\mathfrak{f}-2}{2}+2 = \binom{\mathfrak{f}-3}{2}+\mathfrak{f}-1 \le \operatorname{ht} P\mathcal{R}_{t_1} = \operatorname{ht} P. \qquad \square$$

Proposition 4.3. *Adopt the language of 2.12 and 2.13. If $2 \le \mathfrak{f}$ and R_0 is a Cohen–Macaulay domain, then there is an exact sequence of $\mathcal{A}$-modules*:

$$\mathbb{A}:\quad 0 \to \mathcal{A} \xrightarrow{\tau} \mathcal{N} \xrightarrow{\tau(\xi)} \mathcal{A} \to \mathcal{R}/J \to 0. \tag{4.3.1}$$

The map $\tau : \mathcal{A} \to \mathcal{N}$ sends the element 1 of $\mathcal{A}$ to the class of τ in

$$\mathcal{N} = \mathcal{A} \otimes_{\mathcal{R}} \operatorname{coker}\left(\xi : \textstyle\bigwedge^3 \mathcal{F}^* \to \mathcal{F}^*\right).$$

If ϕ_1 is in $\mathcal{F}^*$, then the map $\tau(\xi) : \mathcal{N} \to \mathcal{A}$ sends the class of ϕ_1 in $\mathcal{N}$ to the class of $[\tau(\xi)](\phi_1)$ in $\mathcal{A} = \mathcal{R}/(I \cdot \mathcal{R})$. The map $\mathcal{A} \to \mathcal{R}/J$ is the natural quotient map

$$\mathcal{A} = \mathcal{R}/(I \cdot \mathcal{R}) \to \mathcal{R}/(I \cdot \mathcal{R} + K) = \mathcal{R}/J.$$

The proof of Proposition 4.3 is given in 4.7.

Remarks 4.4.

(a) After we prove Theorem 4.8, we are able to improve Proposition 4.3. In the improved version, R_0 is allowed to be an arbitrary commutative Noetherian ring. See Proposition 5.5.

(b) The exact sequence $0 \to \mathcal{A} \to \mathcal{N} \to J\mathcal{A} \to 0$, which is a consequence of (4.3.1), exhibits $J\mathcal{A}$ as a Bourbaki ideal of $\mathcal{N}$, in the sense of [2,26,4,30].

(c) The map $\tau(\xi)$ of (4.3.1) is well-defined. Indeed, if $\phi_3 \in \bigwedge^3 \mathcal{F}^*$, then $\xi(\phi_3)$ represents 0 in $\mathcal{N}$ and $[\tau(\xi)](\xi(\phi_3))$, which is equal to $\xi^{(2)}(\phi_3 \wedge \tau)$ by (2.8.1) and (2.10.1), is equal to 0 in $\mathcal{A}$.

(d) It is not difficult to see that (4.3.1) is a complex of $\mathcal{A}$-modules.

(e) If $\mathfrak{f} = 2$, then $\mathcal{R} = \mathcal{A}$ and, in the language of Remark 2.14, the complex (4.3.1) is

$$0 \to \mathcal{R} \xrightarrow{\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}} \mathcal{R}^2 \xrightarrow{\begin{bmatrix} -t_2x_{1,2} & t_1x_{1,2} \end{bmatrix}} \mathcal{R} \to \mathcal{R}/(t_1x_{1,2}\ ,\ t_2x_{1,2}) \to 0,$$

which is exact, see Remark 4.2.

(f) If $\mathfrak{f} = 3$, then $\mathcal{R} = \mathcal{A}$ and, in the language of Remark 2.14, the complex (4.3.1) is

$$0 \to \mathcal{R} \xrightarrow{\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}} \frac{\mathcal{R}^3}{\left(\begin{bmatrix} x_{2,3} \\ -x_{1,3} \\ x_{1,2} \end{bmatrix}\right)} \xrightarrow{\begin{bmatrix} -t_2x_{1,2} - t_3x_{1,3} & t_1x_{1,2} - t_3x_{2,3} & t_1x_{1,3} + t_2x_{2,3} \end{bmatrix}} \mathcal{R}$$

$$\to \mathcal{R}/(-t_2x_{1,2} - t_3x_{1,3}\ ,\ t_1x_{1,2} - t_3x_{2,3}\ ,\ t_1x_{1,3} + t_2x_{2,3}) \to 0,$$

which is exact; see Remark 4.2.

(g) In the language of Remarks 2.14 and 2.15, the exact sequence (4.3.1) is equal to

$$0 \to \frac{\mathcal{R}}{\mathrm{Pf}_4(\mathbf{X})} \xrightarrow{\mathbf{t}^{\mathrm{T}}} \frac{\left(\frac{\mathcal{R}}{\mathrm{Pf}_4(\mathbf{X})}\right)^{\mathfrak{f}}}{\operatorname{im} d_1} \xrightarrow{\mathbf{tX}} \frac{\mathcal{R}}{\mathrm{Pf}_4(\mathbf{X})} \to \frac{\mathcal{R}}{\mathrm{Pf}_4(\mathbf{X}) + I_1(\mathbf{tX})} \to 0,$$

where the $\binom{\mathfrak{f}}{3}$ generators of $\operatorname{im} d_1$ are listed in (2.15.2).

Observation 4.5 and Lemma 4.6 are used in the proof of Proposition 4.3.

Observation 4.5. *Retain the hypotheses of Proposition 4.3. The complex (4.3.1) is exact at $\mathcal{R}/J$ and at both copies of $\mathcal{A}$.*

Proof. It is clear that (4.3.1) is exact at $\mathcal{R}/J$ and at the right hand $\mathcal{A}$. We prove that (4.3.1) is exact at the left hand $\mathcal{A}$. Let $r \in \mathcal{R}$ with $r \cdot \tau \equiv \xi(\phi_3) \mod I\mathcal{F}$ for some $\phi_3 \in \bigwedge^3 \mathcal{F}^*$. Apply $r\tau$ to ξ and use Observation 2.10.a to learn that

$$r \cdot \tau(\xi) \equiv [\xi(\phi_3)](\xi) \equiv \phi_3(\xi^{(2)}) \in I\mathcal{F}.$$

It follows that $r \cdot K \subseteq I$. The ideal I is prime and degree considerations show that $K \nsubseteq I$. It follows that $r \in I$. Thus, $\tau : \mathcal{A} \to \mathcal{N}$ is an injection. □

Lemma 4.6. *Adopt the language of 2.12 and 2.13. Let ϕ_1, ϕ_1' be elements of $\mathcal{F}^*$ with the property that the element $\phi_1 \wedge \phi_1'$ is part of a basis for $\mathcal{F}^*$ and let x be the element $\xi(\phi_1 \wedge \phi_1')$ of $\mathcal{R}$. Then the following statements hold.*

(a) *If the base ring R_0 is a commutative Noetherian domain, then the localization $\mathbb{A}_x$ of the complex (4.3.1) at x is isomorphic to*

$$0 \to \mathcal{A}_x \xrightarrow{\begin{bmatrix} -[\tau(\xi)](\phi_1') \\ [\tau(\xi)](\phi_1) \end{bmatrix}} \mathcal{A}_x \oplus \mathcal{A}_x \xrightarrow{\begin{bmatrix} [\tau(\xi)](\phi_1) & [\tau(\xi)](\phi_1') \end{bmatrix}} \mathcal{A}_x \longrightarrow \frac{\mathcal{A}_x}{([\tau(\xi)](\phi_1)\ ,\ \ [\tau(\xi)](\phi_1'))\mathcal{A}_x} \to 0.$$

(b) *If R_0 is a Cohen–Macaulay domain, then the localization $\mathbb{A}_x$ is exact.*

Remark 4.6.2. Once we prove Theorem 4.8, then a much stronger version of Lemma 4.6 is also true, see Proposition 5.5.

Proof. (a) The element x in $\mathcal{R}$ is a non-zero element of $\mathcal{R}_{(1,0)}$. The ideal $I \cdot \mathcal{R}$ of $\mathcal{R}$ is a prime ideal generated by elements of $\mathcal{R}_{(2,0)}$; hence x is a non-zero-divisor in $\mathcal{A} = \mathcal{R}/(I \cdot \mathcal{R})$. Consider the map

$$\mathcal{A}_x \oplus \mathcal{A}_x \longrightarrow \mathcal{N}_x, \tag{4.6.2}$$

which sends $[a_1 \quad a_2]^{\mathrm{T}}$ to the class of $a_1\phi_1 + a_2\phi_1'$. This map is onto because, if $\phi_1'' \in \mathcal{F}$, then the equation

$$0 = \xi(\phi_1 \wedge \phi_1' \wedge \phi_1'') = x \cdot \phi_1'' - \xi(\phi_1 \wedge \phi_1'') \cdot \phi_1' + \xi(\phi_1' \wedge \phi_1'') \cdot \phi_1 \tag{4.6.3}$$

holds in $\mathcal{N}$ (see Observation 2.10.b); and therefore the class of ϕ_1'' in $\mathcal{N}_x$ is in the image of the map (4.6.2). Let (0) be the prime ideal (0) in the domain $\mathcal{A}$ and L be the kernel of (4.6.2). We know from Lemma 3.3 that $\mathcal{N}_{(0)} = \mathcal{A}_{(0)} \oplus \mathcal{A}_{(0)}$; hence $L_{(0)} = 0$. On the other hand, L is a submodule of a free $\mathcal{A}_x$-module and $\mathcal{A}_x$ is a domain; thus, $L = 0$ and (4.6.2) is an isomorphism.

Apply (4.6.3), with τ in place of ϕ_1'', to see that the composition

$$\mathcal{A}_x \xrightarrow{x} \mathcal{A}_x \xrightarrow{\tau} \mathcal{N}_x$$

sends $1 \in \mathcal{A}_x$ to

$$x\tau = [\tau(\xi)](\phi_1) \cdot \phi_1' - [\tau(\xi)](\phi_1') \cdot \phi_1$$

in $\mathcal{N}_x$; and therefore, the composition

$$\mathcal{A}_x \xrightarrow{x} \mathcal{A}_x \xrightarrow{\tau} \mathcal{N}_x \xrightarrow{(4.6.2)^{-1}} \mathcal{A}_x \oplus \mathcal{A}_x$$

sends $1 \in \mathcal{A}_x$ to

$$\begin{bmatrix} -[\tau(\xi)](\phi_1') \\ [\tau(\xi)](\phi_1) \end{bmatrix} \in \mathcal{A}_x \oplus \mathcal{A}_x.$$

It is clear that the composition

$$\mathcal{A}_x \oplus \mathcal{A}_x \xrightarrow{(4.6.2)} \mathcal{N}_x \xrightarrow{\tau(\xi)} \mathcal{A}_x$$

sends

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto [\tau(\xi)](\phi_1) \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto [\tau(\xi)](\phi_1').$$

This completes the proof of (a).

(b) We know from (a) that the ideal $J\mathcal{A}_x$ is generated by $[\tau(\xi)](\phi_1)$ and $[\tau(\xi)](\phi_1')$ and we know from Lemma 4.1 that $2 \le \mathrm{ht}(J\mathcal{A})$. The ring $\mathcal{A}$ is Cohen–Macaulay; so,

$$2 \le \mathrm{ht}(J\mathcal{A}) = \operatorname{grade} J\mathcal{A} \le \operatorname{grade} J\mathcal{A}_x.$$

It follows that $\mathbb{A}_x$, which, according to (a), is isomorphic to the augmented Koszul complex on the generating set $\{[\tau(\xi)](\phi_1)\ ,\ [\tau(\xi)](\phi_1')\}$ of $J\mathcal{A}_x$, is exact. $\square$

4.7. The proof of Proposition 4.3. In light of Remark 4.4.e, we may assume that $4 \le \mathfrak{f}$. We know from Observation 4.5 that (4.3.1) is a complex of $\mathcal{A}$-modules which is exact everywhere except possibly at $\mathcal{N}$. Let H be the homology of (4.3.1) at $\mathcal{N}$. We argue by contradiction. Assume that $\mathrm{H} \neq 0$. Let P be an associated prime of H. Lemma 4.6 shows that $\mathrm{H}_x = 0$ for every x in $\mathcal{R}$ of the form

$$x = \xi(\phi_1 \wedge \phi_1') \text{ where } \phi_1 \text{ and } \phi_1' \text{ are in } \mathcal{F}^* \text{ with } \phi_1 \wedge \phi_1' \text{ part of a basis for } \textstyle\bigwedge^2 \mathcal{F}^*. \quad (4.7.1)$$

The fact that $\mathrm{H}_x = 0$ and $\mathrm{H}_P \neq 0$ forces x to be an element of P. The R_0-module $\mathcal{R}_{(1,0)}$ is generated by elements x of the form (4.7.1); therefore, $\mathcal{R}_{(1,0)} \subseteq P$.

Consider the complex (4.3.1). Let B be the image of $\tau : \mathcal{A} \to \mathcal{N}$ and Z be the kernel of $\tau(\xi) : \mathcal{N} \to \mathcal{A}$. Combine the exact sequences

$$\begin{aligned} &0 \to \mathcal{A} \to B \to 0 && \text{from Observation 4.5, and} \\ &0 \to B \to Z \to \mathrm{H} \to 0 \end{aligned}$$

in order to obtain the exact sequence

$$0 \to \mathcal{A} \to Z \to \mathrm{H} \to 0. \quad (4.7.2)$$

The $\mathcal{R}$-modules $\mathcal{A}$ and $\mathcal{N}$ are both perfect and their annihilators have grade $\binom{\mathfrak{f}-2}{2}$; see Corollary 3.13. The ring $\mathcal{R}$ is Cohen–Macaulay; so, $\mathcal{A}_P$ and $\mathcal{N}_P$ are both Cohen–Macaulay $\mathcal{R}_P$-modules with

$$\operatorname{depth} \mathcal{N}_P = \dim \mathcal{N}_P = \dim \mathcal{A}_P = \operatorname{depth} \mathcal{A}_P;$$

and this common number is equal to $\dim \mathcal{R}_P - \binom{\mathfrak{f}-2}{2}$. Furthermore, the ideal $(\mathcal{R}_{1,0})$ of $\mathcal{R}$, which is prime of height $\binom{\mathfrak{f}}{2}$, is contained in P. It follows that

$$2 \le \binom{\mathfrak{f}}{2} - \binom{\mathfrak{f}-2}{2} \le \dim \mathcal{A}_P.$$

(The left most inequality holds because $3 \le \mathfrak{f}$.) The module Z_P is a non-zero submodule of $\mathcal{N}_P$; so $1 \le \operatorname{depth} Z_P$. We have chosen P with $\mathrm{H}_P \neq 0$ and $\operatorname{depth} \mathrm{H}_P = 0$. The usual argument about the growth of depth in a short exact sequence shows that the exact sequence

$$0 \to \mathcal{A}_P \to Z_P \to \mathrm{H}_P \to 0,$$

which is obtained by localizing the short exact sequence (4.7.2) at P, is impossible; see, for example, [5, Prop. 1.2.9]. This contradiction establishes the result. $\square$

Theorem 4.8. *Adopt the language of* 2.12 *and* 2.13*. If $4 \le \mathfrak{f}$ and R_0 is an arbitrary commutative Noetherian ring, then J is a perfect Gorenstein ideal of $\mathcal{R}$ of grade $\binom{\mathfrak{f}-2}{2}+2$. In particular, if R_0 is a Gorenstein ring, then $\mathcal{R}/J$ is a Gorenstein ring.*

Proof. We employ the theory of generic perfection as described at the beginning of 3.12. It suffices to prove the result when R_0 is equal to the ring of integers and when R_0 is a field. In particular, we may assume that R_0 is a Cohen–Macaulay domain. Proposition 4.3 (see also Remark 4.4.g) guarantees that there exists an exact sequence of $\mathcal{R}$-modules

$$0 \to \mathcal{A} \to \mathcal{N} \to \mathcal{A} \to \mathcal{R}/J \to 0$$

and Corollary 3.13 ensures that $\mathcal{A}$ and $\mathcal{N}$ have free resolutions of length $\binom{\mathfrak{f}-2}{2}$; furthermore, the back Betti number in the resolution of $\mathcal{A}$ is one. Resolve $\mathcal{A}$ and $\mathcal{N}$ and form the iterated mapping cone in order to find a free resolution of $\mathcal{R}/J$ of length $\binom{\mathfrak{f}-2}{2}+2$. The back Betti number in the resolution of $\mathcal{R}/J$ is one. We see that

$$\binom{\mathfrak{f}-2}{2}+2 \le \operatorname{grade} J \le \operatorname{pd}_{\mathcal{R}} \mathcal{R}/J \le \binom{\mathfrak{f}-2}{2}+2.$$

(The first inequality is Lemma 4.1 and the second inequality is (2.5.1).) Thus, equality holds throughout and the proof is complete. □

5. Consequences of the main result

In this section, especially in Corollary 5.3, we prove some consequences of the fact that J is a perfect ideal in $\mathcal{R}$. We begin by identifying some relations on the generators of J. These relations are used in the proof of Corollary 5.3.b that $(\mathcal{R}/J)_{x_{i,j}}$ is a polynomial ring over $R_0[x_{i,j}\ ,\ x_{i,j}^{-1}]$.

Definition 5.1. Adopt the language of 2.12 and 2.13. Define the maps and modules

$$\mathbb{E}_2 \xrightarrow{D_2} \mathbb{E}_1 \xrightarrow{D_1} \mathbb{E}_0$$

by

$$\mathbb{E}_2 = \ker\left(\begin{matrix} \bigwedge^3 \mathcal{F}^* \\ \oplus \\ \mathcal{F}^* \otimes \bigwedge^5 \mathcal{F}^* \\ \oplus \\ \bigwedge^3 \mathcal{F}^* \otimes \bigwedge^3 \mathcal{F}^* \end{matrix} \xrightarrow{\text{multiplication}} \bigwedge^6 \mathcal{F}^* \right), \quad \mathbb{E}_1 = \begin{matrix} \mathcal{F}^* \\ \oplus \\ \bigwedge^4 \mathcal{F}^* \end{matrix}, \quad \mathbb{E}_0 = \mathcal{R},$$

$$D_2\left(\begin{bmatrix} \phi_3 \\ 0 \\ \phi_3' \otimes \phi_3'' \end{bmatrix}\right) = \begin{bmatrix} \xi(\phi_3) \\ \tau \wedge \phi_3 + \xi(\phi_3') \wedge \phi_3'' - \phi_3' \wedge \xi(\phi_3'') \end{bmatrix},$$

$$D_1\left(\begin{bmatrix} \phi_1 \\ \phi_4 \end{bmatrix}\right) = [\tau(\xi)](\phi_1) + \xi^{(2)}(\phi_4),$$

and the middle component of D_2 is induced by the map $F^* \otimes \bigwedge^5 \mathcal{F}^* \to \bigwedge^4 \mathcal{F}^*$ which sends $\phi_1 \otimes \phi_5$ to $[\phi_1(\xi)](\phi_5)$.

Observation 5.2. *The maps and modules of Definition 5.1 form a complex and the image of D_1 is the ideal J of 2.13.*

Proof. We verify that $D_1 \circ D_2 = 0$. We use (2.8.1), (2.10.1), Observation 2.10.a, and the module action of $\bigwedge^\bullet \mathcal{F}$ and $\bigwedge^\bullet \mathcal{F}^*$ on one another to compute

$$\begin{aligned}
(D_1 \circ D_2)(\phi_3) &= D_1\left(\begin{bmatrix} \xi(\phi_3) \\ \tau \wedge \phi_3 \end{bmatrix}\right) = [\tau(\xi)](\xi(\phi_3)) + \xi^{(2)}(\tau \wedge \phi_3) \\
&= [\tau(\xi^{(2)})](\phi_3) - (\phi_3 \wedge \tau)(\xi^{(2)}) = 0, \\
(D_1 \circ D_2)(\sum_i \phi_{1,i} \otimes \phi_{5,i}) &= \sum_i D_1([\phi_{1,i}(\xi)](\phi_{5,i})) = \sum_i \xi^{(2)}([\phi_{1,i}(\xi)](\phi_{5,i})) \\
&= \sum_i [\phi_{1,i}(\xi^{(3)})](\phi_{5,i}) = (\sum_i \phi_{5,i} \wedge \phi_{1,i})(\xi^{(3)}) = 0, \quad \text{and} \\
(D_1 \circ D_2)(\phi_3' \otimes \phi_3'') &= D_1\big(\xi(\phi_3') \wedge \phi_3'' - \phi_3' \wedge \xi(\phi_3'')\big) \\
&= \xi^{(2)}\big(\xi(\phi_3') \wedge \phi_3'' - \phi_3' \wedge \xi(\phi_3'')\big) \\
&= \xi^{(2)}\big(\xi(\phi_3') \wedge \phi_3''\big) - \xi^{(2)}\big(\phi_3' \wedge \xi(\phi_3'')\big).
\end{aligned}$$

Furthermore, we compute

$$\begin{aligned}
\xi^{(2)}\big(\xi(\phi_3') \wedge \phi_3''\big) &= -[\phi_3'' \wedge \xi(\phi_3')](\xi^{(2)}) = -\phi_3''\big([\xi(\phi_3')](\xi^{(2)})\big) = -\phi_3''\big([\xi(\phi_3')](\xi) \wedge \xi\big) \\
&= -\phi_3''\big(\phi_3'(\xi^{(2)}) \wedge \xi\big) = -[\phi_3'(\xi^{(2)})](\xi(\phi_3'')) = -[\xi(\phi_3'') \wedge \phi_3'](\xi^{(2)}) \\
&= [\phi_3' \wedge \xi(\phi_3'')](\xi^{(2)}) = \xi^{(2)}[\phi_3' \wedge \xi(\phi_3'')];
\end{aligned}$$

and therefore, $(D_1 \circ D_2)(\phi_3' \otimes \phi_3'') = 0$. □

Corollary 5.3. *Adopt the language of 2.12, 2.13, and 2.14. Assume that $4 \le \mathfrak{f}$ and R_0 is an arbitrary commutative Noetherian ring. The following statements hold.*

(a) *The elements $x_{1,2}$, $x_{1,3}$ form a regular sequence on $\mathcal{R}/J$.*
(b) *For each pair i, j with $1 \le i < j \le \mathfrak{f}$, the localization of the ring $\mathcal{R}/J$ at the element $x_{i,j}$ is isomorphic to a polynomial ring over $R_0[x_{i,j}\ ,\ x_{i,j}^{-1}]$.*
(c) *The ring $\mathcal{R}/J$ is a domain if and only if R_0 is a domain.*
(d) *If R_0 is a domain, then $(x_{1,2})\mathcal{R}/J$ is a prime ideal in $\mathcal{R}/J$.*
(e) *The ring $\mathcal{R}/J$ is normal if and only if R_0 is normal.*
(f) *If R_0 is a normal domain, then the divisor class group of R_0 is isomorphic to the divisor class group of $\mathcal{R}/J$. In particular, R_0 is a unique factorization domain if and only if $\mathcal{R}/J$ is a unique factorization domain.*

Proof. (a) We employ the method of proof that is described in Corollary 3.10. It suffices to show that

$$\binom{\mathfrak{f}-2}{2}+3 \le \operatorname{grade} PR_P \text{ for all } P \in \operatorname{Spec}\mathcal{R} \text{ with } J+(x_{1,2}) \subseteq P \qquad \text{situation 1, and}$$
$$\binom{\mathfrak{f}-2}{2}+4 \le \operatorname{grade} PR_P \text{ for all } P \in \operatorname{Spec}\mathcal{R} \text{ with } J+(x_{1,2}\ ,\ x_{1,3}) \subseteq P \qquad \text{situation 2}$$

Fix a prime P from situation 1 or situation 2. There are two cases. Assume first that $t_\mathfrak{f} \notin P$. The ring $\mathcal{R}_P$ is a localization of $\mathcal{R}_{t_f}$ and $\mathcal{R}_{t_f}$ is equal to the polynomial ring

$$(R_0[t_1\ ,\ \dots\ ,\ t_\mathfrak{f}\ ,\ t_\mathfrak{f}^{-1}, \{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}-1\}])[(\mathbf{tX})_1\ ,\ \dots\ ,\ (\mathbf{tX})_{\mathfrak{f}-1}].$$

Let $\mathbf{X}'$ represent $\mathbf{X}$ with row and column $\mathfrak{f}$ deleted. Apply Corollary 3.10. In situation 1, the ideal $(x_{1,2}, J)\mathcal{R}_{t_\mathfrak{f}}$ contains the grade $\binom{\mathfrak{f}-3}{2}+1$ ideal $(x_{1,2}\ ,\ \operatorname{Pf}_4(\mathbf{X}')$ of

$$R_0[t_1\ ,\ \dots\ ,\ t_\mathfrak{f}\ ,\ t_\mathfrak{f}^{-1}\ ,\ \{x_{i,j} \mid 1 \le i < j \le \mathfrak{f}-1\}] \tag{5.3.1}$$

as well as the $\mathfrak{f}-1$ indeterminates $(\mathbf{tX})_1, \dots, (\mathbf{tX})_{\mathfrak{f}-1}$. Thus,

$$\binom{\mathfrak{f}-2}{2}+3 = \left(\binom{\mathfrak{f}-3}{2}+1\right)+(\mathfrak{f}-1) \le \operatorname{grade}(x_{1,2}\ ,\ J)\mathcal{R}_{t_\mathfrak{f}}.$$

Similarly, in situation 2, Corollary 3.10 guarantees that

$$\binom{\mathfrak{f}-3}{2}+2 \le \operatorname{grade}(x_{1,2}\ ,\ x_{1,3}\ ,\ \operatorname{Pf}_4(\mathbf{X}')) \cdot (5.3.1);$$

so, $\binom{\mathfrak{f}-2}{2}+4 \le \operatorname{grade}(x_{1,2}\ ,\ x_{1,3}\ ,\ J)\mathcal{R}_{t_\mathfrak{f}}$. The same argument works if $t_{\mathfrak{f}-1} \notin P$. The second case is $t_\mathfrak{f}$ and $t_{\mathfrak{f}-1}$ are both in P. In this case, Corollary 3.10 yields

$$(t_\mathfrak{f}\ ,\ t_{\mathfrak{f}-1})+\operatorname{Pf}_4(\mathbf{X})+(x_{1,2}) \subseteq P \quad \text{and } \binom{\mathfrak{f}-2}{2}+3 \le \operatorname{grade} P \quad \text{in situation 1, and}$$
$$(t_\mathfrak{f}\ ,\ t_{\mathfrak{f}-1})+\operatorname{Pf}_4(\mathbf{X})+(x_{1,2}\ ,\ x_{1,3}) \subseteq P \quad \text{and } \binom{\mathfrak{f}-2}{2}+4 \le \operatorname{grade} P \quad \text{in situation 2.}$$

(b) It is notationally convenient to prove the result for $(i,j)=(1,2)$. Let S_1 and S_2 be the following subsets of $\mathcal{R}$:

$$\begin{aligned} S_1 &= \{x_{i,j} | 1 \le i \le 2\ ,\ 3 \le j \le \mathfrak{f}\} \cup \{t_j \mid 3 \le j \le \mathfrak{f}\} \quad \text{and} \\ S_2 &= \{x_{1,2}x_{i,j} - x_{1,i}x_{2,j} + x_{1,j}x_{2,i} \mid 3 \le i < j \le \mathfrak{f}\} \\ &\quad \cup \Big\{x_{1,2}t_2 + \sum_{j=3}^{\mathfrak{f}} x_{1,j}t_j\ ,\ x_{1,2}t_1 - \sum_{j=3}^{\mathfrak{f}} x_{2,j}t_j\Big\}. \end{aligned} \tag{5.3.2}$$

Notice that

(A) $S_1 \cup S_2$ is a set of indeterminates over the ring $R_0[x_{1,2}\ ,\ x_{1,2}^{-1}]$,
(B) $\big(R_0[x_{1,2}\ ,\ x_{1,2}^{-1}]\big)[S_1 \cup S_2] = \mathcal{R}[x_{1,2}^{-1}]$, and
(C) $J\mathcal{R}[x_{1,2}^{-1}] = (S_2)\mathcal{R}[x_{1,2}^{-1}]$.

Assertions (A) and (B) are obvious. Once (C) is established, we will have shown that

$$(\mathcal{R}/J)_{x_{1,2}} \text{ is the polynomial ring } R_0[x_{1,2}\ ,\ x_{1,2}^{-1}][S_1] \text{ over } R_0[x_{1,2}\ ,\ x_{1,2}^{-1}] \\ \text{for } S_1 \text{ given in (5.3.2).} \tag{5.3.3}$$

We now prove (C). Observe first that $S_2 \subset J$. Indeed, in the language of Observation 5.2 and Remark 2.14, the ideal $S_2\mathcal{R}$ is the image, under D_1, of the submodule

$$W = \mathcal{R}e_1^* \oplus \mathcal{R}e_2^* \oplus \mathcal{R}(e_1^* \wedge e_2^*) \wedge \bigwedge^2 \mathcal{F}^*$$

of $\mathbb{E}_1$. We show that

$$\begin{array}{rcl} x_{1,2}\mathcal{F}^* & \subseteq & W + \operatorname{im} D_2 \\ x_{1,2}\mathcal{R}(e_1^*\ ,\ e_2^*) \wedge \bigwedge^3 \mathcal{F}^* & \subseteq & W + \operatorname{im} D_2, \quad \text{and} \\ x_{1,2} \bigwedge^4 \mathcal{F}^* & \subseteq & W + \mathcal{R}(e_1^*\ ,\ e_2^*) \wedge \bigwedge^3 \mathcal{F}^* + \operatorname{im} D_2. \end{array} \tag{5.3.4}$$

Once (5.3.4) is established, then iteration of (5.3.4) gives $x_{1,2}^2\mathbb{E}_1 \subseteq W + \operatorname{im} D_2$; hence, $x_{1,2}^2 J$ is contained in $S_2\mathcal{R}$ and (C) holds.

If $\phi_1 \in \mathcal{F}^*$, then use Observation 2.10.b to see that

$$\begin{aligned} x_{1,2}\phi_1 &= \xi(e_2^* \wedge e_1^*) \cdot \phi_1 = \xi(\phi_1 \wedge e_2^* \wedge e_1^*) + \xi(\phi_1 \wedge e_1^*) \cdot e_2^* - \xi(\phi_1 \wedge e_2^*) \cdot e_1^* \\ &= D_2(\phi_1 \wedge e_2^* \wedge e_1^*) - \tau \wedge \phi_1 \wedge e_2^* \wedge e_1^* + \xi(\phi_1 \wedge e_1^*) \cdot e_2^* - \xi(\phi_1 \wedge e_2^*) \cdot e_1^* \\ &\in W + \operatorname{im} D_2. \end{aligned}$$

If $\phi_3 \in \bigwedge^3 \mathcal{F}^*$, then

$$\begin{aligned} &x_{1,2}e_1^* \wedge \phi_3 = \xi(e_2^* \wedge e_1^*) \cdot e_1^* \wedge \phi_3 = [e_1^*(\xi)](e_2^* \wedge e_1^* \wedge \phi_3) + \text{an element of } W \\ &= D_2(e_1^* \otimes e_2^* \wedge e_1^* \wedge \phi_3) + \text{an element of } W \in W + \operatorname{im} D_2. \end{aligned}$$

The calculation $x_{1,2}e_2^* \wedge \phi_3 \in W + \operatorname{im} D_2$ is similar.

If $\phi_1 \in \mathcal{F}^*$ and $\phi_3 \in \bigwedge^3 \mathcal{F}^*$, then

$$\begin{aligned} &x_{1,2}\phi_1 \wedge \phi_3 = \xi(e_2^* \wedge e_1^*) \cdot \phi_1 \wedge \phi_3 \\ &= \xi(\phi_1 \wedge e_2^* \wedge e_1^*) \wedge \phi_3 + \xi(\phi_1 \wedge e_1^*) \cdot e_2^* \wedge \phi_3 - \xi(\phi_1 \wedge e_2^*) \cdot e_1^* \wedge \phi_3 \\ &= D_2((\phi_1 \wedge e_2^* \wedge e_1^*) \otimes \phi_3) + \phi_1 \wedge e_2^* \wedge e_1^* \wedge \xi(\phi_3) + \text{an element of } \mathcal{R}(e_1^*\ ,\ e_2^*) \wedge \bigwedge^3 \mathcal{F}^* \\ &\in W + \mathcal{R}(e_1^*\ ,\ e_2^*) \wedge \bigwedge^3 \mathcal{F}^* + \operatorname{im} D_2. \end{aligned}$$

This completes the proof of (5.3.4) and hence the proof of (b).

(c) Apply (a) and then (b) to see that $\mathcal{R}/J$ is a domain if and only if $(\mathcal{R}/J)_{x_{1,2}}$ is a domain if and only if R_0 is a domain.

(d) Suppose α and β are elements of $\mathcal{R}/J$ with $\alpha\beta \in (x_{1,2}) \cdot \mathcal{R}/J$. We know from (b) that $(x_{1,2}) \cdot (\mathcal{R}/J)_{x_{1,3}}$ is a prime ideal; so, one of the elements α or β (say, α) is in $(x_{1,2}) \cdot (\mathcal{R}/J)_{x_{1,3}}$. It follows that $x_{1,3}^s \alpha \in (x_{1,2}) \cdot \mathcal{R}/J$, for some s. Apply (a) to see that α is in $(x_{1,2}) \cdot \mathcal{R}/J$.

(e) ($\Leftarrow$) We apply the Serre criteria for normality in order to prove that $\mathcal{R}/J$ is normal. It suffices to prove that $(\mathcal{R}/J)_P$ is normal for all primes P with $\operatorname{depth}(\mathcal{R}/J)_P \le 1$. If $\operatorname{depth}(\mathcal{R}/J)_P \le 1$, then (a) guarantees that at least one of the elements $x_{1,2}$ or $x_{1,3}$ is not in P. Thus, we know from (b) that $(\mathcal{R}/J)_P$ is a localization of a polynomial ring over R_0; hence, $(\mathcal{R}/J)_P$ is a normal domain.

(e) ($\Rightarrow$) The hypothesis that $\mathcal{R}/J$ is normal guarantees that $\mathcal{R}/J$ is reduced; and therefore, R_0 is reduced. The localization $(\mathcal{R}/J)_{x_{1,2}}$ is also normal. Recall from (5.3.3) that $(\mathcal{R}/J)_{x_{1,2}}$ is equal to $T[x_{1,2}^{-1}]$ where T is the polynomial ring $R_0[x_{1,2}\ ,\ S_1]$ and S_1 is the list of indeterminates given in (5.3.2). Apply Lemma 5.4, with $y = x_{1,2}$, to conclude that T is normal. Now a standard argument yields that R_0 is also normal.

(f) Avramov's proof [3] that $R/\operatorname{Pf}_{2t}(\mathbf{X})$ is a unique factorization domain may be applied without change. In other words, there are isomorphisms of the following divisor class groups:

$$\operatorname{Cl}(\mathcal{R}/J) \xrightarrow{\alpha} \operatorname{Cl}((\mathcal{R}/J)_{x_{1,2}}) \xrightarrow{\beta} \operatorname{Cl}(R_0[S_1\ ,\ x_{1,2}^{-1}]) \xleftarrow{\gamma} \operatorname{Cl}(R_0[S_1]) \xleftarrow{\delta} \operatorname{Cl}(R_0).$$

The element $x_{1,2}$ generates a prime ideal in $\mathcal{R}/J$ by (d); so the isomorphism α is Nagata's Lemma [11, Cor. 7.3]. We proved in (5.3.3) that $(\mathcal{R}/J)_{x_{1,2}}$ is equal to the polynomial ring $R_0[S_1\ ,\ x_{1,2}^{-1}]$, where S_1 is the list of indeterminates given in (5.3.2); so the isomorphism β is the identity map. The isomorphism γ is again Nagata's Lemma and the isomorphism δ is Gauss' Lemma [11, Thm. 8.1]. $\square$

We have used the following normality criterion which appears as [6, Lemma 16.24]. The result follows quickly from Serre's normality criterion.

Lemma 5.4. *Let T be a Noetherian ring, and y be a regular element of T such that T/Ty is reduced and $T[y^{-1}]$ is a normal ring. Then T is a normal ring.*

Now that we know that J is a perfect ideal, we are able to improve some of the results that we used in order to prove that J is perfect. Notice that there are no hypotheses on the ring R_0.

Proposition 5.5. *Adopt the language of 2.12, 2.13 and 2.14. Let R_0 be an arbitrary commutative Noetherian ring.*

(a) *The maps and modules of (4.3.1) form an exact sequence.*
(b) *The maps and modules of (3.3.1) form an exact sequence.*

(c) *The ideal* $J\mathcal{A}_{x_{1,2}}$ *is generated by the regular sequence* $(\mathbf{tX})_1$, $(\mathbf{tX})_2$.
(d) *The element* $x_{1,2}$ *of* R *is regular on both* A *and* N *and* $N_{x_{1,2}} \cong A_{x_{1,2}} \oplus A_{x_{1,2}}$.

Proof. (a) Let $\mathcal{A}$, $\mathcal{N}$, $\mathcal{R}$, and J be the relevant modules built over R_0 and $\mathcal{A}_{\mathbb{Z}}$, $\mathcal{N}_{\mathbb{Z}}$, $\mathcal{R}_{\mathbb{Z}}$, and $J_{\mathbb{Z}}$ be the relevant modules built over $\mathbb{Z}$. We have shown in Proposition 4.3 that

$$0 \to \mathcal{A}_{\mathbb{Z}} \xrightarrow{\tau} \mathcal{N}_{\mathbb{Z}} \xrightarrow{\tau(\xi)} \mathcal{A}_{\mathbb{Z}} \to \mathcal{R}_{\mathbb{Z}}/J_{\mathbb{Z}} \to 0 \tag{5.5.1}$$

is an exact sequence. We know from Corollary 3.13 and Theorem 4.8 that $\mathcal{A}_{\mathbb{Z}}$, $\mathcal{N}_{\mathbb{Z}}$, and $\mathcal{R}_{\mathbb{Z}}/J_{\mathbb{Z}}$ are generically perfect $\mathbb{Z}[X]$-modules in the sense of [6, Prop. 3.2 and Thm. 3.3]; and so, in particular, these modules are flat $\mathbb{Z}$-modules. Apply $R_0 \otimes_{\mathbb{Z}} -$ to the constituent short exact sequences of (5.5.1) in order to learn that $\operatorname{Tor}_1^{\mathbb{Z}}(R_0, J_{\mathbb{Z}}\mathcal{A}_{\mathbb{Z}}) = 0$ and

$$R_0 \otimes_{\mathbb{Z}} (5.5.1),$$

which is isomorphic to (4.3.1), is exact.

(b) The proof from (a) also works for (b) because the $\mathbb{Z}[X]$-modules A and A', built over $\mathbb{Z}$, are also generically perfect, see Lemma 3.7.

(c) The proof of Corollary 5.3.b shows that $\mathcal{A}_{x_{1,2}}$ is equal to the polynomial ring

$$R_0[x_{1,2}\ ,\ x_{1,2}^{-1}][S_1\ ,\ (\mathbf{tX})_1\ ,\ (\mathbf{tX})_2],$$

where S_1 is the list of indeterminates given in (5.3.2); furthermore, $J\mathcal{A}_{x_{1,2}}$ is generated by the two variables $(\mathbf{tX})_1$ and $(\mathbf{tX})_2$.

(d) We saw in Corollary 3.10 that $x_{1,2}$ is regular on A. Recall from Lemmas 3.1 and 3.7 that the ring A and the A-module N are perfect R-modules, and their annihilators (as R-modules) have the same grade. It follows that $\operatorname{Ass} N \subseteq \operatorname{Ass} A$ and that $x_{1,2}$ is also regular on N. The final assertion is obtained by localizing (3.3.1), which is exact by (b), at $x_{1,2}$. $\square$

6. Remarks and questions

The definition of N, as given in Notation 2.13.aiv, is that

$$N = \tfrac{R}{I} \otimes_R \operatorname{coker}(d_1 : \textstyle\bigwedge^3 F^* \to F^*).$$

However, if 2 is a unit in R_0, then the next result shows that it is not necessary to apply the functor $\frac{R}{I} \otimes_R -$.

Observation 6.1. *Adopt the language of* 2.12.a, 2.13.a *and* (2.15.1). *If* 2 *is a unit in* R_0, *then*

$$\operatorname{coker}(d_1 : \textstyle\bigwedge^3 F^* \to F^*)$$

is an R/I module; so, in particular $N = \operatorname{coker}(d_1 : \bigwedge^3 F^* \to F^*)$.

Proof. If $\phi_4 \in \bigwedge^4 F^*$ and $\phi_1 \in F^*$, then

$$\begin{aligned} \xi^{(2)}(\phi_4) \cdot \phi_1 &= [\phi_1(\xi^{(2)})](\phi_4) + \xi^{(2)}(\phi_1 \wedge \phi_4) && \text{Proposition 2.9} \\ &= [\phi_1(\xi) \wedge \xi](\phi_4) + \tfrac{1}{2}\xi(\xi(\phi_1 \wedge \phi_4)) && (2.8.1) \\ &= \xi\Big([\phi_1(\xi)](\phi_4) + \tfrac{1}{2}\xi(\phi_1 \wedge \phi_4)\Big), \end{aligned}$$

which represents 0 in N. $\square$

Remarks 6.2. Adopt the language of 2.12 and 2.13.

(a) The hypothesis "2 is a unit in R_0" is essential in Observation 6.1. For example, if R_0 is the field $\mathbb{Z}/(2)$, then

$$\begin{bmatrix} 0 & 0 & 0 & 0 & x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3} \end{bmatrix}^{\mathrm{T}}$$

is zero in N, but is not in image of d_1. So, in particular, if R_0 is a field, then the first Betti number of N, as a module over R, depends on the characteristic of R_0, even when $\mathfrak{f} = 5$. We recall that the first Betti number of A, as a module, over R depends on the characteristic R_0, but not until $\mathfrak{f} = 8$; see, for example, [19,20,13].

(b) Assume R_0 is a field. Suppose that $\mathbb{F} : \ \dots \to F_i \to \dots$ and $\mathbb{G} : \ \dots \to G_i \to \dots$ are minimal homogeneous resolutions of A and N by free R-modules with $F_i = \bigoplus R(-j)^{\beta_{i,j}}$ and $G_i = \bigoplus R(-j)^{\gamma_{i,j}}$. Then the proof of Theorem 4.8 shows that the minimal bi-homogeneous resolution of $\mathcal{R}/J$ by free $\mathcal{R}$-modules is $\mathbb{L} : \ \dots \to L_i \to \dots$, with

$$L_i = \bigoplus \mathcal{R}(-j-1,-2)^{\beta_{i-2,j}} \oplus \bigoplus \mathcal{R}(-j-1,-1)^{\gamma_{i-1,j}} \oplus \bigoplus \mathcal{R}(-j,0)^{\beta_{i,j}}.$$

Indeed, the iterated mapping cone associated to

$$\begin{tikzcd} \mathcal{R}(-1,-2) \otimes_R \mathbb{F}[-2] \arrow[r] & \mathcal{R}(-1,-2) \otimes_R A \arrow[d, "\tau"] \\ \mathcal{R}(-1,-1) \otimes_R \mathbb{G}[-1] \arrow[r] & \mathcal{R}(-1,-1) \otimes_R N \arrow[d, "\tau(\xi)"] \\ \mathcal{R} \otimes_R \mathbb{F} \arrow[r] & \mathcal{R} \otimes_R A \arrow[d] \\ & \mathcal{R}/J \end{tikzcd}$$

is a bi-homogeneous resolution of $\mathcal{R}/J$ and consideration of the t-degree shows that this resolution is minimal.

(c) Retain the language of (b). If the characteristic of R_0 is zero, then the resolution $\mathbb{F}$ is given in Theorem 6.4.1 and Exercises 31–33 on page 222 in [32]. Can the geometric method of [32] also be used to obtain the minimal homogeneous equivariant resolution of N by free R-modules?

(d) One consequence of (b) is that the minimal homogeneous resolution $\mathbb{G}$ of N by free R-modules is self-dual. Is this fact obvious for some other reason?

(e) Is the resolution $\mathbb{X}$ of N by free A-modules from Observation 3.14.e a linear complex?

Added in proof. We have learned that an alternate approach to some of the ideas in this paper may be found in [17].

References

[1] P. Aluffi, Shadows of blow-up algebras, Tohoku Math. J. (2) 56 (2004) 593–619.
[2] M. Auslander, Remarks on a theorem of Bourbaki, Nagoya Math. J. 27 (1966) 361–369.
[3] L. Avramov, A class of factorial domains, Serdica 5 (1979) 378–379.
[4] J. Brennan, J. Herzog, B. Ulrich, Maximally generated Cohen–Macaulay modules, Math. Scand. 61 (1987) 181–203.
[5] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.
[6] W. Bruns, U. Vetter, Determinantal Rings, Lecture Notes in Math., vol. 1327, Springer-Verlag, Berlin, 1988.
[7] D. Buchsbaum, D. Eisenbud, Generic free resolutions and a family of generically perfect ideals, Adv. Math. 18 (1975) 245–301.
[8] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447–485.
[9] L. Christensen, Gorenstein Dimensions, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000.
[10] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995.
[11] R. Fossum, The Divisor Class Group of a Krull Domain, Ergeb. Math. Grenzgeb., vol. 74, Springer-Verlag, New York, 1973.
[12] T. Gulliksen, G. Levin, Homology of Local Rings, Queen's Papers in Pure and Appl. Math., vol. 20, Queen's University, Kingston, Ont., 1969.
[13] M. Hashimoto, Relations on Pfaffians: number of generators, J. Math. Kyoto Univ. 35 (1995) 495–533.
[14] M. Hochster, Topics in the Homological Theory of Modules Over Commutative Rings, CBMS Reg. Conf. Ser. Math., vol. 24, Amer. Math. Soc., Providence, RI, 1975.
[15] M. Hochster, Grassmannians and their Schubert subvarieties are arithmetically Cohen–Macaulay, J. Algebra 25 (1973) 40–57.
[16] C. Huneke, B. Ulrich, Divisor class groups and deformations, Amer. J. Math. 107 (1985) 1265–1303.
[17] N. Ilten, C. Turo, Unobstructed Stanley–Reisner degenerations for dual quotient bundles on $G(2, n)$, J. Pure Appl. Algebra 221 (2017) 119–134.
[18] H. Kleppe, D. Laksov, The algebraic structure and deformation of Pfaffian schemes, J. Algebra 64 (1980) 167–189.
[19] K. Kurano, Relations on Pfaffians. I. Plethysm formulas, J. Math. Kyoto Univ. 31 (1991) 713–731.
[20] K. Kurano, Relations on Pfaffians. II. A counterexample, J. Math. Kyoto Univ. 31 (1991) 733–742.
[21] A. Kustin, The minimal free resolutions of the Huneke–Ulrich deviation two Gorenstein ideals, J. Algebra 100 (1986) 265–304.
[22] A. Kustin, The deviation two Gorenstein rings of Huneke and Ulrich, in: Commutative Algebra, Trieste, 1992, World Sci. Publ., River Edge, NJ, 1994, pp. 140–163.

[23] A. Kustin, Huneke–Ulrich almost complete intersections of Cohen–Macaulay type two, J. Algebra 174 (1995) 373–429.
[24] A. Kustin, C. Polini, B. Ulrich, A matrix of linear forms which is annihilated by a vector of indeterminates, J. Algebra 469 (2017) 120–187.
[25] D. Laksov, The arithmetic Cohen–Macaulay character of Schubert schemes, Acta Math. 129 (1972) 1–9.
[26] M. Miller, Bourbaki's theorem and prime ideals, J. Algebra 64 (1980) 29–36.
[27] C. Musili, Postulation formula for Schubert varieties, J. Indian Math. Soc. (N.S.) 36 (1972) 143–171.
[28] A. Nejad, A. Simis, The Aluffi algebra, J. Singul. 3 (2011) 20–47.
[29] Z. Ramos, A. Simis, An analogue of the Aluffi algebra for modules, preprint, 2015.
[30] A. Simis, B. Ulrich, W. Vasconcelos, Rees algebras of modules, Proc. Lond. Math. Soc. (3) 87 (2003) 610–646.
[31] H. Srinivasan, Minimal algebra resolutions for cyclic modules defined by Huneke–Ulrich ideals, J. Algebra 137 (1991) 433–472.
[32] J. Weyman, Cohomology of Vector Bundles and Syzygies, Cambridge Tracts in Math., vol. 149, Cambridge University Press, Cambridge, 2003.