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Max Noether's Theorem for integral curves



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ABSTRACT

We study a celebrated result of Max Noether on global sections of the n -dualizing sheaf of a smooth nonhyperelliptic curve in the case where the curve is integral. We reduce the proof of the statement in such a case to a purely numerical condition, which we show that holds if the non-Gorenstein points are bibranch at worst. This is our main result. We also extend the notion of a canonical embedding for integral curves with unibranch non-Gorenstein points at worst, in a way that we can express the dimensions of the components of the ideal in terms of the main invariants of the curve as well. Afterwards we focus on gonality, Clifford index and Koszul cohomology of non-Gorenstein curves by allowing torsion free sheaves of rank 1 in their definitions. We find an upper bound for the gonality, which agrees with Brill–Noether's one for a rational and unibranch curve. We characterize curves of genus 5 with Clifford index 1, and, finally, we study Green's conjecture for a certain class of curves, called nearly Gorenstein.

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1. Introduction

In 1880, Max Noether established in [27] a remarkable result which, in modern language, can be stated as follows.

Max Noether’s Theorem. *If C is a smooth, nonhyperelliptic curve which is complete over an algebraic closed field, and ω its dualizing sheaf, then the maps*

$$\mathrm{Sym}^n H^0(C, \omega) \longrightarrow H^0(C, \omega^n) \quad (1)$$

are surjective for all $n \geq 1$.

One of its first applications, just taking $n = 2$, is that a smooth canonical curve which is not trigonal or a quintic on the plane is the set-theoretic intersection of $(g-2)(g-3)/2$ linearly independent quadrics, where g is the genus of C . In the late 1910’s Enriques [15] proved that a canonical nonhyperelliptic curve is the intersection of quadrics, unless the smooth curve is trigonal or isomorphic to a plane quintic (a result also proved by Babbage in [4]). A complete description of the canonical ideal in terms of equations, based on Noether’s dimension counts and following Enriques’ division into cases, was done by Petri [28] in the early 1920’s, as presented in [2, p. 131]. Later on, new approaches to Petri’s analysis were carried out by Arbarello–Sernesi [3], Mumford [26], Saint-Donat [31], Shokurov [33] and Stoehr–Viana [34].

A proof of Max Noether’s Theorem can also be found in [2, p. 117], where one can note that it is a consequence of projective normality of extremal (Castelnuovo) curves. Indeed, extremal curves are projectively normal, which is a general fact proved in [2, pp. 113–117] for smooth curves. But since Riemann–Roch and Clifford’s Theorems have versions for singular curves ([14, App.], [30, pp. 186–191], [35, p. 108]), the same proof holds for all integral curves as well. So if we assume that C is Gorenstein, i.e., its dualizing sheaf ω is invertible, then ω defines a morphism $\kappa : C \rightarrow \mathbb{P}^{g-1}$. Let $C' := \kappa(C)$ be the canonical model of C . Based on his Ph. D. thesis under Zariski, Rosenlicht proved in [30] that C' is extremal and that κ is an isomorphism if C is nonhyperelliptic. Therefore Max Noether’s Theorem holds actually for all Gorenstein nonhyperelliptic curves. An application of Max Noether’s Theorem for Gorenstein curves, and also Petri’s analysis, was pointed out by Mumford in [26] to construct certain moduli spaces of curves with prescribed Weierstrass semigroup, which was done by Stoehr in [36], reproved and explored by him and the second named author in [10].

Although Max Noether’s statement is purely intrinsic, its proof for smooth, and more generally Gorenstein, curves is not. As said above, the result is a straightforward consequence of the fact that canonical curves are extremal, a strongly extrinsic argument. On the other hand, if the concern is proving for general integral curves, it is likely difficult to avoid some hard local algebra. A step forward was done by Rosenlicht in his main theorem [30, Cor. and Thm. 17, p. 189] in the late 1950’s. In order to state it, one

needs to extend a notion just introduced above, i.e., if \overline{C} is the normalization of C and $\overline{\kappa} : \overline{C} \rightarrow \mathbb{P}^{g-1}$ the morphism induced by the dualizing sheaf of C , one calls $C' := \overline{\kappa}(\overline{C})$ the *canonical model* of C .

Rosenlicht’s Theorem. *Let C be an integral, nonhyperelliptic curve, which is complete over an algebraic closed field. Then there exists a birational morphism*

$$C' \longrightarrow C \quad (2)$$

which is an isomorphism if and only if C is Gorenstein.

For the sake of simplicity, we will refer to the surjectivity of the morphisms in (1) as “Max Noether’s statement”, and to the existence of a birational morphism like (2) as “Rosenlicht’s statement”, no matter the hypotheses on the curve are.

According to [22, Int.], the Gorenstein part of the result, i.e., the isomorphism $C \cong C'$, was, later on, successively reproved by several authors in many different ways: Deligne–Mumford [11] in 1969, Mumford–Saint-Donat [25] in 1973, Sakai [32] in 1977, Catanese [9, p. 51] in 1982, Fujita [19, p. 39, Thm. (A1)] in 1983, and Hartshorne [21, Thm. 1.6, p. 379] in 1986. More recently, Rosenlicht’s Theorem was also reproved within a modern language and refined version by Kleiman, with the third named author, in [22]: if \widehat{C} is the blowup of C along its dualizing sheaf, then $\widehat{C} \cong C'$.

A connection between Max Noether’s and Rosenlicht’s statements in the general integral case appears in [23, Rem. 2.8], where it is proved that the former implies the latter. Therefore, since Noether’s statement is stronger, the technique of computing values of differentials, which is the core of Rosenlicht’s proof, becomes even harder if applied to prove Max Noether’s Theorem. An attempt of doing so was made in [26, Thm. 3.7], but it assumes all non-Gorenstein points are unibranch, which simplifies the combinatorial part of the proof. Removing this hypothesis is exactly what we do here. The significant effort took to pass from the unibranch to the multibranch case can be seen in Lemmas 3.1 and 3.2. For instance, the content of the former holds for a unibranch non-Gorenstein point by its very definition. So we derive Max Noether’s statement for any integral curve up to a condition on the semigroups of its non-Gorenstein points (see Lemma 3.1), the first part of which we prove in general, and the second for cusps and nodes. As a consequence, we get the following result.

Theorem 1. *Let C be an integral, nonhyperelliptic curve, which is complete over an algebraically closed field, and ω its dualizing sheaf. If the non-Gorenstein points of C are bibranch at worst, then the homomorphisms*

$$\mathrm{Sym}^n H^0(C, \omega) \longrightarrow H^0(C, \omega^n)$$

are surjective for $n \geq 1$.

Max Noether's Theorem also corresponds positively to the case $p = 0$ of Green's famous conjecture on canonical curves. This naturally led us to the study of Koszul cohomology and Clifford index, allowing torsion free sheaves of rank 1 in their definitions. In [Theorem 4.1](#) we relate the following five conditions: (i) C is nonhyperelliptic; (ii) Rosenlicht's statement holds; (iii) Max Noether's statement holds; (iv) $K_{0,2}(C, \omega) = 0$; (v) $\text{Cliff}(C) > 0$ or C is rational nearly normal (see [Definition 2.1](#)).

As mentioned above, once Max Noether's Theorem is introduced, it is natural to compute the dimensions of the homogeneous components of the ideal of the canonical curve. It can be trivially read off from Noether's result. Our version for this in the case of certain non-Gorenstein curves is the following result, obtained by blowdown procedures, as can be checked from its proof.

Theorem 2. *Let C be an integral nonhyperelliptic curve of genus g which is complete over an algebraically closed field. Let also $\pi : \overline{C} \rightarrow C$ be the normalization map, and $\tilde{\pi} : \tilde{C} \rightarrow C$ be the partial desingularization of the non-Gorenstein points. Set $\mathcal{O} := \mathcal{O}_C$, the structure sheaf, $\overline{\mathcal{O}} := \pi_*(\mathcal{O}_{\overline{C}})$ and $\tilde{\mathcal{O}} := \tilde{\pi}_*(\mathcal{O}_{\tilde{C}})$. Assume C has n non-Gorenstein points which, if any, are all unibranch. Then there exists an embedding $\kappa : C \hookrightarrow \mathbb{P}^{g+2\rho+n+\varepsilon-2}$ such that, for $r \geq 2$,*

$$\dim(I_r(C)) = \binom{r+g+2\rho+n+\varepsilon-2}{r} + g(1-2r) - r(2\rho+n-2) - 1$$

where $\rho = h^0(\mathcal{O}/\mathcal{H}\text{om}(\overline{\mathcal{O}}, \mathcal{O})) - h^0(\tilde{\mathcal{O}}/\mathcal{H}\text{om}(\overline{\mathcal{O}}, \tilde{\mathcal{O}}))$, the parameter $\varepsilon = 1$ if C is Gorenstein, and $\varepsilon = 0$ otherwise. In particular,

$$\dim(I_2(C)) = \frac{g^2 + (4\rho + 2n + 2\varepsilon - 7)g + (4\rho(\rho - n + 1) + n(n - 5) + 6)}{2}.$$

If C is Gorenstein, then κ corresponds to the canonical embedding and, since $\varepsilon = 1$ and $\rho = n = 0$, we have $\dim(I_2(C)) = (g^2 - 5g + 6)/2 = (g - 2)(g - 3)/2$.

As we did for Clifford index and Koszul cohomology, we also allow torsion free sheaves of rank 1 in the definition of gonality. From a geometric perspective, this corresponds to replace morphisms by pencils. Our results concerning these three concepts are summarized in the following statement.

Theorem 3. *Let C be a non-Gorenstein integral curve of genus g which is complete over an algebraically closed field.*

(i) *It holds*

$$2 \leq \text{gon}(C) \leq g$$

and if the upper bound is attained, then C is Kunz with only one non-Gorenstein point, and either C is rational or \overline{C} is elliptic.

(ii) If C is rational with a unique non-Gorenstein point, which is unibranch, then

$$\text{gon}(C) \leq \left\lfloor \frac{g+3}{2} \right\rfloor.$$

(iii) For $g = 5$, $\text{Cliff}(C) = 1$ if and only if C is trigonal or there exists a torsion free sheaf \mathcal{F} of rank 1 on C such that $\deg(\mathcal{F}) = 5$ and $h^0(\mathcal{F}) = 3$.

(iv) If C is nearly Gorenstein, then $K_{p,2}(C, \omega) = 0$ for every $p < \eta$. Moreover, there exists a family of curves $\{C_p\}_{p \geq 1}$ such that $\text{Cliff}(C_p) = 1$ and $K_{p,2}(C_p, \omega) = 0$.

The terms *nearly Gorenstein*, *Kunz* and the parameter η used above are introduced in Definition 2.1; their relevance appears right after in Remark 2.2. As the equivalence (iv) \Leftrightarrow (v) of Theorem 4.1 shows that Green's assertion fails to hold if $p = 0$, the family constructed in the item (iv) above shows that it fails to hold for arbitrary $p \geq 1$ as well.

2. Preliminaries

Let C be a complete integral curve of arithmetic genus g defined over an algebraically closed field with structure sheaf \mathcal{O}_C , or simply \mathcal{O} . A *linear system of dimension r on C* is a set of the form

$$\mathcal{L} = \mathcal{L}(\mathcal{F}, V) := \{x^{-1}\mathcal{F} \mid x \in V \setminus \{0\}\}$$

where \mathcal{F} is a coherent fractional ideal sheaf on C and V is a vector subspace of $H^0(\mathcal{F})$ of dimension $r + 1$.

The notion of linear systems on curves presented here is characterized by interchanging bundles by torsion free sheaves of rank 1. This is a meaningful approach since they may possess *non-removable* base points, see for instance M. Coppens' [12].

The *degree* of the linear system \mathcal{L} is the integer $d := \deg \mathcal{F} := \chi(\mathcal{F}) - \chi(\mathcal{O})$, where χ denotes the Euler characteristic. Note, in particular, that if $\mathcal{O} \subset \mathcal{F}$ then

$$\deg \mathcal{F} = \sum_{P \in C} \dim(\mathcal{F}_P / \mathcal{O}_P).$$

The notation g_d^r stands for a linear system of degree d and dimension r . The linear system is said to be *complete* if $V = H^0(\mathcal{F})$, in this case one simply writes $\mathcal{L} = |\mathcal{F}|$.

Recall that a point $P \in C$ is *Gorenstein* if the stalk ω_P is a free \mathcal{O}_P -module, where ω stands for the dualizing sheaf on C . The curve C is said to be *Gorenstein* if all of its points are Gorenstein, or equivalently, if ω is invertible.

According to E. Ballico's [5, p. 363, Dfn. 2.1 (3)], the gonality of C is the smallest d for which there exists a g_d^1 on C , or equivalently, a torsion free sheaf \mathcal{F} of rank 1 on C

with degree d and $h^0(\mathcal{F}) \geq 2$. A geometric motivation for this definition can be found, for instance, in [29] for Gorenstein curves and [17, Ex. 2.2] for non-Gorenstein ones.

Given a sheaf \mathcal{G} on C , if $\varphi : \mathcal{X} \rightarrow C$ is a morphism from a scheme \mathcal{X} to C , we set $\mathcal{O}_{\mathcal{X}}\mathcal{G} := \varphi^*\mathcal{G}/\text{Torsion}(\varphi^*\mathcal{G})$. For each coherent sheaf \mathcal{F} on C we set $\mathcal{F}^n := \text{Sym}^n\mathcal{F}/\text{Torsion}(\text{Sym}^n\mathcal{F})$. In particular, if \mathcal{F} is invertible then clearly $\mathcal{F}^n = \mathcal{F}^{\otimes n}$.

Consider the normalization map $\pi : \overline{C} \rightarrow C$. In [30, p. 188 top] Rosenlicht showed that the linear system $\mathcal{L}(\mathcal{O}_{\overline{C}}\omega, H^0(\omega))$ is base point free. He then considered the induced morphism $\psi : \overline{C} \rightarrow \mathbb{P}^{g-1}$ and called its image $C' := \psi(\overline{C})$ the canonical model of C . Rosenlicht also proved [30, Thm. 17] that if C is nonhyperelliptic, then the map $\pi : \overline{C} \rightarrow C$ factors through a map $\pi' : C' \rightarrow C$. Set $\mathcal{O}' := \pi'_*(\mathcal{O}_{C'})$ in this case.

Let $\widehat{C} := \text{Proj}(\oplus \omega^n)$ be the blowup of C along ω and $\widehat{\pi} : \widehat{C} \rightarrow C$ be the natural morphism. Set $\widehat{\mathcal{O}} = \widehat{\pi}_*(\mathcal{O}_{\widehat{C}})$ and $\widehat{\mathcal{O}}\omega := \widehat{\pi}_*(\mathcal{O}_{\widehat{C}}\omega)$. In [22, Dfn. 4.9] one finds another characterization of the canonical model C' : it is the image of the morphism $\widehat{\psi} : \widehat{C} \rightarrow \mathbb{P}^{g-1}$ defined by the linear system $\mathcal{L}(\mathcal{O}_{\widehat{C}}\omega, H^0(\omega))$. By Rosenlicht's Theorem, since ω is generated by global sections, one deduces that $\widehat{\psi} : \widehat{C} \rightarrow C'$ is an isomorphism if C is nonhyperelliptic [22, Thm. 6.4].

The sheaf $\overline{\mathcal{O}}\omega := \pi_*(\mathcal{O}_{\overline{C}}\omega)$ can be generated by the global sections of ω , see [30, p. 188 top]. Since there are only a finite number of singular points on C and the ground field is infinite, it follows that there is a differential $\zeta \in H^0(\omega)$ such that $(\overline{\mathcal{O}}\omega)_P = \zeta \cdot \overline{\mathcal{O}}_P$ for every singular point $P \in C$, where $\overline{\mathcal{O}} := \pi_*(\mathcal{O}_{\overline{C}})$. This leads us naturally to the quotient

$$\mathcal{W} = \mathcal{W}_{\zeta} := \omega/\zeta.$$

Letting $\mathcal{C} := \mathcal{H}\text{om}(\overline{\mathcal{O}}, \mathcal{O})$ be the conductor of $\overline{\mathcal{O}}$ into \mathcal{O} , there is a chain of inclusions

$$\mathcal{C}_P \subset \mathcal{O}_P \subset \mathcal{W}_P \subset \widehat{\mathcal{O}}_P = \mathcal{O}'_P \subset \overline{\mathcal{O}}_P$$

for every singular point $P \in C$, where the third inclusion can be deduced, for instance, from the proof of [22, Lem. 6.1] and the equality makes sense if and only if C is nonhyperelliptic.

Definition 2.1. Let $P \in C$ be any point. Set

$$\eta_P := \dim(\mathcal{W}_P/\mathcal{O}_P) \qquad \mu_P := \dim(\widehat{\mathcal{O}}_P/\mathcal{W}_P)$$

and also

$$\eta := \sum_{P \in C} \eta_P \qquad \mu := \sum_{P \in C} \mu_P$$

Following [8, pp. 418, 433, Prps. 21, 28] call P *Kunz* if $\eta_P = 1$ and, accordingly, say C is *Kunz* if all of its non-Gorenstein points are Kunz. Following [22, Dfn. 5.7] call C

nearly Gorenstein if $\mu = 1$. Finally, following [22, Dfn. 2.15] call C *nearly normal* if $h^0(\mathcal{O}/\mathcal{C}) = 1$.

Remark 2.2. The importance of the concepts above are summarized below:

- (i) C is nearly Gorenstein if and only if it is non-Gorenstein and C' is projectively normal, owing to [22, Thm. 6.5].
- (ii) C is nearly normal if and only if C' is arithmetically normal, owing to [22, Thm. 5.10].
- (iii) P is Gorenstein iff $\eta_P = \mu_P = 0$, and P is non-Gorenstein iff $\eta_P, \mu_P > 0$ owing to [8, p. 438 top]. Besides, if $\eta_P = 1$ then $\mu_P = 1$, owing to [8, Prp. 21]. In particular, a Kunz curve with only one non-Gorenstein point is as close to being Gorenstein as it gets.

Let $P \in C$ be a point and $\overline{P}_1, \dots, \overline{P}_s$ the points in \overline{C} over P . We say that P is *monomial* provided that the completion $\widehat{\mathcal{O}}_P = k[[t_1^{n_{11}} \dots t_s^{n_{s1}}, \dots, t_1^{n_{1r}} \dots t_s^{n_{sr}}]]$, where t_1, \dots, t_s are local parameters at $\overline{P}_1, \dots, \overline{P}_s$.

Finally, we also recall the concepts of Clifford index, and Koszul cohomology, applied here for curves in a slightly broader sense than usual. Let \mathcal{F} be a torsion free sheaf of rank 1 on C . According to [5, p. 363, Dfn. 2.2 (7)], we introduce the Clifford Index C as:

$$\text{Cliff}(C) = \min\{\deg \mathcal{F} - 2(h^0(\mathcal{F}) - 1); h^0(\mathcal{F}) \geq 2 \text{ and } h^1(\mathcal{F}) \geq 2\}.$$

According to [1,20], consider the complex

$$\wedge^{p+1} H^0(\mathcal{F}) \otimes H^0(\mathcal{F}^{q-1}) \xrightarrow{\phi_{p,q}^1} \wedge^p H^0(\mathcal{F}) \otimes H^0(\mathcal{F}^q) \xrightarrow{\phi_{p,q}^2} \wedge^{p-1} H^0(\mathcal{F}) \otimes H^0(\mathcal{F}^{q+1}).$$

The quotient

$$K_{p,q}(C, \mathcal{F}) := \ker(\phi_{p,q}^2) / \text{im}(\phi_{p,q}^1)$$

is the (p, q) -th Koszul cohomology of \mathcal{F} .

We recall Green's conjecture for smooth curves [20]:

$$K_{p,2}(C, \omega) = 0 \Leftrightarrow p < \text{Cliff}(C).$$

One can find a deep study of the whole problem, for instance, in [1]. The conjecture was proved for general regular curves by C. Voisin in [37,38] and even for a class of singular ones as can be seen, for example, in [6,18] and references therein.

We will see later on, when dealing with Green's conjecture, that we need to allow torsion free sheaves of rank 1 on this definition since ω is not a bundle if C is non-Gorenstein.

3. Max Noether theorem

In this section, as said in the Introduction, we deduce Max Noether's statement for any integral curve up to a numerical condition (see [Lemma 3.1](#)), the first part of which we prove in general, and the second for non-Gorenstein points which are bibranch at worst. In particular, we get [Theorem 1](#) as a consequence. We first recall some basic material on valuations.

Let $P \in C$ be a point and $\overline{P}_1, \dots, \overline{P}_s$ the points on \overline{C} over P . Given any non identically null meromorphic function $x \in k(C)$ the *order of x at P* is the s -tuple of integers $v_P(x) := (v_{\overline{P}_1}(x), \dots, v_{\overline{P}_s}(x)) \in \mathbb{Z}^s$, where $v_{\overline{P}_i}$ is the valuation of the discrete valuation ring $\mathcal{O}_{\overline{C}, \overline{P}_i}$. Similarly, given any differential $\lambda \in \Omega_{k(C)/k}$, we define the s -tuple $v_P(\lambda)$ in the same way.

The *semigroup of values* of P is $S := v_P(\mathcal{O}_P)$. Since \mathcal{O}_P is a ring, S is a sub-semigroup of \mathbb{Z}^s , i.e., it is closed under addition and the zero-element $(0, \dots, 0)$ belongs to S . Additionally, one can verify the following: let $a = (a_1, \dots, a_s)$ and $b = (b_1, \dots, b_s)$ be elements in \mathbb{N}^s ; then

- (i) if $a, b \in S$ then $\min(a, b) := (\min(a_1, b_1), \dots, \min(a_s, b_s)) \in S$.
- (ii) if $a, b \in S$ and $a_i = b_i$ then there exists $\varepsilon \in S$ such that $\varepsilon_i > a_i = b_i$ and $\varepsilon_j \geq \min(a_j, b_j)$ with equality if and only if $a_j \neq b_j$.

We also distinguish the following elements of S

$$\alpha := \min(S \setminus \{0\}) \quad \text{and} \quad \beta := \min(v(\mathcal{C}_P)).$$

The partial order we consider here is the natural one: $a \leq b$ if and only if $a_i \leq b_i \forall i$. Note that the elements α and β are well defined.

Now given any $a := (a_1, \dots, a_s) \in \mathbb{Z}^s$ we denote

$$|a| := a_1 + \dots + a_s$$

and given any subset E of \mathbb{Z}^n one defines

$$\begin{aligned} \Delta^E(a) &:= \{b \in E \mid b_i = a_i \text{ for some } i, \text{ and } b_j > a_j \text{ if } j \neq i\}, \\ E^* &:= \{a \in E \mid a \leq \beta\} \quad \text{and} \quad E^\circ := \{a \in E \mid a < \beta\}. \end{aligned}$$

The *Frobenius vector* of S is $\gamma := \beta - (1, \dots, 1)$ and one defines

$$K = K_P := \{a \in \mathbb{Z}^s \mid \Delta^S(\gamma - a) = \emptyset\}.$$

The importance of K will soon become apparent.

Lemma 3.1. *Let $P \in C$ be an s -branch non-Gorenstein point with semigroup of values S , then there exist $d_1, d_2 \in K^\circ$ such that*

- (i) $d_1 + d_2 = \beta - e_\ell$ for some $\ell \in \{1, \dots, s\}$, where $\{e_1, \dots, e_s\}$ is the canonical basis of \mathbb{N}^s . In particular, $d_1, d_2 \notin S$.
- (ii) If m is the largest integer such that $(m+1)\alpha \leq \beta$ and m_i the largest integer such that $d_i + m_i\alpha < \beta$ for $i = 1, 2$; and if $s \leq 2$, then

$$m_1 + m_2 \geq m.$$

Proof. First of all note that if P is unibranch, the existence of such d_1, d_2 satisfying (i) is equivalent to P being a non-Gorenstein point. Note also that none of them can be in S otherwise $\beta - e_\ell \in K$ which cannot happen since $\Delta^K(\gamma) = \emptyset$. Now suppose P has s -branches. Take $d \in K^\circ \setminus S$ minimal, i.e., such that there is no element in $K^\circ \setminus S$ smaller than d . We will show that $\beta - d - e_\ell \in K^\circ$ for some ℓ . In order to prove that, it is enough to show that, for some $\ell = 1, \dots, s$, we have

$$\Delta^S(\gamma - (\beta - d - e_\ell)) = \Delta^S(d - (1, \dots, 1, 0, 1, \dots, 1)) = \emptyset,$$

where 0 is in the ℓ -th coordinate. Suppose, for the sake of argument, that for every $\ell \in \{1, \dots, s\}$ there exists an element $b_\ell \in \Delta^S(d - (1, \dots, 0, \dots, 1))$. In particular, $b_\ell \in S$. Now, such an element is necessarily one of two types:

- (I) b_ℓ is such that $b_{\ell\ell} = d_\ell$ and $b_{\ell i} \geq d_i$ for $i \neq \ell$, i.e.,

$$b_\ell = (d_1 + x_1, \dots, d_\ell, \dots, d_s + x_s),$$

with $x_i \geq 0$ not simultaneously zero since $d \notin S$.

- (II) b_ℓ is such that $b_{\ell\ell} > d_\ell$; besides, $b_{\ell j} = d_j - 1$ for some $j \neq \ell$; and also $b_{\ell i} \geq d_i$ for $i \neq \ell, j$, i.e.,

$$b_\ell = (d_1 + x_1, \dots, d_j - 1, \dots, d_s + x_s),$$

with $x_\ell > 0$ and $x_i \geq 0$ for $i \neq \ell, j$.

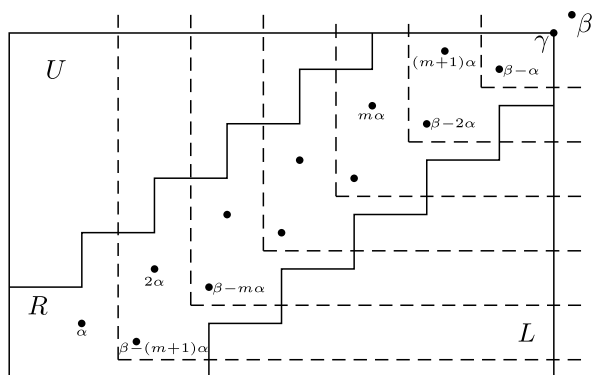
Now, from [7, Prp. 2.14.(i)], we have that K also satisfies properties (i) and (ii) of semigroups mentioned above. So, in the latter case, let

$$s_\ell = \min(b_\ell, d) = (d_1, \dots, d_j - 1, \dots, d_s) \in K^\circ.$$

Because d is minimal, we have $s_\ell \in K^\circ \cap S$. As b_ℓ and s_ℓ have the same j -th coordinate, there is an element $s'_\ell = (d_1 + y_1, \dots, d_\ell, \dots, d_s + y_s) \in S$, with $y_i \geq 0$ not simultaneously zero since $d \notin S$.

Define

$$c_\ell = \begin{cases} b_\ell & \text{if } b_\ell \text{ is of type (I)} \\ s'_\ell & \text{if } b_\ell \text{ is of type (II)}. \end{cases}$$

Fig. 1. Partition of Γ .

Therefore, $d = \min(c_1, \dots, c_s) \in S$ and we have a contradiction. Take $d_1 := d$ and $d_2 := \beta - d - e_\ell$ and (i) is proved.

To prove (ii), we start by analyzing the general case where $s \in \mathbb{N}$. Consider the block

$$B = \{a \in \mathbb{N}^s \mid a < \beta - m\alpha\}$$

and the region

$$R = \left(\bigcup_{n \in \mathbb{N}} (n\alpha + B) \right) \cap \Gamma$$

where $\Gamma := \{a \in \mathbb{N}^s \mid a \leq \gamma\}$.

We claim that if a vector $d_1 \in R$ then for any $i \in \{1, \dots, s\}$, we have that the vector $d_2 := \beta - d_1 - e_i$ is such that $m_1 + m_2 \geq m$ where the m_i are defined in (ii). We will proceed by induction on the n such that $d_1 \in n\alpha + B$. If $n = 0$, then clearly $m_1 \geq m$ since $d_1 + m\alpha < \beta - m\alpha + m\alpha = \beta$ and hence $m_1 + m_2 \geq m$. Now assume the result holds for $n - 1$, then set $d'_1 := d_1 - \alpha$ and $d'_2 := \beta - d'_1 - e_i$. By the induction hypothesis, if m'_j is the largest integer such that $d'_j + m'_j\alpha < \beta$ then $m'_1 + m'_2 \geq m$. Set $d_2 := d'_2 - \alpha$. We have that $d_1 + d_2 = \beta - e_i$, $m_1 = m'_1 - 1$ and $m_2 = m'_2 + 1$. Thus $m_1 + m_2 \geq m$ and the claim follows.

Note that R is as smaller as β tends to be a multiple of α , in which case $B = \{a \in \mathbb{N}^s \mid a < \alpha\}$ is the smallest possible block, in any other case B contains $\{a \in \mathbb{N}^s \mid a < \alpha\}$.

So, in order to prove (ii), it suffices to find $d \in R$ such that both d and $\beta - d - e_\ell \in K^\circ$ for some $\ell \in \mathbb{N}^s$. The statement is immediate if $s = 1$ since $R = \{a \in \mathbb{N} \mid a \leq \gamma\}$ and we will prove it for $s = 2$.

Note that, if so, R is a diagonal set crossing the set Γ as in Fig. 1, where the dotted lines are elements which are not in K , a fact which will be helpful later on.

We may take a partition of Γ as the union of R , its upper and lower side, i.e., one may write $\Gamma := U \cup R \cup L$, which is a disjoint union as sketched in Fig. 1. Formally, we may define

$$\begin{aligned} U &:= \{(a_1, a_2) \in \Gamma \setminus R \mid (a_1, a'_2) \in R \text{ for some } a'_2 < a_2\} \\ &= \{(a_1, a_2) \in \Gamma \setminus R \mid (a'_1, a_2) \in R \text{ for some } a'_1 > a_1\} \end{aligned}$$

and similarly

$$\begin{aligned} L &:= \{(a_1, a_2) \in \Gamma \setminus R \mid (a_1, a'_2) \in R \text{ for some } a'_2 > a_2\} \\ &= \{(a_1, a_2) \in \Gamma \setminus R \mid (a'_1, a_2) \in R \text{ for some } a'_1 < a_1\}. \end{aligned}$$

The strategy of the proof will be the following: suppose $f = (f_1, f_2)$ is such that both f and $\beta - f - e_\ell \in K^\circ$ for some $\ell \in \{1, 2\}$, and suppose also that f is in the upper side U , then we will show that there exists $f' = (f'_1, f'_2)$ with this same property, i.e., f' and $\beta - f' - e_\ell \in K^\circ$ for some $\ell \in \{1, 2\}$, with $f'_1 \geq f_1$ and $f'_2 \leq f_2$ (at least one of the inequalities strict) and f' is either still in U or already in the region R .

So we start by the element d_1 we found out in item (i). If $d_1 \in R$, we are done. If d_1 is not smaller than γ , then either d_1 or d_2 has a zero coordinate; without loss in generality, say d_1 has such. Set $d_3 := \min(d_1, \alpha)$. Then $d_3 \in K$ since it is a minimum of two elements in K , and also $d_3 \notin S$ because it has a zero coordinate. Set $Q := \{a \in K \setminus S \mid a < \alpha\}$. We have that $Q \neq \emptyset$ since d_3 is in there. So take $d_4 \in Q$ which is minimal, i.e., there is no element in Q smaller than d_4 . In particular, d_4 is also minimal respect to being in K° . So $\beta - d_4 - e_\ell \in K^\circ$ for some ℓ because minimality is just what we used to prove item (i). On the other hand, $d_4 \in R$ because $Q \subset B \subset R$. So we are done again.

So assume $d_1 \leq \gamma$ and does not belong to R . Since d_1 and d_2 play dual roles, we may change coordinates, if necessary, and assume $d_1 \in U$ and that the slope of α is greater or equal to the one of β , i.e., if one writes the division $\beta_i = q_i \alpha_i + r_i$ for $i = 1, 2$, then $q_2 \leq q_1$. Set $f := d_1$. This is the first step of our procedure. For the general step, we are not allowed to use the minimality of f .

So let us find out the desired f' by means of f . For simplicity, we will use the following notation: for any $c = (c_1, c_2)$ let

$$L_i(c) := \{(a_1, a_2) \in \mathbb{N}^2 \mid a_i = c_i \text{ and } a_j > c_j \text{ for } j \neq i\}.$$

With this in mind, the properties that f satisfies can be restated as follows: (i) $f \in K$; (ii) either $L_i(f_1 - 1, f_2)$ does not meet S for $i = 1, 2$ or $L_i(f_1, f_2 - 1)$ does not meet S for $i = 1, 2$.

Assume first that $L_1(f)$ meets S and take $b = (b_1, b_2) \in L_1(f) \cap S$, it is pictured in Fig. 2 as a “ \times ”, because it does not appear in the second part of the argument. We claim that, if there is such a b , then $L_2(f)$ meets K but does not meet S . In fact, as we have seen above, according to [7, Prp. 2.14.(i)] we have that K also satisfies properties (i) and (ii) of semigroups; so, by property (ii), $L_2(f)$ meets K in an element, say, $a = (a_1, a_2)$, because $f_1 = b_1$. But this a cannot be in S because $f = \min(a, b)$ and since $b \in S$, if a is in S so is f , which contradicts the fact that $f \notin S$, since f and $\beta - f - e_\ell \in K$, and the claim follows. Now let n_0 be the smallest integer such that $n_0 \alpha > f$. Since $f \in U$, by our

is minimal respect to c being in K . We claim that if $c \in S$ then $L_2(c)$ does not meet S because $f_2 - c_2$ is minimal. In fact, if $L_2(c)$ meets S and $c \in S$ then, by property (ii) of semigroups, there exists $c' = (c'_1, c'_2) \in S \cap L_1(c)$ and $c_2 < c'_2 < f_2$ because $L_1(f)$ does not meet S , but this contradicts the minimality of c , and the claim follows. Now we claim that if $c \in S$ we can make c play the same role of f in the prior paragraph to find the desired f' . Indeed, the key facts of the prior paragraph to find f' were (replacing f by c) the following: (a) $L_2(c)$ does not meet S ; (b) $L_2(c)$ meets K in an element, say a ; (c) if n_2 is the smallest integer such that $n_2\alpha > c$ then $\min(a, n_2\alpha) \in U \cup R$. Now (a) was just proved right above; (b) follows from property (ii) applied to $c, f \in K$; and to check (c), note that since $c_1 = f_1$ and $f_2 - c_2 < \beta_2$ then clearly $c \in U \cup R$ since $f \in U$, and also $n_2 = n_0$, so if a is not smaller than $n_0\alpha$ then $\min(a, n_0\alpha) \in B' \cap \Gamma$ and the claim follows. Finally, if $c \notin S$ then we just make c play the role of v in the prior paragraph. \square

For the next result we make two conventions. Let $P \in C$ be a point and $\overline{P}_1, \dots, \overline{P}_s$ the points on \overline{C} over P . Fix local parameters $t_{\overline{P}_1}, t_{\overline{P}_2}, \dots, t_{\overline{P}_s}$ and for $a = (a_1, a_2, \dots, a_s) \in \mathbb{Z}^s$ we set

$$t^a := t_{\overline{P}_1}^{a_1} t_{\overline{P}_2}^{a_2} \cdots t_{\overline{P}_s}^{a_s} \in k(C).$$

Besides, we consider in a natural way the vector space

$$H^0(\mathcal{W})^n := \left\{ \sum_i f_{i1} f_{i2} \cdots f_{in} \mid f_{ij} \in H^0(\mathcal{W}) \right\}$$

which is not to be confused with $H^0(\mathcal{W})^{\oplus n}$, which is different.

With this in mind, we will prove the following result, which is a generalization of [23, Lem. 3.2]. This is the key point of the proof of Theorem 1 and the reader should note how harder the problem gets when the singular point admits more than one branch. Indeed, the elements of the semigroup are now vectors, the minimum and the conductor may not share a common direction, and, specially, the behaviour of the set K is, so to say, unexpected as we pass from one to many components. In other words, it is a new problem to deal with, whose solutions do not come naturally from the ones in the unibranch case. On the other hand, here, we tried to write down a proof within a little bit lighter form making use of more intuitive notation and opting for dealing with values (in \mathbb{Z}^s) rather than functions and differentials, when possible.

Lemma 3.2. *Let $P \in C$ be a non-Gorenstein point for which the conditions of Lemma 3.1 hold. Then there exists $\epsilon \in \mathbb{N}^s$, such that the natural morphisms*

$$H^0(\mathcal{W})^n \longrightarrow \mathcal{W}_P^n / t^{-\epsilon} \mathcal{C}_P^n$$

are surjective for every $n \geq 2$ and

- (i) $|\epsilon| = 2n - 1$;
- (ii) $|\epsilon| = 1$, if there is $\lambda_0 \in H^0(\omega)$ with $v_P(\lambda_0) = 0$;
- (iii) $|\epsilon| = 0$, if there are $\lambda_0, \lambda_1 \in H^0(\omega)$ with $v_P(\lambda_0) = 0$, and $v_P(\lambda_1) > 0$ with $|v_P(\lambda_1)| = 1$ or 2 .

Proof. Consider the sequence

$$\mathcal{W}_P^n \supset \mathcal{C}_P \supset t^{\beta-\alpha} \mathcal{C}_P = t^{-\alpha} \mathcal{C}_P^2 \supset \mathcal{C}_P^2 \supset \mathcal{C}_P^n. \quad (3)$$

First of all, we claim that $H^0(\mathcal{W})^n$ surjects $\mathcal{W}_P^n/\mathcal{C}_P$. In fact, it can be read off from the proof of [22, Lem. 6.1] that $\mathcal{W}_P \subset H^0(\mathcal{W}) + \mathcal{C}_P$. Since \mathcal{C}_P and $H^0(\mathcal{W})$ are clearly contained in \mathcal{W}_P , we have the equality

$$\mathcal{W}_P = H^0(\mathcal{W}) + \mathcal{C}_P. \quad (4)$$

Now $H^0(\mathcal{W}) \subset \overline{\mathcal{O}}_P$ since, by construction, $\mathcal{W}_P \subset \overline{\mathcal{O}}_P$. But \mathcal{C}_P is an $\overline{\mathcal{O}}_P$ -module, and hence closed under multiplication by functions in $H^0(\mathcal{W})$. So

$$\mathcal{W}_P^n = H^0(\mathcal{W})^n + \mathcal{C}_P \quad (5)$$

which proves the claim. In particular

$$H^0(\mathcal{W})^n \twoheadrightarrow \mathcal{W}_P^n/\mathcal{C}_P. \quad (6)$$

Now we will show that

$$\mathcal{C}_P/t^{\beta-\alpha} \mathcal{C}_P = \overline{H^0(\mathcal{W})^2 \cap \mathcal{C}_P}, \quad (7)$$

that is, $\mathcal{C}_P/t^{\beta-\alpha} \mathcal{C}_P$ agrees with the set of classes of elements in $H^0(\mathcal{W})^2 \cap \mathcal{C}_P$ mod $t^{\beta-\alpha} \mathcal{C}_P$. This is the core of our proof. To begin with, we recall that from [35, Thm. 2.11] or [7, Prp. 2.14.(iv)] we have $v_P(\mathcal{W}_P) = K$. So it suffices to prove that there exists a sequence

$$a_1 = \beta < a_2 < a_3 < \dots < a_{|\beta-\alpha|} < 2\beta - \alpha \quad (8)$$

for which every a_i belongs to the set $G := \{a+b \mid a, b \in K^\circ\}$. In fact, if $a \in K^\circ$ then there is an $f \in H^0(\mathcal{W})$ such that $v_P(f) = a$ owing to (4). Besides, a sequence like (8) implies the existence of $|\beta - \alpha|$ elements in $H^0(\mathcal{W})^2 \cap \mathcal{C}_P$ which are linearly independent mod $t^{\beta-\alpha} \mathcal{C}_P$ owing to [7, Prp. 2.11.(iii)]. But $|\beta - \alpha|$ is precisely the dimension of $\mathcal{C}_P/t^{\beta-\alpha} \mathcal{C}_P$.

We start by assuming $\alpha < \beta$ for otherwise the sequence is empty. Call

$$\alpha^i := \min(i\alpha, \beta)$$

for $i \in \mathbb{N}$. Let m be the largest integer such that $\alpha^{m+1} = (m+1)\alpha$, i.e., such that $(m+1)\alpha \leq \beta$. Let also r be the smallest integer such that $(r+2)\alpha > \beta$. Write $\beta =$

$\alpha^{r+1} + u$, with $0 \leq u < \alpha$, where $0 = (0, \dots, 0) \in \mathbb{N}^s$. So we will build a sequence as in (8) considering its elements as of the three kinds below.

(I) The element is written as

$$a = \beta + i\alpha + v_j$$

for $0 \leq i < m$ and $0 = v_0 < v_1 < \dots < v_{|\alpha|-1} = \alpha - e_\ell$, where the index ℓ comes from the statement of Lemma 3.1.

(II) The element is written as

$$a = \beta + \alpha^{i+1} - \alpha + v_{ij}$$

for $m \leq i < r$ and $0 = v_{i0} < v_{i1} < \dots < v_{i|\alpha^{i+2}-\alpha^{i+1}|-1} < \alpha^{i+2} - \alpha^{i+1}$.

(III) The element is written as

$$a = \beta + \alpha^{r+1} - \alpha + u_j$$

for $0 = u_0 < u_1 < \dots < u_{|u|-1} < u$.

Since the integers in (I), (II) and (III) are ordered, starting from β , and, by construction, the amount is $|\beta - \alpha|$, we just have to prove that they are all in G .

In order to verify so, note that, from the definition of K and since α is the smallest positive element in S , it is easily checked the following inclusion

$$A := \{\beta - \alpha + b \mid 0 \leq b < \alpha \text{ with } |b| \leq |\alpha| - 2\} \subset K^\circ. \quad (9)$$

We first show that the numbers like the ones in (II) are in G . In fact, if $m \leq i < r$ then $0 < \alpha^{i+2} - \alpha^{i+1} < \alpha$. So, $\beta - \alpha + v_{ij} \in K^\circ$ due to (9). On the other hand, $\alpha^{i+1} \in S^\circ \subset K^\circ$, so this proves the assertion. Similarly, one can see that the numbers like those in (III) are in G as well.

Now, let us prove that the integers of the kind (I) are in G . Using (9), note that we just have to show that $\beta + i\alpha - e_\ell \in G$ for $1 \leq i \leq m$. Take d_1, d_2 as in Lemma 3.1 and write

$$\begin{aligned} \beta + i\alpha - e_\ell &= d_1 + d_2 + i\alpha \\ &= (d_1 + q_{1i}\alpha) + (d_2 + q_{2i}\alpha) \end{aligned}$$

with $q_{i1} + q_{i2} = i$ and $q_{i1} \leq m_1$ and $q_{i2} \leq m_2$. Since \mathcal{W}_P is an \mathcal{O}_P -module, and $q_{ij} \leq m_j$ for $j = 1, 2$, it follows that $d_j + q_{ij}\alpha \in K^\circ$ for $j = 1, 2$. And since $m_1 + m_2 \geq m$ we can always find these q_{ij} for any $i \in \{1, \dots, m\}$.

Now take $\sigma \in \mathbb{N}^s$ such that $\beta - \alpha < \sigma < \beta - e_\ell$ and $|\beta - e_\ell - \sigma| = 1$. So $\beta - e_\ell - \sigma = e_\kappa$ for some $\kappa \in \{1, \dots, s\}$, that is, an element of the basis of \mathbb{N}^s . Set $\delta := \beta - \sigma + e_\ell$. Note that $|\delta| = 3$. Let us show that

$$t^{-\alpha}\mathcal{C}_P^2/t^{-\delta}\mathcal{C}_P^2 = \overline{H^0(\mathcal{W})^2 \cap t^{-\alpha}\mathcal{C}_P^2}. \quad (10)$$

In fact, note that

$$\begin{aligned} 2\beta - \delta &= 2\beta - (\beta - \sigma + e_\ell) \\ &= \beta + \sigma - e_\ell \\ &= 2\sigma + (\beta - e_\ell - \sigma) \\ &= 2\sigma + e_\kappa \end{aligned}$$

and consider the set

$$B_1 := \{a \in \mathbb{N}^s \mid 2\beta - \alpha \leq a \leq 2\sigma\}.$$

It can easily be checked that

$$B_1 \subset \{a + b \mid a, b \in A\}$$

which is enough to obtain (10) and this implies (i) of the lemma if $n = 2$.

If (ii) or (iii) holds, we claim that

$$t^{-\alpha}\mathcal{C}_P^2/t^{-\epsilon}\mathcal{C}_P^2 = \overline{H^0(\mathcal{W})^2 \cap t^{-\alpha}\mathcal{C}_P^2}. \quad (11)$$

Indeed, if $y_0 \in H^0(\omega)$ then $y_0/\zeta \in H^0(\mathcal{W})$. Setting $\overline{\omega} := \pi_*(\omega_{\overline{\mathcal{C}}})$, we have that $\overline{\omega}_P = \mathcal{C}_P\zeta$ owing to [22, Lem. 2.8] and $v_P(\overline{\omega}_P) = \mathbb{N}^s$ owing to [35, Thm. 2.12] applied to a nonsingular point. This implies $v_P(\zeta) = -\beta$. So if $v_P(y_0) = 0$, then $v_P(y_0/\zeta) = \beta$ and hence $\beta \in v_P(H^0(\mathcal{W}))$. Now consider the set

$$B_2 := \{a \in \mathbb{N}^s \mid 2\beta - \delta \leq a < 2\beta \text{ and } |2\beta - a| > 1\}.$$

Trivially, $B_2 \subset A + \beta$, so (11) holds if (ii) does due to this inclusion and (10).

If, besides, there is $y_1 \in H^0(\omega)$ with $\nu := v_P(y_1) > 0$ and $|\nu| = 1$ or 2 then, similarly, $\beta + \nu \in v_P(H^0(\mathcal{W}))$. One can check that

$$\{a \in \mathbb{N}^s \mid 2\beta - \alpha \leq a < 2\beta \text{ and } 0 < |2\beta - a| \leq 1\} \subset A + \beta + \nu$$

and the whole claim of (11) follows.

So we have proved the lemma for $n = 2$ by means of (6) applied to $n = 2$, and also (7), (10) and (11). Now we prove it for $n = 3$. Take a sequence

$$\beta = b_{|\beta|} < b_{|\beta|+1} < \dots < b_{2|\beta|-5} < b_{2|\beta|-4} = 2\sigma \quad (12)$$

such that $|b_i| = i$. All the b_i are in G since $\mathcal{C}_P/t^{-\delta}\mathcal{C}_P^2 = \overline{H^0(\mathcal{W})^2 \cap \mathcal{C}_P}$. Then the sequence

$$2\sigma + e_\kappa = \beta - e_\ell + \sigma < b_{|\beta|} + \sigma < b_{|\beta|+1} + \sigma < \dots < b_{2|\beta|-4} + \sigma = 3\sigma \quad (13)$$

is contained in $G' := \{a + b + c \mid a, b, c \in K^\circ\}$ since σ and $\beta - e_\ell$ are in K° . Set $\delta' = 2(\beta - \sigma) + e_\ell$. Then

$$\begin{aligned} 3\beta - \delta' &= \beta + 2\sigma - e_\ell \\ &= 2\beta - (\beta - \sigma + e_\ell) + \sigma \\ &= 2\beta - \delta + \sigma \\ &= 2\sigma + e_\kappa + \sigma \\ &= 3\sigma + e_\kappa. \end{aligned}$$

The union of the sequences (12) and (13) yields $\mathcal{C}_P/t^{-\delta'}\mathcal{C}_P^3 = \overline{H^0(W)^3 \cap \mathcal{C}_P}$. Besides, $|\delta'| = 2|\beta - \sigma| + |e_\ell| = 2 \times 2 + 1 = 5$. So (i) is proved for $n = 3$.

To prove (i) for a general n , first note that the elements of W_P^n/\mathcal{C}_P can be reached by $H^0(W)^n$ due to (6). In order to take care of the elements of $\mathcal{C}_P/\mathcal{C}_P^n$, assume one can find a sequence

$$\beta = b_{|\beta|} < b_{|\beta|+1} < \dots < b_{(n-1)|\beta|-(2(n-1)-1)-1} = (n-1)\sigma \quad (14)$$

such that $|b_i| = i$, and also $b_i = \sum_{j=1}^{n-1} a_{ij}$ with $a_{ij} \in K^\circ$. Now, if $n \geq 4$, add σ to the last $|\beta| - 2$ elements of (14), take the union of this sequence with (14) forming a new sequence linking β to $n\sigma$. Since $\sigma \in K^\circ$ all its elements will satisfy $b_i = \sum_{j=1}^n a_{ij}$ with $a_{ij} \in K^\circ$. Thus we have proved by induction that $\mathcal{C}_P/t^{-\epsilon}\mathcal{C}_P^n = \overline{H^0(W)^n \cap \mathcal{C}_P}$ where $\epsilon = (n-1)(\beta - \sigma) + e_\ell$. But since $|\epsilon| = (n-1)|\beta - \sigma| + |e_\ell| = (n-1) \times 2 + 1 = 2n-1$ we get (i) for a general n . For (ii) and (iii) the argument is similar replacing σ by β which we know is in $v_P(H^0(W))$ in these cases. \square

Now we are able to accomplish Max Noether's statement up to the conditions of Lemma 3.1 and, in particular, having Theorem 1. So follow the proof of [23, Thm. 3.7], and note that the number of branches has no special role up to the part where the local parameter appears, which can easily be fixed using Lemma 3.2. On the other hand, [23, Lems. 3.2 to 3.6] are recalled. First, [23, Lems. 3.3, 3.4] do not mention any unibranch hypothesis while [23, Lem. 3.6] barely depends on this. Indeed, the key fact is just that the singular point is at least triple. Regarding [23, Lem. 3.5], the assumption is needed just at the very last paragraph. First, instead of taking the point of the normalization which lies over the singularity, just choose one of them and force the function a which appears at the end of the proof of Lemma [23, Lem. 3.3] to have a zero on it, then one may find the desired differentials, where $v_P(y_i)$ should be replaced by $|v_P(y_i)|$; the rest of the proof is carried out using [23, Lem. 3.2.(iii)]. But [23, Lem. 3.2.] is generalized by Lemma 3.2. We are finally done.

4. Applications

We start with the following result.

Theorem 4.1. *Consider the statements:*

- (i) *C is nonhyperelliptic;*
- (ii) *There exists a birational morphism $C' \rightarrow C$, i.e., Rosenlicht's statement holds;*
- (iii) *The maps $\mathrm{Sym}^n H^0(\omega) \rightarrow H^0(\omega^n)$ are surjective, i.e., Max Noether's statement holds;*
- (iv) *$K_{0,2}(C, \omega) = 0$;*
- (v) *$\mathrm{Cliff}(C) > 0$ or C is rational nearly normal.*

Then (i) \Leftrightarrow (ii) \Leftrightarrow (v); (iii) \Leftrightarrow (iv) \Rightarrow (i); and if the non-Gorenstein points of C are bibranch at worst, then (i) \Rightarrow (iii).

Proof. We know that (i) \Leftrightarrow (ii) by Rosenlicht [30, Thm. 17]; now, according to [14, App.], $\mathrm{Cliff}(C) = 0$ if and only if C is hyperelliptic or rational nearly normal; since nearly normal curves are non-Gorenstein, they are nonhyperelliptic as well, so (i) \Leftrightarrow (v). On the other hand, (iv) corresponds to Max Noether's statement for $n = 2$, and hence, (iii) \Rightarrow (iv); conversely, use the proof of Lemma 3.2 to see that if (iii) holds for $n = 2$, then it holds for any $n \in \mathbb{N}$, so (iv) \Rightarrow (iii). One can check, for instance, the first paragraph of the proof of [23, Lem. 3.6] to see that nonhyperelliptic curves do not satisfy (iii) for $n = 2$, so (iv) \Rightarrow (i). By Theorem 1, (i) \Rightarrow (iii) with the hypothesis made. \square

As mentioned in the Introduction, one of the first consequences of the regular version of Max Noether's Theorem, which is also valid for Gorenstein curves, is that a canonical curve C lies in the intersection of some quadrics, more precisely:

Let $I_r(C)$ be the vector space of r -forms vanishing on a smooth nonhyperelliptic canonical curve C . We have

$$\dim(I_2(C)) = \frac{(g-2)(g-3)}{2}.$$

In order to generalize this result to non-Gorenstein curves, we use the extrinsic part of the proof of Max Noether's Theorem for nearly Gorenstein curves presented in [23, Thm. 2.6]. Let us fix some required notation.

Let \tilde{C} be the curve obtained by the desingularization of all non-Gorenstein points of C through successive blowups. Thus we obtain a sequence of birational morphisms $\overline{C} \rightarrow \tilde{C} \rightarrow \hat{C} \rightarrow C$. As usual, if $\tilde{\pi} : \tilde{C} \rightarrow C$ is the natural birational morphism, then we set $\tilde{\mathcal{O}} := \tilde{\pi}_*(\mathcal{O}_{\tilde{C}})$ and $\tilde{\mathcal{C}} := \mathcal{H}\mathrm{om}(\overline{\mathcal{O}}, \tilde{\mathcal{O}})$.

Now, our Theorem 2 corresponds to the second item below:

Theorem 4.2. *Let C be a nonhyperelliptic curve of genus g .*

(i) *There exists an embedding $\widehat{C} \hookrightarrow \mathbb{P}^{g+\mu+\varepsilon-2}$ such that, for $r \geq 2$,*

$$\dim(I_r(\widehat{C})) = \binom{r+g+\mu+\varepsilon-2}{r} + g(1-2r) + \eta(r-1) - \mu + 2r - 1$$

where $\varepsilon = 1$ if C is Gorenstein, and $\varepsilon = 0$ otherwise. In particular,

$$\dim(I_2(\widehat{C})) = \frac{g^2 + (2\mu + 2\varepsilon - 7)g + (\mu^2 - 3\mu + 2\eta + 6)}{2}.$$

(ii) *Assume C has n non-Gorenstein points which, if any, are all unibranch. Then there exists an embedding $C \hookrightarrow \mathbb{P}^{g+2\rho+n+\varepsilon-2}$ such that, for $r \geq 2$,*

$$\dim(I_r(C)) = \binom{r+g+2\rho+n+\varepsilon-2}{r} + g(1-2r) - r(2\rho+n-2) - 1$$

where $\rho = h^0(\mathcal{O}/\mathcal{C}) - h^0(\widetilde{\mathcal{O}}/\widetilde{\mathcal{C}})$. In particular,

$$\dim(I_2(C)) = \frac{g^2 + (4\rho + 2n + 2\varepsilon - 7)g + (4\rho(\rho - n + 1) + n(n - 5) + 6)}{2}.$$

Proof. To prove (i), from [22, Prop. 4.5], we have that $\mathcal{O}_{\widehat{C}}\omega$ is an invertible sheaf on \widehat{C} spanned by $H^0(\omega)$. Consider the complete linear system $|\mathcal{O}_{\widehat{C}}\omega|$, which is base point free since $H^0(\omega) \subset H^0(\mathcal{O}_{\widehat{C}}\omega)$. It defines an embedding of \widehat{C} at the space \mathbb{P}^m , where $m = h^0(\mathcal{O}_{\widehat{C}}\omega) - 1$.

If C is nonhyperelliptic, then by the proof of [23, Thm. 2.6], \widehat{C} is projectively normal. Thus

$$\dim(I_r(\widehat{C})) = \binom{r+m}{r} - h^0((\mathcal{O}_{\widehat{C}}\omega)^r).$$

By Riemann–Roch and [22, Prp. 2.14], we have

$$\begin{aligned} h^0((\mathcal{O}_{\widehat{C}}\omega)^r) &= \deg((\mathcal{O}_{\widehat{C}}\omega)^r) + 1 - \widehat{g} + h^1((\mathcal{O}_{\widehat{C}}\omega)^r) \\ &= r(2g - 2 - \eta) + 1 - (g - \eta - \mu) + h^1((\mathcal{O}_{\widehat{C}}\omega)^r). \end{aligned}$$

Then, $m = g + \mu + h^1((\mathcal{O}_{\widehat{C}}\omega)) - 2$. From [22, Lem. 5.1.(3)], one sees that $h^1((\mathcal{O}_{\widehat{C}}\omega)^r) = 0$ for all $r \geq 1$ unless C is Gorenstein and $r = 1$, in which case $h^1(\mathcal{O}_{\widehat{C}}\omega) = h^1(\omega) = 1$. So we set $\varepsilon := h^1(\mathcal{O}_{\widehat{C}}\omega)$, which is 1 if C is Gorenstein, and vanishes otherwise. Thus, for $r \geq 2$, we have

$$\begin{aligned}
 \dim(I_r(\widehat{C})) &= \binom{r+g+\mu+\varepsilon-2}{r} - r(2g-2-\eta) + g - \eta - \mu - 1 \\
 &= \binom{r+g+\mu+\varepsilon-2}{r} + g(1-2r) + \eta(r-1) - \mu + 2r - 1.
 \end{aligned}$$

In particular, for $r = 2$,

$$\begin{aligned}
 \dim(I_2(\widehat{C})) &= \binom{g+\mu+\varepsilon}{2} + \eta - 3g - \mu + 3 \\
 &= \frac{(g+\mu+\varepsilon)(g+\mu+\varepsilon-1)}{2} + \eta - 3g - \mu + 3 \\
 &= \frac{g^2 + (2\mu+2\varepsilon-7)g + (\mu^2-3\mu+2\eta+\varepsilon(2\mu+\varepsilon-1)+6)}{2} \\
 &= \frac{g^2 + (2\mu+2\varepsilon-7)g + (\mu^2-3\mu+2\eta+6)}{2},
 \end{aligned}$$

because, note, $\varepsilon(2\mu+\varepsilon-1) = 0$ since if $\varepsilon = 1$ then $\mu = 0$.

To prove (ii), assume first that C has just one singular point, say P , which is non-Gorenstein, with singularity degree δ and semigroup of values S , whose gaps are $\mathbb{N} \setminus S = \{\ell_1, \dots, \ell_\delta\}$. Consider the curve C^* , with just one singular point P^* , with semigroup of values given by

$$S^* = \{0\} \cup \{2\beta - \ell_i \mid i = 1, \dots, \delta\} \cup \{a \in \mathbb{N} \mid a \geq 2\beta + 1\}$$

and such that there exists a birational morphism $C \rightarrow C^*$. Note that S^* is in fact a semigroup of values: $(2\beta - \ell_i) + (2\beta - \ell_j) = 4\beta - (\ell_i + \ell_j)$, but $\ell_i + \ell_j < 2\beta$, so $(2\beta - \ell_i) + (2\beta - \ell_j) \geq 2\beta + 1$ and thus it is in S^* . By construction, $C = \widehat{C^*}$, i.e., the blowup of C^* along ω_{C^*} . Furthermore, the genus of C^* can be expressed as

$$\begin{aligned}
 g^* &= \overline{g} + 2\beta - \delta = g + 2(\beta - \delta) \\
 &= g + 2 \dim(\mathcal{O}_P/\mathcal{C}_P)
 \end{aligned}$$

whereas $\eta^* = 2 \dim(\mathcal{O}_P/\mathcal{C}_P) - 1$ and $\mu^* = 1$. Note that this is a local procedure and if C has many singular points we may blowdown the non-Gorenstein ones in the same way, in order to obtain such a C^* . If so, it is easily checked that its main invariants turn into $g^* = g + 2\rho$, $\eta^* = 2\rho - n$ and $\mu^* = n$, where ρ and n are as in the statement of the theorem. We point out that we could have done the same local construction for Gorenstein points, but, if so, we would not be able to generalize the canonical embedding. So if C is Gorenstein, set $C^* := C$. Then $g^* = g$ and $\eta^* = \mu^* = 0$, and these numbers match the ones obtained in the non-Gorenstein case, since $\rho = n = 0$ if the curve is Gorenstein. Therefore we may use (i) in any case to obtain

$$\begin{aligned}\dim(I_r(C)) &= \binom{r + g^* + \mu^* + \varepsilon - 2}{r} + g^*(1 - 2r) + \eta^*(r - 1) - \mu^* + 2r - 1 \\ &= \binom{r + g + 2\rho + n + \varepsilon - 2}{r} + g(1 - 2r) - r(2\rho + n - 2) - 1\end{aligned}$$

and if $r = 2$ we use (i) again to express $\dim(I_2(C)) = p(g)$ where

$$\begin{aligned}p(g) &= \frac{(g^*)^2 + (2\mu^* + 2\varepsilon - 7)g^* + ((\mu^*)^2 - 3\mu^* + 2\eta^* + 6)}{2} \\ &= \frac{g^2 + (4\rho + 2n + 2\varepsilon - 7)g + (4\rho(\rho - n + 1) + 4\varepsilon\rho + 4n(n - 5) + 6)}{2} \\ &= \frac{g^2 + (4\rho + 2n + 2\varepsilon - 7)g + (4\rho(\rho - n + 1) + n(n - 5) + 6)}{2}\end{aligned}$$

because, note, $\varepsilon\rho = 0$ since if $\varepsilon = 1$ then $\rho = 0$. We are done. \square

5. On gonality, Clifford index and Green's conjecture

This section is devoted to the study of gonality, Clifford index and Koszul cohomology, in particular Green's conjecture on canonical curves, by allowing torsion free sheaves in their definitions.

Proposition 5.1. *Let C be a non-Gorenstein curve of genus g .*

- (i) *If $\text{gon}(C) < g$ then $\text{Cliff}(C) \leq \text{gon}(C) - 2$.*
- (ii) *$\text{Cliff}(C) = 0$ if and only if $\text{gon}(C) = 2$.*
- (iii) *If C is trigonal and $g \geq 4$, then $\text{Cliff}(C) = 1$.*

Proof. Let \mathcal{F} be a sheaf on C which computes its gonality. Then we have that $\deg(\mathcal{F}) = \text{gon}(C)$ and $h^0(\mathcal{F}) \geq 2$. By Riemann–Roch,

$$h^1(\mathcal{F}) = h^0(\mathcal{F}) + (g - \text{gon}(C)) - 1$$

so $h^1(\mathcal{F}) \geq 2$ if $\text{gon}(C) < g$. Therefore \mathcal{F} contributes to the Clifford index, and hence

$$\begin{aligned}\text{Cliff}(C) &\leq \deg(\mathcal{F}) - 2h^0(\mathcal{F}) + 2 \\ &= \text{gon}(C) - 2h^0(\mathcal{F}) + 2 \\ &\leq \text{gon}(C) - 2\end{aligned}$$

and the item (i) follows.

To prove (ii), as we have already noted, $\text{Cliff}(C) = 0$ if and only if C is hyperelliptic or rational nearly normal. But according to [22, Thm. 3.4] or [26, Thm. 2.1], these are precisely the curves with gonality 2. Item (iii) follows directly from (i) and (ii). \square

Now we prove our last result.

Proof of Theorem 3. To prove (i), let $\overline{\mathcal{F}} := \mathcal{O}_{\overline{C}}\langle 1, x \rangle$ be a sheaf which computes gonality on \overline{C} . We may suppose it is supported outside any regular point over a singular point of C . Set also

$$\mathcal{F} := \mathcal{O}_C\langle 1, x \rangle.$$

For any singular point $P \in C$, write

$$\mathcal{O}_P = k \oplus ky_1 \oplus \cdots \oplus ky_n \oplus \mathcal{C}_P$$

with $y_i \in \mathfrak{m}_P$ for all $i = 1, \dots, n$. Hence

$$\mathcal{O}_P + x\mathcal{O}_P = kx + kxy_1 + \cdots + kxy_n + \mathcal{O}_P$$

and so

$$\deg_P \mathcal{F} = \dim((\mathcal{O}_P + x\mathcal{O}_P)/\mathcal{O}_P) \leq \dim(\mathcal{O}_P/\mathcal{C}_P).$$

Thus

$$\begin{aligned} \text{gon}(C) &\leq \deg \mathcal{F} \\ &= \deg \overline{\mathcal{F}} + \sum_{P \in C_{\text{sing}}} \deg \mathcal{F}_P \\ &= \text{gon}(\overline{C}) + h^0(\mathcal{O}/\mathcal{C}) \\ &\leq \lfloor (\overline{g} + 3)/2 \rfloor + g - \overline{g} - \eta \\ &= g + 1 - \lfloor \overline{g}/2 \rfloor - \eta \end{aligned}$$

so the result follows and we have equality only when $\eta = 1$ and $\overline{g} = 0$ or 1 .

To prove (ii), let $m := \dim(\overline{\mathcal{O}}_P/\mathfrak{m}_P\overline{\mathcal{O}}_P)$ be the multiplicity of P , where $3 \leq m \leq g$. As C is rational, write $k(C) = k(t)$ where t is now the identity function at finite distance of $\mathbb{P}^1 = k \cup \{\infty\}$. Assume the singular point P lies under 0 . Then $t^m u \in \mathcal{O}_P$ for some unit u in $\overline{\mathcal{O}}_P$ since m is the multiplicity of P . Now we know that u admits an m -th root u' in the completion of $\overline{\mathcal{O}}_P$ by the same argument of a Puiseux parametrization. But since u is rational, so is u' . Replacing t by tu' as the identity function at finite distance, we may assume $t^m \in \mathcal{O}_P$. Then $\mathcal{O}_C\langle 1, t^m \rangle$ has degree m at the point under infinity and zero at other points of C . So

$$\text{gon}(C) \leq m.$$

On the other hand, let n be the number of elements in \mathbb{N} between m and β outside the semigroup of values S , i.e., the number of gaps of S between m and β . So, the multiplicity

of P is $m = g + 1 - n$ and, furthermore, the sheaf $\mathcal{O}_C\langle 1, t \rangle$ has degree 1 at the point under infinity, at most $n + 1$ at P and zero at other points of C . Therefore,

$$\text{gon}(C) \leq n + 2.$$

As m and $n + 2$ are inversely proportional, the gonality of C increases as m approaches $n + 2$. The maximum occurs when

$$g + 1 - n = n + 2 \iff n = \frac{g - 1}{2}$$

and $m = g + 1 - n = (g + 3)/2$. As the gonality is an integer number, it follows that

$$\text{gon}(C) \leq \lfloor \frac{g + 3}{2} \rfloor$$

and we are done.

Let us prove (iii). The reciprocal follows directly from the definition of Clifford index and [Proposition 5.1](#). Now, let \mathcal{F} be a sheaf which computes the Clifford index, then

$$\deg(\mathcal{F}) = 1 + 2h^0(\mathcal{F}) - 2. \quad (15)$$

On the other hand, by Clifford's Theorem [\[14, App.\]](#), since $\text{Cliff}(C) \neq 0$, we have

$$h^0(\mathcal{F}) + h^1(\mathcal{F}) \leq 5. \quad (16)$$

If $h^0(\mathcal{F}) = 2$, then C is trigonal by [\(15\)](#). If not, $h^0(\mathcal{F}) = 3$ by [\(16\)](#), and hence $\deg(\mathcal{F}) = 5$ by [\(15\)](#). In order to see that the latter condition is needed, consider the projectively closure of

$$\text{Spec } k[t(t-1)^5, t^2(t-1)^3, t^2(t-1)^6, t^2(t-1)^7].$$

It has genus 5 and is not trigonal by [\[17\]](#), and the sheaf

$$\mathcal{F} := \mathcal{O}_C\langle 1, t(t-1)^3, t^2(t-1)^3 \rangle$$

has degree 0 elsewhere but the infinity where it has degree 5. So $\deg(\mathcal{F}) = 5$ and, by construction, $h^0(\mathcal{F}) = 3$.

To prove (iv), let C' be the canonical model of C . Since C is nearly Gorenstein we verify

$$\begin{aligned} \deg(C') &= 2g - 2 - \eta \\ &= 2(g' + \mu + \eta) - 2 - \eta \\ &= 2(g' + 1 + \eta) - 2 - \eta \\ &= 2g' + \eta. \end{aligned}$$

Moreover, if C' is nearly Gorenstein, it is defined by a complete linear system owing to [22, Lem. 5.8]. Then, by [13, Thm. 8.8.1],

$$K_{p,2}(C', \mathcal{O}_{C'}\omega) = 0 \quad (17)$$

if $p < \eta$ and C' is smooth. But if C is nearly Gorenstein then C' is projectively normal. So one is able to adjust the proof of [13, Thm. 8.8.1] to remove the hypothesis that C' should be smooth.

Now, since C is nearly Gorenstein, then $H^0(\mathcal{O}_{C'}\omega) = H^0(\omega)$ because C' is given by a complete linear system. Moreover, $\pi'_*((\mathcal{O}_{C'}\omega)^q) = \omega^q$ for any $q \geq 2$ according to the proof of [23, Thm. 2.7]. In particular, $H^0((\mathcal{O}_{C'}\omega)^q) = H^0(\omega^q)$ for any $q \geq 1$ which implies $K_{p,q}(C', \mathcal{O}_{C'}\omega) = K_{p,q}(C, \omega)$ for every p, q , and the result follows due to (17).

In order to build the family, for every $p \geq 1$, consider the curve C_p which is the projective closure of

$$\text{Spec } k[t^{p+3}, t^{p+5}, t^{p+6}, \dots, t^{2p+7}].$$

Note that C_p is trigonal since $\mathcal{O}_{C_p}\langle 1, t \rangle$ has degree 1 at the infinity and 2 at the unique singular point of C_p . Besides C_p is nearly Gorenstein with $\eta = p + 1$. Since C_p is trigonal with genus greater than 4 we have $\text{Cliff}(C_p) = 1$, and since C_p is nearly Gorenstein with $p < \eta$ we have $K_{p,2}(C_p, \omega) = 0$. \square

There are many trigonal non-Gorenstein curves of genus 3 since not all of them are nearly normal, which can be easily seen from its very definition. On the other hand, it was proved in [24] that any non-Gorenstein curve of genus 4 is at most trigonal so the bound of (i) is not attained. For genus 5, in [17] one shows that the curve given by the projective closure of

$$\text{Spec } k[tu^2 + tu^3, tu^4 + t^2u^5, t^2u^2 + t^3u^7, t^3u^2, t^4u^2, tu^9, t^2u^9]$$

where $u := t - 1$ has genus and gonality 5. The following example shows that the bound of (ii) is sharp in low genus.

Example 5.1. In the proof of [17, Thm. 2], is shown that the curve with genus 5 given by the projective closure of $\text{Spec } k[t^4, t^5 + t^7, t^{10}, t^{11}]$ has gonality $4 = \lfloor \frac{5+3}{2} \rfloor$. In the same proof, one can observe that, for genus 5, the rational curves whose unique singular point is monomial have gonality at most 3, i.e., the upper limit can only be reached if the singular point is non-monomial.

Now, let C be a rational curve with genus 6, semigroup of values $S^* = \{0, 4, 7, 8, 10\}$ and given by the projective closure of

$$\text{Spec } k[t^4, t^7, t^{10}, t^{12}, t^{13}].$$

We will show that C has gonality $4 = \lfloor \frac{6+3}{2} \rfloor$. Let P be the unique singular point of C . Any sheaf that computes gonality is generated by global sections and can be taken containing the structure sheaf. So we need to prove that any sheaf of the form $\mathcal{G} = \mathcal{O}_C\langle 1, f \rangle$ where $f \in k(t)$ on C has degree at least 4. For this, write $f = t^r h$, where h is an unit at $\overline{\mathcal{O}}_P$. If h has no poles at ∞ , then h does not affect the degree of \mathcal{G} , since their poles at finite points of \mathbb{P}^1 make up for losses at infinity; and if h has a pole at infinity, it only can add degree to \mathcal{G} . Thus, we can assume $\mathcal{G} = \mathcal{O}_C\langle 1, t^r \rangle$. Note that, for $r = 1$, $\mathcal{F} = \mathcal{O}_C\langle 1, t \rangle$ has degree 1 at infinity, zero at the other points but P and 3 at P , since

$$0 < 1 < 4 < 5 < 7 < 8 < 9 < 10$$

is a saturated sequence of elements of $A := v_P(\mathcal{F}_P) = v_P(\mathcal{O}_P + t\mathcal{O}_P)$ linking the minimum element of A to the conductor β of S , with $|A \setminus S| = 3$. According to [7, Prp. 2.11.(iii)], we have $3 = \dim(\mathcal{F}_P/\mathcal{O}_P) = \deg_P(\mathcal{F})$. Therefore, $\deg(\mathcal{F}) = 4$ and $\text{gon}(C) \leq 4$. On the other hand, if $r \geq 4$, then $\deg_\infty(\mathcal{G}) \geq 4$; $r = 3$ implies that $\deg_P(\mathcal{G}) = 1$ and $\deg_\infty(\mathcal{G}) = 3$; if $r = 2$, then $\deg_P(\mathcal{G}) = 3$ and $\deg_\infty(\mathcal{G}) = 2$; finally, if $r \leq -1$, so $\deg_P(\mathcal{G}) \geq 4$. Thus, $\text{gon}(C) = 4$.

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References

- [1] M. Aprodu, G. Farkas, Koszul cohomology and applications to moduli, *Clay Math. Proc.* 14 (2011) 25–50.
- [2] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of Algebraic Curves*, Springer-Verlag, 1985.
- [3] E. Arbarello, E. Sernesi, Petri's approach to the study of the ideal associated to a special divisor, *Invent. Math.* 49 (1978) 99–119.
- [4] D.W. Babbage, A note on the quadrics through a canonical curve, *J. Lond. Math. Soc.* 14 (1939) 310–315.
- [5] E. Ballico, Brill–Noether theory for rank 1 torsion free sheaves on singular projective curves, *J. Korean Math. Soc.* 37 (2000) 359–369.
- [6] E. Ballico, C. Fontanari, L. Tasin, Koszul cohomology and singular curves, *arXiv:0909.2964v3 [math.AG]*, 29 Sep 2009.
- [7] V. Barucci, M. D'Anna, R. Fröberg, Analytically unramified one-dimensional semilocal rings and their value semigroups, *J. Pure Appl. Algebra* 147 (2000) 215–254.
- [8] V. Barucci, R. Fröberg, One-dimensional almost Gorenstein rings, *J. Algebra* 88 (1997) 418–442.

- [9] F. Catanese, Pluricanonical-Gorenstein-curves, in: Enumerative Geometry and Classical Algebraic Geometry, Nice, 1981, in: Progr. Math., vol. 24, 1982, pp. 51–95.
- [10] A. Contiero, K.-O. Stoehr, Upper bounds for the dimension of moduli spaces of curves with symmetric Weierstrass semigroups, J. Lond. Math. Soc. 88 (2013) 580–598.
- [11] P. Deligne, D. Mumford, The irreducibility of the space of curves of a given genus, Publ. Math. Inst. Hautes Études Sci. 36 (1969) 75–109.
- [12] M. Coppens, Free linear systems on integral Gorenstein curves, J. Algebra 145 (1992) 209–218.
- [13] D. Eisenbud, The Geometry of Syzygies, Springer-Verlag, 2005.
- [14] D. Eisenbud, J. Harris, J. Koh, M. Stillman, Determinantal equations for curves of high degree, Amer. J. Math. 110 (1988) 513–539.
- [15] F. Enriques, Sulle curve canoniche di genera p cello spazio a $p - 1$ dimensioni, Rend. Accad. Sci. Ist. Bologna 23 (1919) 80–82.
- [16] L. Feital, Gonalidade e o teorema de Max Noether para curvas não-Gorenstein, Ph.D. Thesis www.mat.ufmg.br/intranet-atual/pgmat/TesesDissertacoes/uploaded/Tese47.pdf.
- [17] L. Feital, R.V. Martins, Gonality of non-Gorenstein curves of genus five, Bull. Braz. Math. Soc. 45 (4) (2014) 1–22.
- [18] M. Franciosi, E. Tenni, Green’s conjecture for binary curves, arXiv:1402.5780v1.
- [19] T. Fujita, On hyperelliptic polarized varieties, Tohoku Math. J. 35 (1983) 1–44.
- [20] M. Green, Koszul cohomology and the geometry of projective varieties, J. Differential Geom. 19 (1984) 125–171.
- [21] R. Hartshorne, Generalized divisors on Gorenstein curves and a theorem of Noether, J. Math. Kyoto Univ. 26 (3) (1986) 375–386.
- [22] S.L. Kleiman, R.V. Martins, The canonical model of a singular curve, Geom. Dedicata 139 (2009) 139–166.
- [23] R.V. Martins, A generalization of Max Noether’s theorem, Proc. Amer. Math. Soc. 140 (2012) 377–391.
- [24] R.V. Martins, Trigonal non-Gorenstein curves, J. Pure Appl. Algebra 209 (2007) 873–882.
- [25] D. Mumford, B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Math., vol. 339, Springer-Verlag, Berlin–Heidelberg–New York, 1973, p. 209, VIII.
- [26] D. Mumford, Curves and Their Jacobians, The University of Michigan Press, Ann Arbor, 1975.
- [27] M. Noether, Über die invariante Darstellung algebraischer Funktionen, Math. Ann. 17 (1880) 263–284.
- [28] K. Petri, Über die invariante Darstellung algebraischer Funktionen einer Veränderlichen, Math. Ann. 88 (1922) 242–289.
- [29] R. Rosa, K.-O. Stoehr, Trigonal Gorenstein curves, J. Pure Appl. Algebra 174 (2002) 187–205.
- [30] M. Rosenlicht, Equivalence relations on algebraic curves, Ann. of Math. 56 (1952) 169–191.
- [31] B. Saint-Donat, On Petri’s analysis of the linear system of quadrics through a canonical curve, Math. Ann. 206 (1973) 157–175.
- [32] F. Saki, Canonical models of complements of stable curves, in: Proc. Int. Symp. on Algebraic Geometry, Kyoto, 1977, pp. 643–661.
- [33] V.V. Shokurov, The Noether–Enriques theorem on canonical curves, Mat. Sb. 86 (1972) 367–408.
- [34] K.-O. Stoehr, P. Viana, A variant of Petri’s analysis of the canonical ideal of an algebraic curve, Manuscripta Math. 61 (1988) 223–248.
- [35] K.-O. Stoehr, On the poles of regular differentials of singular curves, Bull. Braz. Math. Soc. 24 (1993) 105–135.
- [36] K.-O. Stoehr, On the moduli spaces of Gorenstein curves with symmetric Weierstrass semigroups, J. Reine Angew. Math. 441 (1993) 189–213.
- [37] C. Voisin, Green’s generic syzygy conjecture for curves of even genus lying on a K3 surface, J. Eur. Math. Soc. 4 (2002) 363–404.
- [38] C. Voisin, Green’s canonical syzygy conjecture for generic curves of odd genus, Compos. Math. 141 (2005) 1163–1190.