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# Reflective Lorentzian lattices of signature $(5, 1)$

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## ABSTRACT

In this paper we give a complete classification of strongly square-free reflective  $\mathbb{Z}$ -lattices of signature  $(5, 1)$ . This is done by reducing the classification of Lorentzian lattices to those of positive-definite lattices. The classification of totally-reflective genera breaks up into two parts. The first part consists of classifying the square free, totally-reflective, primitive genera by calculating strong bounds on the prime factors of the determinant of positive-definite quadratic forms (lattices) with this property. We achieve these bounds by combining the Minkowski–Siegel mass formula with the combinatorial classification of reflective lattices accomplished by Scharlau & Blaschke. In a second part, we use a lattice transformation that goes back to Watson, to generate all totally-reflective, primitive genera when starting from the square free case.

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## 1. Introduction

With this work, we wish to contribute to the problem of classifying arithmetic reflection groups on the hyperbolic space. More precisely, we analyze the connection between reflective Lorentzian lattices and maximal arithmetic reflection groups with non-compact fundamental domain. We refer to [3] for a detailed survey of known results and an overview of the historical development. Here, we only want to mention that contribu-

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tions to this topic can be found in [31], [30], [34] [32], [17], [18], [33], [23], [27], [7], [10], [19], [20], [21], [1].

To sketch the notion of arithmeticity, consider a totally real number field  $F$  and its ring of integers  $\mathfrak{o}_F$ . Let  $E$  be a  $\mathfrak{o}_F$ -lattice of signature  $(n, 1)$  such that for every non-identity embedding  $\sigma : F \rightarrow \mathbb{R}$  the lattice  ${}^\sigma E$  is positive-definite. The isometry group  $O^+(E)$  can be considered as a discrete subgroup of the full isometry group of the hyperbolic space of dimension  $n$ . Finite-index subgroups of groups obtained in this manner are called *arithmetic*. By definition, the group  $O^+(E)$  is always arithmetic. If it is generated up to finite index by reflections, then the lattice  $E$  is called *reflective*.

Using arithmetic theory of quadratic forms and their connection to root systems, we reduced the classification of Lorentzian lattices to those of totally-reflective genera by proving that a strongly square-free reflective lattice  $E$  of signature  $(5, 1)$  can always be written as  $E = {}^\alpha \mathbb{H} \perp L$ , with  $L$  totally-reflective and  ${}^\alpha \mathbb{H} \cong \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ . Here, a positive-definite lattice is called totally-reflective if every lattice in its genus has a root system of full rank. We should mention that we are also interested in totally-reflective lattices as objects in their own right, thus the following result is more general than needed for the application to reflective lattices of signature  $(5, 1)$ . The strategy leading to the classification is as follows:

- Step 1: Let  $L$  be a strongly square-free totally-reflective lattice with  $\dim L = 4$  (resp.  $\dim = 3$ ). Hence  $\det L$  is of the form  $\det L = p_1^2 \cdots p_r^2 \cdot q_1 \cdots q_s$  (resp.  $r = 0$ ). Using the mass formula and the combinatorial description of lattices with full-rank root system by Scharlau & Blaschke (cf. [25]), we prove that  $r \leq 9$  and  $s \leq 8 - r$  (resp.  $s \leq 9$ ).
- Step 2: Then, we show that there are bounds  $c_i$  and  $d_j$  (one for every prime factor) depending only on the number of prime factors such that  $p_i \leq c_i$  and  $q_j \leq d_j$ . Thus the number of local invariants that need to be taken into account is effectively bounded and the enumeration is computationally feasible.
- Step 3: After finishing the strongly square-free classification, we obtain all square-free, primitive totally-reflective genera by partial dualization.
- Step 4: The last step consists in dropping the assumption “square-free” by determining the pre-images of square-free genera under the Watson transformation.

It is worth mentioning that the primes which actually appear in the determinant are surprisingly low. The biggest prime is 23. After carrying out all the steps, we obtain the following result.

### Theorem.

- a) *In dimension 3, there are 1234 primitive totally-reflective genera of which 289 are square-free and 52 strongly square-free.*

- b) In dimension 4, there are 930 primitive totally-reflective genera of which 230 are square-free and 88 strongly square-free.

Up to here, the procedure gives us a list of candidates for reflective lattices of signature  $(5, 1)$  and allow a subsequent application of an algorithm due to Vinberg. At the end, we prove:

**Theorem.** *There are, up to isometry, 80 strongly square-free reflective  $\mathbb{Z}$ -lattices of signature  $(5, 1)$ .*

Additionally, we gave various geometric invariants of the corresponding fundamental polyhedron, such as the number of faces, the number of cusps and other.

## 2. Background

### 2.1. Integral lattices

Let  $R$  be a principal ideal domain and  $K := \text{Quot}(R)$  its quotient field. A *lattice* over  $R$  is a pair  $(L, b)$ , where  $L$  is a free  $R$ -module of finite rank and  $b : L \times L \rightarrow K$  a symmetric bilinear form. We write  $L$  instead of  $(L, b)$  if the bilinear form is clear from the context. As usual,  $V := L \otimes_R K$  means the enveloping  $K$ -space of  $L$ . By  $O(L)$  we denote the isometry group of  $(L, b)$ . The *determinant* of  $L$  is the determinant of any Gram matrix of  $(L, b)$  which is well defined modulo squares of units in  $R$ .

We say that  $L$  is *integral* if  $b(L, L) \subseteq R$ . An  $R$ -lattice is called *even* if  $b(x, x) \in 2R$  for all  $x \in L$ , and *odd* otherwise. By  ${}^\alpha L$  we mean the lattice  $(L, \alpha b)$  obtained by scaling the bilinear form ( $\alpha \in K$ ). An integral lattice  $L$  is said to be *primitive* if  $L = {}^\alpha K$ , with  $K$  an integral lattice, implies  $\alpha \in R^*$ . We denote by  $L^\#$  the *dual lattice* of  $L$  which is defined as  $L^\# := \{v \in V \mid \forall x \in L : b(x, v) \in R\}$ . Clearly,  $L$  is integral iff  $L \subseteq L^\#$ . For an integral  $\mathbb{Z}$ -lattice, the group  $L^\# / L$  has order  $\det L$  and is called the *discriminant group* of  $L$ . A lattice is *unimodular* if  $L = L^\#$ . More generally, for  $\alpha \in K$ , we say  $L$  is  $\alpha$ -*modular* if  $L = \alpha L^\#$ . It is easy to see that any  $\alpha$ -modular lattice  $K$  can be written as  $K = {}^\alpha L$  where  $L$  is unimodular (cf. [11], Proposition 5.2.1.).

We say a lattice is *indecomposable* if it is not the orthogonal sum of two non-zero sublattices. A non-zero  $\mathbb{Z}$ -lattice can be decomposed as

$$L = K_1 \perp \cdots \perp K_r,$$

with every  $K_i$  being indecomposable. If  $L$  is positive-definite, then such a decomposition is unique up to the order of the  $K_i$  (cf. [15], Satz (27.2)).

Lattices over  $\mathbb{Z}_p$  can be decomposed in a different manner. We start with some basic definitions. Let  $\mathbb{P}$  be the set of all prime numbers. The *localization* of  $L$  at a prime spot  $p \in \mathbb{P} \cup \{\infty\}$  is abbreviated as  $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$  (with  $\mathbb{Z}_\infty = \mathbb{R}$ ). Two  $\mathbb{Z}$ -lattices  $L$  and  $K$

are in the same *genus* if  $L_p \cong K_p$  for all  $p \in \mathbb{P} \cup \{\infty\}$ . It is well known that any genus consists of finitely many isometry classes (cf. [15], Satz (21.3)). We write  $\mathcal{G}(L)$  for the set of all isometry classes in the genus of  $L$  and define  $h(L) := \#\mathcal{G}(L)$  to be the *class number* of  $L$ .

A complete system of invariants for local isometry can be extracted from a decomposition of  $L_p$  in modular sublattices. Every integral  $\mathbb{Z}$ -lattice  $L$  possesses a decomposition

$$L_p = L_0 \perp {}^p L_1 \perp \cdots \perp {}^{p^r} L_r,$$

where all  $L_i$  are unimodular (possibly zero-dimensional). In the literature, a splitting of the above form is called *Jordan decomposition*. We refer to  $L$  as *square-free* if the Jordan decomposition of  $L$  is of the form  $L_0 \perp {}^p L_1$  at every prime  $p$ , and as *strongly square-free* if additionally  $\dim L_0 \geq \dim L_1$  holds. For  $p \neq 2$  the Jordan decomposition is unique up to isometry. At the prime spot  $p = 2$  this is not true in general and only the following data remains invariant:  $\dim L_i$ ,  $2^i$  and the parity of every  $L_i$ . To address this problem, Conway and Sloane have introduced the notion of the so-called *canonical Jordan decomposition*. Among all splittings they have marked out one of a particular easy shape and have showed that it is uniquely determined for every  $\mathbb{Z}_2$ -lattice (cf. [9], Chapter 15, 7.6). The invariants given by the Jordan decomposition are encoded in the *genus symbol*. A detailed introduction to this handy notation, in particular the quite technical realization for  $p = 2$ , can be found in [9], Chapter 15. Here, we only mention that this symbol is a list of local symbols for each prime  $p$  dividing  $2 \det L$ . Assuming a Jordan decomposition as above, the local symbol at the prime  $p \neq 2$  is the formal product

$$\prod_{i=0}^r (p^i)^{\varepsilon_i, n_i}, \text{ where } \varepsilon_i := \left( \frac{\det L_i}{p} \right) \text{ and } n_i := \dim L_i.$$

We will be using this notation for the enumeration of all primitive totally-reflective lattices in dimension 3 and 4.

## 2.2. The mass formula

Throughout this section we assume  $L$  to be a positive-definite  $\mathbb{Z}$ -lattice. Hence  $O(L)$  is a finite group and the following definition makes sense. The *mass* of  $L$  is defined as

$$m(L) := \sum_{M \in \mathcal{G}(L)} \frac{1}{|O(M)|}.$$

It is a deep result from the analytic theory of quadratic forms that the mass of a lattice can be calculated with the local invariants introduced in the previous section. In particular, the knowledge of the whole genus is not required. To amplify this, let  $\Gamma$  denote the gamma function,  $\zeta$  the Riemann zeta function and  $\zeta_D$  the  $L$ -function

**Table 1**

The standard mass in low dimensions.

$n$	2	3	4
$\text{std}(n, D)$	$\frac{2\zeta_D(1)}{\pi}$	$\frac{1}{6}$	$\frac{\zeta_D(2)}{6\pi^2}$

$$\zeta_D(s) = \begin{cases} \prod_{p \in \mathbb{P}} \left(1 - \left(\frac{D}{p}\right) \frac{1}{p^s}\right)^{-1}, & n \text{ even,} \\ 1, & n \text{ odd,} \end{cases}$$

where  $n, D \in \mathbb{N}$  and  $s := \lceil \frac{n}{2} \rceil$ . The fraction of the mass related to the odd prime numbers that do not divide the determinant is stored in the following quantity.

**Definition 2.1.** Let  $n, D \in \mathbb{N}$  and  $s := \lceil \frac{n}{2} \rceil$ . We refer to

$$\text{std}(n, D) := 2\pi^{-n(n+1)/4} \cdot \prod_{j=1}^n \Gamma\left(\frac{j}{2}\right) \cdot \zeta(2)\zeta(4) \cdots \zeta(2s-2)\zeta_D(s),$$

as the *standard mass* (with respect to  $n$  and  $D$ ).

As an example, we mention the dimensions we will be working with in Table 1.

**Lemma 2.2.** Let  $D, s \in \mathbb{N}$  and  $s \geq 2$ . Then

- a)  $\zeta_{-D}(1) \leq 1 + \frac{1}{2} \ln(D)$ .
- b)  $\zeta_D(s) \geq \frac{\zeta(2s)}{\zeta(s)}$ .

**Proof.** Part a) goes back to Watson and can be found in [38], (5.10). To prove part b) the main idea is to use the Liouville function  $\lambda(n) := (-1)^{\Omega(n)}$ , where  $\Omega(n)$  is the number of prime factors of  $n$  counted with multiplicity, as a link between  $\zeta(s)$  and  $\zeta_D(s)$ . We have

$$\zeta_D(s) = \prod_{p \in \mathbb{P}} \left(1 - \left(\frac{D}{p}\right) \frac{1}{p^s}\right)^{-1} \geq \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},$$

and the well-known identity  $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$  implies the assertion (cf. [16]).  $\square$

The actual mass of  $L$  is gained from the standard mass by multiplying with certain correction factors, one for every prime  $p$  dividing  $2 \det L$ . Unlike the standard mass, these correction factors depend on the local structure of  $L$ . A detailed description of how to calculate the mass of a lattice can be found in [8].

**Lemma 2.3.** Let  $L$  be an integral  $\mathbb{Z}$ -lattice.

a) If  $\dim L = 2$  and  $L$  strongly square-free, then

$$m(L) \leq \frac{2}{\pi} \left( 1 + \frac{1}{2} \ln(\det L) \right) \cdot \frac{1}{2} \cdot \prod_{\substack{p|\det L, \\ p \neq 2}} \frac{1}{2} \sqrt{p}.$$

b) If  $\dim L = 2$  and  $L$  square-free, then

$$m(L) \leq \frac{2}{\pi} \left( 1 + \frac{1}{2} \ln(\det L) \right) \cdot \frac{1}{2} \cdot \prod_{\substack{p|\det L, \\ v_p(\det L)=2 \\ p \neq 2}} \frac{p}{p-1} \prod_{\substack{p|\det L, \\ v_p(\det L)=1 \\ p \neq 2}} \frac{1}{2} \sqrt{p}.$$

c) If  $\dim L = 3$  and  $L$  strongly square-free, then

$$\frac{1}{6} \cdot \frac{1}{8} \cdot \prod_{\substack{p|\det L, \\ p \neq 2}} \frac{p-1}{2} \leq m(L).$$

d) If  $\dim L = 4$  and  $L$  strongly square-free, then

$$\frac{1}{90} \cdot \frac{1}{24} \cdot \prod_{\substack{p|\det L, \\ v_p(\det L)=2 \\ p \neq 2}} \frac{p^2(p-1)}{2p+2} \prod_{\substack{p|\det L, \\ v_p(\det L)=1 \\ p \neq 2}} \frac{1}{2} p^{\frac{3}{2}} \leq m(L).$$

### 2.3. Positive-definite, reflective lattices and totally-reflective genera

Throughout this section we assume  $(L, b)$  to be a positive-definite  $\mathbb{Z}$ -lattice. Each non-zero vector  $v \in L$  gives rise to an Euclidean reflection of the enveloping vector space,

$$s_v : V \longrightarrow V, \quad x \longmapsto x - \frac{2b(x, v)}{b(v, v)} v.$$

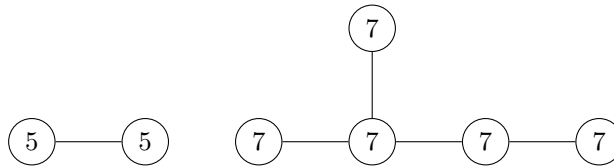
**Definition 2.4.** Let  $L$  be a positive-definite  $\mathbb{Z}$ -lattice.

- a) A vector  $v \in L$  is called *root* of  $L$  if  $v$  is primitive, that is  $v/m \notin L$  for all integers  $m > 1$ , and  $s_v(L) = L$ .
- b) The set  $R(L) := \{v \in L \mid v \text{ is a root of } L\}$  is called the *root system* of  $L$ .
- c) The subgroup  $W(L) \leq O(L)$  generated by all reflections  $s_v$ , with  $v \in R(L)$ , is called *Weyl group* of  $L$ .

It is easy to see that  $R(L)$  is indeed a crystallographic root system (cf. [25], proposition 1.2). The root system of a lattice inherits the quadratic form, thus  $R(L)$  decomposes into scaled irreducible components

$${}^{\alpha}A_n, {}^{\alpha}B_n, {}^{\alpha}C_n, {}^{\alpha}D_n, {}^{\alpha}E_6, {}^{\alpha}E_7, {}^{\alpha}E_8, {}^{\alpha}F_4, {}^{\alpha}G_2.$$

Weyl groups are not affected by the scaling since a reflection  $s_v$  does not depend on the length of  $v$ . The *Dynkin diagram* of such a “scaled” root system is the usual Dynkin diagram of the “unscaled” root system with the addition that every vertex is weighted with the scaling factor of the corresponding root. For example, the Dynkin diagram of  ${}^5A_2{}^7D_5$  is



Unlike Weyl groups, the automorphism groups of Dynkin diagrams are affected by the scaling. Consider, for example, the root systems  $A_2A_2$  and  ${}^3A_2{}^5A_2$  with the corresponding diagrams



and



The automorphism group of the first one is the dihedral group  $Di_4$  of order 8, and  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  in the second case.

Let  $R(L)$  be the root system of a lattice  $L$  and  $U := KR(L)$  the subspace of  $V$  generated by  $R(L)$ . The *symmetry group* of  $R(L)$  is the stabilizer of  $R(L)$  in  $O(U)$  and will be denoted by  $O(R(L))$ . Every graph automorphism of the Dynkin diagram induces an element of the symmetry group of the root system, simply by permuting the roots in the same way the corresponding vertices are permuted in the diagram. One has  $O(R(L)) \cong W(R(L)) \rtimes D(R(L))$  with  $D(R(L))$  being the automorphism group of the Dynkin diagram of  $R(L)$  (cf. [25]).

A positive-definite lattice is called *reflective* if, roughly spoken, its root system is large (and thus the level of symmetry under reflections high). More precisely:

**Definition 2.5.** A positive-definite lattice  $L$  is called *reflective* if  $R(L)$  generates a sublattice of the same rank.

As we will see in Chapter 4, there is also a notion of reflectivity for indefinite lattices of signature  $(n, 1)$ .

The procedure in the next chapter is heavily based on the work of Scharlau & Blaschke (cf. [25]). They classified all indecomposable reflective lattices in low dimensions by pairs

$(R, \mathcal{L})$ , where  $R$  is a scaled root system and  $\mathcal{L}$  the so-called glue code (a subgroup of the discriminant group of  $\langle R \rangle$ ). Given a pair  $(R, \mathcal{L})$ , the associated lattice  $L$  is constructed by  $L = \langle R \rangle + \langle x \mid \bar{x} \in \mathcal{L} \rangle$ . As  $R = R(L)$  is preserved by  $O(L)$ , we have

$$O(L) = \{\varphi \in O(R) \mid \forall \bar{x} \in \mathcal{L} : \varphi(\bar{x}) \in \mathcal{L}\}.$$

Here, we give an excerpt from their result by listing all indecomposable reflective lattices of dimension 4. For the results in dimension 2 and 3 we refer to [25], proposition 4.4 and theorem 4.5.

As the first application we will determine the isometry groups of reflective lattices in small dimensions. It turns out that  $O(L)$  only depends on the combinatorial class of the root system of  $L$ .

**Lemma 2.6.** *Let  $L$  be a reflective lattice with  $\dim L \in \{2, 3, 4\}$ . Assume further that  $L$  is indecomposable if  $\dim L \in \{3, 4\}$ . Then  $O(L)$  only depends on the combinatorial class of  $R(L)$ . In particular,  $O(L)$  depends on neither the glue code nor the scaling. Referring to [25] we have*

a) in dimension 2:

	(a)	(b)	(c)	(d)
$ O(L) $	4	12	4	8

b) in dimension 3:

	(a)	(b)	(c)	(d)	(e)
$ O(L) $	8	8	16	48	48

c) in dimension 4:

	(a), (b), (c)	(d), (e)	(f), (g), (i)	(h)	(j), (k)	(l)
$ O(L) $	16	32	96	72	240	1152

**Proof.** Let  $R(L)$  be decomposed as  $R(L) = {}^{\alpha_1}R_1 \cdots {}^{\alpha_k}R_k$  with  $R_i$  irreducible. The structure of the symmetry group  $O(R(L)) \cong W(R(L)) \rtimes D(R(L))$ , where  $D(R(L))$  is the automorphism group of the Dynkin diagram of  $R(L)$ , and the reflectivity of  $L$  imply the relation  $W(R(L)) \subseteq O(L) \subseteq W(R(L)) \rtimes D(R(L))$ .

All Dynkin diagrams that appear in dimension 2 and 3 (cf. [25]) have a trivial automorphism group, thus  $O(L) = W(R(L))$ . Furthermore,  $W(R(L))$  does not depend on the scaling since  $W({}^{\alpha_1}R_1 \cdots {}^{\alpha_k}R_k) \cong W({}^{\alpha_1}R_1) \times \cdots \times W({}^{\alpha_k}R_k) \cong W(R_1) \times \cdots \times W(R_k)$ .

In dimension 4, one can use the same argument, except in the cases (j), (k) and (h), where  $D(R(L)) \neq \{\text{id}\}$ . There we have



$$D(R(L)) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{in } (j) \text{ and } (k), \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{in } (h). \end{cases}$$

In both cases one easily checks the following: for  $(j)$ ,  $(k)$ , the non-trivial automorphism of the Dynkin diagram preserves both, the glue code of  $(j)$  and  $(k)$ , thus  $O(L) = W(R(L)) \rtimes \mathbb{Z}/2\mathbb{Z}$  depends on neither the glue code nor the scaling. For  $(h)$ , only one of the three non-trivial automorphisms of the Dynkin diagram preserves the corresponding glue code, thus  $O(L) = W(R(L)) \rtimes \mathbb{Z}/2\mathbb{Z}$  (cf. [25] for more details on the glue codes). Weyl groups of unscaled irreducible root systems, particularly their orders, are well known and can be found in [6].  $\square$

**Definition 2.7.** Let  $L$  be an integral lattice and  $\mathcal{G}$  its genus.

- a) We call  $\mathcal{G}$  *totally-reflective* if each lattice in  $\mathcal{G}$  is reflective.
- b) The integral lattice  $L$  is called *totally-reflective* if its genus  $\mathcal{G}$  is totally-reflective.

We will see in the next section that totally-reflective lattices play a major role in the understanding of hyperbolic reflection groups. Nevertheless, they are interesting objects of study in their own right. One can deduce from the work of Biermann [5] that there are only finitely many primitive totally-reflective genera in any dimension  $> 3$  (cf. [27], Theorem 1.4). Furthermore, Esselmann proved in [10] that 20 is the largest dimension of totally-reflective genera, thus a classification is possible (at least) in principle. With the present work we contribute to this problem by classifying the dimensions 3 and 4.

### 3. Classification of totally-reflective lattices

#### 3.1. Bounds for the determinant of strongly square free, totally-reflective genera

**Definition 3.1.** Let  $L$  be an integral lattice. We refer to

$$m_{\text{ref}}(L) := \sum_{\substack{M \in \mathcal{G}(L), \\ M \text{ is reflective}}} \frac{1}{|O(M)|}$$

as the *reflective part* of the mass.

We obtain our bounds by showing that the reflective part of the mass grows more slowly than the whole mass (with increasing determinant). The growth of  $m(L)$  will be regulated by the mass formula which was introduced in the last chapter (cf. Lemma 2.3). The behavior of  $m_{\text{ref}}(L)$  is controlled using the combinatorial classification of reflective lattices from [25].

For this, it will be helpful to distinguish the lattices in  $\mathcal{G}(L)$  by the type of the decomposition in indecomposable sublattices. For a 4-dimensional lattice  $L$  we write

$$\mathcal{G}_4(L) = \{M \in \mathcal{G}(L) \mid M \text{ indecomposable}\},$$

$$\mathcal{G}_3(L) = \{M \in \mathcal{G}(L) \mid M = M_1 \perp M_2, \dim M_1 = 3 \text{ and } M_1 \text{ indecomposable}\},$$

$$\mathcal{G}_2(L) = \{M \in \mathcal{G}(L) \mid M = M_1 \perp M_2, \dim M_1 = 2\}.$$

Analogously, if  $\dim L = 3$  we define

$$\mathcal{G}_3(L) = \{M \in \mathcal{G}(L) \mid M \text{ indecomposable}\},$$

$$\mathcal{G}_2(L) = \{M \in \mathcal{G}(L) \mid M = M_1 \perp M_2, \dim M_1 = 2\}.$$

The reflective part of the mass can now be written as  $m_{\text{ref}}(L) = m_{\text{ref}4}(L) + m_{\text{ref}3}(L) + m_{\text{ref}2}(L)$ , with

$$m_{\text{ref}i}(L) := \sum_{\substack{M \in \mathcal{G}_i(L), \\ M \text{ reflective}}} \frac{1}{|O(M)|},$$

$i = 2, 3, 4$ . Obviously,  $m_{\text{ref}4}$  is omitted when  $\dim L = 3$ . Because we do not want to overload the notation, the dimension of the lattice we are dealing with is not included in the notation and will always be clear from the context. We write  $\omega(d)$  for the number of distinct prime factors of  $d$  and  $\Omega(d)$  for the number of prime factors of  $d$  counted by multiplicities. The divisor set of  $d \in \mathbb{N}$  is  $D(d) := \{x \in \mathbb{N} \mid x \text{ divides } d\}$  and  $a(d) := \#D(d)$  is the number of distinct divisors of  $d$ .

**Lemma 3.2.** *Let  $L$  be an integral  $\mathbb{Z}$ -lattice with determinant  $d$ .*

- a) *For  $\dim L = 3$  we have*
  - 1)  $m_{\text{ref}3}(L) \leq \frac{17}{48} \cdot 3^{\Omega(d)},$
  - 2)  $m_{\text{ref}2}(L) \leq \frac{17}{48} \sum_{x|d} 2^{\Omega(x)}.$
- b) *For  $\dim L = 4$  we have*
  - 1)  $m_{\text{ref}4}(L) \leq \frac{611}{1920} \cdot 4^{\Omega(d)},$
  - 2)  $m_{\text{ref}3}(L) \leq \frac{17}{96} \sum_{x|d} 3^{\Omega(x)},$
  - 3)  $m_{\text{ref}2}(L) \leq \frac{17}{24} \cdot a(d) \cdot 2^{\Omega(d)}.$

**Proof.** We prove b) in detail to illustrate the basic idea. Part a) is proven analogously.

Estimating  $m_{\text{ref}4}(L)$ : First we give an upper bound for the number of possible isometry classes for every type (with type we mean the cases (a) – (l) in Table 2) of 4-dimensional, indecomposable reflective lattices. To this end, it is sufficient to estimate how many scalings of  $R$  lead to the determinant  $d$ . According to Table 2, the determinant  $d$  of a lattice  $L = \langle R \rangle + \langle x \mid \bar{x} \in \mathcal{L} \rangle$  is a product  $d = c \cdot \alpha_1 \alpha_2 \alpha_3 \alpha_4$ , where  $\alpha_i$  is a scaling factor of  $R$  and  $c \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, 4, 5\}$ . The prime factors of  $d$  can be distributed among the  $\alpha_i$  in, at most,  $4^{\Omega(d)}$  different ways (for simplicity, the restrictions for the scaling factors described in Table 2 are ignored). Thus an upper bound for the number of isometry

**Table 2**  
Indecomposable reflective lattices in dimension 4.

	$R$	$\mathcal{L}$	Restrictions	Determinant
(a)	${}^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}A_1$	$\mathcal{L}_{4,1}$	$\alpha < \beta < \gamma < \delta$ $\alpha + \beta + \gamma + \delta \equiv 0(2)$	$4\alpha\beta\gamma\delta$
(b)	${}^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}A_1$	$\mathcal{L}_{4,2}$	$\alpha < \beta, \gamma < \delta$ $\alpha + \beta \equiv 0(2), \gamma \equiv \delta \equiv 0(2)$	$\alpha\beta\gamma\delta$
(c)	${}^{\alpha}A_1^{\beta}A_1^{\gamma}A_1^{\delta}A_1$	$\mathcal{L}_{4,3}$	$\alpha < \beta < \gamma < \delta$ $\alpha \equiv \beta \equiv \gamma \equiv \delta \equiv 0(2)$	$\alpha\beta\gamma\delta/4$
(d)	${}^{\alpha}A_1^{\beta}A_1^{\gamma}C_2$	$\mathcal{L}_{3,1}$	$\alpha < \beta, \alpha \neq \gamma \neq \beta$	$4\alpha\beta\gamma^2$
(e)	${}^{\alpha}A_1^{\beta}A_1^{\gamma}C_2$	$\mathcal{L}_{3,2}$	$\alpha < \beta, \alpha \neq 2\gamma \neq \beta$ $\alpha \equiv \beta \equiv 0(2)$	$\alpha\beta\gamma^2$
(f)	${}^{\alpha}A_1^{\beta}B_3$	$\mathcal{L}_{2,1}$	$2\alpha \neq \beta, \beta \equiv 0(2)$	$\alpha\beta^3/2$
(g)	${}^{\alpha}A_1^{\beta}C_3$	$\mathcal{L}_{2,1}$	$\alpha \neq 2\beta, \alpha \equiv 0(2)$	$4\alpha\beta^3$
(h)	${}^{\alpha}A_2^{\beta}A_2$	$\mathcal{L}_{2,1}$	$\alpha < \beta, 2\alpha \neq \beta$ $\alpha + \beta \equiv 0(3)$	$\alpha^2\beta^2$
(i)	${}^{\alpha}C_2^{\beta}C_2$	$\mathcal{L}_{2,1}$	$\alpha < \beta$	$4\alpha^2\beta^2$
(j)	${}^{\alpha}A_4$	0	no	$5\alpha^4$
(k)	${}^{\alpha}A_4$	$\neq 0$	$\alpha \equiv 0(5)$	$\alpha^4/5$
(l)	${}^{\alpha}F_4$	0	$\alpha \equiv 0(2)$	$4\alpha^4$

classes of reflective lattices of determinant  $d$  which arise from  $(R, \mathcal{L})$  by changing the scaling of  $R$ , is  $4^{\Omega(d)}$ . Combining this with Lemma 2.6, we get

$$m_{\text{ref}4}(L) \leq 4^{\Omega(d)} \left( \frac{3}{16} + \frac{2}{32} + \frac{2}{72} + \frac{3}{96} + \frac{2}{240} + \frac{1}{1152} \right) = \frac{611}{1920} \cdot 4^{\Omega(d)}.$$

Estimating  $m_{\text{ref}2}(L)$ : We use [25], proposition 4.4 and the fact that  $|O(M_2)| \geq 4$  for binary reflective lattices (cf. Lemma 2.6): The same argumentation as in the case  $m_{\text{ref}4}(L)$  shows that an upper bound for the number of isometry classes of binary reflective lattices of determinant  $x$  is  $4 \cdot 2^{\Omega(x)}$ ; there are 4 types of  $(R, \mathcal{L})$  and  $2^{\Omega(x)}$  possibilities to change the scale of  $R$  in  $(R, \mathcal{L})$ . Assuming a 4-dimensional reflective lattice decomposed as  $M = M_1 \perp M_2$ , with  $\dim M_1 = 2$  and  $d = \det M$ ,  $x = \det M_1$ , each of the  $4 \cdot 2^{\Omega(x)}$  isometry classes of  $M_1$  can be combined with  $4 \cdot 2^{\Omega(d/x)}$  isometry classes of  $M_2$ , leading to

$$\begin{aligned} m_{\text{ref}2}(L) &= \sum_{\substack{M \in \mathcal{G}_2(L), \\ M \text{ reflective}}} \frac{1}{|O(M)|} \leq \frac{1}{4} \sum_{x|d} 2^{\Omega(x)} \left( \frac{1}{4} + \frac{1}{12} + \frac{1}{4} + \frac{1}{8} \right) \cdot 4 \cdot 2^{\Omega(d/x)} \\ &= \frac{17}{24} \sum_{x|d} 2^{\Omega(d)} = \frac{17}{24} \cdot a(d) \cdot 2^{\Omega(d)}. \end{aligned}$$

Estimating  $m_{\text{ref}3}(L)$ : Assume a 4-dimensional reflective lattice decomposed as  $M = M_1 \perp M_2$  with  $\dim M_1 = 3$ ,  $M_1$  indecomposable and  $d = \det M$ ,  $x = \det M_1$ . Unlike in the  $m_{\text{ref}2}$  case, the  $4 \cdot 3^{\Omega(x)}$  isometry classes of  $M_1$  can be combined with only one isometry class of  $M_2$ . This follows from the fact that  $M_1 \perp M_2 \cong M_1 \perp M'_2$  implies  $\det M_2 = \det M'_2$  and thus  $M_2 \cong M'_2$ , because of  $\dim M_2 = 1 = \dim M'_2$ . Hence

$$m_{\text{ref}3}(L) = \sum_{\substack{M \in \mathcal{G}_3(L), \\ M \text{ reflective}}} \frac{1}{|O(M)|} \leq \frac{1}{2} \sum_{x|d} 2^{\Omega(x)} \left( \frac{1}{4} + \frac{1}{12} + \frac{1}{4} + \frac{1}{8} \right) = \frac{17}{48} \sum_{x|d} 2^{\Omega(x)}. \quad \square$$

For strongly square-free lattices the upper bound for  $m_{\text{ref}3}$  can be significantly sharpened (this was already known by Berger, cf. [4], Bemerkung 3.5.1).

**Lemma 3.3.** *Let  $L$  be a strongly square-free lattice of determinant  $d$ . Then, in both dimensions, we have  $m_{\text{ref}3}(L) = 0$ .*

**Proof.** Dimension 3: From [25], theorem 4.5, it follows that 4 divides the determinant of a 3-dimensional, indecomposable reflective lattice, which is never the case for strongly square-free lattices.

Dimension 4: We show that a 4-dimensional reflective lattice cannot have a 3-dimensional, indecomposable orthogonal summand. Assume a reflective

$$L = L' \perp L'',$$

with  $\dim L' = 3$ ,  $\dim L'' = 1$  and  $L'$  indecomposable. Theorem 4.5 in [25] implies  $4 \mid \det L'$ , and  $L$  being strongly square-free implies  $8 \nmid \det L'$ . Notice that a lattice is even iff the unimodular component of the 2-adic Jordan decomposition is even (trivial), and that unimodular  $\mathbb{Z}_2$ -lattices only exist in even dimensions (cf. [22], 93:15). So if  $L'$  is even, then the 2-adic Jordan decomposition of  $L'$  is of the form  $L' \otimes \mathbb{Z}_2 = L'_0 \perp {}^4L'_2$  with  $\dim L'_0 = 2$  and  $\dim L'_2 = 1$ . This is a contradiction to  $L$  being strongly square-free. If  $L'$  is odd, then [25], theorem 4.7 implies that  $L'$  can only be of the shape [25], theorem 4.5 (a), which over  $\mathbb{Z}$  always possesses a Gram matrix of the form  $\begin{pmatrix} 2\alpha & 0 & \alpha \\ 0 & 2\beta & \beta \\ \alpha & \beta & \delta \end{pmatrix}$ , with  $\delta = \frac{\alpha+\beta+\gamma}{2}$ , in some basis  $v_1, v_2, v_3 \in L$ . Given that  $L'$  is odd, either  $\alpha$  or  $\beta$  is odd, so we may assume without loss of generality that  $2 \nmid \beta$ . Over  $\mathbb{Z}_2$ , the two vectors  $v_2 - \frac{\beta}{\delta}v_3, v_3$  ( $\delta$  is odd, thus a 2-adic unit) generate an unimodular sublattice of  $L' \otimes \mathbb{Z}_2$  with Gram matrix  $\begin{pmatrix} 2\beta - \frac{\beta^2}{\delta} & 0 \\ 0 & \delta \end{pmatrix}$ ; unimodular because  $\beta$  is odd. As a unimodular sublattice, it is an orthogonal summand of  $L' \otimes \mathbb{Z}_2$  (cf. [22], 82:15), thus  $L' \otimes \mathbb{Z}_2$  must have a 1-dimensional 4-modular Jordan component. Again, this forms a contradiction to  $L$  being strongly square-free. It follows  $m_{\text{ref}3}(L) = 0$ .  $\square$

Let us make the following abbreviation for strongly square-free lattices  $L$ . If the dimension is 4, we write

$$M_{\text{ref}4}(L) = \frac{611}{1920} \cdot 4^{\Omega(d)}, \quad M_{\text{ref}2}(L) = \frac{17}{24} \cdot a(d) \cdot 2^{\Omega(d)},$$

hence

$$M_{\text{ref}}(L) = M_{\text{ref}4}(L) + M_{\text{ref}2}(L)$$

is an upper bound for  $m_{\text{ref}}(L)$ . The lower bound for the full mass from Lemma 2.3 will be abbreviated as

$$M(L) = \frac{1}{90} \cdot \frac{1}{24} \cdot \prod_{\substack{p|\det L, \\ v_p(\det L)=2 \\ p \neq 2}} \frac{p^2(p-1)}{2p+2} \prod_{\substack{p|\det L, \\ v_p(\det L)=1 \\ p \neq 2}} \frac{1}{2} p^{\frac{3}{2}}.$$

If the dimension is 3, we write

$$M_{\text{ref}}(L) = M_{\text{ref}2}(L) = \frac{17}{48} \sum_{x|d} 2^{\Omega(x)}$$

for the upper bound for  $m_{\text{ref}}(L)$  and

$$M(L) = \frac{1}{6} \cdot \frac{1}{8} \cdot \prod_{\substack{p|\det L, \\ p \neq 2}} \frac{p-1}{2}$$

for the lower bound of  $m(L)$ .

By combining Lemma 2.3 and Lemma 3.2 we see that a strongly square-free totally-reflective lattice satisfies the condition  $M_{\text{ref}}(L)/M(L) \geq 1$ . Actually, the estimates on  $m_{\text{ref}}(L)$  and  $m(L)$  depend only on the determinant of  $L$ , so we may write  $M(L) = M(\det L)$  and  $M_{\text{ref}}(L) = M_{\text{ref}}(\det L)$ .

**Lemma 3.4.** *Let  $d \in \mathbb{N}$  and  $q \in \mathbb{P}$ .*

- a) *In dimension 3: For  $q \geq 7$  and  $q \nmid d$  we have  $\frac{M_{\text{ref}}(d)}{M(d)} \geq \frac{M_{\text{ref}}(dq)}{M(dq)}$ .*
- b) *In dimension 4: For  $q \geq 7$  and  $q \nmid d$  we have  $\frac{M_{\text{ref}}(d)}{M(d)} \geq \frac{M_{\text{ref}}(dq^2)}{M(dq^2)}$ . For  $q \geq 5$  and  $q \nmid d$  we have  $\frac{M_{\text{ref}}(d)}{M(d)} \geq \frac{M_{\text{ref}}(dq)}{M(dq)}$ .*

**Proof.** We prove the first part of b) in detail so that the general idea is clear. The rest is proven analogously. From the mass formula it follows that

$$M(dq^2) = \frac{q^2(q-1)}{2(q+1)} \cdot M(d),$$

thus the mass grows basically like a quadratic polynomial if the determinant is extended by a quadratic prime factor. The growth of  $M_{\text{ref}}$  is much slower; there are constants  $c_1$  and  $c_2$  independent of  $q$  (rather than a quadratic polynomial) such that

$$M_{\text{ref}4}(dq^2) = c_1 \cdot M_{\text{ref}4}(d), \quad M_{\text{ref}2}(dq^2) = c_2 \cdot M_{\text{ref}2}(d).$$

The first statement follows immediately from the definition with  $c_1 := 4^2$ . For the second statement, if taking into account that the divisors of  $dq^2$  are the divisors of  $d$  multiplied by 1,  $q$  and  $q^2$ , we get  $a(dq^2) = 3 \cdot a(d)$  and thus

$$\begin{aligned} M_{\text{ref}2}(dq^2) &= \frac{17}{24} \cdot a(dq^2) \cdot 2^{\Omega(dq^2)} = \frac{17}{24} \cdot 3 \cdot a(d) \cdot 2^2 \cdot 2^{\Omega(d)} \\ &= 12 \cdot M_{\text{ref}2}(d). \end{aligned}$$

To prove the lemma, it is now sufficient to show that

$$\frac{M_{\text{ref}4}(d)}{M(d)} \geq \frac{4^2 \cdot M_{\text{ref}4}(d)}{M(dq^2)}, \quad (1)$$

$$\frac{M_{\text{ref}2}(d)}{M(d)} \geq \frac{12 \cdot M_{\text{ref}2}(d)}{M(dq^2)}. \quad (2)$$

Statement (2.1) is equivalent to  $\frac{q^2(q-1)}{2(q+1)} \geq 4^2$ , and statement (2.2) is equivalent to  $\frac{q^2(q-1)}{2(q+1)} \geq 12$ . Both are true for  $q \geq 7$ .  $\square$

**Lemma 3.5.** *In both dimensions  $\frac{M_{\text{ref}}(d)}{M(d)}$  is monotonically decreasing in each prime factor of  $d$ .*

**Proof.** The enumerator  $M_{\text{ref}}(d)$  only depends on the number of prime factors of  $d$  and, as can be seen on the last two pages, the denominator  $M(d)$  is monotonically increasing in each prime factor.  $\square$

The main theorem of this subsection is now a direct consequence of Lemma 3.4 and Lemma 3.5.

**Theorem 3.6.** *Let  $L$  be a strongly square-free totally-reflective lattice.*

- a) *Let  $\dim L = 3$  and  $\det L = q_1 \cdots q_s$ . Then  $s \leq 9$ .*
- b) *Let  $\dim L = 4$  and  $\det L = p_1^2 \cdots p_r^2 q_1 \cdots q_s$ . Then  $r \leq 8$  and  $s \leq 8 - r$ .*

**Proof.** We have to decide when the necessary condition  $\frac{M_{\text{ref}}}{M} \geq 1$  is violated.

a) Let  $L$  be a strongly square-free totally-reflective lattice with  $\det L = q_1 \cdots q_s$  and  $s \geq 10$ . Assume the prime factors are ordered such that  $q_1 < \cdots < q_s$ . We start with the observation

$$\frac{M_{\text{ref}}(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29)}{M(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29)} < 1.$$

Using the monotony statement of Lemma 3.5, we get

$$\frac{M_{\text{ref}}(q_1 \cdot q_2 \cdot q_3 \cdot q_4 \cdot q_5 \cdot q_6 \cdot q_7 \cdot q_8 \cdot q_9 \cdot q_{10})}{M(q_1 \cdot q_2 \cdot q_3 \cdot q_4 \cdot q_5 \cdot q_6 \cdot q_7 \cdot q_8 \cdot q_9 \cdot q_{10})} < 1.$$

Now we apply Lemma 3.4 a) and see that

$$\frac{M_{\text{ref}}(q_1 \cdots q_{10} \cdot q_{11} \cdots q_s)}{M(q_1 \cdots q_{10} \cdot q_{11} \cdots q_s)} = \frac{M_{\text{ref}}(\det L)}{M(\det L)} < 1.$$

Thus  $L$  is not totally-reflective. Part b) is proven similarly.  $\square$

In the next theorem we give upper bounds for the prime factors of the determinant. The tables below should be read as follows:  $p_1 \leq c_1$  means that the smallest prime factor of the determinant of a totally-reflective lattice is at most  $c_1$ ,  $p_2 \leq c_2$  means that if the determinant has at least two prime factors then the second smallest is at most  $c_2$ , and so on. Thus, in dimension 4, for example,  $p_7$  is at most 449 regardless of whether there are 7, 8 or 9 linear prime factors (and regardless how many quadratic prime factors there are).

**Theorem 3.7.** *Let  $L$  be a strongly square-free totally-reflective lattice.*

a) *Let  $\dim L = 3$  and  $\det L = q_1 \cdots q_s$  with  $s \leq 9$ . Assume  $q_1 < \cdots < q_s$ . Then*

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$	$q_6$	$q_7$	$q_8$	$q_9$
$\leq$	103	307	919	1373	1373	827	409	151	47

b) *Let  $\dim L = 4$  and  $\det L = p_1^2 \cdots p_r^2 q_1 \cdots q_s$  with  $r \leq 9$  and  $s \leq 8 - r$ . Assume  $p_1 < \cdots < p_r$  and  $q_1 < \cdots < q_s$ . Then*

	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$	$p_6$	$p_7$	$p_8$	$p_9$
$\leq$	191	661	1601	2069	1831	997	449	157	47

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$	$q_6$	$q_7$	$q_8$
$\leq$	11287	6427	3613	1597	653	229	67	19

**Proof.** By using Lemma 3.5 we can repeatedly increase a prime factor (and thus decrease the function  $M_{\text{ref}}/M$ ) until the necessary condition

$$M_{\text{ref}}(\det L)/M(\det L) \geq 1$$

is violated.  $\square$

**Remark 3.8.**

a) With the help of Theorem 3.6 and Theorem 3.7, the enumeration of all strongly square-free totally-reflective genera can be carried out computationally. All calcula-

tions were performed using MAGMA, cf. [2]. First, we construct all genus symbols of strongly square-free lattices up to the given bounds. Avoiding redundant calculations by considering all the restrictions for genus symbols described in [9], Chapter 15, this can be carried out in around 48 hours on the author's laptop computer. Then, to decide if a genus is totally-reflective, we enumerate all lattices within the genus, calculated their root system and checked whether it is of full rank. The non-trivial part here is the enumeration of the whole genus. Calculating the root system and its rank is a matter of milliseconds. The theory behind the enumeration of a genus is given by Kneser's neighborhood method, cf. [14]. This method is implemented in MAGMA. There is also an implementation in C called TwoNeighbours (because the prime  $p = 2$  is used) going back to Scharlau & Hemkemeier, cf. [26]. The algorithm generates the whole genus if only one lattice from the genus is known. This one lattice that is required to start the neighborhood method (for a given genus symbol) is constructed according to [12], Section 5.2. Even in low dimension such as 3 and 4, enumerating a whole genus can be very time consuming if the determinant has large prime factors. Therefore, the neighborhood method has sometimes been interrupted after a couple of genus representatives have been found to check whether a non-reflective lattice has already occurred. At the end a computing time of 96 hours was necessary.

b) It turns out that the largest occurring value for the number of prime factors is

$$(r, s) = \begin{cases} (3, 3), & \text{in dimension 4,} \\ (0, 4), & \text{in dimension 3.} \end{cases}$$

The largest prime factor  $p$  occurring in dimension 4 is

$$p = \begin{cases} 13, & \text{if } v_p(\det) = 2, \\ 17, & \text{if } v_p(\det) = 1. \end{cases}$$

and  $p = 23$  in dimension 3.

### 3.2. Reduction to the strongly square-free case

**Definition 3.9.** Let  $L$  be a square-free lattice and  $p \in \mathbb{P}$ . The *partial dual* of  $L$  at  $p$  is defined as  $D_p(L) := {}^p(\frac{1}{p}L \cap L^\#)$ .

In contrast to the usual dual operator, the partial dual operator only dualizes the lattice at the prime spot  $p$ . That means

$$D_p(L) \otimes_{\mathbb{Z}} \mathbb{Z}_q = \begin{cases} {}^p(L_q^\#), & \text{if } q = p, \\ {}^pL_q, & \text{if } q \neq p. \end{cases}$$



This has the following effect on the Jordan decomposition of a square-free lattice  $L \otimes_{\mathbb{Z}} \mathbb{Z}_p = L_0 \perp {}^p L_1$ :

$$D_p(L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L_1 \perp {}^p L_0.$$

Thus, starting with a strongly square-free lattice, one can construct a (not necessarily strongly) square-free primitive lattice by applying  $D_p$  for  $p \mid \det L$  (and vice versa). For a set of primes  $I := \{p_1, \dots, p_k\} \subseteq \mathbb{P}$  we use the abbreviation  $D_I := D_{p_1} \circ \dots \circ D_{p_k}$  (where  $D_{\emptyset} := \text{id}$ ), which is well-defined since two partial dual operators with respect to different primes commute. Clearly,  $D_p$  extends to a well-defined bijective function  $\mathcal{G}(L) \rightarrow D_p(\mathcal{G}(L)) = \mathcal{G}(D_p(L))$ .

**Proposition 3.10.**

- a) Let  $L$  be an integral lattice. Then  $L$  is totally-reflective if and only if  $D_p(L)$  is totally-reflective.
- b) Let  $\mathcal{T}_n$  be the set of all strongly square-free totally-reflective genera in dimension  $n \in \{3, 4\}$ . Let  $\mathcal{P}(d)$  be the power set of the set of all prime factors of  $d := \det \mathcal{G}$ . Then

$$\bigcup_{\mathcal{G} \in \mathcal{T}_n} \bigcup_{I \in \mathcal{P}(d)} \{D_I(\mathcal{G})\}$$

is the set of all square-free, primitive totally-reflective genera in dimension  $n$ .

The techniques we will use now are based on the following definition going back to Watson, cf. [36], [37]. Let  $L$  be an integral lattice and  $p \in \mathbb{P}$ . The *Watson transformation* of  $L$  at  $p$  is defined as  $E_p(L) := L + (\frac{1}{p}L \cap pL^{\#})$ . The usefulness of  $E_p$  becomes clear when we consider its effect on the Jordan decomposition. Let  $L$  be an integral lattice with  $L_p = L_0 \perp {}^p L_1 \perp \dots \perp {}^{p^r} L_r$ . Then

$$E_p(L) \otimes_{\mathbb{Z}} \mathbb{Z}_q = \begin{cases} (L_0 \perp L_2) \perp {}^p(L_1 \perp L_3) \perp {}^{p^2} L_4 \perp \dots \perp {}^{p^{r-2}} L_r, & \text{if } q = p, \\ L \otimes_{\mathbb{Z}} \mathbb{Z}_q, & \text{if } q \neq p. \end{cases}$$

Hence, after repeatedly applying the Watson transformation, a primitive lattice transforms into a square-free primitive lattice. Similar to the partial dual,  $E_p$  extends to a well-defined surjective function  $\mathcal{G}(L) \rightarrow E_p(\mathcal{G}(L)) = \mathcal{G}(E_p(L))$ .

**Lemma 3.11.** If  $L$  is totally-reflective lattice, then so is  $E_p(L)$ .

**Proof.** The assertion implies that  $W(L)$  has no non-zero fixed vectors, thus neither has  $W(E_p(L))$  since  $W(L) \subseteq W(E_p(L))$ . Hence the assertion follows from the surjectivity of  $E_p$ .  $\square$

It may happen that prime factors disappear from the determinant after applying the Watson transformation. Thus, when calculating pre-images under  $E_p$ , one has to decide which primes  $p$  to consider (besides the prime factors of the determinant). An answer to this question is given by the following two lemmata.

**Lemma 3.12.** *Let  $L$  be an integral lattice,  $p$  an odd prime with  $p \nmid \det L$  and  $K \in E_p(L)^{-1}$ .*

a) *If  $\dim L = 3$  then,*

$$m(K) \geq \left( \left( \frac{1}{1+p^{-1}} \right)^2 p^2 (1-p^{-2}) \right) \cdot m(L).$$

b) *If  $\dim L = 4$  then,*

$$m(K) \geq \left( \frac{\zeta(4)}{2\zeta(2)^2} \left( \frac{1}{1+p^{-1}} \right)^2 p^3 (1-p^{-2}) \right) \cdot m(L).$$

**Proof.** This follows from the mass formula and Lemma 2.2.  $\square$

**Lemma 3.13.** *Let  $L$  be an integral lattice,  $p$  an odd prime with  $p \nmid \det L$  and  $K \in E_p(L)^{-1}$ .*

a) *If  $\dim L = 3$ , then*

$$M_{\text{ref}}(K) \leq 81 \cdot M_{\text{ref}}(L).$$

b) *If  $\dim L = 4$ , then*

$$M_{\text{ref}}(K) \leq 4096 \cdot M_{\text{ref}}(L).$$

**Proof.** b) Since  $L$  is not necessarily strongly square-free, we need to consider the upper bound for  $m_{\text{ref}3}$  as well. Let  $d := \det L$ . The assumption implies  $K \otimes_{\mathbb{Z}} \mathbb{Z}_p = K_0 \perp {}^{p^2}K_2$ , in particular  $\det K = \det L \cdot p^{2n_2}$  where  $n_2 := \dim K_2 \in \{0, 1, 2, 3\}$ . Thus

$$M_{\text{ref}4}(K) = \frac{611}{1920} \cdot 4^{\Omega(dp^{2n_2})} = 4^{2n_2} \cdot M_{\text{ref}4}(L).$$

For the other two cases we consider the decomposition of the divisor set

$$D(\det K) = \bigcup_{i=0}^{2n_2} p^i D(d).$$

Hence

$$M_{\text{ref}3}(K) = \sum_{i=0}^{2n_2} \sum_{x|D(d)} \frac{17}{96} \cdot 3^{\Omega(x)} \cdot 3^i = \left( \frac{1-3^{2n_2+1}}{1-3} \right) \cdot M_{\text{ref}3}(L)$$

and

$$M_{\text{ref}2}(K) = \frac{17}{24} \cdot a(dp^{2n_2}) \cdot 2^{\Omega(dp^{2n_2})} = (2n_2 + 1) \cdot 2^{n_2} \cdot M_{\text{ref}2}(L).$$

Finally note that  $\max\{4^{2n_2}, \left(\frac{1-3^{2n_2+1}}{1-3}\right), (2n_2 + 1) \cdot 2^{n_2}\} \leq 4096$  for  $n_2 \in \{1, 2, 3\}$ .

Part a) of this lemma is proven analogously.  $\square$

The following corollary serves to answers which prime numbers are relevant to the calculation of pre-images under the Watson-transformation.

**Corollary 3.14.** *Let  $L$  be an integral lattice,  $p$  an odd prime with  $p \nmid \det L$  and  $K \in E_p^{-1}(\{L\})$ .*

a) *If  $K$  is totally-reflective and  $\dim L = 3$ , then*

$$81 \cdot \frac{M_{\text{ref}}(L)}{m(L)} \cdot \left( \left( \frac{1}{1+p^{-1}} \right)^2 p^2 (1-p^{-2}) \right)^{-1} \geq 1.$$

b) *If  $K$  is totally-reflective and  $\dim L = 4$ , then*

$$4096 \cdot \frac{M_{\text{ref}}(L)}{m(L)} \cdot \left( \frac{\zeta(4)}{2\zeta(2)^2} \left( \frac{1}{1+p^{-1}} \right)^2 p^3 (1-p^{-2}) \right)^{-1} \geq 1.$$

**Proof.** Combine Lemma 3.12 and Lemma 3.13.  $\square$

Since the  $p$ -term in the above inequalities depends monotonically decreasingly on  $p$  (for  $p \nmid 2 \det L$ ) and  $M_{\text{ref}}(L)/m(L)$  does not depend on  $p$  at all, it is straightforward to decide when the statement of Corollary 3.14 is satisfied.

### 3.3. Classification result

Given the set of all square-free, primitive totally-reflective genera, one can produce all totally-reflective primitive genera by using Corollary 3.14 and Lemma 3.11. First, Corollary 3.14 tells us which (finitely many) primes one has to consider when calculating pre-images under  $E_p$ . Then, during the process of repeatedly generating lattices  $K \in E_p^{-1}(\{L\})$ , Lemma 3.11 tells us that we can stop and proceed with the next lattice when a not totally-reflective lattice occurs. Eventually this process will terminate since the number of totally-reflective genera is finite. Furthermore, Lemma 3.11 also implies that every totally-reflective genus will be produced this way. We have implemented this procedure in MAGMA and it takes around 24 hours to find all totally-reflective genera.

**Theorem 3.15.**

- a) In dimension 3, there are 1234 primitive totally-reflective genera of which 289 are square-free and 52 strongly square-free.
- b) In dimension 4, there are 930 primitive totally-reflective genera of which 230 are square-free and 88 strongly square-free.

The complete list of all totally-reflective genera in dimension 3 and 4 is available separately as [28] and in the appendix of [29].

**4. Reflective Lorentzian lattices and arithmetic reflection groups**

For the following, we assume some basic knowledge of hyperbolic geometry as can be found in [24], Chapter 3 to 7. Throughout this chapter let  $(V, (\cdot, \cdot))$  be a Lorentzian space of signature  $(n, 1)$ . The set  $\{v \in V \mid (v, v) = -1\}$  falls into two connected components. We pick an arbitrary component  $H^n$  and stick with it from now on. This is our model for the *hyperbolic  $n$ -space*. The full isometry group of  $H^n$  is a subgroup of  $O(V)$  of index 2 and will be denoted as  $O^+(V)$ . Discrete subgroups of  $O^+(V)$  generated by reflections are called *hyperbolic reflection groups*. An integral  $\mathbb{Z}$ -lattice of signature  $(n, 1)$  is called a *Lorentzian lattice*. In order to investigate the connections between arithmetic reflection groups and Lorentzian lattices, we need to adapt the notion of *root systems* to the indefinite case. Let  $E$  be a Lorentzian lattice. A primitive vector  $v \in E$  is called *root* of  $E$  if  $s_v(E) = E$  and  $(v, v) > 0$ . The set  $R(E) := \{v \in E \mid v \text{ is a root of } E\}$  is called *root system* of  $E$ . The subgroup  $W(E) \leq O(E)$  generated by all reflections  $s_v$ , with  $v \in R(E)$ , is called *Weyl group* of  $E$ . Obviously, the condition  $(v, v) > 0$  is redundant in the positive-definite case but here it is made for the following reason:  $(v, v) > 0$  iff  $s_v \in O^+(V)$ . In contrast to the positive-definite case,  $R(E)$  is not a root system in the sense of Lie algebra; in general  $R(E)$  is not a finite set. Nevertheless, the major concept of fundamental roots can be transferred to the Lorentzian case. A set of *fundamental roots* for  $R(E)$  is by definition obtained as follows. The set of all reflecting hyperplanes  $h_v := \{x \in H^n \mid (x, v) = 0\}$ ,  $v \in R(E)$ , is locally finite and divides  $H^n$  into connected components. Actually, the group  $W(E)$  acts transitively on the connected components of  $H^n \setminus \bigcup h_v$  (the so-called *open chambers*). Pick an open chamber  $P(E)^\circ$  and let  $P(E)$  be its closure. The set  $P(E)$  is of the form

$$P(E) = \bigcap_{v \in \tilde{R}(E)} h_v^+$$

for a unique minimal subset  $\tilde{R}(E) \subset R(E)$  (consider the roots belonging to the supporting hyperplanes of  $P(E)$ , cf. [24], Theorem 6.3.2).

**Definition 4.1.** Let  $E$  be a Lorentzian lattice. The elements of  $\tilde{R}(E)$  constructed as above are called *fundamental roots* of  $E$  (depending on the choice of  $P(E)$ ).

Recall that a hyperbolic reflection group  $W$  is called *maximal* if there is no other hyperbolic reflecting group  $W'$  such that  $W \subsetneq W' \subseteq O^+(V)$ . As it will turn out, Lorentzian lattices with a finite set of fundamental roots are precisely those which can be used to describe maximal arithmetic reflection groups. For now, we make the following

**Definition 4.2.** A Lorentzian lattice  $E$  is called *reflective* if  $W(E)$  is of finite index in  $O^+(E)$ .

**Lemma 4.3** (cf. [31]). *Let  $E$  be a reflective Lorentzian lattice. Then, for every isotropic vector  $v \in E$ , the positive-definite lattice  $v^\perp/\mathbb{Z}v$  is reflective.*

In the following, we will write  $\mathbb{H}$  for the *hyperbolic plane*, that is the integral  $\mathbb{Z}$ -lattice with Gram matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Corollary 4.4.** *Let  $E$  be a reflective Lorentzian lattice of the form*

$$E = {}^\alpha\mathbb{H} \perp L,$$

*with  $L$  being a positive-definite  $\mathbb{Z}$ -lattice and  $\alpha \in \mathbb{Q}$ . Then  $L$  is totally-reflective.*

**Proof.** Notice that  $E = {}^\alpha\mathbb{H} \perp L$  only depends on the genus of  $L$ . By this we mean that

$${}^\alpha\mathbb{H} \perp L \cong {}^\alpha\mathbb{H} \perp L',$$

for every  $L' \in \mathcal{G}(L)$ . This follows from

$$({}^\alpha\mathbb{H} \perp L) \otimes \mathbb{Z}_p \cong ({}^\alpha\mathbb{H} \perp L') \otimes \mathbb{Z}_p,$$

for all  $p \in \mathbb{P} \cup \{\infty\}$ , and the fact that lattices of the form  $\mathbb{H} \perp L$  (with  $\dim L \geq 1$ ) are always of class number one (cf. [13]). The assertion follows then from Vinberg's lemma with  $v$  chosen to be an isotropic basis vector of  $\mathbb{H}$ .  $\square$

**Theorem 4.5.**

- a) *Let  $E$  be a strongly square-free reflective Lorentzian lattice of signature  $(5, 1)$ . Then  $E$  is of the form  $E = {}^\alpha\mathbb{H} \perp L$  with  $L$  a square-free totally-reflective lattice of dimension 4 and  $\alpha \in \{1, 2\}$ . The scaling factor  $\alpha = 2$  only occurs if the 2-adic symbol of  $E$  is of the form  $I_{5,1} \left( 1^{\varepsilon_0, 4} 2_{\text{II}}^{\varepsilon_1, 2} \right)$ .*
- b) *Let  $W$  be a maximal arithmetic reflection group on the hyperbolic 5-space. Then  $W$  is of the form  $W = W({}^\alpha\mathbb{H} \perp L)$  with  $L$  a square-free totally-reflective lattice of dimension 4 and  $\alpha \in \{1, 2\}$ .*

**Proof.** a) Let  $E$  be a strongly square-free Lorentzian lattice of signature  $(5, 1)$ . We want to show that  ${}^\alpha\mathbb{H}$  splits off in  $E$ . Because the class number of  $E$  is one, it is sufficient to show that  ${}^\alpha\mathbb{H}$  splits off at every prime  $p \in \mathbb{P}$ . The Jordan decomposition of  $E$  is of the form  $E \otimes \mathbb{Z}_p = E_0 \perp {}^pE_1$ , with  $\dim E_0 \in \{3, 4, 5, 6\}$  and  $\dim E_1 = 6 - \dim E_0$ . If  $p \neq 2$  then [22], 92 : 1, implies that  $\mathbb{H}$  and  ${}^2\mathbb{H}$  split off in  $E_0$  and thus in  $E \otimes \mathbb{Z}_p$ . Now let  $p = 2$ . Dimensionwise, the following 2-adic symbols are possible:

$(\dim E_0, \dim E_1)$	
$(6, 0)$	(a1) $\text{II}_{5,1} (1^{\varepsilon_0, 6})$ (a2) $\text{I}_{5,1} (1^{\varepsilon_0, 6})$
$(5, 1)$	(b) $\text{I}_{5,1} (1^{\varepsilon_0, 5} 2_1^{\varepsilon_1, 1})$
$(4, 2)$	(c1) $\text{II}_{5,1} (1^{\varepsilon_0, 4} 2_{\text{II}}^{\varepsilon_1, 2})$ (c2) $\text{I}_{5,1} (1^{\varepsilon_0, 4} 2_{\text{II}}^{\varepsilon_1, 2})$
	(c3) $\text{II}_{5,1} (1^{\varepsilon_0, 4} 2_1^{\varepsilon_1, 2})$ (c4) $\text{I}_{5,1} (1^{\varepsilon_0, 4} 2_1^{\varepsilon_1, 2})$
$(3, 3)$	(d) $\text{I}_{5,1} (1^{\varepsilon_0, 3} 2_1^{\varepsilon_1, 3})$

In the following,  $A_2$  is the binary even lattice with Gram matrix  $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

“(a1), (a2), (b), (c1), (c3)”: [11], Theorem 5.2.5 and Proposition 5.2.3 imply that  $\mathbb{H}$  splits off in  $E_0$ .

“(c2)”: We want to show that  ${}^2\mathbb{H}$  splits off. The unimodular Jordan component is of the form  $E_0 = \mathbb{H} \perp \langle a, b \rangle$  or  $E_0 = A_2 \perp \langle a, b \rangle$  with  $a, b \in \mathbb{Z}_2$  odd. The 2-modular component is of the form  $E_1 = \mathbb{H}$  or  $E_1 = A_2$ . If  $E_1 = \mathbb{H}$  then  ${}^2\mathbb{H}$  splits off in  $E \otimes \mathbb{Z}_2$ . If  $E_1 = A_2$  then the following isometries show that, again,  ${}^2\mathbb{H}$  splits off in  $E \otimes \mathbb{Z}_2$ :

$$E \otimes \mathbb{Z}_2 = \begin{cases} A_2 \perp \langle a, b \rangle \perp {}^2A_2 & \cong A_2 \perp \langle a + 4, b \rangle \perp {}^2\mathbb{H}, \\ \mathbb{H} \perp \langle a, b \rangle \perp {}^2A_2 & \cong \mathbb{H} \perp \langle a + 4, b \rangle \perp {}^2\mathbb{H}. \end{cases}$$

“(c4)”: We want to show that  $\mathbb{H}$  splits off. The lattice  $E \otimes \mathbb{Z}_2$  is of the form  $E \otimes \mathbb{Z}_2 = \mathbb{H} \perp \langle a, b \rangle \perp {}^2\langle x, y \rangle$  or  $E \otimes \mathbb{Z}_2 = A_2 \perp \langle a, b \rangle \perp {}^2\langle x, y \rangle$  with  $a, b, x, y \in \mathbb{Z}_2$  odd. In the first case the assertion is clear. In the second case the assertion follows from

$$A_2 \perp \langle a, b \rangle \perp {}^2\langle x, y \rangle \cong \mathbb{H} \perp \langle a, b \rangle \perp {}^2\langle x + 4, y \rangle.$$

“(d)”: We want to show that  $\mathbb{H}$  splits off. The unimodular Jordan component is of the form  $E_0 = \mathbb{H} \perp \langle a \rangle$  or  $E_0 = A_2 \perp \langle a \rangle$  with  $a \in \mathbb{Z}_2$  odd. In the first case the assertion is clear. In the second case, let  $E_1$  be either  $\mathbb{H} \perp \langle b \rangle$  or  $A_2 \perp \langle b \rangle$  with  $b \in \mathbb{Z}_2$  odd. Then the following isometries imply the assertion:

$$E \otimes \mathbb{Z}_2 = \begin{cases} A_2 \perp \langle a \rangle \perp {}^2(\mathbb{H} \perp \langle b \rangle) & \cong \mathbb{H} \perp \langle a + 4 \rangle \perp {}^2(A_2 \perp \langle b + 4 \rangle), \\ A_2 \perp \langle a \rangle \perp {}^2(A_2 \perp \langle b \rangle) & \cong \mathbb{H} \perp \langle a + 4 \rangle \perp {}^2(\mathbb{H} \perp \langle b + 4 \rangle). \end{cases}$$

All isometries above are defined over  $\mathbb{Z}_2$  and, thus, can be checked by calculating the 2-adic symbol. In every case,  $E$  can be written as  $E = {}^\alpha\mathbb{H} \perp L$  with  $\alpha \in \{1, 2\}$ . Corollary 4.4 implies that  $L$  is totally-reflective.

b) Since  $W$  is maximal and arithmetic one can find a Lorentzian lattice  $E$  with  $W = W(E)$ . After repeated use of  $D_p$  and  $E_p$  the lattice  $E$  can be transformed into a strongly square-free lattice without changing  $W$ ; like in the proof of Lemma 3.11 use the relation  $W(E) \subseteq W(E_p(E))$  and  $W(E) = W(D_p(E))$ . The assertion then follows from a).  $\square$

The classification of strongly square-free, reflective Lorentzian lattice has been carried out computationally in MAGMA. We have picked a lattice  $L$  from every square-free totally-reflective genus and have applied Vinberg's algorithm to  $E = {}^\alpha\mathbb{H} \perp L$ . As shown in [30] and [33], if the algorithm terminates, then  $E$  is proven to be reflective. If the algorithm does not terminate (we waited for around  $10^3$  roots to be found), then non-reflectivity is proven with the methods discussed below.

**Theorem 4.6.** *The Lorentzian lattices of signature  $(5, 1)$  in the Table 3 below are reflective. Every strongly square-free, reflective Lorentzian lattice of signature  $(5, 1)$  is isometric to one in that table.*

**Corollary 4.7.** *Every maximal arithmetic reflection group  $W$  on  $H^5$  is of the form  $W = W(E)$  with  $E$  being a Lorentzian lattice from the table below.*

**Proof.** This follows from the structure Theorem 4.5 and above classification.  $\square$

The notation  ${}^\alpha\mathbb{H} \perp \text{Genus}$  means that  $L$  can be chosen arbitrarily within the given 4-dimensional genus. The combinatorial structure of the fundamental polyhedron is given as follows:

- $r$  = Number of fundamental roots = Number of 4-dimensional faces,
- $f_3$  = Number of 3-dimensional faces,
- $f_2$  = Number of 2-dimensional faces,
- $e$  = Number of edges,
- $v$  = Number of vertices,
- $c$  = Number of cusps (vertices at infinity).

The lattices are ordered by the parities of the 2-adic Jordan components of  $L$ .

As mentioned above, if our MAGMA implementation of Vinberg's algorithm produces, say, more than  $10^3$  fundamental roots then we can expect the lattice to be non-reflective. We will now present two methods to prove this rigorously. The first one consists of realizing non-reflective lattices of smaller dimension as orthogonal summands.

**Theorem 4.8.** *Let  $E$  be a reflective Lorentzian lattice. Then every Lorentzian orthogonal summand of  $E$  is reflective.*

**Proof.** See [7], §2.  $\square$

Table 3

Strongly square-free reflective lattices of signature (5, 1).

No.	– det	Lattice	$r$	$f_3$	$f_2$	$e$	$v$	$c$
1	5	$\mathbb{H} \perp \text{II}(5^{+1})$	6	15	20	15	5	1
2	9	$\mathbb{H} \perp \text{II}(3^{+2})$	7	21	33	27	9	1
3	21	$\mathbb{H} \perp \text{II}(3^{-1}7^{+1})$	9	32	57	51	18	1
4	25	$\mathbb{H} \perp \text{II}(5^{-2})$	9	33	61	57	21	1
5	45	$\mathbb{H} \perp \text{II}(3^{-2}5^{+1})$	12	50	98	92	30	4
6	49	$\mathbb{H} \perp \text{II}(7^{+2})$	16	80	176	176	64	2
7	125	$\mathbb{H} \perp \text{II}(5^{+3})$	10	40	80	80	30	2
8	1	$\mathbb{H} \perp \text{I}(1_4^{+4})$	6	15	20	15	5	1
9	3	$\mathbb{H} \perp \text{I}(3^{-1})$	7	21	33	27	9	1
10	3	$\mathbb{H} \perp \text{I}(3^{+1})$	8	25	40	34	12	1
11	5	$\mathbb{H} \perp \text{I}(5^{+1})$	8	25	40	34	11	2
12	5	$\mathbb{H} \perp \text{I}(5^{-1})$	9	32	57	51	18	1
13	7	$\mathbb{H} \perp \text{I}(7^{+1})$	11	42	77	70	24	2
14	9	$\mathbb{H} \perp \text{I}(3^{+2})$	8	28	50	44	14	2
15	9	$\mathbb{H} \perp \text{I}(3^{-2})$	9	32	57	51	16	3
16	15	$\mathbb{H} \perp \text{I}(3^{+1}5^{+1})$	16	74	153	148	52	3
17	15	$\mathbb{H} \perp \text{I}(3^{-1}5^{-1})$	15	66	131	122	40	4
18	15	$\mathbb{H} \perp \text{I}(3^{+1}5^{-1})$	12	54	114	113	42	1
19	25	$\mathbb{H} \perp \text{I}(5^{-2})$	21	120	282	288	102	5
20	25	$\mathbb{H} \perp \text{I}(5^{+2})$	14	67	144	142	46	7
21	27	$\mathbb{H} \perp \text{I}(3^{+3})$	9	34	64	58	18	3
22	27	$\mathbb{H} \perp \text{I}(3^{-3})$	9	34	64	58	18	3
23	75	$\mathbb{H} \perp \text{I}(3^{+1}5^{-2})$	86	672	1788	1902	660	42
24	125	$\mathbb{H} \perp \text{I}(5^{+3})$	20	115	280	295	100	12
25	2	$\mathbb{H} \perp \text{I}(2_1^{+1})$	7	20	30	24	8	1
26	6	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{+1})$	9	32	57	51	18	1
27	6	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{-1})$	9	31	53	45	14	2
28	10	$\mathbb{H} \perp \text{I}(2_1^{+1}5^{-1})$	12	49	94	86	28	3
29	14	$\mathbb{H} \perp \text{I}(2_1^{+1}7^{+1})$	15	67	135	127	42	4
30	18	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{-2})$	14	62	125	116	36	5
31	18	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{+2})$	12	51	101	93	30	3
32	54	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{+3})$	16	75	156	145	42	8
33	54	$\mathbb{H} \perp \text{I}(2_1^{+1}3^{-3})$	18	99	230	231	78	6
34	4	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2})$	6	15	20	15	5	1
35	20	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{+2}5^{+1})$	8	25	40	34	11	2
36	20	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}5^{-1})$	8	25	40	34	12	1
37	36	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}3^{-2})$	7	21	33	27	8	2
38	36	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{+2}3^{+2})$	8	28	50	44	14	2
39	84	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}3^{+1}7^{+1})$	16	74	153	148	52	3
40	100	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{+2}5^{+2})$	9	33	61	57	19	3
41	100	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{+2}5^{-2})$	21	120	282	288	102	5
42	180	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}3^{+2}5^{+1})$	16	80	177	178	64	3
43	196	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}7^{-2})$	15	72	156	159	54	8
44	500	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{+2}5^{+3})$	20	115	280	295	100	12
45	500	$\mathbb{H} \perp \text{II}(2_{\text{II}}^{-2}5^{-3})$	12	56	124	126	44	4
46	12	$\mathbb{H} \perp \text{II}(2_2^{+2}3^{-1})$	7	21	33	27	9	1
47	12	$\mathbb{H} \perp \text{II}(2_6^{+2}3^{+1})$	8	25	40	34	12	1
48	28	$\mathbb{H} \perp \text{II}(2_2^{+2}7^{+1})$	11	42	77	70	24	2
49	36	$\mathbb{H} \perp \text{II}(2_0^{+2}3^{+2})$	10	38	69	59	17	3
50	60	$\mathbb{H} \perp \text{II}(2_2^{+2}3^{+1}5^{+1})$	16	74	153	148	52	3

(continued on next page)



Table 3 (continued)

No.	– det	Lattice	$r$	$f_3$	$f_2$	$e$	$v$	$c$
51	60	$\mathbb{H} \perp \text{II}(2_6^{+2}3^{-1}5^{-1})$	15	66	131	122	40	4
52	60	$\mathbb{H} \perp \text{II}(2_6^{+2}3^{+1}5^{-1})$	12	54	114	113	42	1
53	108	$\mathbb{H} \perp \text{II}(2_6^{+2}3^{-3})$	9	34	64	58	18	3
54	108	$\mathbb{H} \perp \text{II}(2_3^{+2}3^{+3})$	9	34	64	58	18	3
55	300	$\mathbb{H} \perp \text{II}(2_2^{+2}3^{+1}5^{-2})$	86	672	1788	1902	660	42
56	$2^2 \cdot 1$	${}^2\mathbb{H} \perp \text{I}(1_4^{+4})$	6	15	20	15	4	2
57	$2^2 \cdot 3$	${}^2\mathbb{H} \perp \text{I}(3^{-1})$	8	26	43	36	11	2
58	$2^2 \cdot 3$	${}^2\mathbb{H} \perp \text{I}(3^{+1})$	8	26	43	36	11	2
59	$2^2 \cdot 5$	${}^2\mathbb{H} \perp \text{I}(5^{+1})$	11	40	70	62	20	3
60	$2^2 \cdot 7$	${}^2\mathbb{H} \perp \text{I}(7^{+1})$	16	70	138	128	40	6
61	$2^2 \cdot 9$	${}^2\mathbb{H} \perp \text{I}(3^{+2})$	11	47	94	87	27	4
62	$2^2 \cdot 9$	${}^2\mathbb{H} \perp \text{I}(3^{-2})$	9	33	60	53	15	4
63	$2^2 \cdot 25$	${}^2\mathbb{H} \perp \text{I}(5^{+2})$	29	176	420	432	132	29
64	$2^2 \cdot 27$	${}^2\mathbb{H} \perp \text{I}(3^{+3})$	10	41	82	77	24	4
65	$2^2 \cdot 27$	${}^2\mathbb{H} \perp \text{I}(3^{-3})$	10	41	82	77	24	4
66	4	$\mathbb{H} \perp \text{I}(2_2^{+2})$	7	21	33	27	8	2
67	12	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{-1})$	9	33	59	50	14	3
68	12	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{+1})$	11	42	77	68	20	4
69	20	$\mathbb{H} \perp \text{I}(2_2^{+2}5^{-1})$	16	74	151	140	44	5
70	36	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{-2})$	16	83	184	177	54	8
71	36	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{+2})$	11	47	94	87	27	4
72	108	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{+3})$	16	85	193	188	56	10
73	108	$\mathbb{H} \perp \text{I}(2_2^{+2}3^{-3})$	16	85	193	188	56	10
74	8	$\mathbb{H} \perp \text{I}(2_3^{+3})$	8	25	40	33	10	2
75	24	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{+1})$	11	43	80	71	22	3
76	24	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{-1})$	11	43	80	71	22	3
77	72	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{-2})$	24	128	284	274	84	12
78	72	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{+2})$	14	66	140	134	44	4
79	216	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{+3})$	24	138	324	324	104	12
80	216	$\mathbb{H} \perp \text{I}(2_3^{+3}3^{-3})$	24	138	324	324	104	12

The question whether a non-reflective lattice of smaller dimension is embeddable into a larger one can be answered in a simple manner by our structure Theorem 4.5.

**Example 4.9.** Consider the genus  $\text{II}(2_6^{+2}3^{-3}5^{-3})$ . It contains three lattices,

$$L_1 \cong \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 30 \end{pmatrix}, \quad L_2 \cong \begin{pmatrix} 10 & -5 & 0 & 0 \\ -5 & 10 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 30 \end{pmatrix}, \quad L_3 \cong \begin{pmatrix} 4 & 2 & -2 & -2 \\ 2 & 16 & -1 & -1 \\ -2 & -1 & 16 & 1 \\ -2 & -1 & 1 & 16 \end{pmatrix},$$

with the full-rank root systems

$$R(L_1) = {}^3\text{A}_1{}^5\text{A}_1{}^{30}\text{B}_2,$$

$$R(L_2) = {}^3\text{A}_1{}^{15}\text{A}_1{}^{15}\text{G}_2,$$

$$R(L_3) = {}^2\text{A}_1{}^{15}\text{C}_3.$$

This genus is totally-reflective and  $E := \mathbb{H} \perp L_1$  is a candidate for a reflective lattice of signature  $(5, 1)$ . In this case Vinberg's algorithm produces over 1300 fundamental roots in a short amount of time, thus one can expect this lattice to be non-reflective. The latter property can be proven rigorously with Theorem 4.8 if we can find a non-reflective orthogonal summand of  $E$ . Considering the Gram matrix of  $L_1$ , the 5-dimensional lattice

$$\tilde{E} := \mathbb{H} \perp \tilde{L}_1$$

with  $\tilde{L}_1 \cong \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$  is easily recognizable as a orthogonal summand of  $E$ . The classification result in [35] implies that  $\tilde{E}$  is not a reflective Lorentzian lattice, thus neither is  $E$ .

The second method consists in finding symmetries of infinite order of the polyhedron defined by some so-far-found fundamental roots. This is done by inspecting the automorphism group of the Coxeter diagram.

**Lemma 4.10.** *Let  $H \leq O^+(V)$  be a discrete subgroup. Let*

$$\text{Fix}(H) := \{v \in V \mid \forall \varphi \in H : \varphi(v) = v\}$$

*be a set of fixed vectors of  $H$  and  $C := \{v \in V \mid (v, v) < 0\}$ . The group  $H$  is infinite iff  $\text{Fix}(H) \cap C = \emptyset$ .*

**Proof.** See [7], Lemma 3.1.  $\square$

We can apply the above lemma to  $H := \text{Sym}(P(E)) \cap O^+(E)$ , where  $P(E)$  is the fundamental polyhedron of  $W(E)$ . Elements of  $H$  can be found with the following theorem.

**Theorem 4.11.** *Let  $E$  be a Lorentzian lattice and  $\{v_1, \dots, v_m\} \subseteq \tilde{R}(E)$  a finite set of fundamental roots of  $R(E)$  generated by Vinberg's algorithm. Let  $\Gamma_m$  be the corresponding subgraph of the Coxeter diagram. If the following conditions hold*

- 1) *the vectors  $v_1, \dots, v_m$  generate  $E$ ,*
- 2) *there is a subset  $J \subseteq \{1, \dots, m\}$  such that  $\bigcap_{j \in J} h_{v_j}^+$  is a vertex or a cusp,*

*then every non-trivial graph automorphism of  $\Gamma_m$  induces an non-trivial element of  $\text{Sym}(P(E)) \cap O^+(E)$ .*

**Proof.** See [7], Lemma 3.3.  $\square$

We have automated and implemented the calculation of the automorphism group of a Coxeter diagram and the search for suitable automorphisms in MAGMA.

**Example 4.12.** Consider the totally-reflective genus  $\text{II}(2_{\text{II}}^{-2}17^{+1})$ . It consists of two lattices

$$L_1 \cong \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 18 \end{pmatrix}, \quad L_2 \cong \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 10 \end{pmatrix}$$

which have the root systems

$$R(L_1) = {}^{34}\text{A}_1\text{C}_3, \\ R(L_2) = {}^{17}\text{A}_1{}^2\text{B}_3.$$

Vinberg's algorithm, applied to  $E := \mathbb{H} \perp L_1$ , did not terminate after 100 hours of runtime, hence we expected  $E$  to be non-reflective. Notice that both lattices are indecomposable, thus embedding smaller lattices does not seem to be promising. We will show that  $E$  is not reflective by constructing an symmetry of infinite order of  $P(E)$  using Theorem 4.11. It then follows from

$$O^+(E) = W(E) \rtimes H$$

with  $H := \text{Sym}(P(E)) \cap O^+(E)$  that  $W(E)$  is not of finite index in  $O^+(E)$  (cf. [31], Proposition 3). The following set of fundamental roots of  $E$  satisfies the condition of Theorem 4.11:

$$\begin{aligned} v_1 &= (0, 0, -1, 0, 1, 0), & v_2 &= (0, 0, 1, -1, 1, 0), \\ v_3 &= (0, 0, 0, 1, -1, 0), & v_4 &= (-1, 0, 0, -1, 0, 0), \\ v_5 &= (-4, 2, 0, 0, -1, -1), & v_6 &= (-8, 1, 1, 0, -1, -1), \\ v_7 &= (-11, 3, 1, 1, -1, -2). \end{aligned}$$

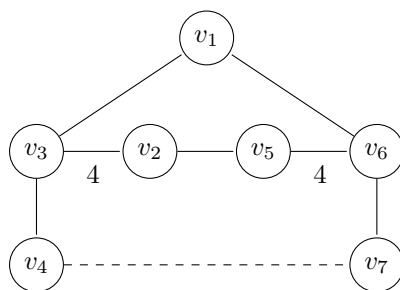
The first condition requires the roots to generate  $E$ , which can be checked with standard routines in MAGMA. Regarding the second condition, we recognize

$$h_{v_1}^+ \cap h_{v_2}^+ \cap h_{v_3}^+ \cap h_{v_6}^+ \cap h_{v_7}^+$$

as a vertex since the normalized Gram matrix of the system  $(v_1, v_2, v_3, v_6, v_7)$ ,

$$\begin{pmatrix} 1 & 0 & -\cos(\frac{\pi}{3}) & -\cos(\frac{\pi}{3}) & 0 \\ 0 & 1 & -\cos(\frac{\pi}{4}) & 0 & 0 \\ -\cos(\frac{\pi}{3}) & -\cos(\frac{\pi}{4}) & 1 & 0 & 0 \\ -\cos(\frac{\pi}{3}) & 0 & 0 & 1 & -\cos(\frac{\pi}{3}) \\ 0 & 0 & 0 & -\cos(\frac{\pi}{3}) & 1 \end{pmatrix}$$

is positive-definite (this is sufficient according to [30]). Let  $\Gamma_7$  be the Coxeter diagram of the vector system  $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ ,



Theorem 4.11 implies that every non-trivial graph automorphism induces a non-trivial symmetry from  $Sym(P(E)) \cap O^+(E)$ . As can be seen above, the permutation

$$(2, 5)(3, 6)(4, 7)$$

is a graph automorphism of  $\Gamma_7$  and the symmetry  $\varphi$  that permutes the corresponding  $v_i$  has infinite order. The latter statement follows from Lemma 4.10; the set of fixed vectors of  $\varphi$  is the one-dimensional subspace of  $E \otimes \mathbb{Q}$  generated by  $(1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4})$ . Since this generator is of length  $\frac{1}{4}$ , the subspace has an empty intersection with the light cone (the set of all negative-norm vectors) and thus the subgroup of  $Sym(P)$  generated by  $\varphi$  has infinite order.

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