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TWISTS OF QUANTUM BOREL ALGEBRAS

CRIS NEGRON

Abstract. We classify Drinfeld twists for the quantum Borel subalgebra $u_q(\mathfrak{b})$ in the Frobenius-Lusztig kernel $u_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and q an odd root of unity. More specifically, we show that alternating forms on the character group of the group of grouplikes for $u_q(\mathfrak{b})$ generate all twists for $u_q(\mathfrak{b})$, under a certain algebraic group action. This implies a simple classification of finite-dimensional Hopf algebras whose categories of representations are tensor equivalent to that of $u_q(\mathfrak{b})$. We also show that cocycle twists for the corresponding De Concini-Kac algebra are in bijection with alternating forms on the aforementioned character group.

1. Introduction

In this paper we classify Drinfeld twists for the quantum Borel subalgebra $u_q(\mathfrak{b})$ in the Frobenius-Lusztig kernel $u_q(\mathfrak{g})$, for a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ at an odd root of unity q. The algebra $u_q(\mathfrak{g})$ is also known as the *small quantum group*. We adopt some additional, minor, restrictions on the order of q which depend on the Dynkin type of $\mathfrak{g}$ (see Section 2).

Recall that Drinfeld twists for a given finite dimensional Hopf algebra H correspond to tensor structures on the forgetful functor from $\mathrm{rep}(H)$ to $Vect$. Here $\mathrm{rep}(H)$ denotes the tensor category of finite dimensional H-modules. Two twists are said to be gauge equivalent if their corresponding functors are naturally isomorphic. We let $\mathrm{Tw}(H)$ denote the set of gauge equivalence classes of twists for H. There is a group of *twisted autoequivalences* of H which acts on $\mathrm{Tw}(H)$, and the resulting quotient parametrizes isomorphism classes of Hopf algebras K which admit a tensor equivalence $\mathrm{rep}(K) \xrightarrow{\sim} \mathrm{rep}(H)$. (See Sections 3 and 8.1.)

For the small quantum Borel, it is known that the unipotent algebraic group $\mathbb{U}$, corresponding to the nilpotent subalgebra $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}$, acts on the collection of twists for $u_q(\mathfrak{b})$ by way of twisted automorphisms [9, 30]. Basic considerations also establish an embedding $\mathrm{Alt}(G^\vee) \to \mathrm{Tw}(u_q(\mathfrak{b}))$, where $\mathrm{Alt}(G^\vee)$ denotes the set of alternating bilinear forms on the character group $G^\vee$ of the Cartan subgroup $G = G(u_q(\mathfrak{b}))$. We show below that the set of alternating forms on $G^\vee$ generates *all* twists for $u_q(\mathfrak{b})$ under the aforementioned action of $\mathbb{U}$.

Theorem A (8.2)**.** *There is an equality* $\mathrm{Tw}\,(u_q(\mathfrak{b})) = \mathbb{U} \cdot \mathrm{Alt}(G^\vee)$.

We also show in Proposition 6.3 that the De Concini-Kac algebra $U_q^{DK}(\mathfrak{b})$ admits no *cocycle* twists up to gauge equivalence, save for those coming from the group of grouplikes. A version of Theorem A can also be shown to hold for quantum Kac-Moody algebras. In this case one should complete $u_q(\mathfrak{b})$ relative to its grading by the root lattice.

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As a consequence of Theorem A we derive the following classification result for Hopf algebras which are tensor equivalent to a small quantum Borel algebra.

Theorem B (8.3). *Suppose H is a finite dimensional Hopf algebra such that* $\mathrm{rep}(H)$ *is equivalent to* $\mathrm{rep}(u_q(\mathfrak{b}))$ *as a tensor category. Then there is an alternating form B on $G^\vee$ so that $H \cong u_q(\mathfrak{b})^B$ as Hopf algebras. When* $\mathrm{ord}(q)$ *is coprime to the determinant of the Cartan matrix for $\mathfrak{g}$, the form B is uniquely determined by H.*

Here $u_q(\mathfrak{b})^B$ denotes the Hopf algebra twisted by B (see Section 3). We note that Theorem B is well-known to hold in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In this case the quantum Borel is a Taft algebra T_q, there are no nontrivial alternating forms on the group of grouplikes, and hence any Drinfeld twist T_q^J will be Hopf isomorphic to T_q (see also [27, 17]). We also note that, even though the Hopf algebras $u_q(\mathfrak{b})^B$ are not quantum Borel algebras in general, they are still pointed with abelian group of grouplikes. Such Hopf algebras are very well-understood, via an extensive classification program carried out over the past two decades (see [5, 6, 1, 8, 22]).

The present work is motivated primarily by a conjecture regarding Drinfeld twists for the small quantum group $u_q(\mathfrak{g})$. Recall that $u_q(\mathfrak{g})$ is generated by the positive and negative Borel subalgebras $u_q(\mathfrak{b}_\pm) \subset u_q(\mathfrak{g})$. We ask in [13, 30] if there is an equality

$$\mathrm{Tw}(u_q(\mathfrak{g})) = \mathbb{G} \cdot \mathrm{BD}(u_q(\mathfrak{g})), \tag{1}$$

where $\mathrm{BD}(u_q(\mathfrak{g}))$ denotes the collection of *Belavin-Drinfeld twists* for $u_q(\mathfrak{g})$, and $\mathbb{G}$ is the simply-connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. As with $\mathbb{U}$ above, $\mathbb{G}$ acts by twisted automorphisms on $\mathrm{Tw}(u_q(\mathfrak{g}))$.

Belavin-Drinfeld twists for $u_q(\mathfrak{g})$ can be described vaguely as follows: Each twist in $\mathrm{BD}(u_q(\mathfrak{g}))$ is specified by a choice of combinatorial data on the Dynkin diagram for $\mathfrak{g}$. This data is called a *Belavin-Drinfeld triple* [11]. Such a choice of data specifies an interaction between the positive and negative roots for $\mathfrak{g}$, and hence an interaction between $u_q(\mathfrak{b}_+)$ and $u_q(\mathfrak{b}_-)$ in $u_q(\mathfrak{g})$. We use the prescribed interaction between the positive and negative Borels, along with the R-matrix for $u_q(\mathfrak{g})$, to produce an especially non-trivial twist for $u_q(\mathfrak{g})$. For example, one can drastically change the group of grouplikes after applying such a twist to $u_q(\mathfrak{g})$ (see [30]), and in an extreme *dynamical* case one can make $u_q(\mathfrak{g})$ self-dual after twisting (see [16]).

Hence, if we consider only the positive or negative Borel $u_q(\mathfrak{b})$ in $u_q(\mathfrak{g})$, we expect that there should be no such interesting twists. Specifically, we expect from the conjectural equality (1) that all twists for $u_q(\mathfrak{b})$ should be induced by twists on the group of grouplikes, up to gauge equivalence and Hopf isomorphism. Theorems A and B verify that this is in fact the case.

A description of the proof of Theorem A. As our general methods may be of interest to readers, we describe here the proof of Theorem A.

Take $u_q = u_q(\mathfrak{b})$. Below we make heavy use of a natural grading on u_q by the root lattice for $\mathfrak{g}$, which we denote by Q. Let Q^+ denote the $\mathbb{Z}_{\geq 0}$-span of the positive roots, minus 0. Given a twist J for u_q we can write $J = B + \sum_{\mu \in Q^+} J_\mu$, where each J_μ of Q-degree μ and B a twist for $\mathbb{C}[G]$. In general, one would like to eliminate the terms J_μ in some orderly fashion to produce an equivalence between J and the corresponding degree 0 term B.

We show that the nonvanishing terms J_μ appearing in J specify degree 2 cohomology classes $[J_\mu] \in H^2((u_q^*)_B)$, where $(u_q^*)_B$ is the cocycle twist of the dual u_q^* by the degree 0 term B. Furthermore, the term J_μ can be eliminated via a gauge

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transformation in u_q if and only if the corresponding cohomology class $[J_\mu]$ vanishes (see Section 5.2).

As a consequence of a natural identification between $H^2((u_q^*)_B)$ and the nilpotent subalgebra $\mathfrak{n}$ in $\mathfrak{b}$ (Proposition 6.5), we find that there are only finitely many problem terms J_γ in J which *cannot* be eliminated via gauge transformation in u_q, up to scaling. Namely, there is one such class for each positive root of $\mathfrak{g}$.

One shows that these problem terms *can* in fact be eliminated via gauge transformation by a distinguished collection of elements $\{\mathbf{v}_\gamma(\lambda)\}_{\gamma\in\Phi^+,\lambda\in\mathbb{C}}$ in a completion $\widehat{U}_q(\mathfrak{b})$ of Lusztig's big quantum Borel algebra. One shows further that the collective gauge actions of the $\mathbf{v}_\gamma(\lambda)$ reproduce a known action of $\mathbb{U}$ on $\mathrm{Tw}(u_q)$ (Section 8.1, [9]). It follows that any twist J for u_q can be reduced to the corresponding grouplike term B via gauging in u_q and the action of $\mathbb{U}$.

A formalization of the above information, and reference to a known equality $\mathrm{Tw}(\mathbb{C}[G]) = \mathrm{Alt}(G^\vee)$ [18, 21], produces the identification $\mathrm{Tw}(u_q) = \mathbb{U}\cdot\mathrm{Alt}(G^\vee)$.

Outline. Sections 2 and 3 provide background information on quantum Borel algebras and twists. Section 4 provides a duality between Lusztig's divided power algebra, and the corresponding De Concini-Kac algebra. Section 5 establishes an essential relationship between twists for quantum Borel algebras and cohomology classes for the dual algebra. Section 6 is dedicated to a presentation of the cohomologies of the relevant dual algebras. In Section 7 we prove a primordial version of Theorems A and B. In Section 8 we recall some information from [9, 30], and prove Theorems A and B. In Section 9 we provide proofs of some technical results from Section 6.

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2. Quantum Borel algebras

Given any simple Lie algebra $\mathfrak{g}$ we will consider a root of unity q of a fixed odd order l. In type G_2 we also require l to be coprime to 3, in accordance with [26]. In specific types we make the following additional assumptions on the order of q:

- In the simply laced case $l \neq 3$.
- In types B_n, C_n, and F_4, $l \neq 3, 5$.
- In type G_2, $l \neq 7$.

We make these assumptions so that certain cohomology groups will vanish (see the proofs of Propositions 6.1 and 6.5, cf. [4]).

2.1. Small quantum Borel algebras.

Let $\mathfrak{g}$ be a simple Lie algebra with fixed Cartan $\mathfrak{h}$, Dynkin diagram Γ, and $\mathfrak{b} = \mathfrak{b}_+ \subset \mathfrak{g}$ the positive Borel subalgebra. We let Φ denote the collection of roots for $\mathfrak{g}$, and $(?, ?) : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$ denote the unique scaling of the Killing form so that $(\alpha, \alpha) = 2$ whenever α is a short simple root.

If we take D to be the ratio of the lengths of a long root in Γ to that of a short root, then for the Cartan integers we have

$$\langle\alpha, \beta\rangle = D\langle\beta, \alpha\rangle, \quad \text{whenever } \beta \text{ is long and } \alpha \text{ is short.}$$

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In this case (α,β) is equal to the Cartan integer $\langle\alpha,\beta\rangle$. The number D is $1,2$, or 3 and the matrix $[(\alpha,\beta)]_{\alpha,\beta\in\Gamma}$ representing the Killing form is integer valued with nonpositive entries off the diagonal.

For α a simple root we define

$$q_\alpha = q^{(\alpha,\alpha)/2} = \begin{cases} q^D & \text{when } \alpha \text{ is a long,} \\ q & \text{when } \alpha \text{ is a short.} \end{cases}$$

We also adopt the standard expressions

$$[n]_{q_\alpha} = (q_\alpha^n - q_\alpha^{-n})/(q_\alpha - q_\alpha^{-1}) \quad \text{and} \quad \left[\begin{array}{c} n \\ m \end{array} \right]_{q_\alpha} = \frac{[n]_{q_\alpha}\dots[n-m+1]_{q_\alpha}}{[m]_{q_\alpha}\dots[1]_{q_\alpha}}.$$

The small quantum Borel algebra is given by generators and relations as

$$u_q(\mathfrak{b}) = \mathbb{C}\langle E_\alpha, K_\alpha^{\pm 1} : \alpha\in\Gamma\rangle/(\text{relations}),$$

where the set of relations are

$$[K_\alpha, K_\beta] = 0, \quad K_\alpha E_\beta = q^{(\alpha,\beta)} E_\beta K_\alpha,$$

$$\sum_{i=0}^{1-\langle\alpha,\beta\rangle} (-1)^i \left[\begin{array}{c} 1-\langle\alpha,\beta\rangle \\ i \end{array} \right]_{q_\alpha} E_\alpha^{1-\langle\alpha,\beta\rangle - i} E_\beta E_\alpha^i = 0, \tag{2}$$

$$K_\alpha^l = 1, \quad E_\mu^l = 0 \ \ \forall\ \mu\in\Phi^+.$$

The E_μ here are given by an action of the Braid group as in [25, 26, 14]. The relations (2) are called the (quantum) *Serre relations*.

The Hopf structure on $u_q(\mathfrak{b})$ is given by $\epsilon(E_\alpha)=0$, $\epsilon(K_\alpha)=1$, and

$$\Delta(K_\alpha) = K_\alpha\otimes K_\alpha, \quad \Delta(E_\alpha) = E_\alpha\otimes 1 + K_\alpha\otimes E_\alpha,$$
$$S(K_\alpha) = K_\alpha^{-1}, \quad S(E_\alpha) = -K_\alpha^{-1}E_\alpha.$$

We let G denote the group of grouplikes in $u_q(\mathfrak{b})$, which happens to be the subgroup generated by the elements K_α. We have $G\cong(\mathbb{Z}/l\mathbb{Z})^{\text{rank}(\mathfrak{g})}$ with free basis over $\mathbb{Z}/l\mathbb{Z}$ given by the K_α, $\alpha\in\Gamma$.

2.2. **Lusztig's divided powers algebra.** Let $U_q(\mathfrak{b})$ denote Lusztig's quantum Borel algebra [25, 26]. We will not give a detailed presentation of $U_q(\mathfrak{b})$ here, but remark on a few specific points. The Hopf algebra $U_q(\mathfrak{b})$ is generated by elements E_α and K_α, for α simple roots, and divided powers $E_\alpha^{(l)}$. There is a Hopf inclusion $u_q(\mathfrak{b})\to U_q(\mathfrak{b})$ which sends the generators for $u_q(\mathfrak{b})$ to the corresponding generators in $U_q(\mathfrak{b})$. The inclusion $u_q(\mathfrak{b})\to U_q(\mathfrak{b})$ identifies $u_q(\mathfrak{b})$ with a normal Hopf subalgebra in $U_q(\mathfrak{b})$, and it fits into a short exact sequence of Hopf algebras

$$1\to u_q(\mathfrak{b})\to U_q(\mathfrak{b})\to U(\mathfrak{n})\to 1,$$

where $U(\mathfrak{n})$ is the usual universal enveloping algebra. The final map $U_q(\mathfrak{b})\to U(\mathfrak{n})$ sends the divided powers $E_\alpha^{(l)}$ to e_α in $\mathfrak{n}=\oplus_{\mu\in\Phi^+}\mathbb{C}e_\alpha$ [26, Lem. 8.9]. By an *exact sequence* we mean that $U_q(\mathfrak{b})$ is faithfully flat over $u_q(\mathfrak{b})$, that the kernel of $U_q(\mathfrak{b})\to U(\mathfrak{n})$ is generated by augmentation ideal for $u_q(\mathfrak{b})$, and that $u_q(\mathfrak{b})$ is the $U(\mathfrak{n})$-coinvariants in $U_q(\mathfrak{b})$, as in [9] (see also [28, Thm. 3.5.3, Prop. 3.4.3]).

The divided power operators $E_\alpha^{(l)}$ are connected with elements in $u_q(\mathfrak{b})$ via the comultiplication

$$\Delta(E_\alpha^{(l)}) = \sum_{0\le i\le l} q^{-i(l-i)} K_\alpha^i E_\alpha^{(l-i)}\otimes E_\alpha^{(i)}$$

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[25, Prop. 4.8].

2.3. **The root lattice and root lattice gradings.** We let $Q = Q(\mathfrak{g})$ denote the root lattice for $\mathfrak{g}$, and Q^+ denote the positive root lattice

$$Q^+ = \left\{ \sum_{\alpha \in \Gamma} n_\alpha \alpha : n_\alpha \geq 0 \text{ for all } \alpha \text{ and } n_\beta > 0 \text{ for some } \beta \right\}.$$

The quotient Q/lQ is identified with G in the obvious manner, $\alpha \mapsto K_\alpha$. The Killing form on $\mathfrak{h}^*$ induces a form on Q/lQ, and hence on G. Whence we have a canonical map from $Q/lQ \cong G$ to the character group $G^\vee$ which sends $\alpha \in Q/lQ$, or $K_\alpha \in G$, to the function $(K_\beta \mapsto q^{(\alpha,\beta)})$. We call this map the *Killing map*, and denote it $\kappa : G \to G^\vee$. In this way we view roots for $\mathfrak{g}$ as characters on G. Note that the Killing map is an isomorphism exactly when the Killing form on G is non-degenerate. This in turn occurs exactly when the determinant of the Cartan matrix is invertible in $\mathbb{Z}/l\mathbb{Z}$.

The coradical gradings on $u_q(\mathfrak{b})$ and $U_q(\mathfrak{b})$ can be seen as induced by a natural gradings by the root lattice

$$\deg_Q(K_\alpha) = 0, \ \deg_Q(E_\alpha) = \alpha, \ \deg_Q(E_\alpha^{(l)}) = l\alpha.$$

We have the obvious group map

$$|?| : Q \to \mathbb{Z}, \quad |\sum_{\alpha \in \Gamma} n_\alpha \alpha| = \sum_{\alpha \in \Gamma} n_\alpha,$$

and see that the coradical gradings on $u_q(\mathfrak{b})$ and $U_q(\mathfrak{b})$ agree with the $\mathbb{Z}$-grading induced by the above map.

We adopt the following partial ordering on Q: For elements $\eta = \sum_\alpha n_\alpha \alpha$ and $\eta' = \sum_\alpha n'_\alpha \alpha$ in the root lattice we write $\eta \leq \eta'$ if $n_\alpha \leq n'_\alpha$ for each α. We write $\eta < \eta'$ if $\eta \leq \eta'$ and there exists a simple β with $n_\beta < n'_\beta$. Note that the above map $|?| : Q \to \mathbb{Z}$ preserves our partial ordering.

3. Basic information for twists

We cover here some basic information on twists, and recall a classification of twists for the group algebra of a finite abelian group, from [18, 21].

3.1. **Twists and gauge equivalence.** We only set our conventions here, and refer the reader to [29, 21, 34, 23, 15] etc. for a more detailed account.

A Drinfeld twist, or just twist, for a (complete topological) Hopf algebra H is a unit $J \in H \otimes H$ satisfying

$$(\Delta \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J) \quad \text{and} \quad (\epsilon \otimes 1)(J) = (1 \otimes \epsilon)(J) = 1.$$

We replace $\otimes$ with $\widehat{\otimes}$ in the topological setting. We let H^J denote the corresponding twisted Hopf algebra, which is just H as an algebra along with the new comultiplication $\Delta^J(h) = J^{-1}\Delta(h)J$, for each $h \in H$. We denote the gauge action of $H^\times$ on the set of twists for H by a dot,

$$v \cdot J = \Delta(v)J(v^{-1} \otimes v^{-1}).$$

We say two twists J and J' are *gauge equivalent* if there is a unit v so that $J' = v \cdot J$, and let $\mathrm{Tw}(H)$ denote the set of gauge equivalence classes of twists for H.

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A *cocycle* twist for H is defined dually, i.e. as a convolution invertible function $\sigma : H \otimes H \to \mathbb{C}$ which satisfies

$$\sigma(h_1 h_1', h'')\sigma(h_2, h_2') = \sigma(h, h_1' h_1'')\sigma(h_2', h_2'')$$

and $\sigma(1, h) = \sigma(h, 1) = \epsilon(h)$ for each $h, h', h'' \in H$. From σ we form the new Hopf algebra H_σ, which is the coalgebra H along with the new multiplication

$$h \cdot_\sigma h' = \sigma^{-1}(h_1, h_1')\sigma(h_3, h_3')h_2 h_2'.$$

There is a dual notion of gauge equivalence as well, and we let $\mathrm{CoTw}(H)$ denote the set of gauge equivalence classes of cocycle twists for H.

Any Drinfeld twist J for H defines a cocycle twist for the dual H^*, and there is an equality of Hopf algebras $(H^J)^* = (H^*)_J$.

3.2. **Twists for finite abelian groups.** Given a finite abelian group Λ we let $\Lambda^\vee$ denote the character group. For any $a \in \Lambda^\vee$ we let

$$P_a = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} a(\lambda^{-1})\lambda$$

denote the corresponding idempotent in $\mathbb{C}[\Lambda]$. We will be considering bilinear forms on the character group $\Lambda^\vee$. By this we mean set maps $B : \Lambda^\vee \times \Lambda^\vee \to \mathbb{C}^\times$ which satisfy $B(ab, c) = B(a, c)B(b, c)$ and $B(a, bc) = B(a, b)B(a, c)$, for each $a, b, c \in \Lambda^\vee$.

We say a form B on $\Lambda^\vee$ is *alternating* if $B(a, b) = B^{-1}(b, a)$ for each $a, b \in \Lambda^\vee$, and we let $\mathrm{Alt}(\Lambda^\vee)$ denote the set of alternating forms on $\Lambda^\vee$.

Under the natural identification $\mathbb{C}[\Lambda^\vee] = \mathbb{C}[\Lambda]^*$ any form B on $\Lambda^\vee$ is identified with an element $B \in \mathbb{C}[\Lambda] \otimes \mathbb{C}[\Lambda]$. This element can be written explicitly as $B = \sum_{a,b \in \Lambda^\vee} B(a, b)P_a \otimes P_b$. Bilinearity implies that any form B on the character group $\Lambda^\vee$ defines a twist for $\mathbb{C}[\Lambda]$. So we get a map

$$\{\text{Bilinear forms on } \Lambda^\vee\} \to \mathrm{Tw}(\mathbb{C}[\Lambda]),$$

which restricts to a map $\mathrm{Alt}(\Lambda^\vee) \to \mathrm{Tw}(\mathbb{C}[\Lambda])$.

Proposition 3.1 ([21, Cor. 5.8])**.** *When Λ is a finite abelian group of odd order, the map $\mathrm{Alt}(\Lambda^\vee) \to \mathrm{Tw}(\mathbb{C}[\Lambda])$ is an isomorphism.*

The interested reader should also consult [21, Prop. 4.2, Prop. 5.1, Thm. 5.5].

4. Completion and duality for big quantum groups

In this section we present some relations between Lusztig's quantum Borel algebra, the small quantum Borel algebra, and the De Concini-Kac quantum Borel algebra. In particular, we present a duality between (a completion of) Lusztig's divided powers algebra and (a modified form of) the De Concini Kac algebra. These relations will be of significance in later sections, where we would like to make some cohomological arguments involving the duals of $U_q(\mathfrak{b})$ and $u_q(\mathfrak{b})$. We expect that the material will be well-known to experts

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4.1. **Duals and completion for $U_q(\mathfrak{b})$.** Take $U_q = U_q(\mathfrak{b})$.

We consider the completion $\widehat{U}_q$ of U_q with respect to the $\mathbb{Z}$-grading:

$$\widehat{U}_q = \varprojlim_N U_q/(\text{elem's of degree } \geq N).$$

Since the comultiplication on U_q respects the grading, the new object $\widehat{U}_q$ is a topological Hopf algebra, with comultiplication landing in the completed tensor product

$$\Delta : \widehat{U}_q \to \widehat{U}_q \mathbin{\widehat{\otimes}} \widehat{U}_q.$$

Definition 4.1. We let $U_q(\mathfrak{b})^{\#}$ denote the ($\mathbb{Z}$-)graded dual of $U_q(\mathfrak{b})$.

The Hopf algebra $U_q^{\#}$ is graded by the root lattice Q, where we take $((U_q)_{-\mu})^* = (U_q^{\#})_\mu$ for each $\mu \in Q$. We can alternately arrive at $U_q(\mathfrak{b})^{\#}$ as the continuous dual of $\widehat{U}_q$. Whence the completed algebra $\widehat{U}_q$ is naturally identified with the (non-graded, non-continuous) dual $\widehat{U}_q = (U_q^{\#})^*$.

The Hopf quotient $U_q \to \mathbb{C}[G]$ dualizes to a Hopf embedding $\mathbb{C}[G^\vee] \to U_q^{\#}$. Furthermore, since the coradical filtration on U_q is exhaustive, with $F_0 U_q = \mathbb{C}[G]$ and

$$F_1 U_q = \mathbb{C}[G] \oplus (\mathbb{C}[G] \cdot \{E_\alpha : \alpha \in \Gamma\}),$$

we have that $U_q^{\#}$ is generated in degree -1 by the functions $(\mathbb{C}[G] \cdot \{E_\alpha : \alpha \in \Gamma\})^*$. We let $X_\alpha \in U_q^{\#}$ denote the degree -1 function

$$X_\alpha : (\mathbb{C}[G] \cdot \{E_\alpha\}_\alpha) \to \mathbb{C}, \quad K_\gamma E_\beta \mapsto \delta_{\alpha,\beta}.$$

Lemma 4.2. *The algebra $U_q^{\#}$ is generated by the grouplikes $G^\vee$ and the functions X_α. Each X_α is skew primitive, with $\Delta(X_\alpha) = X_\alpha \otimes \alpha^{-1} + 1 \otimes X_\alpha$.*

We have abused notation here to identify $\alpha \in \Gamma$ with its corresponding character on G, via the Killing map. Recall that for $\gamma \in G^\vee$ we let $P_\gamma = \frac{1}{|G|} \sum_{g \in G} \gamma(g^{-1}) g$ denote the corresponding idempotent in $\mathbb{C}[G]$.

Proof. We need to show that the elements $\mathbb{C}[G^\vee]\{X_\alpha\}_\alpha$ produce all linear functions from $\mathbb{C}[G]\{E_\alpha\}_\alpha$ to $\mathbb{C}$. For any character $\omega \in G^\vee$ we have

$$\omega X_\alpha(P_\gamma E_\beta) = \sum_{ab=\gamma} \omega(K_\beta)\omega(P_a) X_\alpha(P_b E_\beta) = \omega(K_\beta) X_\alpha(P_{\omega^{-1}\gamma} E_\beta).$$

Note that $X_\alpha(P_\tau E_\beta)$ vanishes when $\beta \neq \alpha$, for every $\tau \in G^\vee$, and that

$$X_\alpha(P_\tau E_\alpha) = \frac{1}{|G|} \sum_{g \in G} \tau(g) = \begin{cases} 1 & \text{when } \tau \text{ is the identity in } G^\vee \\ 0 & \text{else.} \end{cases}$$

Hence

$$\omega(K_\alpha)^{-1} \omega X_\alpha(P_\gamma E_\beta) = \delta_{\omega,\gamma}\delta_{\alpha,\beta}.$$

Since $\{P_\gamma E_\beta : \gamma \in G^\vee, \beta \in \Gamma\}$ provides a basis for the space $\mathbb{C}[G]\{E_\beta\}_\beta$, we see that $\mathbb{C}[G^\vee]\{X_\alpha\}_\alpha$ produces all degree 1 functions.

We need also to compute the comultiplication $\Delta(X_\alpha)$. We have

$$\Delta(X_\alpha)(K_\mu \otimes E_\beta) = X_\alpha(K_\mu E_\beta) = \delta_{\alpha,\beta}$$

and

$$\Delta(X_\alpha)(E_\beta \otimes K_\mu) = q^{-(\mu,\beta)} X_\alpha(K_\mu E_\beta) = q^{-(\mu,\alpha)}\delta_{\alpha,\beta}.$$

Since $\Delta(X_\alpha)$ lies in bidegrees $(0,-1)$ and $(-1,0)$, the above computation show that $\Delta(X_\alpha) = X_\alpha \otimes \alpha^{-1} + 1 \otimes X_\alpha$. □

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Lemma 4.3. *We have* $\omega X_\alpha \omega^{-1} = \omega(K_\alpha) X_\alpha$, *for each* $\omega \in G^\vee$ *and* $\alpha \in \Gamma$.

Proof. The second iteration of the comultiplication gives

$$\omega X_\alpha \omega^{-1}(K_\gamma E_\beta) = \omega(K_\gamma K_\beta)\omega(K_\gamma)^{-1}\delta_{\alpha\beta} = \omega(K_\alpha)\delta_{\alpha,\beta} = \omega(K_\alpha) X_\alpha(K_\gamma E_\beta),$$

for all γ, β. □

4.2. **Identification with the modified De Concini-Kac algebra.** Recall the De Concini-Kac algebra

$$U_{q^{-1}}^{DK}(\mathfrak{b}_-) = \frac{\mathbb{C}\langle K_\alpha, F_\alpha : \alpha \in \Gamma\rangle}{\left(\begin{array}{c} [K_\alpha, K_\beta] = 0,\ K_\alpha F_\beta = q^{(\alpha,\beta)} F_\beta K_\alpha,\ K_\alpha^l = 1, \\ \sum_{i=0}^{1-\langle\alpha,\beta\rangle} (-1)^i \left[\begin{array}{c} 1-\langle\alpha,\beta\rangle \\ i \end{array} \right]_{q_\alpha^{-1}} F_\alpha^{1-\langle\alpha,\beta\rangle - i} F_\beta F_\alpha^i = 0 \end{array} \right)}$$

[14]. The K_α here are grouplike and

$$\Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha.$$

The algebra $U_{q^{-1}}^{DK}$ is graded by the root lattice, with K_α of degree 0 and F_α of degree $-\alpha$.

There is a factoring $U_{q^{-1}}^{DK} = U_{q^{-1}}^- \rtimes \mathbb{C}[G]$, where $U_{q^{-1}}^-$ is the subalgebra generated by the F_α. We define the *modified De Concini-Kac algebra* as the smash product

$$\mathscr{U}^{DK} = \mathscr{U}_{q^{-1}}^{DK} = U_{q^{-1}}^- \rtimes \mathbb{C}[G^\vee],$$

where $\mathbb{C}[G^\vee]$ acts on the negative subalgebra in the natural manner $\omega \cdot F_\alpha = \omega(K_\alpha) F_\alpha$. We give $\mathscr{U}^{DK}$ the obvious Hopf structure, where $G^\vee$ consists of grouplikes and the $\Delta(F_\alpha) = F_\alpha \otimes \alpha^{-1} + 1 \otimes F_\alpha$.

Most of the properties of $U_{q^{-1}}^{DK}$ will be inherited by the modified algebra. For example, $\mathscr{U}^{DK}$ is Q-graded in the obvious way, and Lusztig's basis for $U_{q^{-1}}^-$ produces a basis for $\mathscr{U}^{DK}$.

Lemma 4.4. *There is an isomorphism*

$$\varphi : \mathscr{U}^{DK} \to U_q(\mathfrak{b})^\#, \qquad \begin{array}{l} F_\alpha \mapsto X_\alpha \\ \omega \mapsto \omega, \end{array}$$

of Q-graded Hopf algbras.

Sketch proof. We claim first that $U_q(\mathfrak{b})^\#$ satisfies all of the relations of the modified De Concini-Kac algebra. The only relations left to check are the Serre relations, by Lemma 4.3. In the simply laced case one can readily check the relations by hand. In the non-simply laced case the determinant of the Cartan matrix will be 1 or 2, and hence invertible in $\mathbb{Z}/l\mathbb{Z}$. The small quantum group $u_q(\mathfrak{g})$ will then be factorizable with R-matrix

$$R = \prod_{\mu \in \Phi^+} \left(\sum_{n=0}^{l-1} q^{-n(n+1)/2} \frac{(1-q^2)^n}{[n]_q!} E_\mu^n \otimes F_\mu^n \right) \Omega,$$

where $\Omega \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ is the inverse of the Killing form (see e.g. [35]). The R-matrix provides an isomorphism $u_q(\mathfrak{b})^* \to u_q(\mathfrak{b}_-)^{cop} \cong u_{q^{-1}}(\mathfrak{b}_-)$ which sends each generator X_α to $q^2(q^{-1} - q)F_\alpha$ [33]. Hence the Serre relations hold for $U_q(\mathfrak{b})^\#$ modulo the kernel of the projection $U_q(\mathfrak{b})^\# \to u_q(\mathfrak{b})^*$.

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Since the inclusion $u_q(\mathfrak{b}) \to U_q(\mathfrak{b})$ is an isomorphism in all $\mathbb{Z}$-degrees $< l$, it follows that the graded dual is an isomorphism in all degrees $> -l$. By our assumptions on l, the Serre relations are in degrees $> -l$, and therefore must hold for $U_q(\mathfrak{b})^{\#}$. It follows that the algebra map φ can be defined in the prescribed manner.

By Lemma 4.2 φ is surjective. Also, Lusztig's bases [14, 25, 26] of both $\mathscr{U}^{DK}$ and U_q tell us that the two algebras have the same dimension in each Q-degree. So φ must be an isomorphism. The fact that φ is a Hopf map follows from the fact that comultiplications agree on the generators

$$(\varphi \otimes \varphi)\Delta(F_\alpha) = X_\alpha \otimes \alpha^{-1} + 1 \otimes X_\alpha = \Delta(X_\alpha).$$

□

5. Interactions between Drinfeld twists and cohomology

Fix $\widehat{U}_q = \widehat{U}_q(\mathfrak{b})$, $u_q = u_q(\mathfrak{b})$, and $\mathscr{U}^{DK} = \mathscr{U}^{DK}_{q^{-1}}$ the modified De Concini-Kac algebra of Section 4.2. We identify $\mathscr{U}^{DK}$ with the continuous dual of $\widehat{U}_q$ via the isomorphism of Lemma 4.4.

In this section we cover an association between twists for $\widehat{U}_q$ (resp. u_q) and cohomology classes for $\mathscr{U}^{DK}$ (resp. u_q^*). Namely we show that minimal Q-homogenous terms J_γ in a twists J produce cohomology classes $[J_\gamma] \in H^2(\mathscr{U}^{DK}_B)$ (resp. $[J_\gamma] \in H^2((u_q^*)_B)$), where B is a form on $G^\vee$ associated to J. The class is show to vanish exactly when the term J_γ in J can be eliminated via gauge transformation.

5.1. Basics for twists of u_q and $\widehat{U}_q$.

Lemma 5.1. *Any twist for u_q, or $\widehat{U}_q$, is gauge equivalent to one of the form $J = B + \sum_{\mu \in Q^+} J_\mu$, where $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ is an alternating bilinear form on $G^\vee$ and each J_μ is homogenous of Q-degree μ.*

Proof. We may expand any twist as $J' = J'_0 + \sum_{\mu \in Q^+} J'_\mu$. The degree 0 term J'_0 will be a twist for $\mathbb{C}[G]$, for grading reasons, and by [21, Cor. 5.8] there is a unit $v \in \mathbb{C}[G]$ so that $v \cdot J'_0 = B$ is an alternating bilinear form on $G^\vee$. Hence $J = v \cdot J'$ will have the prescribed form. □

Remark 5.2. We will generally denote the degree 0 term in a twist J by B, although B needn't always restrict to a bilinear form on $G^\vee$. As we are interested in twists up to gauge equivalence, one is free to always assume B is a form on $G^\vee$ in any case.

Since any twist B for $\mathbb{C}[G]$ is also a twist for the ambient algebra $\widehat{U}_q$, we may consider the corresponding twisted Hopf algebra $\widehat{U}_q^B$. Recall that $\widehat{U}_q = \widehat{U}_q^B$ as algebras, and hence the corresponding groups of units are also identified, $(\widehat{U}_q)^\times = (\widehat{U}_q^B)^\times$.

Lemma 5.3. *Given a twist B for $\mathbb{C}[G]$, left multiplication by $B^{\pm 1}$ provides an isomorphism of $(\widehat{U}_q)^\times$-sets*

$$\{\text{Drinfeld twists of } \widehat{U}_q\} \underset{B}{\overset{B^{-1}}{\rightleftarrows}} \{\text{Drinfeld twists of } \widehat{U}_q^B\},$$

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under the gauge actions. Furthermore, if a subgroup $\mathbb{V} \subset (\widehat{U}_q)^\times$ preserves the subset of twists for u_q, under the gauge action, then $\mathbb{V}$ also preserves twists for u_q^B and we have an isomorphism of $\mathbb{V}$-sets

$$\{\text{Drinfeld twists of } u_q\} \underset{B}{\overset{B^{-1}}{\rightleftarrows}} \{\text{Drinfeld twists of } u_q^B\}.$$

Proof. We first prove that there is a bijection of sets for $\widehat{U}_q$, without considering the gauge action. Suppose J is a twist for $\widehat{U}_q$. One simply calculates:

$$\begin{aligned}(\Delta^B \otimes 1)(B^{-1}J)(B^{-1}J \otimes 1) &= (B^{-1} \otimes 1)(\Delta \otimes 1)(B^{-1})(\Delta \otimes 1)(J)(J \otimes 1)\\ &= (1 \otimes B^{-1})(1 \otimes \Delta)(B^{-1})(1 \otimes \Delta)(J)(1 \otimes J)\\ &= (1 \otimes \Delta^B)(B^{-1}J)(1 \otimes B^{-1}J).\end{aligned}$$

So $B^{-1}J$ is a twist for $\widehat{U}_q^B$. A completely similarly calculation shows that BF is a twist for $\widehat{U}_q$ whenever F is a twist for $\widehat{U}_q^B$. Compatibility with the action of $(\widehat{U}_q)^\times$ follows from the equalities

$$B^{-1}\Delta(v)J(v^{-1} \otimes v^{-1}) = \Delta^B(v)(B^{-1}J)(v^{-1} \otimes v^{-1})$$

and $B\Delta^B(v)F(v^{-1} \otimes v^{-1}) = \Delta(v)(BF)(v^{-1} \otimes v^{-1})$. The statement for u_q follows by the above computations and the fact that B is a unit in $u_q \otimes u_q$. □

5.2. **Elimination of low degree terms and cohomology.** Given a twist $J' = B + \sum_\mu J'_\mu$ for $\widehat{U}_q$, we can multiply by B^{-1} to get a twist $J = 1 + \sum_{\mu \in Q^+} J_\mu$ for $\widehat{U}_q^B$. Hence, considering arbitrary twists for $\widehat{U}_q$ is equivalent to considering twists of the form $J = 1 + \sum_{\mu \in Q^+} J_\mu$ for the algebras $\widehat{U}_q^B$, where $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$. Fix such a twist $J \in \widehat{U}_q^B \otimes \widehat{U}_q^B$ and suppose γ is of minimal Q^+-degree such that $J_\gamma \neq 0$.

The equation

$$(\Delta^B \otimes 1)(J)(J \otimes 1) = (1 \otimes \Delta^B)(J)(1 \otimes J)$$

produces in degree γ the equation

$$(1 \otimes J_\gamma) - (\Delta^B \otimes 1)(J_\gamma) + (1 \otimes \Delta^B)(J_\gamma) - (J_\gamma \otimes 1) = 0. \tag{3}$$

Consider now a unit $v = 1 + \sum_{\mu \in Q^+} v_\mu$ in $\widehat{U}_q^B$ with vanishing term $v_{\gamma'}$ for each $\gamma' < \gamma$. Then the inverse $v^{-1} = 1 + \sum_\mu v_\mu^{-1}$ is such that $v_\gamma^{-1} = -v_\gamma$ and has vanishing term $v_{\gamma'}^{-1} = 0$ for each $\gamma' < \gamma$ as well. In this case the gauged twist $v \cdot J$ has vanishing term in degree γ if and only if v_γ solves the equation

$$(1 \otimes v_\gamma) - \Delta^B(v_\gamma) + (v_\gamma \otimes 1) = J_\gamma. \tag{4}$$

We call equation (3) the cocycle equation and equation (4) the coboundary equation. We call a solution v_γ to (4) a *bounding element* for J_γ.

Lemma 5.4. *Suppose $J = 1 + \sum_{\mu \in Q^+} J_\mu$ is a twist for $\widehat{U}_q^B$ and $\gamma \in Q^+$ is a minimal degree such that $J_\gamma \neq 0$. Consider any unit $v = 1 + \sum_{\mu \in Q^+} v_\mu$ with v_γ solving* (4) *and $v_\nu = 0$ whenever $J_{\nu'} = 0$ for all $\nu' \leq \nu$. Then the gauged twist $(v \cdot J)$ has the following properties:*

(i) $(v \cdot J)_\gamma = 0$.
(ii) $(v \cdot J)_\nu = 0$ *whenever* $J_{\nu'} = 0$ *for all* $\nu' \leq \nu$.

In particular, the above conclusions hold when v_γ solves (4) *and we take $v = \exp(v_\gamma)$.*

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Proof. This follows from a direct analysis of the equation

$$\begin{aligned} v \cdot J &= J_\gamma - (1 \otimes v_\gamma - \Delta^B(v_\gamma) + v_\gamma \otimes 1) \\ &\quad + \textstyle\sum_{\mu \neq \gamma} \big(J_\mu + (1 \otimes v_\mu^{-1} + \Delta^B(v_\mu) + v_\mu^{-1} \otimes 1) \big) \\ &\quad + \textstyle\sum_I \Delta^B(v_a) J_b (v_c^{-1} \otimes v_d^{-1}) \\ \\ &= \textstyle\sum_{\mu \neq \gamma} \big(J_\mu + (1 \otimes v_\mu^{-1} - \Delta^B(v_\mu) + v_\mu^{-1} \otimes 1) \big) \\ &\quad + \textstyle\sum_I \Delta^B(v_a) J_b (v_c^{-1} \otimes v_d^{-1}), \end{aligned}$$

where the final sum is indexed over the set I of tuples $(a, b, c, d) \in Q^4$ with at least two nonzero entries. □

The lemma implies that if we can solve (4), for a given minimal term J_γ in a given twist J, then we can get rid of the term J_γ via gauge transformation. Furthermore, we can do so in a manner which does not introduce new terms in low degree.

Consider again a twist $J = 1 + \sum_{\mu \in Q^+} J_\mu$ for $\widehat{U}_q^B$, and take J_γ a nonvanishing term of minimal Q-degree. If we consider J_γ as a function on the cocycle twisted De Concini-Kac algebra $\mathscr{U}_B^{DK}$, and apply (3) to elements $X \otimes Y \otimes Z$ in $\mathscr{U}_B^{DK}$, we arrive at the equation

$$\epsilon(X) J_\gamma(Y, Z) - J_\gamma(X \cdot_B Y, Z) + J_\gamma(X, Y \cdot_B Z) - J_\gamma(X, Y)\epsilon(Z) = 0.$$

Similarly, the coboundary equation becomes

$$\epsilon(X) v_\gamma(Y) - v_\gamma(X \cdot_B Y) + v_\gamma(X)\epsilon(Y) = J_\gamma(X, Y). \tag{5}$$

So, if we let

$$C^\bullet(\mathscr{U}_B^{DK}) = 0 \to \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_B^{DK}, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_B^{DK} \otimes \mathscr{U}_B^{DK}, \mathbb{C}) \to \cdots$$

denote the standard complex computing Hopf cohomology $H^\bullet(\mathscr{U}_B^{DK}) = \mathrm{Ext}_{\mathscr{U}_B^{DK}}(\mathbb{C}, \mathbb{C})$, then J_γ is a degree 2-cocycle in $C^\bullet(\mathscr{U}_B^{DK})$. By equation (5), the function J_γ is bounded by some v_γ if and only if the class $[J_\gamma]$ vanishes in cohomology.

A completely similar analysis holds for the small quantum Borel u_q^B. We record these facts in a lemma.

Lemma 5.5. *Let B be a twist for $\mathbb{C}[G]$ and $J = 1 + \sum_{\mu \in Q^+} J_\mu$ be a twist for $\widehat{U}_q^B$ (resp. u_q^B). If γ is a minimal Q-degree so that $(J-1)_\gamma$ is nonvanishing, then the corresponding function*

$$J_\gamma \in C^2(\mathscr{U}_B^{DK}) \quad \big(\textit{resp. } J_\gamma \in C^2((u_q^*)_B)\big)$$

is a cocycle. There is a homogenous element v_γ in U_q^B (resp. u_q^B) bounding J_γ, in the sense described above, if and only if the cohomology class $[J_\gamma] \in H^2(\mathscr{U}_B^{DK})$ (resp. $[J_\gamma] \in H^2((u_q^*)_B)$) vanishes.

6. Cohomologies of $\mathscr{U}_B^{DK}$ and $(u_q^*)_B$

We provide here a description of the cohomologies $H^i(\mathscr{U}_B^{DK}) = \mathrm{Ext}^i_{\mathscr{U}_B^{DK}}(\mathbb{C}, \mathbb{C})$ and $H^i((u_q^*)_B) = \mathrm{Ext}_{(u_q^*)_B}(\mathbb{C}, \mathbb{C})$ when B is a bilinear form on $G^\vee$, and $i = 1, 2$. The proofs of the most important results, Propositions 6.1 and 6.5, are delayed until Section 9, as they are something of a deviation from our primary objective of classifying twists.

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6.1. **Cohomology for De Concini-Kac algebras.** Our main computation for the De Concini-Kac algebra is

Proposition 6.1. *For any bilinear form B on $G^\vee$ we have $H^1(\mathscr{U}_B^{DK}) = 0$ and $H^2(\mathscr{U}_B^{DK}) = 0$.*

As remarked above, the proof of Proposition 6.1 will be given in Section 9. Of course, we are mostly interested in the vanishing of the second cohomology $H^2(\mathscr{U}_B^{DK})$. However, vanishing of $H^1(\mathscr{U}_B^{DK})$ tells us something which will be quite useful in our classification of twists for the small algebra u_q.

Corollary 6.2. *Consider any form B on $G^\vee$ and twist $J = B + \sum_{\mu \in Q^+} J_\mu$ of the completion $\widehat{U}_q$. Then for any nonvanishing term J_γ of minimal Q-degree, there exists a unique bounding element $v_\gamma \in \widehat{U}_q$.*

Proof. Vanishing of $H^2(\mathscr{U}_B^{DK})$ implies the existence of such a v_γ. Vanishing of $H^1(\mathscr{U}_B^{DK})$ tells us that the map

$$v_\gamma \mapsto (1 \otimes v_\gamma) - \Delta^B(v_\gamma) + (v_\gamma \otimes 1)$$

is injective. Hence the bounding element v_γ is unique. □

As a consequence of Proposition 6.1 we can deduce a classification of cocycle twists for $\mathscr{U}^{DK}$. Note that we have a canonical map $\mathrm{CoTw}(\mathbb{C}[G^\vee]) \to \mathrm{CoTw}(\mathscr{U}^{DK})$ given by restricting along the Hopf projection

$$\mathscr{U}^{DK} \otimes \mathscr{U}^{DK} \to \mathbb{C}[G^\vee] \otimes \mathbb{C}[G^\vee].$$

Recall also the identification $\mathrm{CoTw}(\mathbb{C}[G^\vee]) = \mathrm{Alt}(G^\vee)$ of Section 3. We expect that the following result is known, although we do not know a reference.

Proposition 6.3. *For the modified De Concini-Kac algebra $\mathscr{U}^{DK}$ the restriction map $\mathrm{Alt}(G^\vee) \to \mathrm{CoTw}(\mathscr{U}^{DK})$ is a bijection.*

Proof. The map $\mathrm{CoTw}(\mathscr{U}^{DK}) \to \mathrm{Tw}(\widehat{U}_q)$ given by the identification $\widehat{U}_q = (\mathscr{U}^{DK})^*$ is a bijection. We show that the map $\mathrm{Alt}(G^\vee) \to \mathrm{Tw}(\widehat{U}_q)$ is an isomorphism.

Consider a twist J' for $\widehat{U}_q$. After gauging by an element in $\mathbb{C}[G]$ we may assume the degree zero term for J' is a form B on $G^\vee$. Consider the associated twist $J = B^{-1}J' = 1 + \sum_{\mu \in Q^+} J_\mu$ for U_q^B.

Take an arbitrary enumeration $\{\mu_1, \mu_2, \dots\}$ of Q^+ which is compatible with the total ordering on $\mathbb{Z}$. Specifically, we demand $|\mu_i| \leq |\mu_j|$ whenever $i \leq j$. Define inductively $J(0) = J$ and $J(i) = \exp(v_{\mu_i}) \cdot J(i-1)$, where v_{μ_i} is the unique bounding element for $J(i-1)_{\mu_i}$. Then, by Lemma 5.4, the (convergent) ordered product $v = \prod_{i=1}^{\infty} \exp(v_{\mu_i}) = \dots \exp(v_{\mu_3}) \exp(v_{\mu_2}) \exp(v_{\mu_1})$ will be such that $v \cdot_B J = 1$, where $v \cdot_B -$ denotes the gauging action for U_q^B. Hence, by Lemma 5.3, we have $v \cdot J' = B$. So we see that the map $\mathrm{Alt}(G^\vee) \to \mathrm{Tw}(\widehat{U}_q)$ is surjective. Injectivity can be deduced from Proposition 3.1. □

Remark 6.4. The proof of Proposition 6.1 works perfectly well for the non-modified De Concini-Kac Borel algebra. Hence we will have vanishing of cohomologies $H^1(U_T^{DK}) = H^2(U_T^{DK}) = 0$, when T is a form on G. We therefore get a bijection

$$\mathrm{Alt}(G) \to \mathrm{CoTw}(U_T^{DK}),$$

as above. The role of $\widehat{U}_q$ in this case is be played by a modified algebra $\widehat{\mathscr{U}_q}$.

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6.2. **Cohomology for small quantum Borel algebras.** Let us now consider the small algebra $u_q = u_q(\mathfrak{b})$. Note that the cohomologies $H^i((u_q^*)_B)$ will naturally be graded by the root lattice. One can see this by observing that the Q-grading on u_q^B induces a Q-grading on the complex

$$C^\bullet\left((u_q^*)_B\right) = 0 \to \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}((u_q^*)_B, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}((u_q^*)_B \otimes (u_q^*)_B, \mathbb{C}) \to \cdots$$
$$= 0 \to \mathbb{C} \to u_q^B \to u_q^B \otimes u_q^B \to \cdots .$$

The next result follows by [19] in the specific case $B = 1$. As with Proposition 6.1, we provide the proof in Section 9.

Proposition 6.5 (cf. [19, Thm. 2.5])**.** *Let B be any bilinear form on $G^\vee$. Let $\mathfrak{n}^{(l)} = \oplus_{\mu\in\Phi^+}\mathbb{C}e_\mu$ denote the positive nilpotent subalgerba in $\mathfrak{g}$ equipped with the Q-grading $\deg_Q(e_\mu) = l\mu$. Then we have $H^1\left((u_q^*)_B\right) = 0$ and $H^2((u_q^*)_B) = \mathfrak{n}^{(l)}$ as Q-graded vector spaces.*

The identification $H^2((u_q^*)_B) = \mathfrak{n}^{(l)}$ suggests that there should be non-trivial twists J of the form $J = B + J_{l\mu} +$ (higher deg terms) for each $\mu \in \Phi^+$. Indeed, we can construct such a twist, for $\mu = \alpha$ for example, via gauge transformation in $\widehat{U}_q$, $J = \exp(E_\alpha^{(l)}) \cdot B$. (See Lemma 7.3 below.) In Section 7 we will show that in fact *all* homogeneous classes in $H^2((u_q^*)_B)$ arise as, and can be eliminated by, gauge transformations by such exponents $\exp(\lambda E_\alpha^{(l)})$ in $\widehat{U}_q$.

7. Classifying twists for $u_q(\mathfrak{b})$

The primary result of this section is Theorem 7.5 below. Theorem 7.5, along with some information from [30], will imply Theorem A from the introduction. Recall our notation $u_q = u_q(\mathfrak{b})$, $U_q = U_q(\mathfrak{b})$.

7.1. **The gauge group $\mathbb{V}$.** Let $\mathbb{V}_{small}$ and $\mathbb{V}_{dp}$ be the following two subgroups of $(\widehat{U}_q)^\times$:

$$\mathbb{V}_{small} := \langle 1 + v_+ : v_+ \in (u_q)_{\geq 1}\rangle,$$
$$\mathbb{V}_{dp} := \langle \exp(\lambda E_\alpha^{(l)}) : \alpha \in \Gamma,\ \lambda \in \mathbb{C}\rangle.$$

Note that $\mathbb{V}_{small} \subset (u_q)^\times$.

Definition 7.1. Let $\mathbb{V}$ denote the subgroup of $(\widehat{U}_q)^\times$ generated by $\mathbb{V}_{small}$ and $\mathbb{V}_{dp}$.

We can write any $v \in \mathbb{V}$ as a sum $v = 1 + \sum_{\mu\in Q^+} v_\mu$ with each v_μ homogenous of Q-degree μ, since we can do so for each of the generators of $\mathbb{V}$. It will be helpful to have the following lemma from [24].

Lemma 7.2 ([24, Lem. 4.5])**.** *For each $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}$ the derivation $\mathrm{ad}_{\lambda E_\alpha^{(l)}} : \widehat{U}_q \to \widehat{U}_q$ restricts to a nilpotent derivation on u_q, and the exponential $\exp(\mathrm{ad}_{\lambda E_\alpha^{(l)}})$ restricts to a well-defined algebra automorphism of u_q.*

The first claim follows directly from [24]. The second claim follows from the general fact that exponentiating a nilpotent derivation, of any given algebra, produces an algebra automorphism.

Lemma 7.3. *Fix a twist B for $\mathbb{C}[G]$. Given any twist J for u_q^B, and $v \in \mathbb{V}$, the gauging $(v \cdot J)$ is another twist for u_q^B.*

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Proof. We prove the result in the case $B = 1$. The case when B is arbitrary will follow by Lemma 5.3.

Suppose $B = 1$. Obviously such twists are preserved by the action of $\mathbb{V}_{small}$, so it suffices to prove that $(v \cdot J)$ is a twist for u_q whenever v is a generator $v = \exp(\lambda E_\alpha^{(l)})$ of $\mathbb{V}_{dp}$. We have $\Delta(\lambda E_\alpha^{(l)}) = \lambda E_\alpha^{(l)} \otimes 1 + 1 \otimes \lambda E_\alpha^{(l)} + T$, where T is a square-zero element in $u_q \otimes u_q$ which centralizes $E_\alpha^{(l)} \otimes 1$ and $1 \otimes E_\alpha^{(l)}$. One can write explicitly $T = \lambda \sum_{i=1}^{l-1} q^{-i(l-i)} K_\alpha^i E_\alpha^{(l-i)} \otimes E_\alpha^{(i)}$. Then we have

$$\Delta(v) = \exp(\lambda \Delta(E_\alpha^{(l)})) = (1 + T)(v \otimes v).$$

Hence

$$\Delta(v) J (v^{-1} \otimes v^{-1}) = (1 + T)(v \otimes v) J (v^{-1} \otimes v^{-1}).$$

Now we use the equality

$$(v \otimes v) J (v^{-1} \otimes v^{-1}) = \Big((\exp \mathrm{ad}_{\lambda E_\alpha^{(l)}}) \otimes (\exp \mathrm{ad}_{\lambda E_\alpha^{(l)}}) \Big) (J),$$

and Lemma 7.2, to see that $v \cdot J \in u_q \otimes u_q$. □

Lemma 7.4. *The subgroup $\mathbb{V}_{small} \subset \mathbb{V}$ is normal.*

Proof. Consider any $v = 1 + v_+$ in $\mathbb{V}_{small}$ and any generator $\exp(\lambda E_\alpha^{(l)})$ of $\mathbb{V}_{dp}$. Note that $\mathrm{ad}_{\lambda E_\alpha^{(l)}}$ preserves the subspace of positive degree elements in u_q. Hence $\exp(\mathrm{ad}_{\lambda E_\alpha^{(l)}})(v_+)$ is of positive degrees. It follows that the commutator

$$\exp(\lambda E_\alpha^{(l)}) v \exp(-\lambda E_\alpha^{(l)}) = 1 + \exp(\mathrm{ad}_{\lambda E_\alpha^{(l)}})(v_+)$$

is in $\mathbb{V}_{small}$. Since $\mathbb{V}$ is generated by $\mathbb{V}_{small}$ and the $\exp(\lambda E_\alpha^{(l)})$, it follows that $\mathbb{V}_{small}$ is normal. □

7.2. **A classification of twists for $u_q(\mathfrak{b})$.** We wish to prove the following result.

Theorem 7.5. *Any twist J for $u_q(\mathfrak{b})$ is gauge equivalent to a twist of the form $v \cdot B$, where v is in $\mathbb{V}_{dp}$ and B is an alternating form on $G^\vee$.*

We break the proof down into a sequence of smaller results. For a given form B on $G^\vee$ we let d_B denote the differential on either of the complexes $C^\bullet(\mathscr{U}_B^{DK})$ or $C^\bullet((u_q^*)_B)$. (Since the restriction map $C^\bullet((u_q^*)_B) \to C^\bullet(\mathscr{U}_B^{DK})$ is an inclusion of cochains the notation is unambiguous.) Recall that in each degree i we have $C^i((u_q^*)_B) = (u_q^B)^{\otimes i}$ and $C^i(\mathscr{U}_B^{DK}) = (\widehat{U}_q^B)^{\widehat{\otimes} i}$.

Lemma 7.6. *Let B be a bilinear form on $G^\vee$. If an element $v_\gamma \in \widehat{U}_q \setminus u_q$ is such that $d_B(v_\gamma) \in u_q \otimes u_q$, then the corresponding class $[d_B(v_\gamma)] \in H^2((u_q^*)_B)$ is nonzero.*

Proof. We proceed by contradiction. If $[d_B(v_\gamma)] = 0$ then there is an element $v'_\gamma \in u_q$ with $d_B(v'_\gamma) = d_B(v_\gamma)$. Since $v_\gamma \notin u_q$ the difference $v_\gamma - v'_\gamma$ provides a nonzero element in $C^1(\mathscr{U}_B^{DK})$ with $d_B(v_\gamma - v'_\gamma) = 0$. But this cannot happen as the difference would provide a nonvanishing class in the first cohomology $H^1(\mathscr{U}_B^{DK})$, which we know is 0 by Proposition 6.1. □

Recall the exact sequence of Hopf algebras

$$1 \to u_q(\mathfrak{b}) \to U_q(\mathfrak{b}) \to U(\mathfrak{n}) \to 1$$

from Section 2.2, where $\mathfrak{n} = \oplus_{\mu \in \Phi^+} \mathbb{C} e_\mu$ is the positive nilpotent subalgebra in $\mathfrak{b}$.

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Lemma 7.7. *For any positive root $\mu \in \Phi^+$, and $\gamma = l\mu$, there is an element $v_\gamma \in \widehat{U}_q \setminus u_q$ of Q-degree γ which extends to a family of units $\mathbf{v}_\gamma(\lambda)$ in $\mathbb{V}_{dp}$ of the form*

$$\mathbf{v}_\gamma(\lambda) = 1 + \lambda v_\gamma + (\text{terms of strictly higher } Q\text{-degree}),$$

where λ is arbitrary in $\mathbb{C}$.

Proof. We may write each generator e_μ of the root space in $\mathfrak{n}$ as

$$e_\mu = [e_{\alpha_1}, [\dots [e_{\alpha_{n-1}}, e_{\alpha_n}] \dots]],$$

for some sequence of simple roots $\alpha_1, \dots, \alpha_n$. We claim that for each $\mu \in \Phi^+$ there is a family of units of the form

$$\mathbf{v}_{l\mu}(\lambda) = 1 + \lambda[E^{(l)}_{\alpha_1}, [\dots [E^{(l)}_{\alpha_{n-1}}, E^{(l)}_{\alpha_n}] \dots]] + (\text{higher } Q\text{-degree terms})$$

in $\mathbb{V}_{dp}$. So, $v_{l\mu}$ is the commutator of the $E^{(l)}_{\alpha_i}$ here. We proceed by induction on the length n of the given sequence, i.e. the height of the root. For our base case $n = 1$ we take $\mathbf{v}_{l\alpha}(\lambda) = \exp(\lambda E^{(l)}_\alpha)$.

Suppose the result holds for height n roots and that ν is height $n+1$. Write $e_\nu = [e_\alpha, e_\mu]$ for a simple root α and e_μ as above, and consider

$$v = \mathbf{v}_{l\mu}(1) = 1 + [E^{(l)}_{\alpha_1}, [\dots [E^{(l)}_{\alpha_{n-1}}, E^{(l)}_{\alpha_n}] \dots]] + (\text{higher } Q\text{-degree terms}).$$

Take $v_+ = v - 1$. One now calculates

$$\begin{aligned}
\exp(\lambda E^{(l)}_\alpha) v \exp(\lambda E^{(l)}_\alpha) v^{-1} &= \exp(\lambda \mathrm{ad}_{E^{(l)}_\alpha})(v) v^{-1} \\
&= (v + (\exp(\lambda \mathrm{ad}_{E^{(l)}_\alpha}) - id)(v)) v^{-1} \\
&= 1 + (\exp(\lambda \mathrm{ad}_{E^{(l)}_\alpha}) - id)(1 + v_+) v^{-1} \\
&= 1 + (\exp(\lambda \mathrm{ad}_{E^{(l)}_\alpha}) - id)(v_+) v^{-1} \\
&= 1 + \left(\lambda[E^{(l)}_\alpha, v_{l\mu}] + \text{higher } Q\text{-degree terms} \right) v^{-1}.
\end{aligned}$$

Since $v^{-1} = 1 - (\text{terms of degree } \geq l\mu)$ the final expression reduces to give

$$\exp(\lambda E^{(l)}_\alpha) v \exp(\lambda E^{(l)}_\alpha) v^{-1} = 1 + \lambda[E^{(l)}_\alpha, v_{l\mu}] + \text{higher } Q\text{-degree terms}.$$

Hence

$$\mathbf{v}_{l\mu}(\lambda) := \exp(\lambda E^{(l)}_\alpha) v \exp(\lambda E^{(l)}_\alpha) v^{-1}$$

provides the desired family, with

$$v_{l\nu} = [E^{(l)}_\alpha, v_{l\mu}] = [E^{(l)}_\alpha, [E^{(l)}_{\alpha_1}, [\dots [E^{(l)}_{\alpha_{n-1}}, E^{(l)}_{\alpha_n}] \dots]]].$$

To see that the element $v_{l\nu} \in U_q \subset \widehat{U}_q$ is not in u_q note that its image in $U(\mathfrak{n})$ is $e_\nu (\neq 0)$. □

We use the elements v_γ from the above lemma, in conjunction with Lemma 7.6, to construct all nonvanishing classes in $H^2((u_q^*)_B)$.

Lemma 7.8. *Consider a form B on $G^\vee$ and a twist $J = B + \sum_{\mu \in Q^+} J_\mu$ for u_q. Let d be the minimal positive integer with $J_d \neq 0$, in the $\mathbb{Z}$-grading. Then there is a unit $v \in \mathbb{V}$ so that $(v \cdot J)_{d'} = 0$ for each $0 < d' \leq d$.*

Proof. Let us consider instead a twist $J = 1 + \sum_{\mu \in Q^+} J_\mu$ for u_q^B, with B a form on $G^\vee$. Take d the minimal $\mathbb{Z}$-degree in which $J - 1$ does not vanish. We claim that there exists $v \in \mathbb{V}$ so that $(v \cdot J) - 1$ vanishes in all degrees $\leq d$. To show this it suffices to show that we can eliminate each term J_γ with $|\gamma| = d$, i.e. that for each

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nonvanishing J_γ with $|\gamma| = d$ there exists a unit $v \in \mathbb{V}$ so that $(v \cdot J)_{d''} = 0$ for $0 < d'' < d$ and $(v \cdot J)_d = J_d - J_\gamma$.

For any such J_γ we consider the corresponding class in cohomology $[J_\gamma] \in H^2((u_q^*)_B)$. If $[J_\gamma] = 0$ then there is an element $v_\gamma \in u_q^B$ of degree γ with $d_B(v_\gamma) = J_\gamma$. Consequently, the unit $v = \exp(v_\gamma) \in \mathbb{V}_{small}$ is such that

$$(v \cdot J)_{d''} = 0 \text{ for } 0 < d'' < d$$

and $(v \cdot J)_d = J_d - J_\gamma$, by Lemma 5.4. If $[J_\gamma] \neq 0$ then $\gamma = l\mu$ for some positive root μ, since $H^2((u_q^*)_B)$ is concentrated in Q-degrees $l \cdot \Phi^+$ by Proposition 6.5.

Consider in this case the family $\mathbf{v}_\gamma(\lambda) = 1 + \lambda v_\gamma + \dots$ of Lemma 7.7. We know that each

$$\begin{aligned} \mathbf{v}_\gamma(\lambda) \cdot (1 \otimes 1) &= \Delta(\mathbf{v}_\gamma(\lambda))(\mathbf{v}_\gamma(\lambda)^{-1} \otimes \mathbf{v}_\gamma(\lambda)^{-1}) \\ &= 1 - \lambda d_B(v_\gamma) + \text{higher } Q\text{-degree terms} \end{aligned}$$

is a twist for u_q^B, by Lemma 7.3. It follows that the homogenous degree γ term $\lambda d_B(v_\gamma)$ is in u_q for each $\lambda \in \mathbb{C}$. Since $v_\gamma \notin u_q$, the the cohomology class $[d_B(v_\gamma)]$ is a nonvanishing class of degree γ in $H^2((u_q^*)_B)$, by Lemma 7.6.

Since $\dim\left(H^2((u_q^*)_B)_\gamma\right) = 1$, nonvanishing of $[d_B(v_\gamma)]$ implies that there exists $c \in \mathbb{C}$ so that

$$[d_B(cv_\gamma)] = c[d_B(v_\gamma)] = [J_\gamma].$$

It follows that

$$\mathbf{v}_\gamma(c) \cdot J = 1 + J_d - cd_B(v_\gamma) + \text{higher } \mathbb{Z}\text{-degree terms}$$

is such that $[(\mathbf{v}_\gamma(c) \cdot J)_\gamma] = 0$. As was argued above, there now exists a $v' \in \mathbb{V}_{small}$ so that

$$v' \cdot (\mathbf{v}_\gamma(c) \cdot J) = (v'\mathbf{v}_\gamma(c)) \cdot J$$

satisfies $((v'\mathbf{v}_\gamma(c)) \cdot J)_{d''} = 0$ for $0 < d'' < d$ and $((v'\mathbf{v}_\gamma(c)) \cdot J)_d = J_d - J_\gamma$. If we take in this case $v = v'\mathbf{v}_\gamma(c)$, we arrive at the desired result for twists of u_q^B.

Let us return now to the case of a twist for u_q of the form $J' = B + \sum_{n>0} J'_n$, with B a bilinear form on $G^\vee$ and $J' - B$ vanishing in degrees $< d$. For $J = B^{-1}J'$ we have just seen that there is a unit $v \in \mathbb{V}$ so that gauging J by v produces a twist for u_q^B which vanishes in all degrees $0 < d' \leq d$. By Lemma 5.3 it follows that $B^{-1}(v \cdot J')$ vanishes in all degrees $0 < d \leq d$. Hence $v \cdot J'$ itself vanishes in degrees $0 < d' \leq d$, since $B \in u_q \otimes u_q$ is a degree 0 unit. □

We can now prove the main result.

Proof of Theorem 7.5. After gauging by an element in $\mathbb{C}[G]$ if necessary, we may write our twist J as $J = B + \sum_{n>0} J_n$, where B is an alternating form on $G^\vee$. Take N the maximal $\mathbb{Z}$-degree with $(u_q)_N \neq 0$. By Lemma 7.8 we can define recursively a sequence of units $v_1, \dots, v_{N^2} \in \mathbb{V}$ so that

$$((v_i v_{i-1} \dots v_1) \cdot J) - B$$

has first nonvanishing term in a $\mathbb{Z}$-degree $d_i > i$. Hence, if we define $v_{tot} = (v_{N^2} \dots v_1)$ then $(v_{tot} \cdot J) - B$ vanishes in degrees $\leq N^2$. Since $v_{tot} \cdot J$ is a twist in $u_q \otimes u_q$, by Lemma 7.3, and $u_q \otimes u_q$ vanishes in degree $> N^2$, we see that $(v_{tot} \cdot J) - B = 0$. Rather, $v_{tot} \cdot J = B$.

Since $\mathbb{V}_{small}$ is normal in $\mathbb{V}$, and $\mathbb{V}$ is generated by $\mathbb{V}_{dp}$ and $\mathbb{V}_{small}$, we may write $v_{tot} = v^{-1}v'$ with $v \in \mathbb{V}_{dp}$ and $v' \in \mathbb{V}_{small}$. Since $v_{tot} \cdot J = B$ we have then $v' \cdot J = v \cdot B$. Hence J is gauge equivalent to the twist $v \cdot B$. □

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8. Algebraic group actions and tensor equivalences

In this section we prove Theorems A and B of the introduction.

8.1. **Algebraic groups and twisted automorphisms.** We recall some information from [9, 30].

One takes the graded dual of the exact sequence of Hopf algebras

$$1 \to u_q(\mathfrak{b}) \to U_q(\mathfrak{b}) \to U(\mathfrak{n}) \to 1$$

to arrive at another exact sequence

$$1 \to \mathscr{O}(\mathbb{U}) \to \mathscr{U}^{DK} \to u_q(\mathfrak{b})^* \to 1,$$

where $\mathbb{U}$ the unipotent group corresponding to $\mathfrak{n}$. From such a sequence Arkhipov and Gaitsgory show that there is an equivalence of tensor categories between the de-equivariantization $\mathrm{corep}(\mathscr{U}^{DK})_{\mathbb{U}}$ and $\mathrm{rep}(u_q)$ [9, 7].

The de-equivariantization is a certain monoidal subcategory of $\mathrm{Coh}(\mathbb{U})$, and it inherits an action of $\mathbb{U}$ which is induced by the translation action of $\mathbb{U}$ on $\mathrm{Coh}(\mathbb{U})$. Whence we get a natural action of $\mathbb{U}$ on $\mathrm{rep}(u_q)$. Just as in [30, Thm. 9.1, Prop. 9.4], one can verify that the 1-parameter subgroups $\exp(\lambda e_\alpha) \subset \mathbb{U}$, $\alpha \in \Gamma$, act on $\mathrm{rep}(u_q)$ by way of the *twisted automorphisms*

$$(\exp_\alpha^\lambda, J_\alpha^\lambda),$$

where $\exp_\alpha^\lambda := \exp(\mathrm{ad}_{\lambda E_\alpha^{(l)}}) = \mathrm{Ad}_{\exp(\lambda E_\alpha^{(l)})}$ and

$$J_\alpha^\lambda = \Delta\left(\exp(\lambda E_\alpha^{(l)})\right)\left(\exp(-\lambda E_\alpha^{(l)}) \otimes \exp(-\lambda E_\alpha^{(l)})\right).$$

Recall that a twisted automorphism (ϕ, J) of a Hopf algebra H is a pair consisting of a twist J and a Hopf isomorphism $\phi : H \to H^J$ (see [31, Thm. 2.2] and [23, 12]). Any such pair (ϕ, J) defines an autoequivalence of $\mathrm{rep}(H)$ via the composite $\mathrm{rep}(H) \xrightarrow{J} \mathrm{rep}(H^J) \xrightarrow{\mathrm{res}_\phi} \mathrm{rep}(H)$. We have already seen in Section 7.1 that $\exp_\alpha^\lambda : u_q \to u_q$ is an algebra automorphism and that J_α^λ is a twist for u_q.

The collection of twisted automorphisms of a Hopf algebra H forms a group under the product

$$(\phi, J)(\phi', J') = (\phi\phi', J\phi^{\otimes 2}(J')),$$

and this group acts naturally on the left of the set $\mathrm{Tw}(H)$ of gauge equivalence classes of twists by $(\phi, J) \cdot [J'] = [J\phi^{\otimes 2}(J')]$.[1]

Lemma 8.1. *For any $v \in \mathbb{V}_{dp}$ the pair $(\mathrm{Ad}_v, \Delta(v)(v^{-1} \otimes v^{-1}))$ is a twisted automorphism for u_q. Furthermore, the collection of all such pairs*

$$\mathfrak{V}_{dp} = \{(\mathrm{Ad}_v, \Delta(v)(v^{-1} \otimes v^{-1})) : v \in \mathbb{V}_{dp}\}$$

forms a subgroup in the group of twisted automorphisms for u_q.

Proof. The fact that each pair $(\mathrm{Ad}_v, \Delta(v)(v^{-1} \otimes v^{-1}))$ is a twisted automorphism follows from Lemmas 7.2 and 7.3, and the computation

$$\Delta^{\Delta(v)(v^{-1}\otimes v^{-1})}(\mathrm{Ad}_v(h)) = (v\otimes v)\Delta(v)^{-1}\Delta(vhv^{-1})\Delta(v)(v^{-1}\otimes v^{-1}) = \mathrm{Ad}_{v\otimes v}(\Delta(h)).$$

[1] Our group operations are *opposite* that of some other references. With our conventions, the association $(\phi, J) \mapsto [\phi, J] \in \mathrm{Aut}_\otimes(\mathrm{rep}(H))$ will be group anti-homomorphism.

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For any two twisted automorphisms of the form $\left(\mathrm{Ad}_{v_1}, \Delta(v_1)(v_1^{-1}\otimes v_1^{-1})\right)$ and $\left(\mathrm{Ad}_{v_2}, \Delta(v_2)(v_2^{-1}\otimes v_2^{-1})\right)$, with the $v_i \in \widehat{U}_q$, we have

$$\begin{aligned}
&\left(\mathrm{Ad}_{v_1}, \Delta(v_1)(v_1^{-1}\otimes v_1^{-1})\right)\cdot\left(\mathrm{Ad}_{v_2}, \Delta(v_2)(v_2^{-1}\otimes v_2^{-1})\right)\\
&= \left(\mathrm{Ad}_{v_1v_2},\ \Delta(v_1)(v_1^{-1}\otimes v_1^{-1})\mathrm{Ad}_{v_1\otimes v_1}\left(\Delta(v_2)(v_2^{-1}\otimes v_2^{-1})\right)\right)\\
&= \left(\mathrm{Ad}_{v_1v_2},\ \Delta(v_1)\Delta(v_2)(v_2^{-1}\otimes v_2^{-1})(v_1^{-1}\otimes v_1^{-1})\right)\\
&= \left(\mathrm{Ad}_{v_1v_2},\ \Delta(v_1v_2)((v_1v_2)^{-1}\otimes (v_1v_2)^{-1})\right).
\end{aligned}$$

Since $\mathbb{V}_{dp}$ is a group, the above calculation also shows that $\mathfrak{V}_{dp}$ is a subgroup in the group of twisted automorphisms. □

8.2. **A classification of twists for $u_q(\mathfrak{b})$ in terms of the $\mathbb{U}$-action.**

Theorem 8.2. *Under the above natural action of $\mathbb{U}$ on $\mathrm{Tw}(u_q(\mathfrak{b}))$, there is an equality $\mathrm{Tw}\,(u_q(\mathfrak{b})) = \mathbb{U}\cdot\mathrm{Alt}(G^\vee)$.*

Proof. Take $\mathfrak{V}_{dp}$ as in Lemma 8.1. Since the generating 1-parameter subgroups $\exp(\lambda e_\lambda)$ of $\mathbb{U}$ act on $\mathrm{Tw}(u_q)$ via the generators of $\mathfrak{V}_{dp}$, we have

$$\mathbb{U}\cdot[J] = \mathfrak{V}_{dp}\cdot[J]$$

for any class $[J]\in\mathrm{Tw}(u_q)$. By Theorem 7.5 we have $[J] = [v\cdot B]$ for some $v\in\mathbb{V}_{dp}$ and alternating form B on $G^\vee$. One checks directly that

$$[v\cdot B] = (\mathrm{Ad}_v, \Delta(v)(v^{-1}\otimes v^{-1}))\cdot[B]\in\mathfrak{V}_{dp}\cdot[B].$$

Hence

$$\mathrm{Tw}(u_q) = \mathfrak{V}_{dp}\cdot\mathrm{Tw}(u_q) = \mathfrak{V}_{dp}\cdot[\mathrm{Alt}(G^\vee)] = \mathbb{U}\cdot[\mathrm{Alt}(G^\vee)].$$

From the classification of twists for abelian group algebras [21, Cor. 5.8], and the fact that the Hopf inclusion $\mathbb{C}[G]\to u_q$ is split, one finds that the mapping $\mathrm{Alt}(G^\vee)\to\mathrm{Tw}(u_q)$ is injective. Hence $[\mathrm{Alt}(G^\vee)] = \mathrm{Alt}(G^\vee)$ and $\mathrm{Tw}(u_q) = \mathbb{U}\cdot\mathrm{Alt}(G^\vee)$. □

8.3. **Tensor equivalence and Hopf algebras.** We use Theorem 7.5 to classify all Hopf algebras which are tensor equivalent to u_q, up to Hopf isomorphism.

Theorem 8.3. *Suppose H is a finite dimensional Hopf algebra such that $\mathrm{rep}(H)$ and $\mathrm{rep}(u_q(\mathfrak{b}))$ are equivalent as tensor categories. Then there is an alternating form B on $G^\vee$ such that $H\cong u_q(\mathfrak{b})^B$ as Hopf algebras. When l is coprime to the determinant of the Cartan matrix for $\mathfrak{g}$, the form B is uniquely determined by H. In general, if two forms B and B' admit a Hopf isomorphism $u_q(\mathfrak{b})^B\cong u_q(\mathfrak{b})^{B'}$ then their restrictions to G, along the Killing map, are equal.*

Proof. We verify the existence of such a form B, then address uniqueness.

By Lemma 8.1, for each $v\in\mathbb{V}_{dp}$ the conjugation Ad_v restricts to an algebra automorphism of u_q. Given any twist J for u_q, and $v\in\mathbb{V}$, a simple calculation verifies that Ad_v provides a Hopf isomorphism between the twisted algebras $\mathrm{Ad}_v : u_q^{(v\cdot J)}\to u_q^J$. It follows that any twist J for u_q will admit an alternating form B on $G^\vee$ so that $u_q^J\cong u_q^B$ as a Hopf algebra, by Theorem 7.5.

Consider now any Hopf algebra H with $\mathrm{rep}(H)\xrightarrow{\sim}\mathrm{rep}(u_q)$ as tensor categories. Then there is a Hopf isomorphism between H and u_q^J for some twist J, by [31, Thm. 2.2]. Hence there is a Hopf isomorphism $H\cong u_q^B$ for some alternating B.

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As for our uniqueness claim, suppose there is a Hopf isomorphism $\psi : u_q^{B'} \xrightarrow{\cong} u_q^B$ for two alternating forms B' and B. Note that B', and $(B')^{-1}$, remain twists for $u_q^{B'}$. So we may twist both $u_q^{B'}$ and u_q^B to arrive at a Hopf isomorphism $u_q \xrightarrow{\cong} u_q^{B\psi^{\otimes 2}(B')^{-1}}$. Hence it suffices to assume $B' = 1$ and show that if $u_q \cong u_q^B$ for an alternating form B, then B restricts to the trivial form on G.

We may take the dual of such a isomorphism to get a Hopf isomorphism $\phi : u_q^* \xrightarrow{\cong} (u_q^*)_B$. By Lemma 4.4 u_q^*, which is a Hopf quotient of the modified De Concini-Kac algebra $\mathscr{U}^{DK}$, is generated by the grouplikes $G^\vee$ and skew primitives X_α. By considering the J-adic filtration on u_q, with respect to the Jacobson radical $J = (E_\alpha : \alpha \in \Gamma)$, one can also see that the X_α provide all skew primitives in the dual u_q^*.

Since ϕ preserves grouplikes and skew primitives there is a set bijection $\phi' : \Gamma \to \Gamma$, and scalars $c_\alpha \in \mathbb{C}^\times$ for each $\alpha \in \Gamma$, so that

$$\phi(X_\alpha) = c_\alpha X_{\phi'(\alpha)}.$$

Compatibility with the comultiplication implies $\phi(\alpha) \in G^\vee$ is the image of $\phi'(\alpha) \in Q/lQ$ under the Killing map. After precomposing with a scaling on the generators of u_q^+, we may assume that each $c_\alpha = 1$.

We have now

$$\begin{aligned} q^{(\alpha,\beta)-(\phi'(\alpha),\phi'(\beta))}\phi(X_\alpha) &= q^{-(\phi'(\alpha),\phi'(\beta))}\phi(\beta X_\alpha \beta^{-1}) \\ &= q^{-(\phi'(\alpha),\phi'(\beta))}\phi(\beta) \cdot_B X_{\phi'(\alpha)} \cdot_B \phi(\beta)^{-1} \\ &= \tfrac{B(\phi(\alpha),\phi(\beta))}{B(\phi(\beta),\phi(\alpha))} X_{\phi'(\alpha)} = B^2(\phi(\alpha),\phi(\beta)) X_{\phi'(\alpha)} \end{aligned}$$

for each $\alpha, \beta \in \Gamma$ (see Lemma 9.2). This implies

$$B^2(\phi(\alpha),\phi(\beta)) = q^{(\alpha,\beta)-(\phi'(\alpha),\phi'(\beta))}$$

for each $\alpha, \beta \in \Gamma$. Since the right hand side of the above equation is symmetric, we conclude that B^2 restricted to the image of G in $G^\vee$ is symmetric. Since $2 \nmid l$ we can take the square root of B^2 to find that the restriction of B to G is also symmetric. This forces the restriction of B to be 1, as desired.

When l is coprime to the determinant of the Cartan matrix the killing map $\kappa : G \to G^\vee$ is an isomorphism. So uniqueness follows. □

9. Proofs of Propositions 6.1 and 6.5

We prove Propositions 6.1 and 6.5, which claim equalities

$$H^1(\mathscr{U}_B^{DK}) = H^2(\mathscr{U}_B^{DK}) = 0,$$

$$H^1((u_q^*)_B) = 0 \quad \text{and} \quad H^2((u_q^*)_B) = \mathfrak{n}^{(l)}$$

respectively. Our computations are based on explicit presentations of the algebras $\mathscr{U}_B^{DK}$ and $(u_q^*)_B$, and the majority of the material below is dedicated to exhibiting such presentations. We employ the notion of a minimal generating set of relations for a connected graded algebra $A = T(V)/I$, where I is a graded ideal concentrated in (tensor) degree > 1. We call a set of homogenous generating relations $\{r_i\}_i \subset I$ *minimal* if the r_i are linearly independent in the reduction $\mathbb{C} \otimes_T I \otimes_T \mathbb{C}$.

We fix for the section a bilinear form B on $G^\vee$. Take $u_q = u_q(\mathfrak{b})$ and $\mathscr{U}^{DK}$ as in Section 4.2.

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9.1. **A presentation of the algerba $\mathscr{U}_B^{DK}$.** The left and right $\mathbb{C}[G]$-actions on $\mathscr{U}^{DK}$ are given by

$$g \cdot \omega = \omega \cdot g = g(\omega)\omega, \quad g \cdot X_\alpha = g(\alpha^{-1})X_\alpha, \quad \text{and} \quad X_\alpha \cdot g = X_\alpha,$$

for $g \in G$ and $\omega \in G^\vee$. Hence, for the multiplication $\cdot_B$ of the cocycle twist $\mathscr{U}_B^{DK}$, we have

$$X_\alpha \cdot_B X_\beta = B(\alpha^{-1}, \beta^{-1})X_\alpha X_\beta = B(\alpha, \beta)X_\alpha X_\beta,$$

$$\omega \cdot_B X_\alpha = B(\omega, \alpha)^{-1}\omega X_\alpha,$$

and

$$\omega \cdot_B \chi = \frac{B(\omega, \chi)}{B(\omega, \chi)}\omega\chi = \omega\chi$$

for each $\omega, \chi \in G^\vee$. One establishes by induction

Lemma 9.1. *For any collection of generators $X_{\alpha_1}, \dots, X_{\alpha_m}$ of $\mathscr{U}^{DK}$, and $\omega \in G^\vee$, we have*

$$\omega \cdot_B X_{\alpha_1} \cdot_B \cdots \cdot_B X_{\alpha_m} = B^{-1}(\omega, \alpha_1 \dots \alpha_m)\Big(\prod_{1 \le i < j \le m} B(\alpha_i, \alpha_j)\Big)\omega X_{\alpha_1} \dots X_{\alpha_m}.$$

In particular, $\mathscr{U}_B^{DK}$ is generated as an algebra by $\mathbb{C}[G^\vee]$ and the X_α.

Lemma 9.2. *The generators of the cocycle twist $\mathscr{U}_B^{DK}$ satisfy the commutator relations*

$$\omega \cdot_B X_\alpha \cdot_B \omega^{-1} = \omega(K_\alpha)\frac{B(\alpha, \omega)}{B(\omega, \alpha)}X_\alpha \tag{6}$$

and the B-Serre relations

$$\sum_{i=0}^{1-\langle\alpha,\beta\rangle} (-1)^i \frac{B(\alpha,\alpha)^{\frac{\langle\alpha,\beta\rangle(1-\langle\alpha,\beta\rangle)}{2}}}{B(\alpha,\beta)^{1-\langle\alpha,\beta\rangle-i}B(\beta,\alpha)^i} \left[\begin{array}{c} 1-\langle\alpha,\beta\rangle \\ i \end{array} \right]_{q_\alpha^{-1}} X_\alpha^{1-\langle\alpha,\beta\rangle-i} \cdot_B X_\beta \cdot_B X_\alpha^i = 0. \tag{7}$$

Proof. The commutator relation follows by Lemma 4.3 and the calculation

$$\omega \cdot_B X_\alpha \cdot_B \omega^{-1} = \frac{B(\alpha, \omega)}{B(\omega, \alpha)}\omega X_\alpha \omega^{-1}.$$

The B-Serre relations follow from the the previous lemma and the relations for $\mathscr{U}^{DK}$. □

Proposition 9.3. *The algebra projection*

$$\pi : \frac{\mathbb{C}\langle \bar{\omega}, \bar{X}_\alpha : \omega \in G^\vee, \alpha \in \Gamma\rangle}{(1_{G^\vee} - 1,\ \bar{\omega}\bar{\chi} - \overline{\omega\chi},\ \text{relations (6) and (7)})} \to \mathscr{U}_B^{DK}, \ \left\{ \begin{array}{l} \bar{\omega} \mapsto \omega \\ \bar{X}_\alpha \mapsto X_\alpha \end{array} \right.$$

is an isomorphism.

We adopt an unorthodox notation for the proof. For a Hopf algebra H, twist $J \in H \otimes H$, and H-module algebra R, we can twist the multiplicaiton on R to arrive at a new algebra with multiplication $r \cdot_{new} r' = \text{mult}(J \cdot (r \otimes r'))$. We write this new algebra as $R(J)$, to not confuse with the cocycle or Drinfeld twist of a Hopf algebra.

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Proof. Let R denote the domain of π. We grade R by the root lattice Q by taking $\deg_Q(\bar{\omega}) = 0$ and $\deg_Q(\bar{X}_\alpha) = -\alpha$. Under this grading π is homogenous. Hence to prove that π is an isomorphism it suffices to prove that R and $\mathscr{U}_B^{DK}$ have the same dimension in each Q-degree.

We can give R the $G \times G^{op}$-action defined by $(g,h) \cdot \omega = g(\omega)h^{-1}(\omega)\bar{\omega}$ and $(g,h) \cdot \bar{X}_\alpha = g(\alpha^{-1})\bar{X}_\alpha$. Under this action R becomes a $\mathbb{C}[G \times G^{op}]$-module algebra, and π is a map of $\mathbb{C}[G \times G^{op}]$-module algebras.

Let $\mathbb{B} \in \mathbb{C}[G \times G^{op}]^{\otimes 2}$ be the bilinear form corresponding to $B \otimes B \in \mathbb{C}[G]^{\otimes 2} \otimes \mathbb{C}[G^{op}]^{\otimes 2}$ under the obvious isomorphism $\mathbb{C}[G]^{\otimes 2} \otimes \mathbb{C}[G^{op}]^{\otimes 2} \cong \mathbb{C}[G \times G^{op}]^{\otimes 2}$. We twist π to arrive at a surjection

$$\pi(\mathbb{B}^{-1}) : R(\mathbb{B}^{-1}) \to (\mathscr{U}_B^{DK})(\mathbb{B}^{-1}).$$

(Note that $\pi(\mathbb{B}) = \pi$ as a vector space map.) The algebra $R(\mathbb{B}^{-1})$ now has the relations of the De Concini Kac algebra. So, for each $\mu \in Q$,

$$\dim(R_\mu) = \dim(R(\mathbb{B}^{-1})_\mu) \leq \dim((\mathscr{U}^{DK})_\mu) = \dim((\mathscr{U}_B^{DK})_\mu).$$

The inverse inequality follows by surjectivity or π, and hence $\dim(R_\mu) = \dim((\mathscr{U}_B^{DK})_\mu)$ for each $\mu \in Q$. We conclude that π is an isomorphism. □

Let $\mathscr{U}_B^-$ denote the subalgebra in $\mathscr{U}_B^{DK}$ generated by the $\{X_\alpha : \alpha \in \Gamma\}$. Note that $\mathscr{U}_B^-$ is a Q-graded subalgebra in U_B^{DK}.

Proposition 9.4. *The map*

$$\pi^- : \frac{\mathbb{C}\langle \bar{X}_\alpha : \alpha \in \Gamma \rangle}{(\text{relations (7)})} \to \mathscr{U}_B^-, \quad \bar{X}_\alpha \mapsto X_\alpha,$$

is an algebra isomorphism. Furthermore, if we let $G^\vee$ act on $\mathscr{U}_B^-$ by conjugation in $\mathscr{U}_B^{DK}$, multiplication provides an isomorphism $\mathscr{U}_B^- \rtimes \mathbb{C}[G^\vee] \cong \mathscr{U}_B^{DK}$.

Proof. Surjectivity of the multiplication map $\mathscr{U}_B^- \rtimes \mathbb{C}[G^\vee] \to \mathscr{U}_B^{DK}$ is clear, since $\mathscr{U}_B^{DK}$ is generated by $G^\vee$ and the X_α. Injectivity follows by Lemma 9.1 and a dimension count in each homogenous Q-degree. As for the map π^-, let us denote the domain by R^-. We let $\mathbb{C}[G^\vee]$ act on R^- by "conjugation"

$$\omega \cdot \bar{X}_\alpha = \omega(K_\alpha)\frac{B(\alpha,\omega)}{B(\omega,\alpha)}\bar{X}_\alpha.$$

Take the smash product to get a surjective map

$$\pi^- \rtimes \mathbb{C}[G^\vee] : R^- \rtimes \mathbb{C}[G^\vee] \to \mathscr{U}_B^- \rtimes \mathbb{C}[G^\vee].$$

The algebra $R^- \rtimes \mathbb{C}[G^\vee]$ satisfies all of the relations of the domain R of π, from Proposition 9.3. So, just as in Proposition 9.3, we find that $\pi^- \rtimes \mathbb{C}[G^\vee]$ is an isomorphism. Since $\mathbb{C}[G^\vee]$ is faithfully flat over $\mathbb{C}$, it follows that π^- is an isomorphism. □

9.2. **Proof of Proposition 6.1.** Before presenting the proof we give some background information. Since $\mathscr{U}_B^{DK} = \mathscr{U}_B^- \rtimes \mathbb{C}[G^\vee]$, for any $\mathscr{U}_B^{DK}$-modules M and N we have

$$\mathrm{Hom}_{\mathscr{U}_B^{DK}}(M,N) = \mathrm{Hom}_{\mathscr{U}_B^-}(M,N)^{G^\vee}.$$

Hence we have an equality of cohomologies $H^\bullet(\mathscr{U}_B^{DK}) = H^\bullet(\mathscr{U}_B^-)^{G^\vee}$.

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We are particularly interested in the case in which M and N are G-graded, and the corresponding $G^\vee \subset U_B^{DK}$-action is as follows:

$$\omega \cdot m = \omega(\deg_G m)\frac{B(\deg_G m, \omega)}{B(\omega, \deg_G m)}m,$$

for homogenous m. (We use the Killing map to identify the G-degree with a character in $G^\vee$.) In this case we can decompose the set of morphisms

$$\mathrm{Hom}_{\mathscr{U}_B^-}(M,N) = \oplus_{g\in G}\mathrm{Hom}_{\mathscr{U}_B^-}(M,N)_g,\ \ \mathrm{Hom}_{\mathscr{U}_B^-}(M,N)_g = \left\{ \begin{array}{c} \text{homogenous degree } g \\ \text{maps } f: M \to N \end{array} \right\}.$$

Under this decomposition $G^\vee$ will act on each homogenous $f \in \mathrm{Hom}_{\mathscr{U}_B^-}(M,N)_g$ as

$$\omega \cdot f = \omega(g)\frac{B(\omega, g)}{B(g,\omega)}f.$$

Whence we have an easy basis of eigenvectors with respect to which we can calculate the $G^\vee$-invariants.

Note that $\mathscr{U}_B^-$, with G-grading induced by its Q-grading, admits such a $\mathscr{U}_B^{DK}$-module structure. As do the shifts $\mathscr{U}_B^- \otimes \mathbb{C}_g$ by $g \in G$.

Proof of Proposition 6.1. Via the identification $Q/lQ = G$ we will denote G-degrees by elements in Q/lQ. We let $\bar{X}$ denote formal variables. As $\mathscr{U}_B^-$ is a connected $\mathbb{Z}$-graded algebra, we have the minimal $\mathbb{Z}$-graded resolution

$$P = \cdots \to \mathscr{U}_B^- \otimes V \to \mathscr{U}_B^- \otimes \mathbb{C}\{\bar{X}_\alpha : \alpha \in \Gamma\} \to \mathscr{U}_B^- \to 0$$

of the trivial module, where V is the graded vector space of relations

$$\mathbb{C}\left\{ \sum_{i=0}^{1-\langle\alpha,\beta\rangle} (-1)^i \frac{B(\alpha,\alpha)^{\frac{\langle\alpha,\beta\rangle(1-\langle\alpha,\beta\rangle)}{2}}}{B(\alpha,\beta)^{1-\langle\alpha,\beta\rangle-i}B(\beta,\alpha)^i} \left[\begin{array}{c} 1-\langle\alpha,\beta\rangle \\ i \end{array} \right]_{\zeta_\alpha} \bar{X}_\alpha^{1-\langle\alpha,\beta\rangle-i} \cdot_B \bar{X}_\beta \cdot_B \bar{X}_\alpha^i \right\}_{\alpha,\beta}.$$

(See for example [10, Sect. 2].) We using here the fact that the B-Serre relations are of distinct, incomparable, Q-degree, and hence provide a minimal set of relations for $\mathscr{U}_B^-$. The resolution P is also G-graded, where the $\bar{X}_\alpha$ are of degree α^{-1} and the B-Serre relations are of respective degrees $\alpha^{\langle\alpha,\beta\rangle-1}\beta^{-1}$.

Since P is minimal, the first and second cohomologies are given by the duals $H^1(\mathscr{U}_B^-) = (\mathbb{C}\{\bar{X}_\alpha : \alpha \in \Gamma\})^*$ and $H^2(\mathscr{U}_B^-) = V^*$, as a G-graded spaces. We have $\alpha \cdot \bar{X}_\alpha^* = q^2\bar{X}_\alpha^*$ for each dual function $\bar{X}_\alpha^*$ to the $\bar{X}_\alpha$, and hence $H^1(\mathscr{U}_B^{DK}) = H^1(\mathscr{U}_B^-)^{G^\vee} = 0$. We claim also $H^2(\mathscr{U}_B^{DK}) = 0$.

The space $H^2(\mathscr{U}_B^-) = V^*$ is spanned by functions f of G-degrees $\alpha^{1-\langle\alpha,\beta\rangle}\beta$. Since the fraction $\frac{B(\omega,?)}{B(?,\omega)}$ is alternating, we can act on such a function by its degree to get

$$(\alpha^{1-\langle\alpha,\beta\rangle}\beta)\cdot f = q^{(1-\langle\alpha,\beta\rangle)^2(\alpha,\alpha)+2(1-\langle\alpha,\beta\rangle)(\alpha,\beta)+(\beta,\beta)}f. \tag{8}$$

By considering the possible lengths of roots in Γ we have

$$(1-\langle\alpha,\beta\rangle)^2(\alpha,\alpha)+2(1-\langle\alpha,\beta\rangle)(\alpha,\beta)+(\beta,\beta)$$

$$= \left\{ \begin{array}{ll} 4 \text{ or } 6 & \text{when } \mathfrak{g} \text{ is simply laced} \\ 4,\ 6,\ \text{or } 10 & \text{in types } B_n \text{ or } C_n \\ 4,\ 6,\ 10,\ \text{or } 12 & \text{in type } F_4 \\ 14 & \text{in type } G_2. \end{array} \right.$$

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Since l is odd, the restrictions on l introduced in Section 2 are sufficient to ensure that the order of q avoids each of the above integers. So the coefficient appearing in (8) are never 1. It follows that the invariants vanish, and $H^2(\mathscr{U}_B^{DK}) = H^2(\mathscr{U}_B^-)^{G^\vee} = 0$. □

9.3. **Preliminary information for the proof of Proposition 6.5.** We now consider the small algebra $u_q = u_q(\mathfrak{b})$. If we take the (graded) dual of the Q-graded inclusion $u_q \to U_q$ we get a Q-graded Hopf projection $\mathscr{U}^{DK} \cong U_q^\# \to u_q^*$. Since $\mathscr{U}^{DK}$ is generated by the functions X_α, and $G^\vee$, u_q^* will be generated by these functions as well. Let u^- denote the subalgebra of u_q^* generated by the X_α.

Lemma 9.5. *The projection $\mathscr{U}^{DK} \to u_q^*$ gives u_q^* as the quotient*

$$u_q^* = \mathscr{U}^{DK}/(X_\mu^l : \mu \in \Phi^+) = u^- \rtimes \mathbb{C}[G^\vee].$$

Proof. Note that u_q^* is pointed, coradically graded, and generated by the skew primitives X_α. Hence u_q^* is a bosonization $\mathfrak{B}(V) \rtimes \mathbb{C}[G^\vee]$, where V is the Yetter-Drinfeld $G^\vee$-module spanned by the skew primitives X_α in u_q^* and $\mathfrak{B}(V)$ is the corresponding Nichols algebra [32, 3].

Consider alternatively the span V' of the skew-primitives in $u_{q^{-1}}(\mathfrak{b}_-)$. Since the algebra $\mathfrak{B}(V)$ only depends on the braiding for V, the relations of Lemmas 4.2 and 4.3 imply an algebra isomorphism $u_{q^{-1}}^- = \mathfrak{B}(V') \xrightarrow{\cong} \mathfrak{B}(V)$, $F_\alpha \mapsto X_\alpha$. It follows that the X_α satisfy the same relations as the F_α in $u_{q^{-1}}^-$. In particular $X_\mu^l = 0$ for all $\mu \in \Phi^+$. Whence we have a surjection $\mathscr{U}^{DK}/(X_\mu^l) \to u_q^*$. Agreement of dimensions implies that the surjection is an isomorphism. □

Following our notation for u_q^*, we write u_B^- for the subalgebra of the cocycle twist $(u_q^*)_B$ generated by the X_α.

Corollary 9.6. *For any bilinear form $B \in \mathbb{C}[G] \otimes \mathbb{C}[G]$ we have a minimal presentations*

$$u_B^- = \frac{\mathbb{C}\langle \bar{X}_\alpha : \alpha \in \Gamma \rangle}{(\text{relations } (7),\ \bar{X}_\mu^l :\ \mu \in \Phi^+)}.$$

Proof. One can use Lemma 9.1 to check that $X_\mu^{\cdot_B l} = X_\mu^l$ in $\mathscr{U}_B^{DK}$. So there is no ambiguity in writing X_μ^l here. The presentations now follow from the presentation of $\mathscr{U}_B^-$ given at Propositions 9.4, and Lemma 9.5. Minimality can be argued via centrality of the X_μ^l and by considering Lusztig's basis of $\mathscr{U}_B^-$ [14, 26]. □

9.4. **Proof of Proposition 6.5.** As was mentioned in Section 6.2, the cohomology $H^\bullet((u_q^*)_B)$ is Q-graded. This was proposed to follow from the fact that the standard complex $C^\bullet((u_q^*)_B)$ is Q-graded, which is true. We can, however, also find this Q-grading by identifying the cohomology of $(u_q^*)_B$ with the derived Q-graded Hom functor on the category of Q-graded $(u_q^*)_B$-modules.

Proof of Proposition 6.5. We proceed as in the proof of Theorem 6.1. We want to first give the cohomology of u_B^-, then take the invariants to get $H^\bullet(u_B^-)^{G^\vee} = H^\bullet(u_B^*)$.

We have the minimal Q-graded resolution

$$\cdots \to (u_B^- \otimes W) \oplus (u_B^- \otimes V) \to u_B^- \otimes \mathbb{C}\{\bar{X}_\alpha : \alpha \in \Gamma\} \to u_B^- \to 0$$

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of $\mathbb{C}$ over u_B^-, where V is spanned by the B-Serre relations and $W = \mathbb{C}\{\bar{X}_\mu^l : \mu \in \Phi^+\}$. Note that W lay in Q-degrees $-l\Phi^+$. (See [10, Sect. 2].)

Hence

$$H^1(u_B^-) = \mathbb{C}\{\bar{X}_\alpha : \alpha \in \Gamma\}^*, \quad \text{and} \quad H^2(u_B^-) = W^* \oplus V^*.$$

As we saw in the proof of Proposition 6.1, $H^1(u_B^-)^{G^\vee} = 0$ and $(V^*)^{G^\vee} = 0$. The subspace W^* is $G^\vee$-invariant, since each $\omega \in G^\vee$ acts trivially on elements of Q-degrees lQ. This gives

$$H^1\left((u_q^*)_B\right) = H^1(u_B^-)^{G^\vee} = 0 \quad \text{and} \quad H^2\left((u_q^*)_B\right) = H^2(u_B^-)^{G^\vee} = W^*.$$

We identify $\mathfrak{n}^{(l)}$ with W^* by taking each e_μ to the function $\bar{X}_\nu^l \mapsto \delta_{\mu,\nu}$. □

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