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Cohomology with twisted one-dimensional coefficients for congruence subgroups of $\mathrm{SL}_4(\mathbb{Z})$ and Galois representations [☆]

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We extend the computations in [2–4] to find the cohomology in degree five of a congruence subgroup of $\mathrm{SL}_4(\mathbb{Z})$ with coefficients in a field twisted by a nebentype character, along with the action of the Hecke algebra on the cohomology. This is the top cuspidal degree. For each Hecke eigenclass we find, we produce the unique Galois representation that appears to be attached to it.

The computations require serious modifications to our previous algorithms. Nontrivial coefficients add a layer of complication to our data structures. New possibilities must be taken into account in the Galois Finder, the code that finds the Galois representations. We have improved the Galois Finder to report when the attached Galois representation is uniquely determined by our data.

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1. Introduction

1.1. The cohomology of arithmetic groups plays a fundamental role in number theory, especially algebraic K-theory and the Langlands program. Many people have worked on computing the homology and cohomology of particular arithmetic groups, such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \leqslant 9$, and congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ with small level for $n \leqslant 4$. For a sampling of such work, we refer to [17,16,32,33,25,24,29,13].

The dimensions of these cohomology groups are important. However, to extract more of the number-theoretic information from the cohomology, and in particular to explore the connections between cohomology and Galois representations, it is essential to compute the action of the algebra of Hecke operators on these groups. In this paper, which continues the series [2–4], we present the results of such computations for congruence subgroups of $\mathrm{SL}_4(\mathbb{Z})$, using the sharbly complex to compute the cohomology and the action of the Hecke operators.

Examples of systems of Hecke eigenvalues that occur in the mod p cohomology of a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$, when correlated with Galois representations, shed light on generalizations of Serre's conjecture [1], provide instances of Scholze's results on the mod p cohomology of locally symmetric spaces [30], and when p is a large random prime may illustrate the existence of Galois representations attached to the cuspidal cohomology in characteristic zero, as proved in the work of Harris, Lan, Taylor and Thorne [23].

1.2. We now introduce more notation and state our main result. For any integer $N \geqslant 1$, let $\Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ be the subgroup of matrices with bottom row congruent to $(0, 0, 0, *)$ mod N. As mentioned above, this paper is the next step in our series of papers devoted to the computation of the cohomology of $\Gamma_0(N)$ together with the action of the Hecke operators on the cohomology. In this paper the coefficient modules are twists of a finite field by a nebentype character η. By contrast, our previous papers considered only constant coefficients, i.e., $\eta = 1$. We work with coefficient modules defined over large finite fields, instead of the complex numbers, to avoid the inaccuracy of floating point arithmetic in linear algebra computations. Given a level N, we say that any prime p (respectively, finite field $\mathbb{F}_{p^r}$) that we use for level N is a *proxy prime* (resp., *proxy field*) for that level. That is, the finite field is a proxy for the complex numbers.

A complete account of the results of our computations appears in the tables in Appendix A.1. Each table lists the level N, and the proxy field $\mathbb{F} = \mathbb{F}_{p^r}$ used for N, together with η, a nebentype character taking values in $\mathbb{F}$ with conductor a divisor of N. By $\mathbb{F}_\eta$ we denote the one-dimensional coefficient system where $\mathbb{F}$ is twisted by η. (For details about η and $\mathbb{F}_\eta$, see (2.5)–(2.7).) Let $G_\mathbb{Q}$ denote the absolute Galois group. With this notation, the following theorem summarizes our computational results:

Theorem 1.3. *For each prime level $N \leqslant 41$ and each composite $N \leqslant 28$, and for each nebentype character η taking values in the proxy field $\mathbb{F}$, the tables in Appendix A.1 give* *the dimension of* $H^5(\Gamma_0(N), \mathbb{F}_\eta)$*. When there is no table for a pair* (N, η)*, this dimension is zero.*

Moreover, for each Hecke eigenclass $z \in H^5(\Gamma_0(N), \mathbb{F}_\eta)$*, there exists a Galois representation* $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathbb{F})$ *such that the characteristic polynomial of* $\rho(\mathrm{Frob}_\ell)$ *equals the Hecke polynomial of* z *at* ℓ *for all primes* ℓ *appearing in the tables. This Galois representation is unique among the candidates considered by the algorithm described in Section 4.3. All these Galois representations are given in the tables, and their corresponding Hecke eigenvalues may be deduced from the equality of the characteristic and Hecke polynomials.*

1.4. We now make some remarks about our results. First of all, the reason we compute H^5 and not some other cohomological degree is that H^5 is the most easily accessible group that supports cuspidal cohomology (see Section 3 for details). Furthermore, this group is susceptible to the computation of Hecke operators using the algorithm in [21]. Indeed, our previous papers on $\mathrm{SL}_4(\mathbb{Z})$ [2–4] also investigate H^5 exclusively.

Let us say that a representation ρ of $G_\mathbb{Q}$ is *attached* to a Hecke eigenclass z if, for almost all primes ℓ, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ is equal to the Hecke polynomial at ℓ (see Definition 2.8 below). If one verifies this equality computationally for a finite number of ℓ, then we say that ρ *appears to be attached* to z. Theorem 1.3 states that for each Hecke eigenclass we compute, we find an apparently attached Galois representation that is uniquely determined by our data.

A recent theorem of Scholze [30] (see Theorem 2.9 below) guarantees that for each Hecke eigenclass there is a semisimple Galois representation attached to it, which by the Chebotarev Density Theorem is unique up to equivalence. It is very likely that the ρ given in our tables is indeed this representation. It is possible, but highly unlikely, that the truly attached Galois representation is a different one that shares the same characteristic polynomials of Frob_ℓ for the ℓ's we computed. If one could compute enough Hecke operators, then using Scholze's theorem and the method of Faltings–Serre one could prove that the apparently attached Galois representations we find are truly attached. This has been done computationally in some settings; see for instance [15] for GL_2 over an imaginary quadratic field and [20] for $\mathrm{GL}_3/\mathbb{Q}$. In bad cases, such computations can require one to evaluate Hecke eigenvalues for ℓ in the thousands (cf. [15, 6.3]); this is far beyond current computational abilities for GL_4 (cf. 1.5). However, the existence of attached Galois representations helps corroborate the correctness of our computations. Indeed, it is unlikely that *any* connection to a Galois representation could be made if our computations of the Hecke eigenvalues were randomly erroneous.

Finally, in all the cases we computed, the apparently attached Galois representation ρ is reducible. Thus, as explained in Section 5, our data also raise questions about the geometry of the boundary B_Γ of the Borel–Serre compactification $\overline{X}/\Gamma$ of the locally symmetric space for Γ. Namely, which classes in $H^*(\overline{X}/\Gamma, V)$ restrict to nonzero classes in $H^*(B_\Gamma, V)$, and why?

1.5. We now give some indications of how we carried out the computations, and make some remarks about our computational limits.

To search for Galois representations attached to Hecke eigenclasses, we use the Galois Finder, a computer program described in Section 4. It looks for Galois representations of degree four that appear to be attached to the Hecke eigenclasses we compute. This Galois Finder is a modification of the one we used in [7].

The Galois representations in this paper are all reducible, with constituents drawn from 1-dimensional representations corresponding to Dirichlet characters, 2-dimensional representations coming from classical holomorphic modular forms of weights 2, 3, and 4, symmetric squares of some of these 2-dimensional Galois representations, and 3-dimensional representations attached to cuspidal automorphic representations for congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$ which are not symmetric square lifts from GL_2. The cusp forms we encounter are listed in Appendices A.3 and A.4. We remark that in our previous papers (except for [7]), classical modular forms of weight 3 did not occur, because forms of odd weight can only occur for odd characters η if p is odd.

The scope of the levels N and Hecke primes ℓ we consider is limited by computing time and space. The level N is limited because the numbers of rows and columns of the boundary matrices in our complex grow like $O(N^3)$. The ℓ's in the Hecke operators are limited for two reasons. First, the number of single cosets in $T(\ell, k)$ grows like ℓ^3 for $k = 1, 3$ and like ℓ^4 for $k = 2$. Already for a small prime like $\ell = 37$, one needs to compute over two million Hecke images for each cell in the support of a cycle, which then must be fed into the reduction algorithm [21]. Second, to determine the Hecke action, much more data must be computed than for cohomology alone. To compute the dimensions of cohomology spaces in our setting, one needs to compute Smith normal forms (SNF) of large sparse matrices. There are excellent algorithms and implementations for this; for example, one can use Linbox [14], and can use the improvements of parallelization [13]. However, to compute the Hecke operators, we must compute the change of basis matrices that put a given matrix into SNF. These additional matrices are always dense in practice, even if the original matrix was sparse, and it is not known how to make this computation parallel. If we had been interested only in the dimensions of the cohomology spaces and not in the Hecke operators, we could have gone to much higher levels N.

While we worked to improve the speed of our code, we did not aim for the state of the art in speed in this project. The programs for this paper were designed for correctness and, to some extent, efficient memory usage, as in [4]. We used Sage [12], which conveniently offers many features that we needed (especially modular forms and reduction mod $\mathfrak{P}$ in number fields—see Section 3.4). We did not use parallel programming techniques, for the reasons above. We present no timing results. While our code could reach higher N and ℓ than we report on, we will use the experience provided by these computations to work next with coefficient modules of dimension greater than 1.

1.6. We now give a more detailed overview of the contents of the paper. In Section 2 we define the central objects studied in this paper: the congruence subgroup $\Gamma_0(N) \subset$

$\mathrm{SL}_4(\mathbb{Z})$ and the coefficient modules $\mathbb{F}_\eta$. We also recall the definitions of the Steinberg module and the sharbly complex, which we use to compute $H^5(\Gamma_0(N), \mathbb{F}_\eta)$. We give the exact definition of the concept of an attached Galois representation, and we quote Scholze's theorem on the existence of attached Galois representations in a form tailored to our purposes.

In Section 3 we describe how we calculate the sharbly homology as a Hecke module, with reference to our earlier papers for details, and with the modifications needed to deal with $\mathbb{F}_\eta$-coefficients.

In Section 4 we describe our Galois Finder and how it was modified from [7]. Section 5 offers interpretation of our results, including heuristics referring to the Borel–Serre boundary of the locally symmetric space for congruence subgroups of $\mathrm{SL}_4(\mathbb{Z})$.

The Appendix contains our computational results, beginning with an explanation of the notation.

1.7. Acknowledgments

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2. The sharbly complex, Hecke operators, and Galois representations

2.1. Let $n \geqslant 2$. Let $\mathbb{Q}^n$ denote the space of n-dimensional column vectors.

Definition 2.2. The *sharbly complex* $Sh_\bullet$ is the complex of left $\mathbb{Z}\mathrm{GL}_n(\mathbb{Q})$-modules defined as follows. As an abelian group, Sh_k is generated by symbols $[v_1, \ldots, v_{n+k}]$, where the v_i are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations:

(i) $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^\sigma [v_1, \ldots, v_{n+k}]$ for all permutations σ;

(ii) $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of $\mathbb{Q}^n$; and

(iii) $[v_1, \ldots, v_{n+k}] - [av_1, v_2, \ldots, v_{n+k}]$ for all $a \in \mathbb{Q}^\times$.

The element $g \in \mathrm{GL}_n(\mathbb{Q})$ acts on $Sh_\bullet$ by $g[v_1, \ldots, v_{n+k}] = [gv_1, \ldots, gv_{n+k}]$. The boundary map $\partial_k \colon Sh_k \to Sh_{k-1}$ is

$$\partial_k([v_1, \ldots, v_{n+k}]) = \sum_{i=1}^{n+k} (-1)^i [v_1, \ldots, \widehat{v_i}, \ldots v_{n+k}],$$

where as usual $\widehat{v_i}$ means to omit v_i.

All these objects depend on n, which we suppress from the notation, since we will later work only with $n = 4$.

The sharbly complex

$$\cdots \to Sh_i \to Sh_{i-1} \to \cdots \to Sh_1 \to Sh_0$$

is an exact sequence of $\mathrm{GL}_n(\mathbb{Q})$-modules. We may define the Steinberg module St as the cokernel of $\partial_1 \colon Sh_1 \to Sh_0$ (cf. [6, Theorem 5]).

Let Γ be a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$.

Definition 2.3. Let M be a left Γ-module. The *sharbly homology* of Γ with coefficients in M is $H_*(\Gamma, Sh_\bullet \otimes_\mathbb{Z} M)$, where Γ acts diagonally on the tensor product.

If (Γ, S) is a Hecke pair in $\mathrm{GL}_n(\mathbb{Q})$ and M is a left S-module, the Hecke algebra $\mathcal{H}(\Gamma, S)$ acts on the sharbly homology, since S acts (diagonally) on $Sh_\bullet \otimes_\mathbb{Z} M$ and because the sharbly homology is the homology of the complex $H_0(\Gamma, Sh_\bullet \otimes_Z M)$.

The following theorem is proved in [5].

Theorem 2.4. *For any $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$ and any coefficient module M in which all the torsion primes of Γ are invertible, there is a natural isomorphism of Hecke modules*

$$H_i(\Gamma, Sh_\bullet \otimes_\mathbb{Z} M) \to H^{\binom{n}{2}-i}(\Gamma, M)$$

for all i.

2.5. We now define the groups Γ and the Γ-modules used in this paper.

As explained in the Introduction, we use a finite field $\mathbb{F} = \mathbb{F}_{p^r}$ as a proxy for $\mathbb{C}$. If $p > 5$ and there is no p-torsion in the $\mathbb{Z}$-cohomology, then the $\mathbb{C}$-Betti numbers will equal the dimensions over $\mathbb{F}$ of the mod-p cohomology groups. We generally use p with four or five decimal digits. Both p and the degree r are chosen to meet certain criteria. We choose p for a given N so that the exponent of $(\mathbb{Z}/N\mathbb{Z})^\times$ divides $p-1$. This makes the group of characters $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p^\times$ isomorphic to the group of characters $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. We choose r to ensure that the various Hecke eigenvalues are $\mathbb{F}$-rational (see Section 4).

Definition 2.6. Let $\Gamma_0(N)$ be the subgroup of matrices in $\mathrm{SL}_n(\mathbb{Z})$ whose bottom row is congruent to $(0, \ldots, 0, *)$ modulo N.

Define S_{pN} to be the subsemigroup of integral matrices in $\mathrm{GL}_n(\mathbb{Q})$ satisfying the same congruence condition as $\Gamma_0(N)$ and having positive determinant relatively prime to pN. Let $\mathcal{H}(pN)$, the *anemic Hecke algebra*, be the $\mathbb{Z}$-algebra of double cosets $\Gamma_0(N)S_{pN}\Gamma_0(N)$. Then $\mathcal{H}(pN)$ is a commutative algebra that acts on the cohomology and homology of $\Gamma_0(N)$ with coefficients in any S_{pN}-module. In particular, $\mathcal{H}(pN)$ contains all double cosets of the form $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$, where ℓ is a prime not dividing pN, $0 \leqslant k \leqslant n$, and $D(\ell, k)$ is the diagonal matrix with the first $n-k$ diagonal entries equal to 1 and the last k diagonal entries equal to ℓ. These double cosets generate $\mathcal{H}(pN)$

(cf. [31, Thm. 3.20]). When we consider the double coset generated by $D(\ell,k)$ as a Hecke operator, we call it $T(\ell,k)$.

Let $\eta\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}^\times$ be a character, which we will call the *nebentype* (even if it is trivial).

Definition 2.7. $\mathbb{F}_\eta$ is the S_{pN}-module where a matrix $s \in S_{pN}$ acts on $\mathbb{F}$ via $\eta(s_{nn})$, where s_{nn} is the $*$ in the bottom row congruent to $(0,\dots,0,*)$ mod N.

Definition 2.8. Let V be an $\mathbb{F}[\mathcal{H}(pN)]$-module. Suppose that $v \in V$ is a simultaneous eigenvector for all $T(\ell,k)$ and that $T(\ell,k)v = a(\ell,k)v$ with $a(\ell,k) \in \mathbb{F}$ for all prime $\ell \nmid pN$ and $0 \leqslant k \leqslant n$. If

$$\rho\colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{F})$$

is a continuous representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside pN, and if

$$\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell,k) X^k = \det(I - \rho(\mathrm{Frob}_\ell)X) \tag{1}$$

for all $\ell \nmid pN$, then we say that ρ is *attached* to v.

Here, Frob_ℓ refers to an arithmetic Frobenius element, so that if ε is the cyclotomic character, we have $\varepsilon(\mathrm{Frob}_\ell) = \ell$.

The polynomial in (1) is called the *Hecke polynomial* for v and ℓ.

As mentioned in the introduction, we have the following specialization of a theorem of Scholze:

Theorem 2.9. *Let $N \geqslant 1$. Let v be a Hecke eigenclass in $H^5(\Gamma_0(N), \mathbb{F}_\eta)$. Then there is attached to v a continuous Galois representation*

$$\rho\colon G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{F}).$$

Echoing Definition 2.8, we say that ρ is *apparently attached* to v if condition (1) holds for a finite range of ℓ which we have computed, a range large enough that we are confident ρ really is attached to v.

3. Computing homology and the Hecke action

3.1. Recall Definition 2.6 of $\Gamma_0(N)$. It is known that the *virtual cohomological dimension* (vcd) [10, VIII.11] of $\mathrm{SL}_4(\mathbb{Z})$ is equal to 6. This implies that $H^d(\Gamma_0(N), M)$ vanishes for $d > 6$ for any $\Gamma_0(N)$-module M. Moreover, one knows [28] that the *cuspidal* *cohomology*[1] can occur only in degrees $d = 4, 5$, and that the cuspidal parts of H^4 and H^5 are dual to each other and afford the same systems of Hecke eigenvalues.

Thus we focus on degree five. We use an algorithm due to one of us to compute the Hecke action [21] on the cohomology of an arithmetic subgroup of GL_n in the cohomological degree one below the vcd; in addition to our prior work on $\mathrm{GL}_4/\mathbb{Q}$, this algorithm has been used to compute with GL_n/F for various n and number fields F [18,22,19].

Using Theorem 2.4, we compute the Hecke operators acting on sharbly cycles that are supported on Voronoi sharblies. Theorem 13 of [6] guarantees that the packets of Hecke eigenvalues we compute do actually occur on eigenclasses in $H_1(\Gamma_0(N), Sh_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_\eta) \cong H^5(\Gamma, \mathbb{F}_\eta)$. In the remainder of this section, we define the Voronoi sharblies, recall results we need from [5,6], and explain how those results are modified to work with $\mathbb{F}_\eta$-coefficients.

3.2. The sharbly complex is not finitely generated as a $\mathbb{Z}\mathrm{SL}_n(\mathbb{Z})$-module, which makes it difficult to use in practice to compute homology. To get a finite complex to compute H_1, we use the Voronoi complex. We refer to [6, Section 5] for any unexplained notation in what follows.

Let $X_n^0 \subset \mathbb{R}^{n(n+1)/2}$ be the convex cone of positive-definite real quadratic forms in n variables. This has a partial (Satake) compactification $(X_n^0)^*$ obtained by adjoining rational boundary components, and the compactification is itself a convex cone. The space $(X_n^0)^*$ can be partitioned into cones $\sigma = \sigma(x_1, \dots, x_m)$, called *Voronoi cones*, where the x_i are contained in certain subsets of nonzero vectors from $\mathbb{Z}^n$. (We write elements of $\mathbb{Z}^n$ as column vectors, as we did in Section 2 for $\mathbb{Q}^n$.) The cones are built as follows. Each nonzero $x_i \in \mathbb{Z}^n$ determines a rank-one quadratic form $q(x_i) = x_i x_i^t \in (X_n^0)^*$. Let Π be the closed convex hull of the points $\{q(x) \mid x \in \mathbb{Z}^n, x \neq 0\}$. Then each of the proper faces of Π is a polytope, and the σ are the cones on these polytopes. The indexing sets are constructed in the obvious way: if σ is the cone on $F \subset \Pi$, and F has distinct vertices $q(x_1), \dots, q(x_m)$, then the indexing set is $\{\pm x_1, \dots, \pm x_m\}$. We let Σ denote the set of all Voronoi cones.

3.3. Let X_n^* be the quotient of $(X_n^0)^*$ by homotheties. The images of the Voronoi cones are cells in X_n^*. Let $\mathbb{Z}V_\bullet$ be the oriented chain complex on these cells, graded by dimension. Let $\mathbb{Z}\partial V_\bullet$ be the subcomplex generated by those cells that do not meet the interior of X_n^*. The *Voronoi complex* is then defined to be $\mathcal{V}_\bullet = \mathbb{Z}V_\bullet / \mathbb{Z}\partial V_\bullet$. For our purposes, it is convenient to reindex $\mathcal{V}_\bullet$ by introducing the complex $\mathcal{W}_\bullet$, where $\mathcal{W}_k = \mathcal{V}_{n+k-1}$. The results of [5,6] show that, if $n \leqslant 4$, both $\mathcal{W}_\bullet$ and $Sh_\bullet$ give resolutions of the Steinberg module. In particular, let $\Gamma = \Gamma_0(N)$. If M is a $\mathbb{Z}\Gamma$-module such that the order of all torsion elements in Γ is invertible, then $H_*(\Gamma, \mathcal{W}_\bullet \otimes_{\mathbb{Z}} M) \cong H_*(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}}$

[1] In other words, the cohomology corresponding to cuspidal automorphic forms; see [26] for a discussion of the connection between cohomology of arithmetic groups and automorphic forms.

M), and furthermore by Borel–Serre duality these are isomorphic (after reindexing) to $H^*(\Gamma, M)$.

These two complexes can be related as follows when $n = 4$. Every Voronoi cell in X_4^* of dimension $\leqslant 5$ is a simplex. Thus for $0 \leqslant k \leqslant 2$, we can define a map of $\mathbb{Z}\mathrm{SL}_4(\mathbb{Z})$-modules

$$\theta_k \colon \mathcal{W}_k \longrightarrow Sh_k$$

that takes the Voronoi cell $\sigma(v_1, \ldots, v_{k+4})$ to $\theta_k((v_1, \ldots, v_{k+4})) := [v_1, \ldots, v_{k+4}]$. This allows us to realize Voronoi cycles in these degrees in the sharbly complex. The image of θ_k is the set of *Voronoi sharblies* in degree k. Then $H_1(\Gamma, \mathcal{W}_\bullet \otimes_{\mathbb{Z}} M) \cong H_1(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}} M)$ by [6, Corollary 12].

3.4. We now explain concretely how we compute $H_1(\Gamma, \mathcal{W}_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_\eta)$. We have a body of code in Sage [12] for these computations. The code supports G-modules M, that is, representations of G. Here G is a finite group, or a matrix group like $\Gamma_0(N)$ or S_{pN}. The module M has finite dimension over its base ring. The base ring is $\mathbb{F}$, $\mathbb{Q}$, or $\mathbb{Z}$ in this project, though it could be more general. Morphisms of G-modules are supported, as are kernel, cokernel, image, direct sum, and tensor products of G-modules. When H is a subgroup of G of finite index, we support Res_H^G, Ind_H^G, and Coind_H^G of G-modules, functorially.

The program takes as input the values of N, p, and the nebentype η. (The extension from $\mathbb{F}_p$ to $\mathbb{F}_{p^r}$ comes later, in the Galois Finder.) The nebentype is a mod p Dirichlet character $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p^\times$. Sage makes it automatic to enumerate the Dirichlet characters.

The complex $\mathcal{W}_\bullet$ has only finitely many classes of Voronoi cells modulo $\mathrm{SL}_n(\mathbb{Z})$ [34]. To compute H_1 we only need $\mathcal{W}_0$, $\mathcal{W}_1$, and $\mathcal{W}_2$, so our code truncates away the rest of $\mathcal{W}_\bullet$ for efficiency.

For each class of cells modulo $\mathrm{SL}_n(\mathbb{Z})$, the code maintains a standard representative cell σ as listed in [27]. The stabilizer G_σ of σ in $\mathrm{SL}_n(\mathbb{Z})$ acts on σ with orientation character Z_σ. The code stores G_σ and Z_σ.

Fix right coset representatives r, r', ... for $\Gamma_0(N)\backslash\mathrm{SL}_n(\mathbb{Z})$ once and for all. Since $\Gamma_0(N)$ has finite index in $\mathrm{SL}_n(\mathbb{Z})$, the complex $\mathcal{W}_\bullet$ has only finitely many classes of cells modulo $\Gamma_0(N)$. For each class modulo $\Gamma_0(N)$, we may choose a representative cell $\sigma_1 = r\sigma$, where σ is one of the representative cells modulo $\mathrm{SL}_n(\mathbb{Z})$, and r is one of the standard coset representatives. An awkward fact is that, for two different coset representatives r, r', the cells $r\sigma$ and $r'\sigma$ may be in the same $\Gamma_0(N)$-orbit. This occurs when $r^{-1}r'$ is in the stabilizer $G_\sigma \subset \mathrm{SL}_n(\mathbb{Z})$. For computation, we must choose r or r', not both; say we choose r. A class `CellOrbitStructure` in our code handles these details. σ_1 itself may have a non-trivial stabilizer $G_{\sigma_1} \subset \Gamma_0(N)$; the `CellOrbitStructure` takes care of these stabilizers G_{σ_1} and how their orientation characters Z_{σ_1} interact with the orientation characters Z_σ of G_σ.

Equation (2) below presents a problem we need to solve repeatedly during the homology calculation. Suppose we are given a cell $\tau \in \mathcal{W}_\bullet$, with $\tau = g\sigma$ for σ a standard cell and for $g \in \mathrm{SL}_n(\mathbb{Z})$. Then $g = \gamma_1 r'$ for some coset representative r' and $\gamma_1 \in \Gamma_0(N)$. Since we chose r instead of r', we have $\gamma r' = r g_\sigma$ for some stabilizer element $g_\sigma \in G_\sigma$ and some $\gamma \in \Gamma_0(N)$. Thus

$$g = (\gamma_1 \gamma^{-1}) r g_\sigma. \tag{2}$$

The problem is, given g and σ, to solve for γ_1, γ, r, g_σ, and to compute the orientation characters. The `CellOrbitStructure` has a method `decompose` that solves (2).

Let σ_1 run through all the representatives $r\sigma$ of the classes of cells modulo $\Gamma_0(N)$. During the homology computation, we need, for each σ_1, to restrict the nebentype η to the finite stabilizer group G_{σ_1}, and to tensor the restriction with the orientation character of G_{σ_1}. This tensor product $\eta_{\sigma_1} \colon G_{\sigma_1} \to \mathbb{F}_p^\times$ is called the *local representation* for σ_1. The `CellOrbitStructure` keeps track of the local representations.

3.5. As we explain in [5,6], $H_\bullet(\Gamma, \mathcal{W}_k \otimes_{\mathbb{Z}} \mathbb{F}_\eta)$ is computed by a spectral sequence. The columns are indexed by k, and the j-th row is the direct sum of the homology groups $H_j(G_{\sigma_1}, \eta_{\sigma_1})$. Since the torsion in $\Gamma_0(N)$ has order prime to p for our large proxy primes p, all the homology groups H_j vanish for $j > 0$. The E^1 term has only one row, whose entry in the k-th box is the module of co-invariants

$$E^1_{k,0} = H_0(\Gamma, \mathcal{W}_k \otimes_{\mathbb{Z}} \mathbb{F}_\eta) = \mathcal{W}_k \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta.$$

As σ_1 runs through representatives of the cells modulo $\Gamma_0(N)$, the co-invariant module $\mathcal{W}_k \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta$ breaks up as a direct sum:

$$E^1_{k,0} = \bigoplus_{\sigma_1 \text{ of degree } k} H_0(G_{\sigma_1}, \eta_{\sigma_1}). \tag{3}$$

Each summand $H_0(G_{\sigma_1}, \eta_{\sigma_1})$ is the module of co-invariants for the local representation η_{σ_1}. It is isomorphic to $\mathbb{F}$ if η_{σ_1} is a trivial representation, and is zero otherwise.

The $E^2_{k,0}$ of the spectral sequence is isomorphic to $H_k(\mathcal{W}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta)$. This is computed using the differential $\bar{\partial}_k$ that is the tensor product with η of the differential ∂_k on sharblies in Section 2.2. $\bar{\partial}_k$ is constructed in Sage as a sparse matrix of size $\dim E^1_{k,0} \times \dim E^1_{k-1,0}$. As before, we are computing H_1, so we only compute $\bar{\partial}_2$ and $\bar{\partial}_1$.

We illustrate the sizes of these matrices with the example of $N = 41$, $p = 21881$, and trivial nebentype. Here $\bar{\partial}_2$ is 24590×7100, and $\bar{\partial}_1$ is 7100×746. (This is small compared to [4], where, for $N = 211$ and trivial nebentype, $\bar{\partial}_2$ was about four million by one million. We did not compute the Hecke operators in [4].)

We write the matrices $\bar{\partial}_2$ and $\bar{\partial}_1$ to disk, partly as insurance in case of a computer crash during a long run. The next step is to choose a basis $\{x_i\}$ of the homology, $\ker(\bar{\partial}_1)/\operatorname{im}(\bar{\partial}_2)$. We choose the basis using Sheafhom, a package written by one of us

(MM) in Common Lisp and described in [4]. Sheafhom performs homology calculations by row- and column-reducing large sparse matrices while saving the change-of-basis matrices to disk. It works with base rings $\mathbb{F}_p$ as well as $\mathbb{Z}$. If y is a cycle in the homology, Sheafhom can express it as a linear combination of the homology basis, $y = \sum c_i x_i$, using only a small amount of RAM.

3.6. To compute the Hecke operators, we use the basis $\{x_i\}$ we found for the homology group $H_1(\mathcal{W}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta)$. We identify the x_i with elements $y_i = \theta_{1,*}(x_i) \in H_1(Sh_\bullet \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta)$. Let T be a Hecke operator. Using the algorithm mentioned above, we compute each Hecke translate Ty_i and then find a sharbly cycle z_i such that $z_i = Ty_i$ in $H_1(Sh_\bullet \otimes_{\mathbb{Z}\Gamma} \mathbb{F}_\eta)$ and such that z_i is in the image of the map $\theta_{1,*}$. The inverse images $\theta_{1,*}^{-1}(z_i)$ can be written as linear combinations $\sum c_i x_i$ as in the previous paragraph. This gives a matrix representing the action of T. From this matrix we can find eigenclasses and eigenvalues.

4. Finding attached Galois representations

4.1. By now we have set $n = 4$. We describe how we find Galois representations that are apparently attached to Hecke eigenclasses in the homology. Our Galois Finder program is part of our Sage code.

As in Section 3.6, we compute the action on $V = H_1(\Gamma_0(N), \mathcal{W}_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_\eta)$ of the Hecke operators $T(\ell, k)$ for $k = 1, 2, 3$ and for ℓ ranging through a set

$$L = \{\ell \mid \ell \text{ prime}, \ \ell \leqslant \ell_0, \ \ell \nmid pN\}.$$

Here the upper bound ℓ_0 depends on the level N and the nebentype η, and in this paper $5 \leqslant \ell_0 \leqslant 17$. Two additional Hecke operators contribute to the Hecke polynomial, namely $T(\ell, 0)$ and $T(\ell, 4)$. Both act as scalars, the former with eigenvalue 1 and the latter with eigenvalue $\eta(\ell)$. We remark that to check our work, we always verify that our Hecke operators pairwise commute. We also note that the Hecke operators we compute are semisimple.

One new idea in this paper is that, for the larger ℓ's, we sometimes compute $T(\ell, 1)$ but not $T(\ell, 2)$ or $T(\ell, 3)$. This lets us avoid the $O(\ell^4)$ part of the computation, while still allowing us to eliminate some candidate Galois representations.

As mentioned in the introduction, in the range of our computations all the Galois representations that occur are reducible. (For larger N, irreducible Galois representations would occur.) In this paper it was sufficient for our Galois Finder to work with possible constituents of dimensions 1, 2, and 3:

(1) 1-dimensional constituents come from Dirichlet characters mod N taking values in the cyclotomic field K_0 generated by ζ_N a primitive N-th root of unity.

(2) 2-dimensional constituents come from newforms of level dividing N and weights 2, 3, or 4.
(3) 3-dimensional constituents come either from symmetric squares of 2-dimensional representations or from GL_3-homology classes which are not symmetric squares.

Consider the fields of definition $K_1, K_2, \ldots$ of a list of newforms, together with K_0. The Galois Finder will be computing, not in $\mathbb{F}_p$, but in the residue class fields for the primes $\mathfrak{P}$ over p in the different K_i. We define r to be the smallest integer so that all these residue class fields embed in $\mathbb{F} = \mathbb{F}_{p^r}$. We choose p to make r as small as possible, given the constraint $p < 2^{31}$ (which is helpful for speed). The field $\mathbb{F}$ is recorded at the top of each table.

Let E denote a simultaneous eigenspace of our computed Hecke operators on $V = H_1(\Gamma_0(N), \mathcal{W}_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_\eta) \cong H^5(\Gamma_0(N), \mathbb{F}_\eta)$, where $\mathbb{F} = \mathbb{F}_{p^r}$. We define two notions of multiplicity for E, namely Hecke and Galois multiplicity.

Definition 4.2. The *Hecke multiplicity* of E equals $\dim_{\mathbb{F}} E$.

We will define the Galois multiplicity of E below in Definition 4.7.

To a simultaneous eigenspace E we attach a family of polynomials. The *polynomial system* $\mathcal{F}(E)$ is the mapping that sends $\ell \in L$ to the Hecke polynomial with eigenvalues $a(\ell, k)$ defined in (1), or to a partial Hecke polynomial which we now explain. For small ℓ, we can compute the Hecke eigenvalues $a(\ell, k)$ for all $k = 0, \ldots, 4$, so we know the whole Hecke polynomial (1); call this a *full* Hecke polynomial. For larger ℓ, we compute only $T(\ell, 1)$, and we only know that the Hecke polynomial is $1 - a(\ell, 1)X + O(X^2)$, where $O(X^2)$ means some undetermined linear combination of X^2, X^3, and X^4. We call the latter a *partial* Hecke polynomial. A partial Hecke polynomial is implemented in Sage as an element of the quotient ring $\mathbb{F}[X]/(X^2)$. As a whole, $\mathcal{F}(E)$ contains one or more full polynomials, all of degree 4, and zero or more partial polynomials, whose degree is undefined. We say $\deg \mathcal{F}(E) = 4$.

4.3. The Galois finder uses known Galois representations ρ unramified outside pN, taking values in $\mathrm{GL}_m(\mathbb{F})$ for $m = 1$ or 2. These come from Dirichlet characters and newforms as described in Section 4.5 below. We also use the symmetric squares of the ρ coming from newforms; these take values in $\mathrm{GL}_m(\mathbb{F})$ for $m = 3$. The characteristic polynomial of Frobenius for each of these representations is known and is of degree m for each $\ell \nmid pN$. Define the *polynomial system* $\mathcal{F}(\rho)$ to be the mapping that sends $\ell \in L$ to the characteristic polynomial of Frobenius for ρ at ℓ, and define $\deg \mathcal{F}(\rho) = m$.

Define $\mathcal{F}(\rho_1 \oplus \cdots \oplus \rho_t) = \prod_{i=1}^{t} \mathcal{F}(\rho_i)$, a product of polynomial systems. We also define quotients, but we must be careful about the partial Hecke polynomials. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two polynomial systems with the same L. Say that $\mathcal{F}_1$ *divides* $\mathcal{F}_2$ if, for each $\ell \in L$, the polynomial at ℓ for $\mathcal{F}_1$ divides the polynomial at ℓ for $\mathcal{F}_2$. Implicit in this definition is that $\deg \mathcal{F}_1 \leqslant \deg \mathcal{F}_2$. When one polynomial system divides another, define the *quotient system* in the obvious way. The degree of the quotient system is $\deg \mathcal{F}_2 - \deg \mathcal{F}_1$. For some ℓ we will be dividing a partial Hecke polynomial by a full Hecke polynomial, but we never use a partial polynomial as a divisor. Dividing a partial Hecke polynomial $f_1(x) \pmod{X^2}$ by a full Hecke polynomial $f_2(x)$ is well defined because $f_2(x)$ always has constant term 1, hence $f_2(x)$ projects via $\mathbb{F}[X] \to \mathbb{F}[X]/(X^2)$ to a unit of $\mathbb{F}[X]/(X^2)$. We stop dividing by $\mathcal{F}_1$ as soon as the full polynomials of the quotient reach degree 0.

For a given E, we make a list $\mathcal{R}$ of all the ρ for which $\mathcal{F}(\rho)$ divides $\mathcal{F}(E)$. Then we run through all possible finite subsets of $\mathcal{R}$, say $\{\rho_1, \ldots, \rho_t\}$, and we make a list $\mathcal{R}'$ of all the direct sums $\rho_1 \oplus \cdots \oplus \rho_t$ for which $\mathcal{F}(\rho_1 \oplus \cdots \oplus \rho_t) = \mathcal{F}(E)$. As stated in Theorem 1.3, after computing enough Hecke operators, we obtain a unique representation (up to isomorphism) in $\mathcal{R}'$.

4.4. There are a few exceptions to the statement that $\mathcal{R}'$ has exactly one element. In two cases in the tables, $N = 24$, $\eta = \chi_{24,0}\chi_{24,1}\chi_{24,2}$, $\sigma_{24,2c}$, and $N = 28$, $\eta = \chi_{28,0}\chi_{28,1}^3$, $\sigma_{28,2c}$, we find that a representation with a symmetric square in it coincides with a representation without a symmetric square. In these cases, the three-dimensional representation is the symmetric square of a dihedral Galois representation, and therefore is reducible. So although $\mathcal{R}'$ has two elements in these cases, they are isomorphic.

The other exceptions occurred at level $N = 41$ and the nebentype $\eta = \chi_{41}^{10}$ whose image has order 4. Here $\dim V = 8$, splitting into eight E's of dimension 1, and $\mathcal{R}'$ was empty for two out of the eight E. Darrin Doud, upon our request, using computer programs he developed, found an autochthonous[2] form for GL_3 at level 41. Specifically, he found a three-dimensional Galois representation δ attached to a cohomology class z for a congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$ and with coefficients in $\mathbb{F}_\eta$, such that in the two cases where $\mathcal{R}'$ is empty, the attached Galois representations proved to be $1 \oplus \varepsilon\delta$ and $\varepsilon^3 \oplus \delta$, respectively. The eigenvalues of the class δ are given in Section A.4.

4.5. We now describe in detail the list of Galois representations ρ which our Galois Finder was programmed to use.

First are the Dirichlet characters χ with values in $\mathbb{F}$, which we identify with one-dimensional Galois representations as usual. We take all the characters of conductor N_1 for all $N_1 \mid N$. Sage's class `DirichletGroup` enumerates the χ automatically. The characteristic polynomial of Frobenius at ℓ for χ is $1 + \chi(\ell)X$, for all $\ell \nmid pN$.

Another one-dimensional character is the cyclotomic character ε. We look at ε^w for $w = 0, 1, 2, 3$, because these are the powers predicted by the generalizations of Serre's conjecture for mod p Galois representations [8,1]. The list $\mathcal{L}_1$ of one-dimensional characters is now $\chi \otimes \varepsilon^w$, for all the χ just described and for $w = 0, 1, 2, 3$.

[2] *Autochthonous* means this form appears on GL_3 and is not a functorial lift from a smaller rank group.

Next are the two-dimensional Galois representations ρ coming from newforms for certain congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, reduced modulo a prime above p.

Let $N_1 \mid N$ and ζ_{N_1} be a primitive N_1-th root of unity. Let ψ be a Dirichlet character of conductor N_1 taking values in $\mathbb{C}^\times$. Let f be a newform of weight 2, 3, or 4 for $\Gamma_1(N_1)$ with nebentype character ψ. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{N_1})/\mathbb{Q})$ acts on the ψ's by acting on their values; we use only one ψ from each Galois orbit, since the others give Galois-conjugate newforms.

The coefficients of the q-expansion of f generate a number field K_f, with ring of integers $\mathcal{O}_{K_f}$. Let $\mathfrak{P}$ be a prime of K_f over p. We choose $\mathbb{F}$ so that there is an embedding $\alpha_{\mathfrak{P}}\colon \mathcal{O}_{K_f}/\mathfrak{P} \to \mathbb{F}$. Then $(f, \mathfrak{P})$ gives rise to a Galois representation ρ into $\mathrm{GL}_2(\mathbb{F})$, by reduction mod $\mathfrak{P}$ composed with $\alpha_{\mathfrak{P}}$. For any $\ell \nmid pN$, the characteristic polynomial of Frobenius is $1 - \alpha_{\mathfrak{P}}(a_\ell)X + X^2$, where a_ℓ is the ℓ-th coefficient in the q-expansion of f. If we chose a different prime $\mathfrak{P}$, we would get a Galois-conjugate representation. Computing $\alpha_{\mathfrak{P}}$ is a large problem in its own right; Sage provides most of the solution.

We make a list $\mathcal{L}_2^0$ containing the representation ρ for $(f, \mathfrak{P})$, for all $N_1 \mid N$ and all newforms f of weight 2, 3, or 4 for $\Gamma_1(N_1)$ and all nebentypes ψ. Sage's class `CuspForms`, with its method `newforms`, makes this automatic.

We take all the ρ in $\mathcal{L}_2^0$, and tensor them in all possible ways with the one-dimensional representations from the list $\mathcal{L}_1$ of Dirichlet characters and cyclotomic character powers. This list of tensor products is our final list $\mathcal{L}_2$ of two-dimensional Galois representations.

Our list of three-dimensional Galois representations is the list of symmetric squares of $\rho \in \mathcal{L}_2^0$, tensored in all possible ways with $\mathcal{L}_1$.

We define the *Hodge–Tate (HT) numbers* for ρ as follows. For an element $\chi \otimes \varepsilon^w \in \mathcal{L}_1$, there is a list of one number, $[w]$. If ρ is a representation coming from a newform of weight k, the list of HT numbers is $[0, k-1]$. For $\chi \otimes \varepsilon^w \otimes \rho$, the list is $[w, w+k-1]$. The three-dimensional representation δ has HT numbers $[0, 1, 2]$, and, when δ is tensored by ε^w, these numbers each have w added to them. For direct sums of representations, the lists are concatenated. For each four-dimensional Galois representation we find to fit our data, we always observe that the list is $[0, 1, 2, 3]$ after sorting. This is compatible with the Serre-type conjectures and gives us a check on our computations.

Another check on our computations comes from considering the relationship between the nebentype character and the determinant of the attached representation. For example, consider a Galois representation $\rho = \varepsilon^a \oplus \chi\varepsilon^b \oplus \sigma$ apparently attached to a Hecke eigenclass, where σ is attached to a cusp form of weight k with nebentype character ψ. Then the determinant of ρ is $\varepsilon^{a+b+k-1}\chi\psi$ and by the definition of attachment this must equal $\varepsilon^6\eta$.

4.6. The Galois groups $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ acts on the lists $\mathcal{L}_1$ and $\mathcal{L}_2^0$.

Definition 4.7. If a cohomology group contains Hecke eigenspaces $E_{(1)}, \ldots, E_{(g)}$ which are attached to Galois representations $\rho_{(1)}, \ldots, \rho_{(g)}$, where $\rho_{(1)}, \ldots, \rho_{(g)}$ form an orbit under the Galois action, the *Galois multiplicity* of each of $E_{(1)}, \ldots, E_{(g)}$ equals g.

In the tables in the Appendix, we only list one of the $E_{(i)}$, and we indicate the Galois multiplicity in the first column.

5. Observed regularities in the data

5.1. This section details the regularities we observed in the tables in the Appendix. When we have a reasonable heuristic explanation of a regularity, we give it. Converting any of these heuristics to theorems would require a finer analysis of the Borel–Serre boundary than is presently available and a greater expertise with Eisenstein series than we possess.

In this section, we let $\Gamma_0(a, b)$ denote the subgroup of $\mathrm{GL}_a(\mathbb{Z})$ where the bottom row is congruent to $(0, \dots, 0, *)$ modulo b. Thus $\Gamma_0(N) = \Gamma_0(4, N) \cap \mathrm{SL}_4(\mathbb{Z})$ in our notation. Recall that the nebentype character is denoted η and the cyclotomic character is denoted ε.

We shall refer to a Hecke eigenclass in $H^5(\Gamma_0(N), \mathbb{F}_\eta)$ by the letter z and to its attached Galois representation by ρ.

5.2. The determinant of ρ

We always observe that the determinant of ρ equals $\varepsilon^6\eta$. That this should be true is a tautology from the definition of attachment.

5.3. The parity of ρ

The parity of ρ is always odd. In other words, the eigenvalues of $\rho(c)$ are $+1, -1, +1, -1$, where c denotes complex conjugation. That this must be the case follows from a theorem of Caraiani and LeHung [11].

5.4. Powers of ε

Another observed pattern has to do with the powers of the cyclotomic character that appear in ρ. We defined the HT (Hodge–Tate) numbers above in Section 4.5. A folklore conjecture in the theory of arithmetic cohomology implies that for p sufficiently large a Galois representation attached to a Hecke eigenclass in $H^5(\Gamma, \mathbb{F}_\eta)$ should have HT numbers $0, 1, 2, 3$. This is observed in all of our data and as noted in Section 4 is compatible with the Serre-type conjectures.

5.5. Hecke multiplicity 3

We defined *Hecke multiplicity* in Section 4.1. In every case of our data, the Hecke multiplicity of the eigenspace for a system of Hecke eigenvalues equals either 1 or 3. (As mentioned in Section 4, the Hecke operators we computed are always observed to be semisimple.)

Hecke multiplicity greater than 1 occurs in our data only when N is composite and the components of ρ have conductors strictly dividing N. We don't have a good explanation for this, except it seems to be related to the existence of old forms when N is composite.

5.6. There are a number of patterns involving the weights, nebentypes and levels of the newforms and Eisenstein series whose attached Galois representations appear as constituents of our observed Galois representations.

Since all observed Galois representations in this paper are reducible, it is reasonable to assume that the corresponding cohomology classes "come from" the Borel–Serre boundary B_Γ of the locally symmetric space for Γ, in the following sense.

Let $\Gamma = \Gamma_0(N)$. Let B_Γ be the Borel–Serre boundary of the locally symmetric space $X_\Gamma = \Gamma\backslash \mathrm{SL}_4(\mathbb{R})/\mathrm{SO}(4)$. Then the Borel–Serre boundary B_Γ is the union of faces $F(P)$, where P runs over a set of representatives of Γ-orbits of parabolic subgroups P of $\mathrm{GL}_4(\mathbb{Q})$.

When comparing Hecke eigenclasses on B_Γ and on X_Γ, it is simpler to discuss homology rather than cohomology. The systems of Hecke eigenvalues and hence the attached Galois representations are the same for homology and cohomology.

The injection $B_\Gamma \to X_\Gamma \cup B_\Gamma$ induces a map on homology, for any coefficient system M:

$$H_5(B_\Gamma, M) \longrightarrow H_5(X_\Gamma \cup B_\Gamma, M) = H_5(\Gamma, M).$$

The *boundary homology* is the image of this map. In this paper, every Hecke eigenclass we computed has Hecke eigenvalues compatible with what would be expected of a Hecke eigenclass in $H_5(B_\Gamma, \mathbb{F}_\eta)$.

For each parabolic subgroup P, let $P = LU$, where L is a Levi component of P and U is the unipotent radical of P. Let $\pi\colon P \to P/U$ be the projection. The image of π is isomorphic to L and is a product of GL_{n_i}'s, where $\sum n_i = 4$. If the block sizes are $(n_1, \dots, n_{k+2})$, we call this tuple the "type" of P. The nonnegative integer k equals the codimension of $F(P)$ in B_Γ.

Set $P_\Gamma = P \cap \Gamma$, $U_\Gamma = U \cap \Gamma$, and $L_\Gamma = \pi(P_\Gamma)$. Let X_L denote the symmetric space of $L(\mathbb{R})$. The face $F(P)$ is a fibration with base X_L/L_Γ and fiber $U(\mathbb{R})/U_\Gamma$. The Serre spectral sequence of this fibration degenerates at E^2 (at least if p is sufficiently large). Therefore, if we put a homology class on each block of X_L, whose degrees $i_1, \dots, i_{k+2}$ add to i, with coefficients in $H_j(U(\mathbb{R})/U_\Gamma, M)$, we obtain a class in $H_{i+j}(F(P), M)$.

This class may or may not give rise to a nonzero class $H_{i+j+k}(B_\Gamma, M)$, depending on how it behaves in the Leray spectral sequence for the covering of B_Γ by its faces. (The complete computation of this Leray spectral sequence has not been performed for $GL_n/\mathbb{Q}$ except when $n \leq 3$; it is a difficult problem.) Finally, if there is a nonzero

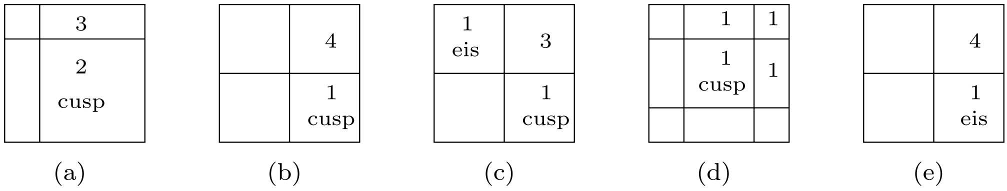

Fig. 1. Schematics of homology classes on faces of the Borel–Serre boundary.

class in $H_{i+j+k}(B_\Gamma, M)$ obtained this way, it may or may not map to a nonzero class in $H_5(\Gamma, M)$, another difficult problem that has not been studied in general.

All of this behaves Hecke-equivariantly. It gives a heuristic for predicting what kind of constituents will occur in the reducible Galois representation attached to a cohomology class that comes from B_Γ, with components corresponding to the homology classes on the blocks of L. In our observations, we always have $i + j + k = 5$ because we computed $H^5(\Gamma, \mathbb{F}_\eta)$.

For each type of boundary homology class we can make a schematic picture of the parameters, as in Fig. 1. Each diagram represents a standard parabolic subgroup conjugate to a P that gives rise to some kind of boundary homology.

A general remark on Dirichlet and nebentype characters: different Γ-orbits of the same type of parabolic subgroups may result in different levels of the components of L_Γ. Therefore if N is composite, various characters can occur in the constituent Galois representations, but they will all have conductor dividing N.

A block of size 1 may support a homology class with attached Galois representation equal to a Dirichlet character times a power of the cyclotomic character. These Dirichlet characters will all have conductor dividing N.

For a 2×2 block L' of L, we use the Eichler–Shimura theorem to interpret the homology of a congruence subgroup $\Gamma_0(2, N)$ of L' with coefficients in $\mathrm{Sym}^g(\mathbb{F}^2) \otimes \chi$ in terms of classical modular forms of weight $g+2$ and nebentype χ. Therefore in this case, the corresponding component of ρ will be attached to such a modular form. Thus a block of size 2 may support a holomorphic cuspform with level dividing N, or an Eisenstein series corresponding to the sum of two characters each with conductors dividing N.

A block L' of size 3 may support a homology class of a congruence subgroup of L' with an irreducible 3-dimensional Galois representation attached.

We now use this heuristic method to motivate the various kinds of Galois representations that occur in our data in the tables in the Appendix. When using the method, remember that the HT numbers must always be $0, 1, 2, 3$.

5.7. GL_3 *classes*

In this case (Fig. 1(a)), P is a $(1, 3)$-parabolic subgroup; $i_1 = 0, i_2 = 2, j = 3$. Then $H_3(U_\Gamma, \mathbb{F}_\eta)$ is a one-dimensional L'-module. We place a cuspidal homology class w from

$H_2(\Gamma_0(3,N),\mathbb{F}_\eta)$ on the second block. This class w can be the symmetric square of a classical cusp form, or a class that is not a symmetric square. The latter occurs in our data only at level 41.

When w is a symmetric square of the cusp form s, the level of s equals N, the nebentype of s equals the nebentype η, and η is the quadratic character. This nebentype is the only one allowing the symmetric square of a cusp form of prime level N to have the same level N as the cusp form.

Writing the symmetric square of the Galois representation attached to w as τ, it always appears twice in our data, as $\rho = \varepsilon^0 \oplus \varepsilon\tau$ and $\rho = \varepsilon^3 \oplus \tau$. This is because there will be two relevant Γ-orbits of P, corresponding to block sizes $(1,3)$ and $(3,1)$ down the diagonal.

5.8. Holomorphic cusp forms of weight 2

In this case (Fig. 1(b)), P is a $(2,2)$-parabolic subgroup; $i_1 = 0, i_2 = 1, j = 4$. Then $H_4(U_\Gamma,\mathbb{F}_\eta)$ is a one-dimensional L'-module. We place a cusp form v of weight 2 on one of the two blocks. The other block supports an H_0, so there is no choice for it—we just put 1 on it. We observe that v always has level N.

In our data, σ always appears twice: once in $\varepsilon^0\oplus\varepsilon^1\oplus\varepsilon^2\chi\sigma$ and once in $\varepsilon^2\oplus\varepsilon^3\oplus\varepsilon^0\chi\sigma$, for some character χ. This is because there will be two relevant Γ-orbits of P, both corresponding to block sizes $(2,2)$; but in the second orbit, v gets placed on the first block instead of the second block. The character χ is the same in both expressions. We can and do always choose the ideal $\mathfrak{P}$ so that $\chi = 1$.

Sometimes these Galois representations appear with multiplicity 1, and sometimes with higher multiplicity. We do not have a heuristic explanation for this variability.

5.9. Holomorphic cusp forms of weight 3

Cusp forms of odd weight can appear only if $p = 2$, as in [7], or if odd nebentypes are available, as in the current paper.

In this case (Fig. 1(c)), P is a $(2,2)$-parabolic subgroup; $i_1 = 1, i_2 = 1, j = 3$. Note that $H_3(U_\Gamma,\mathbb{F}_\eta)$ restricted to either of the 2×2 blocks is a sum of two copies of the standard 2-dimensional GL_2-representation. We place a cusp form v of weight 3 on one of the two blocks and an Eisenstein series u on the other block.

Let σ be the Galois representation attached to v. We observe that v always has level strictly dividing N and always appears in our data four times as follows, each with the same character ψ:

- $\rho = \psi\varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon\sigma$
- $\rho = \varepsilon^0 \oplus \psi\varepsilon^2 \oplus \varepsilon\sigma$
- $\rho = \psi\varepsilon^1 \oplus \varepsilon^3 \oplus \sigma$

- $\rho = \varepsilon^1 \oplus \psi\varepsilon^3 \oplus \sigma$

We have three examples of this phenomenon, at levels $N = 24$, 27 and 28. It doesn't always occur even when N is composite and there is an appropriate v available. For example, there is a weight 3 cusp form of level 7 that contributes to $N = 28$ but it does not contribute when $N = 14$. We have no conjecture to offer as to when a weight 3 cusp form appears for a given (N, η), but it may be related to the special value of some L-function.

5.10. Holomorphic cusp forms of weight 4

In this case (Fig. 1(d)), P is a $(1, 2, 1)$-parabolic subgroup; $i_1 = 0, i_2 = 1, i_3 = 0, j = 3$. Note that $H_3(U_\Gamma, \mathbb{F}_\eta)$ contains an L'-submodule isomorphic to Sym^2 of the standard representation. We place a cusp form v of weight 4 on the second block.

Let σ be the Galois representation attached to v. We observe that $\rho = \varepsilon^1 \oplus \varepsilon^2 \oplus \sigma$ occurs only once in our data, if at all. We observe that in our data, it occurs if and only if the special value $L(v, 1/2)$ of the L-function is 0. For the levels we have computed, this occurs only when $\eta = 1$. The level of v always divides N but need not equal N. This type of homology class on the boundary consists of ghost classes.

5.11. Sums of 4 characters

See Fig. 1(e). Here, as in (5.8), P is a $(2, 2)$-parabolic subgroup; $i_1 = 0, i_2 = 1, j = 4$. We place an Eisenstein series e on one of the two blocks. The two characters ψ and χ associated with e have conductors dividing N and the level e divides N.

If η factors nontrivially as $\eta = \psi\chi$ then either all three of the following or none of the following occur:

- $\rho = \psi\varepsilon^0 \oplus \chi\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
- $\rho = \psi\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi\varepsilon^3$
- $\rho = \varepsilon^0 \oplus \varepsilon^1 \oplus \psi\varepsilon^2 \oplus \chi\varepsilon^3$

For example, when $N = 9$ all three forms occur, and when $N = 13$ none of the three occur. The powers of ε which are not multiplied by a nontrivial Dirichlet character are always consecutive. We do not have an explanation as to why just these patterns occur, nor for which N they should occur, nor for why the unadorned powers of ε are always consecutive. Answers to these questions, and to the similar questions raised above, would require a very fine analysis of the boundary homology as Hecke module, which is not available to us.

Note that ψ and χ can trade places to get another triple, giving 6 such ρ's in total, for example, when $n = 15$.

Factoring of η seems to be important here. For example, when $\eta = 1$ we never get $\rho = \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$. (It is known that for GL_n with n larger than 4, the sum of consecutive powers of ε may be attached to a homology class, for example in the case of a Borel stable class [9, (4.2)]. But we do not know if this can happen for $n = 4$.)

Appendix A

A.1. The tables in this appendix present the results of our computations.

The topmost box in each table gives the level N, the nebentype η, and the field $\mathbb{F}_{p^r}$ that was our proxy for $\mathbb{C}$. We write $GF(p^r)$ instead of $\mathbb{F}_{p^r}$ for readability. We only include one representative for each Galois orbit of nebentype characters.

Next we list the Hecke operators we computed. T_ℓ means we computed $T_{\ell,1}$, $T_{\ell,2}$, and $T_{\ell,3}$. Listing $T_{\ell,1}$ means we computed only that part of T_ℓ.

After the T's, in the right margin, we give Dim, the total dimension of $H^5(\Gamma, \mathbb{F}_\eta)$.

The succeeding rows in each table give the Galois multiplicity (Definition 4.7), the Hecke multiplicity (Definition 4.2), and the Galois representation itself that is found by the Galois Finder to be apparently attached to the given Hecke eigenclass in the cohomology $H^5(\Gamma_0(N), \mathbb{F}_\eta)$.

The characters χ_N or $\chi_{N,i}$ are a basis for the mod p Dirichlet characters $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p$. They are listed in a separate table in Appendix A.2. The cyclotomic character is denoted ε.

The $\sigma_{N,k}$ are classical cuspidal homomorphic newforms of level N and weight k. They are listed in a separate table in Appendix A.3. We use the same symbol $\sigma_{N,k}$ to stand for the two-dimensional Galois representation attached to the cusp form of that name. When we have more than one cusp form for the same N and k, we give them names like $\sigma_{17,2a}$ and $\sigma_{17,2b}$. The symmetric square of σ is denoted $\mathrm{Sym}^2(\sigma)$.

The GL_3 representation δ corresponds to the eigenclass described in Section 4.4; the Hecke eigenvalues of δ are shown in Section A.4.

Level $N = 9$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(12379)$. Computed T_2, T_5, T_7. Dim 3.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_9^3\varepsilon^2 \oplus \chi_9^3\varepsilon^3$
1	1	$\chi_9^3\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_9^3\varepsilon^3$
1	1	$\chi_9^3\varepsilon^0 \oplus \chi_9^3\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 11$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(4001^2)$. Computed T_2, T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{11,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{11,2}$

Level $N = 12$**. Nebentype** $\eta = \chi_{12,0}\chi_{12,1}$**. Field** $\mathbb{F} = GF(5413^2)$**.** Computed T_5, T_7. Dim 6.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{12,0}\varepsilon^2 \oplus \chi_{12,1}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{12,1}\varepsilon^2 \oplus \chi_{12,0}\varepsilon^3$
1	1	$\chi_{12,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{12,1}\varepsilon^3$
1	1	$\chi_{12,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{12,0}\varepsilon^3$
1	1	$\chi_{12,0}\varepsilon^0 \oplus \chi_{12,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{12,1}\varepsilon^0 \oplus \chi_{12,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 13$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12037)$**.** Computed T_2, T_3, T_5, T_7. Dim 1.		
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{13,4}$

Level $N = 13$**. Nebentype** $\eta = \chi_{13}^2$**. Field** $\mathbb{F} = GF(12037)$**.** Computed T_2, T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{13,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{13,2}$

Level $N = 14$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12379^2)$**.** Computed T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{14,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{14,2}$

Level $N = 15$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12037^2)$**.** Computed T_2, T_7, $T_{11,1}$. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{15,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{15,2}$

Level $N = 15$**. Nebentype** $\eta = \chi_{15,0}\chi_{15,1}$**. Field** $\mathbb{F} = GF(12037^2)$**.** Computed T_2, T_7. Dim 6.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{15,0}\varepsilon^2 \oplus \chi_{15,1}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{15,1}\varepsilon^2 \oplus \chi_{15,0}\varepsilon^3$
1	1	$\chi_{15,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{15,1}\varepsilon^3$
1	1	$\chi_{15,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{15,0}\varepsilon^3$
1	1	$\chi_{15,0}\varepsilon^0 \oplus \chi_{15,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{15,1}\varepsilon^0 \oplus \chi_{15,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 16$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(4001^6)$**.** Computed T_3, T_5, T_7. Dim 3.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{16,0}\varepsilon^2 \oplus \chi_{16,0}\varepsilon^3$
1	1	$\chi_{16,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{16,0}\varepsilon^3$
1	1	$\chi_{16,0}\varepsilon^0 \oplus \chi_{16,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 16$**. Nebentype** $\eta = \chi_{16,1}$**. Field** $\mathbb{F} = GF(4001^6)$**.** Computed T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{16,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{16,2}$

Level $N = 17$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(16001^2)$**.** Computed T_2, T_3, T_5, T_7. Dim 3.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{17,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{17,2a}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{17,4}$

Level $N = 17$**. Nebentype** $\eta = \chi_{17}^2$**. Field** $\mathbb{F} = GF(16001^2)$**.** Computed T_2, T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{17,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{17,2b}$

Level $N = 18$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(3637^2)$**.** Computed T_5, T_7. Dim 9.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{18}^3\varepsilon^2 \oplus \chi_{18}^3\varepsilon^3$
1	3	$\chi_{18}^3\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{18}^3\varepsilon^3$
1	3	$\chi_{18}^3\varepsilon^0 \oplus \chi_{18}^3\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 18$**. Nebentype** $\eta = \chi_{18}^2$**. Field** $\mathbb{F} = GF(3637^2)$**.** Computed T_5, T_7, $T_{11,1}$. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{18,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{18,2}$

Level $N = 19$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(3637^6)$**.** Computed T_2, T_3, T_5, T_7. Dim 3.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{19,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{19,2a}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{19,4}$

Level $N = 19$**. Nebentype** $\eta = \chi_{19}^2$**. Field** $\mathbb{F} = GF(3637^6)$**.** Computed T_2, T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{19,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{19,2b}$

Level $N = 20$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12037^{12})$**.** Computed T_3, T_7, $T_{11,1}$, $T_{13,1}$. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{20,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{20,2a}$

Level $N = 20$**. Nebentype** $\eta = \chi_{20,0}\chi_{20,1}$**. Field** $\mathbb{F} = GF(12037^{12})$**.** Computed T_3, T_7, $T_{11,1}$, $T_{13,1}$. Dim 8.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{20,0}\varepsilon^2 \oplus \chi_{20,1}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{20,1}\varepsilon^2 \oplus \chi_{20,0}\varepsilon^3$
1	1	$\chi_{20,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{20,1}\varepsilon^3$
1	1	$\chi_{20,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{20,0}\varepsilon^3$
1	1	$\chi_{20,0}\varepsilon^0 \oplus \chi_{20,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{20,1}\varepsilon^0 \oplus \chi_{20,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{20,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{20,2b}$

Level $N = 21$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12037^6)$**.** Computed T_2, T_5. Dim 3.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{21,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{21,2a}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{21,4}$

Level $N = 21$**. Nebentype** $\eta = \chi_{21,1}^2$**. Field** $\mathbb{F} = GF(12037^6)$**.** Computed T_2, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{21,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{21,2b}$

Level $N = 21$**. Nebentype** $\eta = \chi_{21,0}\chi_{21,1}$**. Field** $\mathbb{F} = GF(12037^6)$**.** Computed T_2, T_5, $T_{11,1}$, $T_{13,1}$. Dim 8.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{21,0}\varepsilon^2 \oplus \chi_{21,1}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{21,1}\varepsilon^2 \oplus \chi_{21,0}\varepsilon^3$
1	1	$\chi_{21,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{21,1}\varepsilon^3$
1	1	$\chi_{21,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{21,0}\varepsilon^3$
1	1	$\chi_{21,0}\varepsilon^0 \oplus \chi_{21,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{21,1}\varepsilon^0 \oplus \chi_{21,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{21,2c}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{21,2c}$

Level $N = 21$**. Nebentype** $\eta = \chi_{21,0}\chi_{21,1}^3$**. Field** $\mathbb{F} = GF(12037^6)$**.** Computed T_2, T_5. Dim 10.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{21,0}\varepsilon^2 \oplus \chi_{21,1}^3\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{21,1}^3\varepsilon^2 \oplus \chi_{21,0}\varepsilon^3$
1	1	$\chi_{21,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{21,1}^3\varepsilon^3$
1	1	$\chi_{21,1}^3\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{21,0}\varepsilon^3$
1	1	$\chi_{21,0}\varepsilon^0 \oplus \chi_{21,1}^3\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{21,1}^3\varepsilon^0 \oplus \chi_{21,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^0 \oplus \chi_{21,0}\varepsilon^2 \oplus \varepsilon^1\sigma_{7,3}$
1	1	$\varepsilon^1 \oplus \chi_{21,0}\varepsilon^3 \oplus \varepsilon^0\sigma_{7,3}$
1	1	$\chi_{21,0}\varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon^1\sigma_{7,3}$
1	1	$\chi_{21,0}\varepsilon^1 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{7,3}$

Level $N = 22$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(16001^2)$**.** Computed T_3, T_5, T_7. Dim 7.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{11,2}$
1	3	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{11,2}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{22,4}$

Level $N = 22$**. Nebentype** $\eta = \chi_{22}^2$**. Field** $\mathbb{F} = GF(16001^2)$**.** Computed T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{22,2}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{22,2}$

Level $N = 23$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(22067^{60})$**.** Computed T_2, T_3, T_5, T_7. Dim 5.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{23,2a}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{23,2a}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{23,4}$

Level $N = 23$**. Nebentype** $\eta = \chi_{23}^2$**. Field** $\mathbb{F} = GF(22067^{60})$**.** Computed T_2, T_3, T_5, T_7. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{23,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{23,2b}$

Level $N = 24$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12379^2)$**.** Computed T_5, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{24,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{24,2a}$

Level $N = 24$**. Nebentype** $\eta = \chi_{24,1}$**. Field** $\mathbb{F} = GF(12379^2)$**.** Computed T_5, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{24,2b}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{24,2b}$

Level $N = 24$**. Nebentype** $\eta = \chi_{24,0}\chi_{24,2}$**. Field** $\mathbb{F} = GF(12379^2)$**.** Computed T_5, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. Dim 18.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{24,0}\varepsilon^2 \oplus \chi_{24,2}\varepsilon^3$
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{24,2}\varepsilon^2 \oplus \chi_{24,0}\varepsilon^3$
1	3	$\chi_{24,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,2}\varepsilon^3$
1	3	$\chi_{24,2}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,0}\varepsilon^3$
1	3	$\chi_{24,0}\varepsilon^0 \oplus \chi_{24,2}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	3	$\chi_{24,2}\varepsilon^0 \oplus \chi_{24,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 24$**. Nebentype** $\eta = \chi_{24,0}\chi_{24,1}\chi_{24,2}$**. Field** $\mathbb{F} = GF(12379^2)$**.** Computed T_5, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$, $T_{17,1}$. Dim 14.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{24,0}\chi_{24,1}\varepsilon^2 \oplus \chi_{24,2}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{24,2}\varepsilon^2 \oplus \chi_{24,0}\chi_{24,1}\varepsilon^3$
1	1	$\chi_{24,0}\chi_{24,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,2}\varepsilon^3$
1	1	$\chi_{24,2}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,0}\chi_{24,1}\varepsilon^3$
1	1	$\chi_{24,0}\chi_{24,1}\varepsilon^0 \oplus \chi_{24,2}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{24,2}\varepsilon^0 \oplus \chi_{24,0}\chi_{24,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{24,2c}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{24,2c}$
1	1	$\varepsilon^0 \oplus \chi_{24,2}\varepsilon^2 \oplus \varepsilon^1 \sigma_{8,3}$
1	1	$\varepsilon^1 \oplus \chi_{24,2}\varepsilon^3 \oplus \varepsilon^0 \sigma_{8,3}$
1	1	$\chi_{24,2}\varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon^1 \sigma_{8,3}$
1	1	$\chi_{24,2}\varepsilon^1 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{8,3}$

Level $N = 25$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(16001^{60})$**.** Computed T_2, T_3. Dim 7.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{25}^{15}\varepsilon^2 \oplus \chi_{25}^{5}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{25}^{5}\varepsilon^2 \oplus \chi_{25}^{15}\varepsilon^3$
1	1	$\chi_{25}^{15}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{25}^{5}\varepsilon^3$
1	1	$\chi_{25}^{5}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{25}^{15}\varepsilon^3$
1	1	$\chi_{25}^{15}\varepsilon^0 \oplus \chi_{25}^{5}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{25}^{5}\varepsilon^0 \oplus \chi_{25}^{15}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{25,4}$

Level $N = 25$**. Nebentype** $\eta = \chi_{25}^2$**. Field** $\mathbb{F} = GF(16001^{60})$**.** Computed T_2, T_3. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{25,2a}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{25,2a}$

Level $N = 25$**. Nebentype** $\eta = \chi_{25}^4$**. Field** $\mathbb{F} = GF(16001^{60})$**.** Computed T_2, T_3. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{25,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{25,2b}$

Level $N = 25$**. Nebentype** $\eta = \chi_{25}^{10}$**. Field** $\mathbb{F} = GF(16001^{60})$**.** Computed T_2, T_3. Dim 6.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{25}^{15}\varepsilon^2 \oplus \chi_{25}^{15}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{25}^{5}\varepsilon^2 \oplus \chi_{25}^{5}\varepsilon^3$
1	1	$\chi_{25}^{15}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{25}^{15}\varepsilon^3$
1	1	$\chi_{25}^{5}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{25}^{5}\varepsilon^3$
1	1	$\chi_{25}^{15}\varepsilon^0 \oplus \chi_{25}^{15}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{25}^{5}\varepsilon^0 \oplus \chi_{25}^{5}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$

Level $N = 26$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12037^{2})$**.** Computed T_3, T_5. Dim 7.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{26,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{26,2a}$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{26,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{26,2b}$
1	3	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{13,4}$

Level $N = 26$**. Nebentype** $\eta = \chi_{26}^2$**. Field** $\mathbb{F} = GF(12037^{2})$**.** Computed T_3, T_5. Dim 6.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{13,2}$
1	3	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{13,2}$

Level $N = 26$. **Nebentype** $\eta = \chi_{26}^4$. **Field** $\mathbb{F} = GF(12037^2)$. Computed T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{26,2c}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{26,2c}$

Level $N = 26$. **Nebentype** $\eta = \chi_{26}^6$. **Field** $\mathbb{F} = GF(12037^2)$. Computed T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{26,2d}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{26,2d}$

Level $N = 27$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(11863^6)$. Computed T_2, T_5, $T_{7,1}$. Dim 12.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{27}^9 \varepsilon^2 \oplus \chi_{27}^9 \varepsilon^3$
1	3	$\chi_{27}^9 \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{27}^9 \varepsilon^3$
1	3	$\chi_{27}^9 \varepsilon^0 \oplus \chi_{27}^9 \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{27,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{27,2a}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{27,4}$

Level $N = 27$. **Nebentype** $\eta = \chi_{27}^2$. **Field** $\mathbb{F} = GF(11863^6)$. Computed T_2, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{27,2b}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{27,2b}$

Level $N = 27$. **Nebentype** $\eta = \chi_{27}^6$. **Field** $\mathbb{F} = GF(11863^6)$. Computed T_2, T_5, $T_{7,1}$. Dim 10.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{27}^{15} \varepsilon^2 \oplus \chi_{27}^9 \varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{27}^9 \varepsilon^2 \oplus \chi_{27}^{15} \varepsilon^3$
1	1	$\chi_{27}^{15} \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{27}^9 \varepsilon^3$
1	1	$\chi_{27}^9 \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{27}^{15} \varepsilon^3$
1	1	$\chi_{27}^{15} \varepsilon^0 \oplus \chi_{27}^9 \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{27}^9 \varepsilon^0 \oplus \chi_{27}^{15} \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\varepsilon^0 \oplus \chi_{27}^9 \varepsilon^2 \oplus \varepsilon^1 \sigma_{9,3}$
1	1	$\varepsilon^1 \oplus \chi_{27}^9 \varepsilon^3 \oplus \varepsilon^0 \sigma_{9,3}$
1	1	$\chi_{27}^9 \varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon^1 \sigma_{9,3}$
1	1	$\chi_{27}^9 \varepsilon^1 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{9,3}$

Level $N = 28$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(12379^{12})$**.** Computed T_3, T_5, $T_{11,1}$, $T_{13,1}$. Dim 7.		
1	3	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{14,2}$
1	3	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{14,2}$
1	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{28,4}$

Level $N = 28$**. Nebentype** $\eta = \chi_{28,1}^2$**. Field** $\mathbb{F} = GF(12379^{12})$**.** Computed T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{28,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{28,2a}$

Level $N = 28$**. Nebentype** $\eta = \chi_{28,0}\chi_{28,1}$**. Field** $\mathbb{F} = GF(12379^{12})$**.** Computed T_3, T_5. Dim 10.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0}\varepsilon^2 \oplus \chi_{28,1}\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,1}\varepsilon^2 \oplus \chi_{28,0}\varepsilon^3$
1	1	$\chi_{28,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{28,1}\varepsilon^3$
1	1	$\chi_{28,1}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{28,0}\varepsilon^3$
1	1	$\chi_{28,0}\varepsilon^0 \oplus \chi_{28,1}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{28,1}\varepsilon^0 \oplus \chi_{28,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{28,2b}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{28,2b}$

Level $N = 28$**. Nebentype** $\eta = \chi_{28,0}\chi_{28,1}^3$**. Field** $\mathbb{F} = GF(12379^{12})$**.** Computed T_3, T_5, $T_{11,1}$, $T_{13,1}$. Dim 14.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0}\varepsilon^2 \oplus \chi_{28,1}^3\varepsilon^3$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,1}^3\varepsilon^2 \oplus \chi_{28,0}\varepsilon^3$
1	1	$\chi_{28,0}\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{28,1}^3\varepsilon^3$
1	1	$\chi_{28,1}^3\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{28,0}\varepsilon^3$
1	1	$\chi_{28,0}\varepsilon^0 \oplus \chi_{28,1}^3\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
1	1	$\chi_{28,1}^3\varepsilon^0 \oplus \chi_{28,0}\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{28,2c}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{28,2c}$
1	1	$\varepsilon^0 \oplus \chi_{28,0}\varepsilon^2 \oplus \varepsilon^1\sigma_{7,3}$
1	1	$\varepsilon^1 \oplus \chi_{28,0}\varepsilon^3 \oplus \varepsilon^0\sigma_{7,3}$
1	1	$\chi_{28,0}\varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon^1\sigma_{7,3}$
1	1	$\chi_{28,0}\varepsilon^1 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{7,3}$

Level $N = 29$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(2297^6)$. Computed T_2, T_3, T_5. Dim 6.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2a}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{29,2a}$
2	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{29,4}$

Level $N = 29$. **Nebentype** $\eta = \chi_{29}^2$. **Field** $\mathbb{F} = GF(2297^6)$. Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2b}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{29,2b}$

Level $N = 29$. **Nebentype** $\eta = \chi_{29}^4$. **Field** $\mathbb{F} = GF(2297^6)$. Computed T_2, T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2c}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{29,2c}$

Level $N = 29$. **Nebentype** $\eta = \chi_{29}^{14}$. **Field** $\mathbb{F} = GF(2297^6)$. Computed T_2, T_3, T_5. Dim 6.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2d}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{29,2d}$
2	1	$\varepsilon^0 \oplus \varepsilon^1 \mathrm{Sym}^2(\sigma_{29,2d})$

Level $N = 31$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(4201^{60})$. Computed T_2, T_3, T_5. Dim 6.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2a}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2a}$
2	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{31,4}$

Level $N = 31$. **Nebentype** $\eta = \chi_{31}^2$. **Field** $\mathbb{F} = GF(4201^{60})$. Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2b}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2b}$

Level $N = 31$. **Nebentype** $\eta = \chi_{31}^6$. **Field** $\mathbb{F} = GF(4201^{60})$. Computed T_2, T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2c}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2c}$

Level $N = 31$. **Nebentype** $\eta = \chi_{31}^{10}$. **Field** $\mathbb{F} = GF(4201^{60})$. Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{31,2d}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{31,2d}$

Level $N = 37$. **Nebentype** $\eta = 1$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5, $T_{7,1}$, $T_{13,1}$. Dim 8.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2a}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2a}$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2b}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2b}$
4	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0\sigma_{37,4}$

Level $N = 37$. **Nebentype** $\eta = \chi_{37}^{2}$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5. Dim 6.		
3	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2c}$
3	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2c}$

Level $N = 37$. **Nebentype** $\eta = \chi_{37}^{4}$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5. Dim 4.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2d}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2d}$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2e}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2e}$

Level $N = 37$. **Nebentype** $\eta = \chi_{37}^{6}$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2f}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2f}$

Level $N = 37$. **Nebentype** $\eta = \chi_{37}^{12}$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5. Dim 2.		
1	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2g}$
1	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2g}$

Level $N = 37$. **Nebentype** $\eta = \chi_{37}^{18,2}$. **Field** $\mathbb{F} = GF(3889^{24})$. Computed T_2, T_3, T_5. Dim 6.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2\sigma_{37,2h}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0\sigma_{37,2h}$
2	1	$\varepsilon^0 \oplus \varepsilon^1\mathrm{Sym}^2(\sigma_{37,2h})$

Level $N = 41$**. Nebentype** $\eta = 1$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 9.		
3	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2a}$
3	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2a}$
3	1	$\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{41,4}$

Level $N = 41$**. Nebentype** $\eta = \chi_{41}^2$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 6.		
3	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2b}$
3	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2b}$

Level $N = 41$**. Nebentype** $\eta = \chi_{41}^4$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2c}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2c}$

Level $N = 41$**. Nebentype** $\eta = \chi_{41}^8$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 4.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2d}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2d}$

Level $N = 41$**. Nebentype** $\eta = \chi_{41}^{10}$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 8.		
3	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2e}$
3	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2e}$
1	1	$\varepsilon^0 \oplus \varepsilon^1 \delta$
1	1	$\varepsilon^3 \oplus \varepsilon^0 \delta$

Level $N = 41$**. Nebentype** $\eta = \chi_{41}^{20}$**. Field** $\mathbb{F} = GF(21881^{60})$**.** Computed T_2, T_3, T_5. Dim 6.		
2	1	$\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{41,2f}$
2	1	$\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{41,2f}$
2	1	$\varepsilon^0 \oplus \varepsilon^1 \mathrm{Sym}^2(\sigma_{41,2f})$

A.2. For each N, the next table specifies the basis that Sage chooses for the group of characters $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p$. If there is one basis element, it is denoted χ_N. If there is more than one, they are denoted $\chi_{N,0}$, $\chi_{N,1}$, etc. The *order* of χ is the smallest positive n so that χ^n is trivial on $(\mathbb{Z}/N\mathbb{Z})^\times$. The *parity* is even if $\chi(-1) = +1$ and odd if $\chi(-1) = -1$.

$\chi_{N,i}$	p	order	parity	definition
χ_7	12037	6	odd	$3 \mapsto -1293$
χ_9	12379	6	odd	$2 \mapsto 5770$
$\chi_{12,0}$	5413	2	odd	$7 \mapsto -1$, $5 \mapsto 1$
$\chi_{12,1}$	5413	2	odd	$7 \mapsto 1$, $5 \mapsto -1$
χ_{13}	12037	12	odd	$2 \mapsto 4019$
$\chi_{15,0}$	12037	2	odd	$11 \mapsto -1$, $7 \mapsto 1$
$\chi_{15,1}$	12037	4	odd	$11 \mapsto 1$, $7 \mapsto 3417$
$\chi_{16,0}$	4001	2	odd	$15 \mapsto -1$, $5 \mapsto 1$
$\chi_{16,1}$	4001	4	even	$15 \mapsto 1$, $5 \mapsto -899$
χ_{17}	16001	16	odd	$3 \mapsto 83$
χ_{18}	3637	6	odd	$11 \mapsto -695$
χ_{19}	3637	18	odd	$2 \mapsto -31$
$\chi_{20,0}$	12037	2	odd	$11 \mapsto -1$, $17 \mapsto 1$
$\chi_{20,1}$	12037	4	odd	$11 \mapsto 1$, $17 \mapsto 3417$
$\chi_{21,0}$	12037	2	odd	$8 \mapsto -1$, $10 \mapsto 1$
$\chi_{21,1}$	12037	6	odd	$8 \mapsto 1$, $10 \mapsto -1293$
χ_{22}	16001	10	odd	$13 \mapsto 3018$
χ_{23}	22067	22	odd	$5 \mapsto 7863$
$\chi_{24,0}$	12379	2	odd	$7 \mapsto -1$, $13 \mapsto 1$, $17 \mapsto 1$
$\chi_{24,1}$	12379	2	even	$7 \mapsto 1$, $13 \mapsto -1$, $17 \mapsto 1$
$\chi_{24,2}$	12379	2	odd	$7 \mapsto 1$, $13 \mapsto 1$, $17 \mapsto -1$
χ_{25}	16001	20	odd	$2 \mapsto 7734$
χ_{26}	12037	12	odd	$15 \mapsto 4019$
χ_{27}	11863	18	odd	$2 \mapsto 5034$
$\chi_{28,0}$	12379	2	odd	$15 \mapsto -1$, $17 \mapsto 1$
$\chi_{28,1}$	12379	6	odd	$15 \mapsto 1$, $17 \mapsto 5770$
χ_{29}	2297	28	odd	$2 \mapsto 1108$
χ_{31}	4201	30	odd	$3 \mapsto -1970$
χ_{37}	3889	36	odd	$2 \mapsto -1338$
χ_{41}	21881	40	odd	$6 \mapsto -10354$

A.3. In the following table we give the q-expansions of the holomorphic cusp forms whose attached Galois representations are referred to in the tables of Appendix A.1. $S_k(N, \chi)$ denotes the space of weight k cusp forms on $\Gamma_0(N)$ with character χ. The notation $\sigma_{N,k}$ for individual cusp forms makes manifest the level N and weight k. The q-expansions were computed using Sage [12].

The field of definition of a cusp form, if not specified, is the field generated by the coefficients we display. For instance, $q + 2iq^2 + 55q^3 + \cdots$ has coefficients in $\mathbb{Q}(i)$. By ζ_m we mean a primitive m-th root of unity. When we must specify the field, it is in the line beginning with "over".

$\sigma_{7,3} = q - 3q^2 + 5q^4 - 7q^7 + O(q^8)$ in $S_3(7, \chi_7^3)$
$\sigma_{8,3} = q - 2q^2 - 2q^3 + 4q^4 + 4q^6 - 8q^8 - 5q^9 + 14q^{11} - 8q^{12} + O(q^{16})$ in $S_3(8, \chi_{24,0}\chi_{24,1})$
$\sigma_{9,3} = q + (-\zeta_6 - 1)\, q^2 + (3\zeta_6 - 3)\, q^3 - \zeta_6 q^4 + O(q^5)$ in $S_3(9, \chi_{27}^{15})$
$\sigma_{11,2} = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + O(q^8)$ in $S_2(11, 1)$
$\sigma_{13,2} = q + (-\zeta_6 - 1)\, q^2 + (2\zeta_6 - 2)\, q^3 + \zeta_6 q^4 + O(q^5)$ in $S_2(13, \chi_{13}^2)$
$\sigma_{13,4} = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 - 13q^7 + O(q^8)$ in $S_4(13, 1)$
$\sigma_{14,2} = q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 + O(q^8)$ in $S_2(14, 1)$
$\sigma_{15,2} = q - q^2 - q^3 - q^4 + q^5 + q^6 + O(q^8)$ in $S_2(15, 1)$
$\sigma_{16,2} = q + (-i - 1)\, q^2 + (i - 1)\, q^3 + 2iq^4 + O(q^5)$ in $S_2(16, \chi_{16,1})$
$\sigma_{17,2a} = q - q^2 - q^4 - 2q^5 + 4q^7 + O(q^8)$ in $S_2(17, 1)$
$\sigma_{17,2b} = q + \left(-\zeta_8^3 + \zeta_8^2 - 1\right) q^2 + \left(\zeta_8^3 - \zeta_8^2 - \zeta_8 - 1\right) q^3 + O(q^4)$ in $S_2(17, \chi_{17}^2)$
$\sigma_{17,4} = q - 3q^2 - 8q^3 + q^4 + 6q^5 + 24q^6 - 28q^7 + O(q^8)$ in $S_4(17, 1)$
$\sigma_{18,2} = q - \zeta_6 q^2 + (\zeta_6 - 2)\, q^3 + (\zeta_6 - 1)\, q^4 + O(q^6)$ in $S_2(18, \chi_{18}^2)$
$\sigma_{19,2a} = q - 2q^3 - 2q^4 + 3q^5 - q^7 + O(q^8)$ in $S_2(19, 1)$
$\sigma_{19,2b} = q + \left(-\zeta_{18,2}^2 + \zeta_{18,2} - 1\right) q^2 + O(q^3)$ in $S_2(19, \chi_{19}^2)$
$\sigma_{19,4} = q - 3q^2 - 5q^3 + q^4 - 12q^5 + 15q^6 + 11q^7 + O(q^8)$ in $S_4(19, 1)$
$\sigma_{20,2a} = q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + O(q^{14})$ in $S_2(20, 1)$
$\sigma_{20,2b} = q + (-i - 1)\, q^2 + 2iq^4 + (i - 2)\, q^5 + O(q^8)$ in $S_2(20, \chi_{20,0}\chi_{20,1})$
$\sigma_{21,2a} = q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + O(q^8)$ in $S_2(21, 1)$
$\sigma_{21,2b} = q + (2\zeta_6 - 2)\, q^2 - \zeta_6 q^3 - 2\zeta_6 q^4 + (-2\zeta_6 + 2)\, q^5 + O(q^6)$ in $S_2(21, \chi_{21,1}^2)$
$\sigma_{21,2c} = q + (-\zeta_6 - 1)\, q^3 + (2\zeta_6 - 2)\, q^4 + O(q^7)$ in $S_2(21, \chi_{21,0}\chi_{21,1})$
$\sigma_{21,4} = q - 3q^2 - 3q^3 + q^4 - 18q^5 + 9q^6 + 7q^7 + O(q^8)$ in $S_4(21, 1)$
$\sigma_{22,2} = q - \zeta_{10} q^2 + \left(-\zeta_{10}^3 + \zeta_{10} - 1\right) q^3 + \zeta_{10}^2 q^4 + O(q^5)$ in $S_2(22, \chi_{22}^2)$
$\sigma_{22,4} = q - 2q^2 - 7q^3 + 4q^4 - 19q^5 + 14q^6 + 14q^7 + O(q^8)$ in $S_4(22, 1)$
$\sigma_{23,2a} = q + b_0 q^2 + (-2b_0 - 1)\, q^3 + (-b_0 - 1)\, q^4 + 2b_0 q^5 + O(q^6)$ in $S_2(23, 1)$ over $\mathbb{Q}[b_0]/(b_0^2 + b_0 - 1)$
$\sigma_{23,2b} = q + \left(\zeta_{22}^9 - \zeta_{22}^6 - \zeta_{22}^4 - 1\right) q^2 + O(q^3)$ in $S_2(23, \chi_{23}^2)$
$\sigma_{23,4} = q - 2q^2 - 5q^3 - 4q^4 - 6q^5 + 10q^6 - 8q^7 + O(q^8)$ in $S_4(23, 1)$
$\sigma_{24,2a} = q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + O(q^{14})$ in $S_2(24, 1)$
$\sigma_{24,2b} = q + b_0 q^2 + (b_0 + 1)\, q^3 + (-2b_0 - 2)\, q^4 + O(q^5)$ in $S_2(24, \chi_{24,1})$ over $\mathbb{Q}[b_0]/(b_0^2 + 2b_0 + 2)$
$\sigma_{24,2c} = q + b_0 q^2 + (-b_0 - 1)\, q^3 - 2q^4 + (-b_0 + 2)\, q^6 + O(q^8)$ in $S_2(24, \chi_{24,0}\chi_{24,1}\chi_{24,2})$ over $\mathbb{Q}[b_0]/(b_0^2 + 2)$
$\sigma_{25,2a} = q + b_0 q^2 + \left(\left(\zeta_{10}^3 + \zeta_{10} - 1\right) b_0 + \zeta_{10}^2 - 1\right) q^3 + O(q^4)$ in $S_2(25, \chi_{25}^2)$ over $(\mathbb{Q}(\zeta_{10}))[b_0]/(b_0^2 + (\zeta_{10} + 1)\, b_0 + \zeta_{10}^2 - 2\zeta_{10} + 1)$
$\sigma_{25,2b} = q + \left(-\zeta_5^3 - \zeta_5 - 1\right) q^2 + \zeta_5 q^3 + \left(-\zeta_5^2 - \zeta_5 - 1\right) q^4 + O(q^5)$ in $S_2(25, \chi_{25}^4)$
$\sigma_{25,4} = q - q^2 - 7q^3 - 7q^4 + 7q^6 - 6q^7 + O(q^8)$ in $S_4(25, 1)$
$\sigma_{26,2a} = q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 + O(q^8)$ in $S_2(26, 1)$
$\sigma_{26,2b} = q + q^2 - 3q^3 + q^4 - q^5 - 3q^6 + q^7 + O(q^8)$ in $S_2(26, 1)$

$\sigma_{26,2c} = q + (-\zeta_3 - 1)\, q^2 + \zeta_3 q^4 - q^5 + 4\zeta_3 q^7 + O(q^8)$ in $S_2(26, \chi_{26}^4)$
$\sigma_{26,2d} = q + b_0 q^2 - q^3 - q^4 - 3b_0 q^5 - b_0 q^6 + 3b_0 q^7 + O(q^8)$ in $S_2(26, \chi_{26}^6)$ over $\mathbb{Q}[b_0]/(b_0^2 + 1)$
$\sigma_{27,2a} = q - 2q^4 - q^7 + O(q^8)$ in $S_2(27, 1)$
$\sigma_{27,2b} = q + b_0 q^2 + \left(\left(\zeta_{18,2}^5 - \zeta_{18,2}\right) b_0 - \zeta_{18,2}^3 + \zeta_{18,2}^2 - \zeta_{18,2}\right) q^3 + O(q^4)$ in $S_2(27, \chi_{27}^2)$ over $(\mathbb{Q}(\zeta_{18,2}))[b_0]/(b_0^2 + \left(\zeta_{18,2}^2 - \zeta_{18,2} + 1\right) b_0 + \zeta_{18,2}^4 - \zeta_{18,2}^3 - \zeta_{18,2}^2 - \zeta_{18,2} + 1)$
$\sigma_{27,4} = q - 3q^2 + q^4 - 15q^5 - 25q^7 + O(q^8)$ in $S_4(27, 1)$
$\sigma_{28,2a} = q - \zeta_6 q^3 + (3\zeta_6 - 3)\, q^5 + (-2\zeta_6 - 1)\, q^7 + O(q^8)$ in $S_2(28, \chi_{28,1}^2)$
$\sigma_{28,2b} = q + b_0 q^2 + ((\zeta_6 - 2)\, b_0 - \zeta_6 - 1)\, q^3 + O(q^4)$ in $S_2(28, \chi_{28,0}\chi_{28,1})$ over $(\mathbb{Q}(\zeta_6))[b_0]/(b_0^2 + 2\zeta_6 b_0 + 2\zeta_6 - 2)$
$\sigma_{28,2c} = q + b_0 q^2 + (-b_0 - 2)\, q^4 + (-2b_0 - 1)\, q^7 + O(q^8)$ in $S_2(28, \chi_{28,0}\chi_{28,1}^3)$ over $\mathbb{Q}[b_0]/(b_0^2 + b_0 + 2)$
$\sigma_{28,4} = q - 10q^3 - 8q^5 - 7q^7 + 73q^9 - 40q^{11} - 12q^{13} + O(q^{14})$ in $S_4(28, 1)$
$\sigma_{29,2a} = q + b_0 q^2 - b_0 q^3 + (-2b_0 - 1)\, q^4 - q^5 + (2b_0 - 1)\, q^6 + O(q^7)$ in $S_2(29, 1)$ over $\mathbb{Q}[b_0]/(b_0^2 + 2b_0 - 1)$
$\sigma_{29,2b} = q + b_0 q^2 + \left(\left(\zeta_{14}^3 + \zeta_{14}^2 + \zeta_{14}\right) b_0 + \zeta_{14}^4 + \zeta_{14}^3 + \zeta_{14}^2 + \zeta_{14}\right) q^3 + O(q^4)$ in $S_2(29, \chi_{29}^2)$ over $(\mathbb{Q}(\zeta_{14}))[b_0]/(b_0^2 + \left(-\zeta_{14}^5 + \zeta_{14}^3 + \zeta_{14} + 1\right) b_0 - \zeta_{14}^5 + \zeta_{14}^4 + \zeta_{14}^2 - \zeta_{14} + 1)$
$\sigma_{29,2c} = q + \left(-\zeta_7^5 - \zeta_7^4 - \zeta_7^3 - \zeta_7 - 1\right) q^2 + \left(-\zeta_7^5 - 1\right) q^3 + O(q^4)$ in $S_2(29, \chi_{29}^4)$
$\sigma_{29,2d} = q + b_0 q^2 - b_0 q^3 - 3q^4 - 3q^5 + 5q^6 + 2q^7 + O(q^8)$ in $S_2(29, \chi_{29}^{14})$ over $\mathbb{Q}[b_0]/(b_0^2 + 5)$
$\sigma_{29,4} = q + b_0 q^2 + (-3b_0 - 8)\, q^3 + (-2b_0 - 7)\, q^4 + (4b_0 - 1)\, q^5 + O(q^6)$ in $S_4(29, 1)$ over $\mathbb{Q}[b_0]/(b_0^2 + 2b_0 - 1)$
$\sigma_{31,2a} = q + b_0 q^2 - 2b_0 q^3 + (b_0 - 1)\, q^4 + q^5 + (-2b_0 - 2)\, q^6 + O(q^7)$ in $S_2(31, 1)$ over $\mathbb{Q}[b_0]/(b_0^2 - b_0 - 1)$
$\sigma_{31,2b} = q + b_0 q^2 + \left(\left(\zeta_{30}^5 - 2\zeta_{30}^3 - \zeta_{30}^2 + \zeta_{30} + 1\right) b_0 + \zeta_{30}^6 + \zeta_{30}^5 - \zeta_{30}^4 - 2\zeta_{30}^3 - \zeta_{30}^2 + \zeta_{30}\right) q^3$ $+ O(q^4)$ in $S_2(31, \chi_{31}^2)$ over $(\mathbb{Q}(\zeta_{30}))[b_0]/(b_0^2 + \left(-\zeta_{30}^3 + 1\right) b_0 + 2\zeta_{30}^6 - \zeta_{30}^4 + \zeta_{30}^3 - \zeta_{30}^2 + 2)$
$\sigma_{31,2c} = q + \left(\zeta_5^3 + \zeta_5^2 + \zeta_5\right) q^2 - \zeta_5^3 q^3 + \left(\zeta_5^3 + 1\right) q^4 + O(q^5)$ in $S_2(31, \chi_{31}^6)$
$\sigma_{31,2d} = q + b_0 q^2 + ((-\zeta_3 - 1)\, b_0)\, q^3 + (-2b_0 - 1)\, q^4 + O(q^5)$ in $S_2(31, \chi_{31}^{10})$ over $(\mathbb{Q}(\zeta_3))[b_0]/(b_0^2 + 2b_0 - 1)$
$\sigma_{31,4} = q + b_0 q^2 + (-2b_0 - 6)\, q^3 + (-5b_0 - 10)\, q^4 + (3b_0 - 5)\, q^5 + O(q^6)$ in $S_4(31, 1)$ over $\mathbb{Q}[b_0]/(b_0^2 + 5b_0 + 2)$
$\sigma_{37,2a} = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + O(q^{12})$ in $S_2(37, 1)$
$\sigma_{37,2b} = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} - 2q^{12} - 4q^{13} + O(q^{14})$ in $S_2(37, 1)$
$\sigma_{37,2c} = q + b_0 q^2$ $+ \left(\left(\zeta_{18,2}^4 + \zeta_{18,2}^2\right) b_0^2 + \left(\zeta_{18,2}^5 + \zeta_{18,2}^4 + \zeta_{18,2}^3 + \zeta_{18,2}^2\right) b_0 - 2\zeta_{18,2}^5 + \zeta_{18,2}^4 - \zeta_{18,2}^3 + \zeta_{18,2}^2 - 1\right) q^3$ $+ O(q^4)$ in $S_2(37, \chi_{37}^2)$ over $(\mathbb{Q}(\zeta_{18,2}))[b_0]/(b_0^3 + \left(-\zeta_{18,2}^4 + \zeta_{18,2}^3 + 2\zeta_{18,2} + 1\right) b_0^2$ $+ \left(-2\zeta_{18,2}^5 + 2\zeta_{18,2}^3 + 2\zeta_{18,2}^2 - 2\zeta_{18,2}\right) b_0 + \zeta_{18,2}^5 - \zeta_{18,2}^4 + \zeta_{18,2}^3 - 2\zeta_{18,2}^2 - \zeta_{18,2} + 1)$

$\sigma_{37,2d} = q + (-\zeta_9 - 1)\, q^2 + \left(-\zeta_9^4 + \zeta_9^3 - \zeta_9^2 + 1\right) q^3 + O(q^4)$ in $S_2(37, \chi_{37}^4)$
$\sigma_{37,2e} = q + \left(-\zeta_9^5 + \zeta_9^4 - \zeta_9^3 + \zeta_9\right) q^2 + \left(\zeta_9^5 + \zeta_9^2 - 1\right) q^3 + O(q^4)$ in $S_2(37, \chi_{37}^4)$
$\sigma_{37,2f} = q + b_0 q^2 + ((-\zeta_6 - 1)\, b_0 + \zeta_6)\, q^3 - \zeta_6 q^4 + O(q^5)$ in $S_2(37, \chi_{37}^6)$ over $(\mathbb{Q}(\zeta_6))[b_0]/(b_0^2 - \zeta_6)$
$\sigma_{37,2g} = q + (-\zeta_3 - 1)\, q^2 - \zeta_3 q^4 + \zeta_3 q^5 + 2\zeta_3 q^7 + O(q^8)$ in $S_2(37, \chi_{37}^{12})$
$\sigma_{37,2h} = q + b_0 q^2 - q^3 - 2q^4 - b_0 q^5 - b_0 q^6 + 3q^7 + O(q^8)$ in $S_2(37, \chi_{37}^{18,2})$ over $\mathbb{Q}[b_0]/(b_0^2 + 4)$
$\sigma_{37,4} = q + b_0 q^2 + \left(-\frac{1}{8} b_0^3 - \frac{9}{8} b_0^2 - \frac{13}{4} b_0 - \frac{11}{4}\right) q^3 + \left(b_0^2 - 8\right) q^4 + O(q^5)$ in $S_4(37, 1)$ over $\mathbb{Q}[b_0]/(b_0^4 + 6b_0^3 - b_0^2 - 16b_0 + 6)$
$\sigma_{41,2a} = q + b_0 q^2 + \left(-\frac{1}{2} b_0^2 - b_0 + \frac{3}{2}\right) q^3 + \left(b_0^2 - 2\right) q^4 + O(q^5)$ in $S_2(41, 1)$ over $\mathbb{Q}[b_0]/(b_0^3 + b_0^2 - 5b_0 - 1)$
$\sigma_{41,2b} = q + b_0 q^2$ $+ \left(\left(\zeta_{20}^3 - \zeta_{20}^2 - \zeta_{20} + 1\right) b_0^2 + \left(-\zeta_{20}^7 + \zeta_{20}^6 + \zeta_{20}^3 - \zeta_{20}^2\right) b_0\right.$ $\left. +2\zeta_{20}^5 - 2\zeta_{20}^4 - \zeta_{20}^3 + \zeta_{20}^2 + 2\zeta_{20} - 2\right) q^3 + O(q^4)$ in $S_2(41, \chi_{41}^{18,2})$ over $(\mathbb{Q}(\zeta_{20}))[b_0]/(b_0^3 + \left(\zeta_{20}^6 + \zeta_{20}^3 + 1\right) b_0^2$ $+ \left(\zeta_{20}^7 - 3\zeta_{20}^6 - \zeta_{20}^5 + 2\zeta_{20}^3 - \zeta_{20}^2 - \zeta_{20} + 1\right) b_0 - 2\zeta_{20}^6 + \zeta_{20}^5 + \zeta_{20}^4 - \zeta_{20}^3 + 2)$
$\sigma_{41,2c} = q + b_0 q^2 + \left(\left(\frac{2}{5}\zeta_{10}^3 + \frac{1}{5}\zeta_{10}^2 - \frac{4}{5}\zeta_{10} + \frac{2}{5}\right) b_0 + \frac{2}{5}\zeta_{10}^3 + \frac{6}{5}\zeta_{10}^2 - \frac{4}{5}\zeta_{10} + \frac{2}{5}\right) q^3$ $+ O(q^4)$ in $S_2(41, \chi_{41}^4)$ over $(\mathbb{Q}(\zeta_{10}))[b_0]/(b_0^2 + (-\zeta_{10} + 1)\, b_0 + \zeta_{10}^2 + \zeta_{10} + 1)$
$\sigma_{41,2d} = q + b_0 q^2 + \zeta_5^2 b_0 q^3 + \left(\left(-2\zeta_5^3 - \zeta_5 - 1\right) b_0 + \zeta_5^2 - \zeta_5 + 1\right) q^4 + O(q^5)$ in $S_2(41, \chi_{41}^8)$ over $(\mathbb{Q}(\zeta_5))[b_0]/(b_0^2 + \left(2\zeta_5^3 + \zeta_5 + 1\right) b_0 - \zeta_5^2 - \zeta_5 - 1)$
$\sigma_{41,2e} = q + b_0 q^2 + \left(\left(\frac{1}{2} i - \frac{1}{2}\right) b_0^2 + \frac{5}{2} i - \frac{5}{2}\right) q^3 + O(q^4)$ in $S_2(41, \chi_{41}^{10})$ over $(\mathbb{Q}(i))[b_0]/(b_0^3 - i b_0^2 + 5b_0 - 3i)$
$\sigma_{41,2f} = q - q^2 + \left(-\frac{1}{2} b_0 - \frac{1}{2}\right) q^3 - q^4 + 2q^5 + \left(\frac{1}{2} b_0 + \frac{1}{2}\right) q^6 + O(q^7)$ in $S_2(41, \chi_{41}^{20})$ over $\mathbb{Q}[b_0]/(b_0^2 + 2b_0 + 33)$
$\sigma_{41,4} = q + b_0 q^2 + \left(-\frac{1}{2} b_0^2 - 3b_0 - \frac{5}{2}\right) q^3 + \left(b_0^2 - 8\right) q^4 + O(q^5)$ in $S_4(41, 1)$ over $\mathbb{Q}[b_0]/(b_0^3 + 3b_0^2 - 5b_0 - 3)$

A.4. In this section we give the eigenvalues for the class δ, which were communicated to us by Darrin Doud. This is a class in $H^3(\Gamma_0(3, 41), \mathbb{F}_\eta)$, the finite field $\mathbb{F}$ has order 21881, and η is the nebentype mapping the primitive root 6 mod 41 to $2408 \in \mathbb{F}$. The eigenvalues here are given in $\mathbb{F}$.

Primes ℓ and the eigenvalues of $T_{\ell,1}, T_{\ell,2}$					
2	19471, 19475	11	11832, 19459	23	12264, 9647
3	21880, 2408	13	17072, 5027	29	4811, 12038
5	17066, 17064	17	16840, 5241	31	17478, 4393
7	14655, 4819	19	4809, 5027	37	7030, 14881

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