

# Semiprime Graded Algebras of Dimension Two

M. Artin

*Department of Mathematics, Massachusetts Institute of Technology,  
Cambridge, Massachusetts 02139  
E-mail: [artin@math.mit.edu](mailto:artin@math.mit.edu)*

and

J. T. Stafford

*Department of Mathematics, University of Michigan,  
Ann Arbor, Michigan 48109  
E-mail: [jts@math.lsa.umich.edu](mailto:jts@math.lsa.umich.edu)*

*Communicated by Michel Van den Bergh*

Received November 20, 1998

Semiprime, noetherian, connected graded  $k$ -algebras  $R$  of quadratic growth are described in terms of geometric data. A typical example of such a ring is obtained as follows: Let  $Y$  be a projective variety of dimension at most one over the base field  $k$  and let  $\mathcal{E}$  be an  $\mathcal{O}_Y$ -order in a finite dimensional semisimple algebra  $A$  over  $K = k(Y)$ . Then, for any automorphism  $\tau$  of  $A$  that restricts to an automorphism  $\sigma$  of  $Y$  and any ample, invertible  $\mathcal{E}$ -bimodule  $\mathcal{B}$ , Van den Bergh constructs a noetherian, “twisted homogeneous coordinate ring”  $B = \oplus H^0(Y, \mathcal{B} \otimes \cdots \otimes \mathcal{B}^{\tau^{n-1}})$ . We show that  $R$  is noetherian if and only if some Veronese ring  $R^{(m)}$  of  $R$  has the form  $k + I$ , where  $I$  is a left ideal of such a ring  $B$  and where  $I = B$  at each point  $p \in Y$  at which  $\sigma$  has finite order. This allows one to give detailed information about the structure of  $R$  and its modules. © 2000 Academic Press

*Key Words:* Noetherian graded rings; Gelfand–Kirillov dimension; twisted homogeneous coordinate rings; noncommutative projective geometry.

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## 0. INTRODUCTION

Noetherian graded domains of quadratic growth over an algebraically closed field were classified in [AS], and this paper extends that classification to semiprime algebras with quadratic growth over arbitrary base fields. It can be thought of as completing the program to describe noncommutative projective curves that was begun in [AS].

To begin, we need to be precise about our definitions. Fix a base field  $k$  and let  $R = \bigoplus_{i \geq 0} R_i$  be a finitely generated, semiprime Goldie, locally finite graded  $k$ -algebra (thus,  $\dim_k R_i < \infty$  for all  $i$ ) such that  $\text{GK-dim}(R/P) = 2$  for all minimal prime ideals  $P$  of  $R$ . Then [NV, Theorems A.I.5.8 and C.I.1.6] shows that the graded ring of fractions  $Q = Q(R)$  of  $R$  has the form  $Q = \bigoplus_{i=1}^t A_i[z_i, z_i^{-1}; \tau_i]$ , where  $\tau_i$  is an automorphism of the semisimple algebra  $A_i$  and multiplication is defined by  $z_i a = a^{\tau_i} z_i$ , for  $a \in A_i$ . Moreover, by [AS, Theorem 1.1 and Remark 1.17] each  $A_i$  is finite dimensional over its center  $K_i = Z(A_i)$ , and each  $K_i$  is a finitely generated field of transcendence degree one over  $k$ . A semiprime Goldie, locally finite graded algebra  $R$ , not necessarily finitely generated, is defined to be a *two-dimensional algebra* if  $Q(R)$  satisfies these properties. A particularly pleasant case is when  $R$  contains a regular element  $z \in R_1$ , since we can then adjust matters so that  $Q = A[z, z^{-1}; \tau]$ , where the semisimple algebra  $A$  equals  $Q_0$  [NV, Lemma A.II.1.5]. In this case we call the two-dimensional algebra  $R$  a *nice algebra*.

By [AS] and Corollary 3.3, a finitely generated, semiprime Goldie, locally finite graded algebra  $R$  is a two-dimensional algebra if and only if  $\text{GK-dim}(R/P) = 2$  for all minimal prime ideals  $P$  of  $R$ . The reason for the extra generality in the definitions is that many of our proofs require passage from  $R$  to a *Veronese subring*  $R^{(n)}$ , defined by  $R_k^{(n)} = R_{nk}$ . Unfortunately, even when  $R$  is prime, or finitely generated, the same need not be true of  $R^{(n)}$  (see Example 7.25 and [AS, Theorem 0.4]). In contrast, the Veronese subring  $R^{(n)}$  of a two-dimensional algebra  $R$  will still be a two-dimensional algebra. Moreover, a two-dimensional algebra  $R$  always contains a regular element  $z$  in some  $R_n$  and, for that  $n$ ,  $R^{(n)}$  will be nice (see Lemma 3.1). The assumption that  $\text{GK-dim}(R/P) = 2$  in these defini-

tions is only included to avoid some trivialities, and can easily be weakened to the assertion that  $\text{GK-dim}(R) \leq 2$ . See Corollary 9.5 for the details.

For the rest of this introduction, we fix a two-dimensional algebra  $S = \bigoplus_{i \geq 0} S_i$ . We pick some nice Veronese subring  $R = S^{(n)} = \bigoplus_{i \geq 0} R_i$  and write  $Q(R) = A[z, z^{-1}; \tau]$ , for  $z \in R_1$  and  $A = Q(R)_0$ . Set  $K = \bar{Z}(A)$ , let  $X$  denote the projective, normal model of  $K/k$  and write  $\sigma$  for the restriction of  $\tau$  to  $K$  and  $X$ . By Lemma 3.1, this geometric data is independent of the choice of  $R$ .

As in [AS], a basic construction is that of a twisted homogeneous coordinate ring, although in this paper we use a more general version from [VdB]. Let  $Y$  be a  $\sigma$ -stable projective model of  $K/k$ ,  $\mathcal{E}$  an  $\mathcal{O}_Y$ -order in  $A$ , and  $\mathcal{B}_1 \subset A$  an ample invertible  $(\mathcal{E}, \mathcal{E}^\tau)$ -bimodule. (The definition of such a bimodule is given in Section 6. It is the natural generalization of an ample invertible  $\mathcal{O}_Y$ -module.) The *twisted homogeneous coordinate ring, or twisting ring* for short, of  $\mathcal{B}_1$  is

$$B = B(\mathcal{E}, \mathcal{B}_1; \tau) = \bigoplus_n H^0(Y, \mathcal{B}_n), \quad (0.1)$$

$$\text{where } \mathcal{B}_n = \mathcal{B}_1 \otimes_{\mathcal{E}^\tau} \mathcal{B}_1^\tau \otimes_{\mathcal{E}^{\tau^2}} \cdots \otimes_{\mathcal{E}^{\tau^{n-1}}} \mathcal{B}_1^{\tau^{n-1}}.$$

As is shown in [VdB] and Sect. 6, the twisting ring  $B$  is noetherian, and the category  $\mathbf{gr}\text{-}B$  of finitely generated graded  $B$ -modules, modulo those of finite length, is equivalent to the category  $\text{mod-}\mathcal{O}_{\mathcal{E}}$  of coherent sheaves over  $\mathcal{E}$ .

This construction is illustrated by the following rings. Let  $Y = \mathbb{P}^1$ ,  $\mathcal{O} = \mathcal{O}_Y$ , and let  $\mathcal{O}(1)$  denote the sheaf of functions on  $Y$  with pole at infinity. Let  $K$  be the function field of  $Y$ , and set  $A = M_2(K)$ . Set  $\mathcal{E} = \mathcal{O} + M_2(\mathcal{O}(-1)) \subset A$  and  $\mathcal{B}_1 = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(1)$ . If  $\tau = \text{Id}$ , then  $B(\mathcal{E}, \mathcal{B}_1, \text{Id}) \cong k[x] + yM_2(k[x, y])$ . In contrast, let  $\tau$  denote conjugation by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Then

$$B(\mathcal{E}, \mathcal{B}_1, \tau) \cong \left\{ \begin{pmatrix} a & x \frac{\partial(a)}{\partial x} \\ 0 & a \end{pmatrix} : a \in k[x, y] \right\} + yM_2(k[x, y]). \quad (0.2)$$

This is one of the standard examples of a noetherian PI ring which is not a finite module over its center [Rw2, Example 5.1.16]. See Example 5.14 for more details.

When  $\sigma$  has infinite order, we require the more general construction given in the theorem below.

**THEOREM 0.3.** *Let  $Y$ ,  $\mathcal{E}$  and  $\mathcal{B}_1$  be as in (0.1). Let  $\mathcal{R}_1$  be an essential left  $\mathcal{E}$ -submodule of  $\mathcal{B}_1$  such that  $1 \in H^0(Y, \mathcal{R}_1)$  and  $\mathcal{R}_1 = \mathcal{B}_1$  locally on every finite orbit of  $\sigma$  on  $Y$ . For  $n \geq 1$ , set  $\mathcal{R}_n = \mathcal{B}_1 \otimes \mathcal{B}_1^\tau \otimes \cdots \otimes \mathcal{B}_1^{\tau^{(n-2)}}$*

$\otimes \mathcal{R}_1^{(n-1)}$  where, as before, the tensor products are over appropriate shifts of  $\mathcal{E}$  by  $\tau$ . Then  $R = k + \oplus_{n \geq 1} H^0(Y, \mathcal{R}_n)$  is a noetherian, nice algebra.

This result is contained in Theorem 7.23 and Corollaries 9.2 and 9.3 and is illustrated by Examples 5.13 and 5.14. In the general case of the theorem, when  $\mathcal{R}_n \neq \mathcal{B}_n$ ,  $R$  is a proper subring of  $B = B(\mathcal{E}, \mathcal{B}_1, \tau)$  which looks very different from its commutative analogues (see Example 5.13). The structure of  $R$  and its modules are, however, still tightly constrained, as is illustrated by the next result, which is proved in Corollary 7.3 and Example 7.4.

**COROLLARY 0.4.** *Let  $R$  be the ring constructed by Theorem 0.3. Then,*

(i)  *$R$  is a subidealizer in  $B = B(\mathcal{E}, \mathcal{B}_1, \tau)$  in the sense that, for some  $r \geq 1$ ,  $V = \oplus_{i \geq r} R_i$  is a left ideal of  $B$ . Moreover,  $B$  will be a finitely generated left  $R$ -module, although it may be infinitely generated as a right  $R$ -module.*

(ii) *the categories  $\overline{\mathbf{gr}}\text{-}R$  and  $\text{mod-}\mathcal{O}_{\mathcal{E}}$  are equivalent.*

The main result of this paper is the following partial converse to Theorem 0.3.

**THEOREM 0.5.** *If  $S$  is a noetherian, two-dimensional algebra then, for some  $n$ , the Veronese subring  $R = S^{(n)}$  of  $S$  has the form described in Theorem 0.3.*

This result is proved in Theorem 7.2 and Corollaries 7.3 and 9.2. The required order  $\mathcal{E}$  and curve  $Y$  are constructed in a fairly natural way; the rings  $\Lambda_u = k\langle R_n u^{-1} \rangle$ , as  $u$  ranges over the regular elements of  $R_n$ , generate a sheaf of algebras  $\mathcal{E}_n \subset \mathcal{A}$  defined over a curve  $Y_n$ . However, the proofs that  $Y_n$  is complete and that  $Y_n$  and  $\mathcal{E}_n$  are independent of  $n \gg 0$  are more subtle (see Remark 1.12). They take up most of Sects. 4 and 5.

An important difference between prime rings and domains is that generation in degree one does not imply that a nice algebra is noetherian (see Examples 2.7 and 2.8). A more subtle geometric condition is needed. To describe this condition, it is convenient to write a nice algebra  $R$  as  $R = \oplus \bar{R}_i z^i$ , so that  $\bar{R}_i \subset A$  for all  $i$ . The fact that  $R$  is a graded ring translates to the inclusions  $\bar{R}_i \bar{R}_j^{\tau} \subseteq \bar{R}_{i+j}$ . Let  $N$  denote the reduced norm map  $A \rightarrow K$ . For a point  $p \in X$  and a  $k$ -subspace  $V$  of  $A$ , we denote by  $\mathcal{P}(V; p)$  the maximum order of pole at  $p$  of  $N(v)$ , for  $v \in V$ . The key condition is

$$\mathcal{P}(\bar{R}_i; p) + \mathcal{P}(\bar{R}_j^{\tau}; p) = \mathcal{P}(\bar{R}_{i+j}; p). \quad (0.6)$$

If  $A = K$ , then (0.6) will hold whenever  $\bar{R}_i \bar{R}_j^{\tau} = \bar{R}_{i+j}$ . However, this is definitely not the case for general two-dimensional algebras (see Examples 2.7 and 2.8). Indeed, modulo passage to a Veronese ring, twisting rings

are characterized as the nice algebras for which (0.6) holds at all points  $p \in X$  (see Corollary 7.3(i) for the precise statement). More generally:

**THEOREM 0.7.** *A finitely generated, two-dimensional algebra  $S$  is right noetherian if and only if it is left noetherian. This is true if and only if there is an integer  $n$  such that  $R = S^{(n)}$  is a nice algebra for which (0.6) holds on all finite orbits of  $\sigma$  on  $X$ .*

(Since we are passing to a Veronese subring anyhow, the integer  $n$  can be chosen so that all the finite orbits of  $\sigma$  are fixed points.) This theorem is proved in Theorem 7.1 and Corollary 9.3. Combined with Corollary 0.4, it can be used to prove the following result, which is proved in Theorem 7.2 and Corollary 9.3.

**COROLLARY 0.8.** *Let  $S = \bigoplus_{i \geq 0} S_n$  be a noetherian, two-dimensional algebra. There exists a  $\sigma$ -stable, projective model  $Y$  of  $K$ , such that, if  $\mathcal{R}_n = \mathcal{O}_Y S_n$ , then  $S_n = H^0(Y, \mathcal{R}_n)$  for all  $n \gg 0$ .*

There are several generalizations of these results that deserve mention. If  $S$  is a noetherian two-dimensional algebra, then the module structure of  $S$  may be more complex than that described in Corollary 0.4; indeed this is true for a commutative ring that is not generated in degree one. Nevertheless, the equivalence nearly extends to this case. Formally, by [AS, Proposition 6.1(iv)], the category of graded, 1-critical  $S$ -modules that are generated in degrees  $n\mathbb{Z}$  is equivalent to the category of 1-critical modules over  $R = S^{(n)}$  and so one may apply Corollary 0.4 via this correspondence. Consequently, the simple objects in  $\overline{\mathbf{gr}}\text{-}S$  that are generated in degrees  $n\mathbb{Z}$  are in one-to-one correspondence with the simple  $\mathcal{E}$ -modules. See Remark 7.21 and Corollary 7.22 for more details.

One problem with working with a two-dimensional algebra  $S$ , rather than a nice algebra, is that  $S$  need not be contained in a twisting ring, simply because there may be no element  $z \in Q(S)_1$  that can be used to define the automorphism  $\tau$ . However, twisting can also be defined by a Morita equivalence (see Example 7.25), and we use this more general notion in the next result (see Proposition 7.24).

**PROPOSITION 0.9.** *Let  $S$  be a two-dimensional algebra that satisfies a polynomial identity and is generated by  $S_1$ . Then,  $S$  is noetherian if and only if, up to a finite dimensional vector space,  $S$  is a generalized twisting ring in the sense alluded to above.*

The final result we mention shows that the non-noetherian graded rings are determined by the behavior of linkages between prime ideals and is proved in Theorem 8.6.

**THEOREM 0.10.** *A finitely generated, semiprime graded Goldie ring  $S$  of Gelfand–Kirillov dimension two is non-noetherian if and only if there exists a linked pair of prime ideals  $(M, P)$ , where  $M$  is a maximal ideal and  $P$  is a prime ideal such that  $\text{GK-dim}(S/P) = 1$ .*

The following notation will be used throughout the paper.

(0.11)

(i) The ground field is  $k$ , which is assumed to be infinite in Sections 1–7.

(ii)  $K = \bigoplus_{i=1}^t K_i$  is a finite product of function fields in one variable over  $k$ .

(iii)  $X = X_1 \cup \cdots \cup X_t$  is the normal projective model of  $K/k$ , a non-singular curve which may have several irreducible components  $X_i$ , ordered so that  $k(X_i) = K_i$  for all  $i$ .

(iv)  $A$  is a semisimple ring whose center is  $K$ , so  $A = \bigoplus A_i$ , where  $A_i$  is a central simple  $K_i$ -algebra.

(v)  $\tau$  is an automorphism of  $A$ , and  $\sigma$  denotes both its restriction to  $K$  and the corresponding automorphism of  $X$ .

(vi)  $Q$  is the twisted Laurent polynomial extension  $A[z, z^{-1}; \tau]$ , graded so that  $\deg z = 1$  and  $\deg a = 0$  for  $a \in A$ .

Our convention is that  $\tau$  acts on the right on  $A$ , and that  $\sigma$  acts on the right on  $K$ , and on the left on  $X$ . Thus  $f^\sigma(p) = f(\sigma(p))$  for  $f \in K$  and  $p \in X$ . As (0.11) suggests, we will assume that  $k$  is infinite in the main body of the paper, since it will allow us to use general position arguments. The fact that our results hold for any ground field  $k$  is verified in Section 9.

## 1. NOTATION AND BACKGROUND MATERIAL

The paper [AS] was largely concerned with the structure of a domain  $R = \bigoplus_{i \geq 0} \bar{R}_i z^i \subseteq A[z, z^{-1}; \tau]$ , where  $A = K = k(X)$  is the function field of a nonsingular curve  $X$ . The way in which that paper obtained geometric data from  $R$  was through the Weil divisors  $D(\bar{R}_i)$  associated to the vector spaces  $\bar{R}_i$ ; thus,  $D = D(\bar{R}_i)$  is the smallest Weil divisor on  $X$  such that  $(f) + D \geq 0$  for all  $f \in \bar{R}_i$ . We employ a similar strategy in this paper, although, as  $A$  is now a semisimple artinian ring, we first have to use the reduced norm to pass from  $A$  to  $K$  before taking divisors.

The aim of this section is to provide the basic material about this construction. Since the problem is really a local one, we begin with a brief discussion of the following situation.

(1.1)  $k$  is an infinite field and  $K$  is an extension field. One is given a central simple  $K$ -algebra  $A$  and a discrete valuation ring  $S$ , with field of fractions  $K$ , maximal ideal  $\mathfrak{m} = tS$ , and associated valuation  $\nu$ . We assume that  $k \subset S$  and that  $S/\mathfrak{m}$  is finite over  $k$ .

As always,  $N$  denotes the reduced norm from  $A$  to  $K$ . The order of pole of a function  $f \in K$  on  $S$  is  $-\nu(f)$ , with  $\nu(0) = \infty$ . If  $V$  is a subset of  $A$ , we define the *pole* of  $V$  on  $S$  to be

$$\mathcal{P}(V) = \max\{-\nu N(v) : v \in V\}. \quad (1.2)$$

Thus  $\mathcal{P}(V)$  is an integer, provided that  $V$  is a finite dimensional subspace which contains a regular element of  $A$ . To extend the definition to all  $V$ , we allow  $\mathcal{P}(V)$  to take one of the values  $\pm\infty$ . An element  $v \in V$  is said to have *maximal pole* on  $S$  if  $-\nu N(v) = \mathcal{P}(V)$ .

LEMMA 1.3. *With the notation as in (1.1), let  $V$  be a  $k$ -subspace of  $A$ , and let  $L = SV$  be the  $S$ -module generated by  $V$ . Then,  $\mathcal{P}(V) = \mathcal{P}(L)$ .*

*Proof.* Let  $w \in L$ , and write  $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$ , with  $\alpha_i \in S$  and  $v_i \in V$ . The reduced norm  $N(x_1 v_1 + \cdots + x_n v_n)$  is a polynomial in the variables  $x_1, \dots, x_n$  with coefficients in  $K$  (see [Co, Ex. 10.8.3]). Moreover, the monomials  $\mathbf{x}^j = x_1^{j_1} \cdots x_n^{j_n}$  are linearly independent functions on the infinite field  $k$ . So, for any polynomial  $f(\mathbf{x}) = \sum_j c_j \mathbf{x}^j \in K[\mathbf{x}] = K[x_1, \dots, x_n]$ , one has

$$\min\{\nu(f(\mathbf{x})) | x_i \in k\} = \min\{\nu(c_j)\}.$$

On the other hand, it is clear that

$$\min\{\nu(c_j)\} \leq \min\{\nu(f(\mathbf{x})) | x_i \in S\} \leq \min\{\nu(f(\mathbf{x})) | x_i \in k\},$$

so these inequalities are equalities. This shows that  $\mathcal{P}(V) = \mathcal{P}(L)$ . ■

LEMMA 1.4. *Let  $C$  be a commutative noetherian domain with field of fractions  $K$ , and let  $\Lambda$  be a  $C$ -order in a central simple  $K$ -algebra  $A$ . Assume either that  $C$  is normal or that  $\Lambda$  is an Azumaya algebra. Then  $\Lambda\langle a^{-1} \rangle = \Lambda\langle N(a)^{-1} \rangle$  for any regular element  $a \in \Lambda$ . In particular, an element  $a \in \Lambda$  is invertible if and only if  $N(a)$  is invertible in  $C$ .*

*Proof.* By [MR, Proposition 13.9.11], the trace ring  $T$  of  $\Lambda$  is a finite  $C$ -module. If  $C$  is normal, this forces  $C = T$ , while if  $\Lambda$  is an Azumaya algebra, then  $C = T$  follows from [MR, Proposition 13.9.8]. In other words the coefficients of the reduced characteristic  $p(a)$  polynomial of  $a$  must lie in  $C$ . Therefore,  $a$  divides the constant coefficient  $N(a)$  of  $p(a)$  in  $\Lambda$  and  $\Lambda\langle a^{-1} \rangle \subseteq \Lambda\langle N(a)^{-1} \rangle$ . The opposite inclusion follows from Cramer's rule. ■

LEMMA 1.5. *Let  $\Lambda$  be a maximal  $S$ -order in  $A$ . If  $a$  is an element of  $A$  such that  $\nu N(a) \leq n$ , then  $t^n \in a\Lambda$ .*

*Proof.* Lemma 1.4 handles the case that  $\nu N(a) = 0$ . If  $\nu N(a) > 0$ , then  $a$  is not a unit. As  $\Lambda$  is a principal ideal ring, [Re, Theorem 18.10], we may pick a cyclic, maximal right ideal  $b\Lambda \supseteq a\Lambda$ ; say  $a = bc$ . Then  $m\Lambda \subseteq b\Lambda$ , so  $t \in b\Lambda$ . Moreover,  $\nu N(a) = \nu N(b) + \nu N(c)$  and  $\nu N(b) > 0$ , so  $\nu N(c) \leq n - 1$ . Thus  $t^{n-1} \in c\Lambda$ , by induction on  $n$ , and so  $bt^{n-1}\Lambda \subset bc\Lambda = a\Lambda$ . Since  $t$  is central, it follows that  $t^n \in bc\Lambda$ . ■

If  $T$  is a subring of  $K$ , then a  $T$ -lattice is defined to be a finitely generated  $T$ -submodule of  $A$  which spans  $A$  as  $K$ -module. In this paper, the word lattice will always mean lattice in  $A$ .

PROPOSITION 1.6. (i) *Let  $\Lambda$  be a maximal  $S$ -order in  $A$ , and let  $v \in A \setminus \Lambda$ . There is an element  $w \in \Lambda$  such that  $(v + w)^{-1}$  is in  $\Lambda$ .*

(ii) *Let  $\Lambda$  be a maximal  $S$ -order in  $A$  and let  $V$  be a  $k$ -subspace of  $A$  which strictly contains  $\Lambda$ . Then  $\mathcal{P}(V) > 0$ .*

(iii) *If  $V_1 \subsetneq V_2 \subsetneq \cdots$  is a strictly ascending chain of  $k$ -subspaces of  $A$  which contain an  $S$ -lattice, then  $\lim_{i \rightarrow \infty} \mathcal{P}(V_i) = \infty$ .*

*Proof.* (i) As  $\Lambda$  is a principal ideal ring [Re, Theorem 18.10],  $\Lambda + v\Lambda = c\Lambda$ , for some  $c \in A$ . Equivalently,  $a\Lambda + b\Lambda = \Lambda$ , where  $a = c^{-1}$  and  $b = c^{-1}v$  lie in  $\Lambda$ . Since  $\Lambda$  is semilocal,  $a\Lambda + b\Lambda$  contains an invertible element, say  $z$  [Ba, Proposition 2.8(i), p.87]. Then for some  $w \in \Lambda$ ,  $w + v = cz$ , hence  $(w + v)\Lambda = c\Lambda$ . Since  $1 \in c\Lambda$ ,  $(w + v)^{-1} \in \Lambda$ .

(ii) By Lemma 1.3, we may assume that  $V$  is an  $S$ -lattice. Let  $v \in V \setminus \Lambda$ . Then the element  $u = w + v$  of (i) is in  $V$ , and  $u^{-1} \in \Lambda$  is not a unit. Hence  $\nu N(u) > 0$ , by Lemma 1.4, and so  $\mathcal{P}(V) > 0$ .

(iii) By [Re, Corollary 10.4],  $A$  contains a maximal  $S$ -order  $\Lambda$ . Since  $V_1$  contains a lattice, it contains  $m^N\Lambda$  for some  $N$ . Replacing the  $V_i$  by  $t^{-N}V_i$ , we may assume that  $\Lambda \subseteq V_1$ . We may also assume that  $V_i/\Lambda$  is finite dimensional for every  $i$ . Then, since  $\dim_k(S/m) < \infty$ ,  $V_iS/\Lambda$  is finite dimensional too, from which it follows that  $V_iS \subsetneq V_{i+1}S$  for infinitely many  $i$ . By Lemma 1.3, we may assume that each  $V_i$  is an  $S$ -module.

Fix an integer  $n \geq 0$ . Since  $t^{-n}\Lambda/\Lambda$  is finite dimensional,  $V_r$  is not contained in  $t^{-n}\Lambda$ , for large  $r$ . By (i), there is an element  $u \in V_r \setminus t^{-n}\Lambda$  whose inverse  $s$  is in  $\Lambda$ . Then  $u \notin t^{-n}\Lambda$  implies that  $t^n \notin \Lambda$ . By Lemma 1.5,  $\nu N(s) \geq n$ , hence  $\mathcal{P}(V_r) \geq n$ . ■

We now pass to the general notation established in (0.11), with  $k$  infinite. Thus,  $X$  is the smooth projective model of  $K = \bigoplus_{i=1}^t K_i$  and  $A = \bigoplus A_i$  is a semisimple ring with center  $K$ . By the *norm*  $N(\alpha)$  of  $\alpha = (\alpha_1, \dots, \alpha_t) \in A$ , we mean the element of  $K$  whose component in the factor  $K_i$  is the reduced



norm of the corresponding factor  $\alpha_i \in A_i$ . Let  $X'$  be an open subscheme of  $X$ . If  $S$  is the local ring of  $X'$  at a point  $p$  and  $V \subseteq A$ , we write  $\nu_p$  for the associated valuation of  $K$  and  $\mathcal{P}(V; p) = \max\{-\nu_p N(\alpha) : \alpha \in V\}$ . Since  $S \subset K_i$ , for some  $i$ , this is consistent with the notation for  $\mathcal{P}(V)$ , as defined in (1.2). If  $V$  is a  $k$ -subspace of  $A$  which contains a regular element, we define the *divisor* of  $V$  on  $X'$  by

$$\operatorname{div} V = \operatorname{div}_{X'} V = \sum_{p \in X'} \mathcal{P}(V; p) p,$$

provided that this sum is finite. The divisor  $\operatorname{div} V$  exists if  $V$  is a finite dimensional  $k$ -space. Similarly, if  $V$  is an  $\mathcal{O}_{X'}$ -lattice in  $A$ , by which we mean a coherent subsheaf which generates  $A$  over  $K$ , then  $\operatorname{div} V$  is defined by the same formula.

The next few results provide some basic properties of the divisor  $\operatorname{div}(V)$ .

LEMMA 1.7. *Let  $U, V$  be subspaces or lattices in  $A$  such that  $\operatorname{div} U$  and  $\operatorname{div} V$  are defined. Then*

- (i) *If  $1 \in V$ , then  $\operatorname{div}(V) \geq 0$ .*
- (ii) *If  $U \subseteq V$  then  $\operatorname{div}(U) \leq \operatorname{div}(V)$ .*
- (iii)  *$\operatorname{div}(U) + \operatorname{div}(V) \leq \operatorname{div}(UV)$ .*
- (iv) *If  $\alpha$  is a regular element of  $A$  and if  $Z$  is the divisor of  $N(\alpha)$ , then  $\operatorname{div}(V\alpha) = \operatorname{div}(V) + Z$ . ■*

Let  $Y$  be a noetherian model of  $K/k$  which is not necessarily projective or normal, and let  $\mathcal{O} = \mathcal{O}_Y$ . If  $\mathcal{L}$  is an  $\mathcal{O}$ -lattice in  $A$ , we define the *left* and *right orders* of  $\mathcal{L}$  by

$$E(\mathcal{L}) = \{\alpha \in A \mid \alpha \mathcal{L} \subseteq \mathcal{L}\} \quad \text{and} \quad E'(\mathcal{L}) = \{\alpha \in A \mid \mathcal{L} \alpha \subseteq \mathcal{L}\}, \quad (1.8)$$

respectively. If  $\mathcal{L}$  is an  $\mathcal{O}$ -lattice, we denote by  $\mathcal{O}\langle\mathcal{L}\rangle$  the subalgebra of  $A$  generated by  $\mathcal{O}$  and  $\mathcal{L}$ . The proofs of the next two lemmas are left to the reader.

LEMMA 1.9. *With the above notation, the following are equivalent:*

- (a)  *$\mathcal{L}$  is an  $\mathcal{O}$ -order,*
- (b)  *$\mathcal{L} = E(\mathcal{L})$ .*

*If 1 is a global section of  $\mathcal{L}$ , then (a), (b) are also equivalent to*

- (c)  *$\mathcal{L}$  is a left  $\mathcal{O}\langle\mathcal{L}\rangle$ -module. ■*

We call an  $\mathcal{O}$ -lattice  $\mathcal{L}$  in  $A$  *invertible* if it is a locally free left  $E(\mathcal{L})$ -module of rank 1.

LEMMA 1.10. (i) *If  $\mathcal{L}$  is an  $\mathcal{O}$ -lattice, then  $E(\mathcal{L})$  is a coherent  $\mathcal{O}$ -algebra.*

(ii) A lattice  $\mathcal{L}$  is an invertible left  $E(\mathcal{L})$ -module if and only if it is a locally free right  $E'(\mathcal{L})$ -module of rank 1.

(iii) If an  $\mathcal{O}$ -lattice  $\mathcal{L}$  is generated as left  $E(\mathcal{L})$ -module by a global section  $v$ , then  $v$  is a regular element of  $A$ , and  $\mathcal{L}v^{-1} = E(\mathcal{L})$ . ■

Let  $\mathcal{L}$  be a lattice, and write  $E(\mathcal{L}) = \mathcal{E}$  and  $E'(\mathcal{L}) = \mathcal{E}'$ . Then the lemma implies that  $\mathcal{L}$  is invertible if and only if it is an invertible  $(\mathcal{E}, \mathcal{E}')$ -bimodule in the usual sense of, for example, [Re, Sect. 37].

As a converse to part (iii) of Lemma 1.10, suppose that  $v$  is a regular global section of  $\mathcal{L}$ , and that  $\mathcal{L}v^{-1}$  is an  $\mathcal{O}$ -order. Then Lemma 1.9 implies that  $\mathcal{L}v^{-1} = E(\mathcal{L}v^{-1}) = E(\mathcal{L})$ , and  $\mathcal{L}$  is invertible. Thus an invertible lattice determines a unique order over which it is a locally principal fractional left ideal.

LEMMA 1.11. *Let  $S$  be a discrete valuation ring with fraction field  $K$ , let  $\mathcal{L}$  be an invertible  $S$ -lattice in  $A$ , let  $v \in \mathcal{L}$ , and let  $E = E(\mathcal{L})$ . Then  $\mathcal{L} = Ev$  if and only if  $v$  has maximal pole on  $S$ .*

*Proof.* If  $\mathcal{L} = Ev$ , then  $v$  is a regular element and  $E = \mathcal{L}v^{-1}$ . Since  $E$  is an order and  $S$  is normal, the reduced norm of every element of  $E$  lies in  $S$ . So  $v$  has maximal pole. Conversely, if  $\mathcal{L} = Ev$  and if  $u \in \mathcal{L}$  has maximal pole, then  $uv^{-1}$  has maximum pole among elements of  $E = \mathcal{L}v^{-1}$ . This implies that  $N(uv^{-1})$  is a unit and  $uv^{-1}$  is invertible in  $E$  (Lemma 1.4). Thus,  $\mathcal{L} = Eu$ . ■

Remark 1.12. Lemma 1.11 can be interpreted as saying that, for an invertible  $S$ -lattice  $\mathcal{L}$ ,  $\mathcal{L}v^{-1}$  is an  $S$ -order for all elements  $v \in \mathcal{L}$  with a maximal pole. This conclusion is rather strong and it does not generalize to non-invertible lattices. Indeed, suppose that a lattice  $\mathcal{L}$  over the discrete valuation ring  $S$  is spanned by a  $k$ -vector space  $V$  and that  $A = M_r(K)$ . If  $v \in V$  has a maximal pole, then  $1 \in W = Vv^{-1}$  and so the determinants of elements of  $W$  lie in  $S$ . As Schelter has explained to us, this implies that  $W$  is an order when  $r = 2$ . It is not enough to guarantee that  $W$  is an order, or even that  $S\langle W \rangle$  is an order, if  $r > 2$ . To see this, assume that  $\text{char } k = 0$  and that  $W$  has basis  $\{1, w_1, \dots, w_n\}$ . Then  $W$  generates an  $S$ -order if and only if the traces of all monomials in  $\{w_i\}$  are in  $S$  [Pr, Chap. 6, Sect. 5], and unless  $r = 2$ , this does not follow from the condition on determinants alone. For example, the transcendence degree of the ring of trace functions in two generic  $r \times r$  matrices  $x, y$  is  $r^2 + 1$ , while the determinants  $\det(a + bx + cy)$ , being polynomials of degree  $r$  in  $a, b, c$ , depend on at most  $\binom{r+2}{2}$  variables. Thus, if  $r \geq 4$ , there exist matrices  $x, y \in A$  so that the determinants of all elements of  $W = \text{Span}\{1, x, y\}$  are in  $S$ , but some other trace function has a pole. Then  $S\langle W \rangle$  is not an  $S$ -order. In contrast, one of the main technical results in this paper, Theorem 4.2, shows

that the lattice  $SV$  spanned by  $V = \overline{R}_n$  is invertible and hence that  $S\langle V \rangle$  is an order, provided that  $R$  satisfies condition (0.6) at all fixed points. It is only after one has such a result that one can define the curve and orders required to state results like Theorem 0.5.

If  $\mathcal{L}, \mathcal{M}$  are invertible lattices, we say that the product lattice  $\mathcal{L}\mathcal{M}$  is a *tensor product* if  $E'(\mathcal{L}) = E(\mathcal{M})$ . In that case,  $\mathcal{L}\mathcal{M} \cong \mathcal{L} \otimes_{E(\mathcal{M})} \mathcal{M}$ . We will use the notation  $\mathcal{L} \cdot \mathcal{M}$  exclusively to denote such a tensor product  $\mathcal{L}\mathcal{M}$ .

LEMMA 1.13. *Let  $\mathcal{L}, \mathcal{M}$  be invertible  $\mathcal{O}$ -lattices such that  $\mathcal{E}' = E'(\mathcal{L}) = E(\mathcal{M})$ . Write  $\mathcal{E} = E(\mathcal{L})$  and  $\mathcal{L}^* = \text{Hom}_{\mathcal{E}}(\mathcal{L}, \mathcal{E})$ . Then*

- (i)  $\mathcal{L} \cdot \mathcal{M}$  is invertible,  $E(\mathcal{L} \cdot \mathcal{M}) = \mathcal{E}$  and  $E'(\mathcal{L} \cdot \mathcal{M}) = E'(\mathcal{M})$ ;
- (ii)  $\mathcal{L}^*$  is invertible,  $\mathcal{L}^* = \text{Hom}_{\mathcal{E}'}(\mathcal{L}, \mathcal{E}')$ ; moreover,  $\mathcal{L} \cdot \mathcal{L}^* = \mathcal{E}$  and  $\mathcal{L}^* \cdot \mathcal{L} = \mathcal{E}'$ ;
- (iii)  $\text{div } \mathcal{L} \cdot \mathcal{M} = \text{div } \mathcal{L} + \text{div } \mathcal{M}$ .

If  $X'$  is an open subset of the smooth projective model  $X$  of  $K$  and  $\mathcal{O} = \mathcal{O}_{X'}$ , a lattice  $\mathcal{L}$  such that  $E(\mathcal{L})$  is a maximal order will be called a *normal* lattice.

PROPOSITION 1.14. *Let  $X'$  be an open subset of  $X$ .*

- (i) *A normal lattice is invertible.*
- (ii) *A lattice  $\mathcal{L}$  is normal if and only if  $E'(\mathcal{L})$  is a maximal order.*
- (iii) *Let  $\mathcal{L}$  and  $\mathcal{M}$  be lattices, with  $\mathcal{L}$  normal. Then  $\mathcal{L}\mathcal{M}$  is a normal lattice and  $E(\mathcal{L}) = E(\mathcal{L}\mathcal{M})$ .*
- (iv) *Let  $\mathcal{L} \subseteq \mathcal{L}_1$  be lattices, and assume that  $\mathcal{L}$  is normal. If  $\text{div}(\mathcal{L}) = \text{div}(\mathcal{L}_1)$ , then  $\mathcal{L} = \mathcal{L}_1$ .*
- (v) *If  $\mathcal{L}, \mathcal{M}$  are normal and  $\text{div } \mathcal{L} + \text{div } \mathcal{M} = \text{div } \mathcal{L}\mathcal{M}$ , then  $E'(\mathcal{L}) = E(\mathcal{M})$ . Thus,  $\mathcal{L}\mathcal{M} = \mathcal{L} \cdot \mathcal{M}$ .*

*Proof.* (iii) Since  $E(\mathcal{L})$  operates on the left on  $\mathcal{L}\mathcal{M}$ , one has  $E(\mathcal{L}) \subseteq E(\mathcal{L}\mathcal{M})$ . Since  $E(\mathcal{L})$  is a maximal order,  $E(\mathcal{L}) = E(\mathcal{L}\mathcal{M})$ , and  $\mathcal{L}\mathcal{M}$  is normal.

(iv) Once again, we may prove this locally, so let  $S$  be a local ring of  $X'$ . Left multiplication of  $\mathcal{L} \subseteq \mathcal{L}'$  and conjugation by a regular element of  $A$  reduces us to the case that  $\mathcal{L} = E(\mathcal{L})$  is a maximal order, and hence that  $\text{div } \mathcal{L}' = \text{div } \mathcal{L} = 0$ ; that is,  $N(\alpha) \in S$  for all  $\alpha \in \mathcal{L}'$ . Then Lemma 1.4 shows that  $\mathcal{L} = \mathcal{L}'$ .

(v)  $\mathcal{L}^* \mathcal{L}\mathcal{M} = E'(\mathcal{L})\mathcal{M} \supseteq \mathcal{M}$ . But,  $\mathcal{L}^* \mathcal{L}\mathcal{M} = \mathcal{L}^* \cdot (\mathcal{L}\mathcal{M})$  has divisor equal to  $\text{div } \mathcal{L}\mathcal{M} - \text{div } \mathcal{L} = \text{div } \mathcal{M}$ . Hence, by (iv),  $E'(\mathcal{L})\mathcal{M} = \mathcal{M}$  and  $E'(\mathcal{L}) = E(\mathcal{M})$ . ■

## 2. THE SEQUENCE OF DIVISORS

We use the notation (0.11), and make the additional assumption that  $k$  is infinite. All lattices are assumed to be  $\mathcal{O}_X$ -lattices in  $\mathcal{A}$ . Given a nice algebra  $R = \bigoplus_{i \geq 0} R_i$ , we fix a regular element  $z \in R_1$  and write  $R_n = \overline{R}_n z^n$ , as in the introduction. Thus,  $R = \bigoplus \overline{R}_n z^n$  and  $1 \in \overline{R}_n \subseteq \mathcal{A}$  for all  $n$ . Moreover,

$$\begin{aligned} \overline{R}_i \overline{R}_j^{\tau^i} &\subseteq \overline{R}_{i+j} & \text{for all } i, j \geq 0, \text{ and} \\ \overline{R}_i^{\tau^r} &\subseteq \overline{R}_j & \text{if } i + r \leq j. \end{aligned} \quad (2.1)$$

In this section, we will study the asymptotic properties of the divisors  $\text{div } \overline{R}_j$ . The main idea is to study where property (0.6) holds and how badly it fails. For example, we show that the structure of  $\text{div } \overline{R}_j$  is tightly constrained on infinite orbits while, for a noetherian algebra  $R$ , property (0.6) holds at all fixed points of  $\sigma$ .

LEMMA 2.2. (i) For any  $\alpha \in \mathcal{A}$ ,  $N(\alpha^\tau) = N(\alpha)^\sigma$ .

(ii)  $\mathcal{P}(V^\tau; p) = \mathcal{P}(V; \sigma(p))$ , and if  $p$  is a fixed point of  $\sigma$ , then  $\mathcal{P}(V^\tau; p) = \mathcal{P}(V; p)$ .

*Proof.* (i) Since the reduced norm is functorial, we have  $N(\alpha^\tau) = N(\alpha)^\tau$ , and since the restriction of  $\tau$  to  $K$  is  $\sigma$ ,  $N(\alpha)^\tau = N(\alpha)^\sigma$ .

(ii) Since  $f^\sigma(p) = f(\sigma(p))$  for  $f \in K$ , we also have  $\nu_p(f^\sigma) = \nu_{\sigma(p)}(f)$ . The assertion follows from (i) by choosing an element  $v$  with maximal pole and substituting  $N(v)$  for  $f$ .

Given the nice algebra  $R$ , we define the *divisor sequence* for  $R$  to be

$$D_n = \text{div}(\overline{R}_n). \quad (2.3)$$

These divisors are the analogues of the ones defined in [AS] in the case that  $\mathcal{A} = K$ . Since  $1 \in \overline{R}_n$ , one has  $\mathcal{P}(\overline{R}_n; p) \geq 0$ , and hence  $D_n \geq 0$  for all  $n \geq 0$ . The inclusion  $\overline{R}_i \overline{R}_j^{\tau^i} \subseteq \overline{R}_{i+j}$  implies that

$$0 \leq \mathcal{P}(\overline{R}_i; p) + \mathcal{P}(\overline{R}_j^{\tau^i}; p) \leq \mathcal{P}(\overline{R}_{i+j}; p), \quad (2.4)$$

and hence that

$$0 \leq D_i + \sigma^{-i} D_j \leq D_{i+j} \quad \text{for all } i, j \geq 0. \quad (2.5)$$

Let  $\{p_i = \sigma^{-i} p \mid i \in \mathbb{Z}\}$  be an infinite  $\sigma$ -orbit and set  $t_i^n = \mathcal{P}(\overline{R}_n; p_i)$ . Formula 2.4 translates to

$$0 \leq t_i^m + t_{i-m}^n \leq t_i^{m+n} \quad \text{for all } m, n, i. \quad (2.6)$$

Unfortunately, even when  $\overline{R}_i \overline{R}_j^{\tau^i} = \overline{R}_{i+j}$ , the right-hand inequality in each of these three formulas does not need to be an equality. Here are two typical examples that illustrate the problem.

EXAMPLE 2.7. Set  $U = k\{x, y\}/(xy - yx - x^2)$  and  $D = k + Uy$ , where  $k = \mathbb{C}$ . Then  $Q(D) = Q(U) = k(w)[y, y^{-1}; \sigma]$  where  $w = xy^{-1}$  and  $\sigma$  is defined by  $w^\sigma = w/(1 + w)$ . The ring that we want is

$$R = \begin{pmatrix} U & Uy \\ U & D \end{pmatrix}.$$

This is a noetherian ring that is generated in degrees zero and one ([SZ, Corollary 2.14]).

Take  $z = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$  and  $w = xy^{-1}$ . Let  $\sigma$  be the automorphism of  $k(w) = k(\mathbb{P}^1)$  defined by  $w^\sigma = w/(1 + w)$  and let  $\tau$  be the natural extension of  $\sigma$  to  $M_2(k(w))$ , defined by  $e_{ij}^\tau = e_{ij}$ . Then, as in [AS, Example 2.5],  $Q(R) = M_2(k(w))[z, z^{-1}; \tau]$  and so, in particular,  $X = \mathbb{P}^1$ . By [AS, Example 2.5],  $\bar{D}_n$  has basis  $\{1, ww^\sigma \cdots w^{\sigma^r} : 0 \leq r \leq n-2\}$  and so  $\bar{R}_n = \sum W_{ij}^n e_{ij}$ , where, for  $n \geq 1$ ,  $W_{11}^n = W_{21}^n$  has basis  $\{1, ww^\sigma \cdots w^{\sigma^r} : 0 \leq r \leq n-1\}$  while  $W_{12}^n = W_{22}^n = W_{11}^{n-1}$ . Let  $p_n = \sigma^{-n}(p_0)$  denote the pole of  $w^{\sigma^n}$ . It follows that

$$\mathcal{P}(R_n; p) = \begin{cases} 2 & \text{if } p = p_m \text{ for } 0 \leq m \leq n-2 \\ 1 & \text{if } p = p_{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathcal{P}(\bar{R}_i; p_{i-1}) + \mathcal{P}(\bar{R}_j^i; p_{i-1}) < \mathcal{P}(\bar{R}_{i+j}; p_{i-1}),$$

even though  $\bar{R}_i \bar{R}_j^i = \bar{R}_{i+j}$ . However, this is the only point at which one has a strict inequality.

EXAMPLE 2.8. One can repeat Example 2.7 with  $U$  replaced by the commutative polynomial ring  $C = \mathbb{C}[x, y]$ . In this case,  $R$  will not be noetherian, although it is still generated in degrees zero and one. The analysis of the last paragraph goes through unchanged except that, as  $\tau = \sigma = Id$ , one finds that  $\mathcal{P}(R_n; p_0) = 2n - 1$ . Thus, one still has a strict inequality in (2.4) at just one point  $p = p_0$ , but  $p$  is a fixed point of  $\sigma$ . Since it is generated in degrees 0, 1, every Veronese of this ring is finitely generated. This contrasts with the case of a domain [AS, Theorem 0.4], where finite generation of every Veronese implies that the ring is noetherian.

These examples suggest the following definition.

DEFINITION 2.9. A  $k$ -algebra  $R$  is said to satisfy  $(\dagger)$  if it is a nice algebra, and if the following conditions are satisfied.

(a) The divisors  $D_n$  are supported on a finite set of  $\sigma$ -orbits in  $X$ , which is independent of  $n$ .

- (b) On an infinite  $\sigma$ -orbit, there are integers  $c_1, c_2$  such that  $t_i^n = 0$  if either  $i < -c_1$  or  $i > n + c_2$ , and moreover the integers  $t_i^n$  are bounded independently of  $n, i$ .
- (c) The finite  $\sigma$ -orbits are fixed points.
- (d) For all fixed points  $p$  and all sufficiently large  $i, j$ ,

$$\mathcal{P}(\overline{R}_i; p) + \mathcal{P}(\overline{R}_j^i; p) = \mathcal{P}(\overline{R}_{i+j}; p). \quad (2.10)$$

DEFINITION 2.11. We say that  $R$  satisfies  $(\dagger\dagger)$  if  $(\dagger)$  holds and, in addition, (2.10) holds for all points  $p$  and for all  $i, j \gg 0$ .

Thus  $(\dagger\dagger)$  and  $(\dagger)$  differ only in case  $\sigma$  has infinite order. One of the main results of the paper will show that  $(\dagger)$  characterizes Veroneses of noetherian two-dimensional algebras, while  $(\dagger\dagger)$  only holds for the twisting rings in the sense of (0.1). The rest of this section is devoted to a preliminary analysis of these conditions.

Remark 2.12. Condition (d) is the most significant one in this definition. Indeed, we are mainly interested in finitely generated algebras in which case, as is shown below, parts (a) and (b) are automatic while part (c) will always hold for an appropriate Veronese ring. However, in our proofs, we need to pass to such a Veronese ring and, a priori, this need not be finitely generated (see [AS, Theorem 4.9]). Since conditions (a) and (b) do pass to Veronese rings, we avoid this problem by simply phrasing our results for rings satisfying  $(\dagger)$ . Of course if the equality of (d) holds for large  $i, j$ , then it holds for all  $i, j$  on some Veronese. After the fact, these technicalities are actually irrelevant, since it follows from one of our main results (Theorem 7.1) that  $(\dagger)$  forces both  $R$  and its Veronese rings to be finitely generated.

LEMMA 2.13. (i) *With the above notation, the validity of  $(\dagger)$  and  $(\dagger\dagger)$  do not depend upon the choice of the regular element  $z \in R_1$ .*

(ii) *An algebra  $R$  satisfies  $(\dagger)$ , respectively  $(\dagger\dagger)$ , if and only if  $R^{\text{op}}$  satisfies  $(\dagger)$ , respectively  $(\dagger\dagger)$ .*

Proof. (i) Let  $z, w$  be regular elements of  $R_1$ . Then  $w = \alpha z$ , for some invertible element  $\alpha \in A$  and  $Q = A[z, z^{-1}; \tau] = A[w, w^{-1}; \psi]$ , where  $\psi$  is the automorphism of  $A$  defined by conjugation by  $w$ . If  $\mu$  denotes conjugation by  $\alpha$ , then  $\psi = \mu \circ \tau$ . Hence  $\tau$  and  $\psi$  restrict to the same automorphism  $\sigma$  of  $K$ . If we write  $R = \bigoplus_{i \geq 0} \overline{R}_i z^i = \bigoplus_{i \geq 0} \overline{S}_i w^i$ , then  $\overline{R}_n = \overline{S}_n \alpha \alpha^\tau \cdots \alpha^{\tau^{n-1}}$ . Set  $E_n = \text{div}(\overline{S}_n)$  and let  $Z$  denote the divisor of  $N(\alpha)$ . Then

$$D_n = E_n + Z + \sigma^{-1}Z + \cdots + \sigma^{n-1}Z.$$

It follows that (2.9) holds for  $\{\overline{R}_n\}$  and  $\tau$  if and only if it holds for  $\{\overline{S}_n\}$  and  $\psi$ .

(ii) The only non-trivial step is to show that (2.10) holds for  $R$  if and only if it holds for  $R^{\text{op}}$ . This is analogous to the proof of [AS, Lemma 5.7] and is left to the reader. ■

LEMMA 2.14. *Let  $L$  be a lattice in  $A$  which contains 1. Set  $L_n = LL^\tau \cdots L^{\tau^{n-1}}$  and  $\Delta_n = \text{div}(L_n)$ . Then*

(i) *the supports of the divisors  $\Delta_n$  are contained in finitely many  $\sigma$ -orbits in  $X$ ;*

(ii) *the multiplicities of points of an infinite orbit  $\{p_\ell = \sigma^{-\ell} p\}$  in  $\Delta_n$  are bounded independently of  $\ell$  and  $n$ ;*

(iii) *with the notation of (ii), there are integers  $c_1, c_2$  such that the multiplicity of  $p_\ell$  in  $\Delta_n$  is zero if  $\ell < -c_1$  or  $\ell > n + c_2$ .*

*Proof.* (i) By Lemma 1.10,  $E(L)$  is a coherent  $\mathcal{O}_X$ -module. Thus, there exists finite set  $S \subset X$ , such that, on  $X \setminus S$ , the ring  $E(L)$  is a maximal order and that  $E(L) = L = E'(L) = E(L)^\tau$ . Thus,  $\Delta_1$  is supported on  $S$  and, except on  $\sigma^{-i}S$ ,  $L^{\tau^i}$  is a normal lattice with  $\text{div}(L) = 0$ . Moreover, the product  $L^{\tau^i} L^{\tau^{i+1}}$  is a tensor product except, possibly, on  $\sigma^{-i}S$ . So the products making up  $LL^\tau \cdots L^{\tau^{n-1}}$  are tensor products outside of the locus  $S_n = S \cup \sigma^{-1}S \cup \cdots \cup \sigma^{-(n-1)}S$ , and Lemma 1.13(iii) shows that  $\Delta_n$  is supported on  $S_n$ . This proves (i).

(ii) Proposition 1.14(iii) shows that  $L_n$  is a normal lattice and that  $E(L_n) = E$ , except possibly on  $S$ . Working similarly on the other side,  $E'(L_n) = E^{\tau^{n-1}}$  and  $L_n$  is a normal lattice, except possibly on  $\sigma^{n-1}S$ . For large  $n$ , the set  $S \cap \sigma^{n-1}S$  does not meet any infinite orbit. So if  $n \gg 0$ ,  $L_n$  is a normal lattice, and hence is invertible, locally on every infinite orbit. Also,  $E'(L_i) = E(L_j^{\tau^i})$  except on  $\sigma^{-i}S$ . So  $L_i L_j^{\tau^i}$  is a tensor product, and  $\Delta_i + \sigma^{-i}\Delta_j = \Delta_{i+j}$ , outside of  $\sigma^{-i}S$ . Then if  $n = ri + j$ ,

$$\Delta_n = \Delta_i + \sigma^{-i}\Delta_i + \cdots + \sigma^{-(r-1)i}\Delta_i + \sigma^{-ri}\Delta_j$$

except on  $S' = \sigma^{-i}S \cup \sigma^{-2i}S \cup \cdots \cup \sigma^{-ri}S$ , and so the multiplicities of  $\Delta_n$  on an infinite orbit are bounded by those of  $\Delta_i$  except on  $S'$ . Similarly,

$$\Delta_n = \Delta_j + \sigma^{-j}\Delta_i + \cdots + \sigma^{-j-(r-1)i}\Delta_i$$

except on  $S'' = \sigma^{-j}S \cup \sigma^{-j-i}S \cup \cdots \cup \sigma^{-j-(r-1)i}S$ , so the multiplicities are bounded by those of  $\Delta_i$  except on  $S''$  as well. For  $i \gg j \gg 0$ ,  $S' \cap S'' = \emptyset$  on any infinite orbit. Thus, for large  $n$ ,  $\Delta_n$  satisfies one of the two displayed equations there and so the multiplicity of  $p_\ell$  in  $\Delta_n$  is bounded independently of  $n$  and  $\ell$ .

(iii) It suffices to pick  $c_1$  and  $c_2$  such that the multiplicity of  $p_\ell$  in  $S$  is zero for  $\ell \notin [c_1, c_2]$ . ■

The next result justifies the comments made in Remark 2.12.

**PROPOSITION 2.15.** (i) *A nice algebra  $R$  which is finitely generated satisfies (2.9a) and (2.9b), and it satisfies (2.9d) at all but a finite set of points.*

(ii) *Let  $R^{(m)}$  be a Veronese of a nice algebra  $R$ . Then  $R$  satisfies (2.9a) and (2.9b) if and only if  $R^{(m)}$  does.*

(iii) *If (2.9c), respectively (2.9d), holds for a nice algebra  $R$ , then it holds for every Veronese ring  $R^{(m)}$  of  $R$ .*

*Proof.* (i) Assume that  $R$  is generated in degrees  $\leq k$ , so that  $\bar{R}_n = \sum_{j=1}^k \bar{R}_{n-j} \bar{R}_j^{\tau_{n-j}}$ , for all  $n \geq 0$ . Since  $1 \in \bar{R}_k$  and  $\bar{R}_j \subseteq \bar{R}_k$ , for all  $j < k$ , this implies that

$$\bar{R}_n \subseteq \bar{R}_k \bar{R}_k^{\tau} \bar{R}_k^{\tau^2} \cdots \bar{R}_k^{\tau^{n-1}}.$$

Taking for  $L$  the lattice  $\mathcal{O}_X \bar{R}_k$  generated by  $\bar{R}_k$ , both assertions of the proposition follow from Lemma 2.14.

Part (ii) is an easy application of (2.6), while (iii) is trivial. ■

The analysis of the divisors  $D_n$  in [AS] assumed that they be a  $\sigma$ -sequence (see [AS, 2.2]). This is a slightly stronger hypothesis than (2.5), but the asymptotic behavior of  $D_n$  on an infinite orbit is the same.

**PROPOSITION 2.16.** *Let  $R$  be a nice algebra satisfying (2.9b). Let  $\{p_i = \sigma^{-i}(p_0)\}$  be an infinite  $\sigma$ -orbit, and let  $t_i^n = \mathcal{P}(\bar{R}_n, p_i)$  denote the maximal order of pole of  $\bar{R}_n$  at the point  $p_i$ . Assume that  $t_i^n \neq 0$  for some  $n, i$ . Then, shifting the indices  $i$  suitably, there exist integers  $a, b, c$ , with  $a \geq c \geq -b \geq 0$  and  $a > 0$ , and integers*

$0 = \mathbf{r}_0 < \mathbf{r}_1 \leq \cdots \leq \mathbf{r}_{a-1} < \mathbf{r}_a = \mathbf{r}_{\max} = \mathbf{s}_{-b} > \mathbf{s}_{1-b} \geq \cdots \geq \mathbf{s}_{c-1} > \mathbf{s}_c = 0$   
such that

(i)  $\mathbf{r}_k + \mathbf{s}_k \leq \mathbf{r}_{\max}$  for  $0 \leq k \leq c$ .

(ii) For all  $n$ ,

$$t_k^n \leq \begin{cases} 0 & \text{for } k \leq 0 \\ \mathbf{r}_k & \text{for } 0 \leq k \leq a \\ \mathbf{r}_{\max} & \text{for } a \leq k \leq n - b \\ \mathbf{s}_{k-n} & \text{for } n - b \leq k \leq n + c \\ 0 & \text{for } k \geq n + c. \end{cases}$$

(iii) The inequality (ii) is an equality for  $n \gg 0$ .

The meaning of this proposition is best understood by assuming equality for some fixed  $n$  and plotting the points  $(k, t_k^n)$  in the plane. One may also check the proposition on Example 2.7.



*Proof.* The proof is obtained by making the natural modifications to the proof of [AS, Proposition 2.11]. Formula 2.6 shows that, for fixed  $k$ , the sequence  $t_k^n$  is increasing. Set  $\mathbf{r}_k = \max_n \{t_k^n\}$  and  $\mathbf{r}_{\max} = \max_k \{\mathbf{r}_k\}$ , which exist by (2.9b). Then (2.6) shows that  $\mathbf{r}_{k-1} = t_{k-1}^n \leq t_k^{n+1} = \mathbf{r}_k$ , for  $n \gg 0$ . So  $\cdots \mathbf{r}_k \leq \mathbf{r}_{k+1} \leq \cdots \leq \mathbf{r}_{\max}$ . Also, (2.9b) tells us that  $\mathbf{r}_k = 0$  for  $k \ll 0$ .

The set of integers  $\{g_i^n = t_{n-i}^n\}$  also satisfies (2.6) and (2.9b), and the integers  $\mathbf{s}_k$  are obtained in the same way as  $\mathbf{r}_k$ , but using  $\{g_i^n\}$ . Finally, we may shift indices so that  $t_i^n = 0$  if  $i \leq 0$  but that  $t_1^n > 0$  for some  $n$  (see 2.9b). ■

DEFINITION 2.17. If  $(\dagger)$  holds for  $R$  then, as in [AS, 2.12–2.16], we define the gap divisor  $\Omega_I$  on the infinite orbit  $I = \{p_i = \sigma^{-i}(p_0)\}$  to be

$$\sum_{i=-b}^a \omega_i p_i,$$

where  $\omega_i = \mathbf{r}_{\max} - \mathbf{r}_i - \mathbf{s}_i$ . The gap divisor is defined to be  $\Omega = \sum_I \Omega_I$ , where the sum is over the infinite orbits  $I$ .

COROLLARY 2.18. (i) Let  $R$  be a nice algebra satisfying (2.9b). Then  $D_i + \sigma^{-i}D_j + \sigma^{-i}\Omega = D_{i+j}$  holds for all  $i, j \gg 0$ .

(ii) Suppose that  $I$  is an infinite orbit of  $\sigma$  and let  $D_n|_I$  denote the restriction of  $D_n$  to  $I$ . If  $a, b$  are defined by Proposition 2.16, then  $\sigma^{-s}(D_n)|_I \geq \sigma^{-n}\Omega_I$ , for all  $n \gg 0$  and all  $a + b \leq s \leq n - a - b$ . ■

### 3. NOETHERIAN ALGEBRAS

In this section we show that  $(\dagger)$  is an appropriate condition for studying noetherian rings. We begin with some subsidiary results.

LEMMA 3.1. Let  $R$  be a locally finite, semiprime, graded Goldie ring, such that  $R$  has no finite dimensional algebra summands, and let  $n$  be a positive integer. Then

(i) The Veronese ring  $R^{(n)}$  is a semiprime Goldie ring, and its graded ring of fractions  $Q(R^{(n)})$  is the Veronese  $Q^{(n)}$  of  $Q = Q(R)$ .

(ii) In particular, if  $Q$  has the form  $Q = A[z, z^{-1}, \tau]$ , with  $z \in R_1$  and  $A = Q_0$ , then  $Q(R^{(n)}) = A[z^n, z^{-n}, \tau^n]$ .

*Proof.* The assumption on finite dimensional summands ensures that  $R$  does have a graded semisimple, graded ring of fractions [NV, Theorem C.I.6(2)]. By [NV, Theorem A.I.5.8],  $Q^{(n)}$  is graded semisimple, and it is routine that  $R^{(n)}$  is an order in  $Q^{(n)}$ . Thus,  $R^{(n)}$  is Goldie. ■

The next result is an extension of [AZ, Proposition 5.10]. For this result, only, we do not assume that the gradation is locally finite.

**PROPOSITION 3.2.** *Let  $R$  be a semiprime, graded Goldie ring. Then*

- (i)  *$R$  is a submodule of a finitely generated right  $R^{(n)}$ -module;*
- (ii)  *$R$  is right noetherian if and only if  $R^{(n)}$  is right noetherian. In this case,  $R$  is a finitely generated right  $R^{(m)}$ -module.*

*Proof.* (i) The direct sum  $\oplus R/P$ , where  $P$  runs through the minimal prime ideals of  $R$ , is a finitely generated overring of  $R$  and each such ideal  $P$  is graded ([NV, Theorem A.II.7.3]). Thus, we may reduce to the case where  $R$  is prime. Since the result is vacuously true otherwise, we may assume that  $R \neq R_0$ . Thus, by [NV, Corollary C.I.1.7 and Theorem A.I.5.8],  $R$  has a graded simple artinian quotient ring  $Q \cong M_r(D[z, z^{-1}; \sigma])$ , where  $D$  is a division ring. The matrix units in  $M_r(D)$  need not have degree zero and the degree of  $z$  may be bigger than one [NV, p.45].

For  $0 \leq i \leq n-1$ , let  $L_i$  denote the sum of the terms  $R_m$  with  $m$  congruent to  $i$  modulo  $n$ , that is,  $L_i = \oplus_{k \geq 0} R_{nk+i}$ . Then  $R = \oplus_{i=0}^{n-1} L_i$  as a right  $R^{(n)}$ -module, and it suffices to exhibit an injective  $R^{(n)}$ -linear map from each  $L_i$  to a finite sum of copies of  $R^{(n)}$ .

Note that  $Q = \oplus L_i Q^{(n)}$  with  $L_i Q^{(n)} = \oplus_{k \in \mathbb{Z}} Q_{nk+i}$ . As a right  $Q^{(n)}$ -module,  $Q$  is generated by the homogeneous elements  $e_{uv} z^t$ , where  $e_{uv}$  runs through the matrix units of  $M_r(D)$  and  $t < n$ . Those  $e_{uv} z^t$  whose degrees are congruent to  $i$  modulo  $n$  generate  $L_i Q^{(n)}$ . Pick  $\alpha = e_{uv} z^t$  with degree congruent to  $i$ . Since  $R$  is Goldie, there is a regular element  $c \in R$  such that  $c z^{-t} e_{vu} \in R$ . Since we may replace  $c$  by  $c^n$ , we may take  $c \in R^{(n)}$ . Note that  $\deg e_{uv} = -\deg e_{vu}$ . So left multiplication by  $\beta = c z^{-t} e_{vu}$  defines a  $Q^{(n)}$ -linear map  $\lambda : L_i Q \rightarrow Q^{(n)}$ . This map is injective on the submodule  $\alpha Q^{(n)}$ . Moreover, since  $\beta \in R$ ,  $\lambda$  maps  $L_i$  to  $R^{(n)}$ . The sum of these maps  $\lambda$ , as  $\alpha$  runs over the generators  $e_{uv} z^t$  whose degrees are congruent to  $i$ , is the required injection  $L_i \rightarrow \oplus R^{(n)}$ .

- (ii) Use [AZ, Proposition 5.10] and part (i). ■

We note that the implication  $\Leftarrow$  in Proposition 3.2 is not true without some hypothesis on  $R$ . See for example [AZ, Remarks after (5.10)].

**LEMMA 3.3.** *If  $R$  is a two-dimensional algebra, then  $\text{GK-dim}(R/P) = 2$ , for all minimal prime ideals  $P$  of  $A$ .*

*Proof.* By [AS, Theorem 1.2] the result is true when  $Q(R) = A[z, z^{-1}; \tau]$ , with  $A$  simple artinian. Thus, we need only reduce to this case.

By Proposition 3.2(i), we may replace  $R$  by a Veronese subring. Thus we may assume that  $R$  is nice and, moreover, that  $Q(R) = A[z, z^{-1}; \tau]$ , where

$A = \oplus A_i$  is a semisimple ring such that each simple factor  $A_i$  of  $A$  is fixed by  $\tau$ . Thus,  $Q(R) \cong \oplus_{i=1}^t (A_i[z_i, z_i^{-1}; \tau_i])$ , for the appropriate  $\tau_i$ . For any minimal prime ideal  $P$  of  $R$ , the quotient ring  $Q(R/P)$  is isomorphic to one of the rings  $A_i[z_i, z_i^{-1}; \tau_i]$ . Thus [AS, Theorem 1.2] applies. ■

LEMMA 3.4. *Let  $\Lambda$  be a right noetherian, locally finite, graded  $k$ -algebra of GK-dimension 1.*

(i) *There is a homogeneous element  $a \in \Lambda$  of positive degree such that  $\dim_k \Lambda/a\Lambda < \infty$ .*

(ii) *Let  $a$  be as in (i), and let  $\rho_a$  denote right multiplication by  $a$  on a finitely generated graded  $\Lambda$ -module  $M$ . The kernel and cokernel of  $\rho_a$  are finite dimensional. Hence  $\rho_a$  is bijective on  $M_n$  for  $n \gg 0$ .*

We note that  $\Lambda$  need not have any regular homogeneous elements of positive degree. A standard example is  $\begin{pmatrix} k & xk[x] \\ 0 & k[x] \end{pmatrix}$ .

*Proof.* (ii) It suffices to show that the kernel is finite dimensional. If  $m \in M$  and if  $ma = 0$ , then  $m\Lambda$  is a quotient of  $\Lambda/a\Lambda$ . So  $m\Lambda$  is finite dimensional. This shows that  $\ker M$  is contained in the maximal finite-dimensional submodule of  $M$ .

(i) The assertion is true when  $\Lambda$  is semiprime. Indeed, in that case one can choose  $a$  to be any homogeneous element which is regular in  $\Lambda/P$  for each prime ideal  $P$  such that  $\text{GK-dim}(\Lambda/P) = 1$ . In the general case, let  $N$  be the nilradical of  $\Lambda$ , and let  $a \in \Lambda$  be chosen such that (i) holds for  $[a + N] \in \Lambda/N$ . Then, part (ii) can be applied with  $[a + N]$  and the  $\Lambda/N$ -modules  $M = N^j/N^{j+1}$ . It follows by that result and induction on  $j$  that  $\Lambda/a\Lambda$  is finite dimensional. ■

THEOREM 3.5. *If  $R$  is a right noetherian, two-dimensional algebra, then  $(\dagger)$  holds for some Veronese ring  $R^{(m)}$ .*

*Proof.* By Lemma 3.1, we may replace  $R$  by a Veronese ring and so we may assume that  $R$  is a nice algebra. Since  $R$  is still right noetherian, it is necessarily finitely generated as a  $k$ -algebra.

For applications in Sect. 9 we first suppose that  $R$  is a finitely generated nice algebra, but not necessarily noetherian. After passing to a further Veronese ring, we may assume, by Proposition 2.15(i), (ii), that (2.9a), (2.9b), and (2.9c) hold. Of course,  $R$  may no longer be finitely generated, but this will not affect the argument. In the notation of (0.11), let  $p \in X_i$  be a fixed point of  $\sigma$ , and let  $S$  denote the local ring of  $X$  at  $p$ , with maximal ideal  $tS$ . Because  $p$  is fixed,  $S = S^\sigma$  and  $t^\sigma = \mu t$ , for some unit  $\mu \in S$ . By passing to a further Veronese ring, we may assume that  $\sigma$  acts trivially on  $k(p)$ .

Let  $\bar{L}_n = S\bar{R}_n = \bar{R}_n S$  be the  $S$ -module spanned by  $\bar{R}_n$ . Since the field of fractions of  $S$  is one of the factors  $K_i$  of  $K$ ,  $\bar{L}_n$  is an  $S$ -lattice in the corresponding factor  $A_i$  of  $A$ . Equation (2.1) implies that

$$\bar{L}_i \bar{L}_j^{\tau^i} \subseteq \bar{L}_{i+j}. \quad (3.6)$$

Thus  $L = \oplus \bar{L}_n z^n$ , with  $z \in R_1$  as in (2.1), is a graded ring; it is just the ring generated by  $S$  and the image in  $A_i[z, z^{-1}; \sigma]$  of  $R$ . Moreover,  $t$  is a normalizing element of  $L$ , of degree zero. Set  $\Lambda = L/tL = L \otimes_S k(p)$ . This is a factor ring of the  $k(p)$ -algebra  $R \otimes_k k(p)$ . Since  $\bar{L}_n$  is a finitely generated  $S$ -submodule of  $A_i$ , it is free, of rank at most  $\dim_{K_i} A_i$ . So  $\dim_k \Lambda_n$  is bounded. This implies that  $\Lambda$  is a locally finite, graded  $k(p)$ -algebra of GK-dimension  $\leq 1$ . Also,  $\Lambda_n \neq 0$  for large  $n$ , since  $\bar{R}_n$  spans  $\bar{L}_n$  and  $\bar{R}_n \neq 0$  for large  $n$ .

If  $R$  is right noetherian, then  $\Lambda$  will also be noetherian, and the theorem follows from the next lemma.

LEMMA 3.7. (i) *For each fixed point  $p \in X$  there is an integer  $n$  such that the algebra  $\Lambda = \Lambda_p$  described above is a factor of  $R^{(n)} \otimes_k k(p)$ .*

(ii) *If  $\Lambda_p$  is right noetherian and  $n$  is as in (i), then (2.9d) holds for  $R^{(n)}$  at  $p$ . Hence if  $(\dagger)$  does not hold for any Veronese, then there is a point  $p$  such that the algebra  $\Lambda_p$  is not right noetherian.*

*Proof.* It remains to prove part (ii). Since  $\Lambda$  is infinite dimensional and finitely generated, it has Gelfand–Kirillov dimension one. We choose  $a \in L$  as in Lemma 3.4, and we pass to a Veronese so that  $a$  becomes of degree 1. In order to prove (2.9d) at  $p$ , Lemma 1.3 and induction show that it is enough to prove that

$$\mathcal{P}(\bar{L}_n; p) = \mathcal{P}(\bar{L}_{n-1}; p) + \mathcal{P}(\bar{L}_1^{\tau^{n-1}}; p) \quad \text{for } n \gg 0. \quad (3.8)$$

To do this, we show that for every  $b \in \bar{L}_n$  and for every positive integer  $s$ , there exist  $c \in \bar{L}_{n-1}$  and  $\beta \in \bar{L}_n$  such that

$$b = ca^{\tau^{n-1}} + t^s \beta. \quad (3.9)$$

Indeed, if this holds, then  $N(b) \equiv N(c)N(a^{\tau^{n-1}})$  modulo  $t^s N(\bar{L}_n)$ . Taking  $s \gg \nu N(b)$ , this implies that  $\mathcal{P}(\bar{L}_n; p) \leq \mathcal{P}(\bar{L}_{n-1}; p) + \mathcal{P}(\bar{L}_1^{\tau^{n-1}}; p)$ . The opposite inequality follows from (3.6).

In order to prove (3.9), we can substitute  $\beta$  back in for  $b$ . So it suffices to treat the case  $s = 1$ , and this case follows from Lemma 3.4(i) for all large  $n$ . ■

One of the main goals of this paper is to prove the converse to Theorem 3.5. This is done in Theorem 7.1.

4. THE  $\mathcal{O}_X$ -LATTICE SPANNED BY  $\bar{R}_n$ 

Throughout this section,  $k$  is assumed to be infinite. Let  $R$  be a nice algebra, let  $X$  denote the smooth model of  $K$  as in (0.11), and let

$$\mathcal{L}_n = \mathcal{O}_X \bar{R}_n = \bar{R}_n \mathcal{O}_X$$

denote the  $\mathcal{O}_X$ -lattice spanned by  $\bar{R}_n$ . Then (2.1) implies that

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \quad \text{and} \quad \mathcal{L}_i \mathcal{L}_j^i \subseteq \mathcal{L}_{i+j}. \quad (4.1)$$

The main aim of this section is to prove the following theorem.

**THEOREM 4.2.** *Let  $R$  be a  $k$ -algebra satisfying  $(\dagger)$ . Then  $\mathcal{L}_n = \mathcal{O}_X \bar{R}_n$  is invertible for all sufficiently large  $n$ . If  $p$  is not a fixed point of  $\sigma$ , then for large  $n$ ,  $\mathcal{L}_n$  is a normal lattice locally at  $p$ .*

For a given  $n \gg 0$ , the lattice  $\mathcal{L}_n$  is normal locally at all but a finite set  $S$  of points and hence, by Proposition 1.14(i), it is invertible away from  $S$ . Also, by (2.9a), the divisors  $D_n$  of  $\mathcal{L}_n$  are supported on finitely many orbits. Let  $T$  denote the union of  $S$  with these orbits. Because  $\{\mathcal{L}_r\}$  is an increasing sequence of lattices, Proposition 1.14(iii), (iv) imply that  $\mathcal{L}_m$  is also normal at every point not in  $T$ , for every  $m \geq n$ . Therefore, finitely many orbits remain to be checked, and it suffices to verify the assertion of the theorem on each one of them separately. Note that, by Lemma 1.3,  $\mathcal{P}(\mathcal{L}_n; p) = \mathcal{P}(\bar{R}_n; p)$  for all  $p \in X$ . Since  $(\dagger)$  holds, the proof for a fixed point therefore follows from the next lemma.

**LEMMA 4.3.** *Let  $\mathcal{O}$  be a  $\sigma$ -invariant, semi-local subring of  $K$  such that  $\dim_k(\mathcal{O}/\mathfrak{m}) < \infty$  for all maximal ideals  $\mathfrak{m}$  of  $\mathcal{O}$ . Let  $\tilde{\mathcal{O}}$  be normalization of  $\mathcal{O}$  and assume that  $\sigma$  acts trivially on the set  $W$  of closed points of  $\tilde{\mathcal{O}}$ . For  $n \gg 0$ , let  $\mathcal{L}_n$  be  $\mathcal{O}$ -lattices in  $A$  such that, for  $i, j \gg 0$ ,  $\mathcal{L}_i \mathcal{L}_j^i \subseteq \mathcal{L}_{i+j}$  and that*

$$\mathcal{P}(\mathcal{L}_i; p) + \mathcal{P}(\mathcal{L}_j^i; p) = \mathcal{P}(\mathcal{L}_{i+j}; p) \quad (4.4)$$

*holds for all points  $p \in W$ . Then, for  $n, i, j \gg 0$ ,*

- (i)  $\mathcal{L}_n$  is an invertible  $\mathcal{O}$ -lattice;
- (ii)  $E = E(\mathcal{L}_n)$  is independent of  $n$ , and  $E'(\mathcal{L}_n) = E^{\tau^n}$ ;
- (iii)  $\mathcal{L}_i \cdot \mathcal{L}_j^i = \mathcal{L}_{i+j}$ ;
- (iv) the center of  $E$  is  $\sigma$ -invariant.

*Proof.* Choose  $i, j \gg 0$  and let  $u_i \in \mathcal{L}_i$  and  $u_j \in \mathcal{L}_j$  have maximal pole at all  $p \in W$ . Then, for  $n \gg 0$ , (4.4) implies that  $u_{i+nj} := u_i u_j^i u_j^{\tau^{i+j}} \cdots u_j^{\tau^{i+(n-1)j}} \in \mathcal{L}_{i+nj}$  has maximal pole at  $p \in W$ . For  $k = j$  or  $k = i + nj$ , let  $\mathcal{M}_k = \mathcal{L}_k u_k^{-1}$ . Let  $\mu_k$  denote conjugation by  $u_k$  on  $A$  and set  $\phi_k = \mu_k \circ \tau^k$ .

SUBLEMMA 4.5. (i)  $1 \in \mathcal{M}_k$  and  $N(\alpha) \in \tilde{\mathcal{O}}$  for all  $\alpha \in \mathcal{M}_k$ .

(ii)  $\mathcal{M}_{i+nj} \mathcal{M}_{rj}^{\phi_{i+nj}} \subseteq \mathcal{M}_{i+(n+r)j}$ .

(iii)  $\mathcal{M}_{i+nj} \subseteq \mathcal{M}_{i+(n+1)j} \subseteq \dots$  and this sequence is constant for  $n \gg 0$ .

*Proof of the sublemma.* Part (i) is true since  $u_k$  has maximal pole in  $\mathcal{L}_k$  (see Lemma 1.7(iv)), part (ii) is a translation of  $\mathcal{L}_i \mathcal{L}_j^{\tau^i} \subseteq \mathcal{L}_{i+j}$ , and the first assertion of part (iii) follows from (i) and (ii). Setting  $\tilde{\mathcal{M}}_n = \tilde{\mathcal{O}} \mathcal{M}_n$ , we have  $\tilde{\mathcal{M}}_{i+nj} \subseteq \tilde{\mathcal{M}}_{i+(n+1)j}$  for all  $n \geq 0$ . By part (i) and Lemma 1.3,  $\mathcal{P}(\mathcal{M}_n; p) = 0$ , for all  $p \in W$ . Thus, Proposition 1.6(iii) implies that  $\tilde{\mathcal{M}}_{i+nj} = \tilde{\mathcal{M}}_{i+(n+1)j}$  for large  $n$ . As  $\dim_k \tilde{\mathcal{O}}/\mathcal{O} < \infty$  it follows that  $\mathcal{M}_{i+nj} = \mathcal{M}_{i+(n+1)j}$  for  $n \gg 0$ . ■

We return to the proof of Lemma 4.3. The sublemma shows that, for  $n, i, j \gg 0$ , the lattices  $\mathcal{L}_{i+nj}$  are mapped isomorphically to each other by right multiplication by regular elements of  $A$ . Hence, all the lattices  $\mathcal{L}_m$  with  $m \gg 0$  are related by right multiplications and  $E(\mathcal{L}_m)$  is independent of  $m \gg 0$ .

We now take  $i = j$  in Sublemma 4.5(iii), to conclude that  $\mathcal{M}_{nj} = \mathcal{M}$  is independent of  $n \gg 0$ . Then, 4.5(ii) becomes

$$\mathcal{M} \mathcal{M}^{\phi_m} \subseteq \mathcal{M}.$$

Since  $1 \in \mathcal{M}$ , we obtain  $\mathcal{M}^{\phi_m} \subseteq \mathcal{M}$ , and  $\mathcal{M} \subseteq \mathcal{M}^{\phi_m^{-1}} \subseteq \mathcal{M}^{\phi_m^{-2}} \subseteq \dots$ . Proposition 1.6 shows that this sequence is also essentially constant. Therefore  $\mathcal{M}^{\phi_m} = \mathcal{M}$ . Thus  $\mathcal{M} \mathcal{M} \subseteq \mathcal{M}$ , and  $\mathcal{M}$  is an order. Since  $\mathcal{L}_m = \mathcal{M} u_m$ ,  $\mathcal{L}_m$  is an invertible lattice, and  $E(\mathcal{L}_m) = \mathcal{M}$  for  $m \gg 0$ .

Since  $E^{\phi_m} = E$ , we have  $E'(\mathcal{L}_m) = u_m^{-1} E u_m = u_m^{-1} E^{\phi_m} u_m = E^{\tau^m}$ , which completes the proof of (ii). Assertion (iii) follows directly from (ii). Finally, since  $E = E^{\phi_m} = u_m E^{\tau^m} u_m^{-1}$ , the center of  $E$  is  $\tau^m$ -invariant for all  $m = (n+1)j$ , with  $n, j \gg 0$ . Hence it is  $\tau$ -invariant. ■

The proof of Theorem 4.2 is completed by the next lemma, which proves the result for infinite orbits.

LEMMA 4.6. *Let  $R$  be an algebra which satisfies  $(\dagger)$ . For large  $n$ , the  $\mathcal{O}_X$ -lattice  $\mathcal{L}_n = \mathcal{O}_X \bar{R}_n$  is a normal lattice locally on any infinite  $\sigma$ -orbit.*

*Proof.* Fix  $i \gg 0$ . As before, there is a finite set  $Z \subset X$ , containing the fixed locus  $F$  of  $\sigma$ , so that  $\mathcal{L}_i$  is a normal lattice at all points  $p \notin Z$ . Let  $T$  be the support of the gap divisor  $\Omega$ , as defined in Definition 2.17. By Lemma 1.3, the divisors of  $\bar{R}_n$  and  $\mathcal{L}_n$  are equal. Thus, for  $j \geq 1$ , Corollary 2.18(i) implies that

$$\mathcal{P}(\mathcal{L}_i; p) + \mathcal{P}(\mathcal{L}_j^{\tau^i}; p) = \mathcal{P}(\mathcal{L}_{i+j}; p) \quad \text{for all } p \notin \sigma^{-i}T \cup Z.$$

Proposition 1.14(iii) shows that  $\mathcal{L}_i \mathcal{L}_j^{\tau^i}$  is a normal lattice for all such  $p$ . Thus, by (4.1) and Proposition 1.14(iv),  $\mathcal{L}_i \mathcal{L}_j^{\tau^i} = \mathcal{L}_{i+j}$  and  $\mathcal{L}_{i+j}$  is a normal

lattice at  $p \in X \setminus (\sigma^{-i}T \cup Z)$ . Applying the same reasoning with left and right interchanged shows that  $\mathcal{L}_j \mathcal{L}_i^{\tau^j} = \mathcal{L}_{i+j}$  and that  $\mathcal{L}_{i+j}$  is a normal lattice at  $p$  if  $p \notin \sigma^{-j}T \cup \sigma^{-j}Z$ . Since  $(\sigma^{-j}Z \cup \sigma^{-j}T) \cap (Z \cup \sigma^{-i}T) = F$  for  $j \gg 0$ , the lemma follows. ■

We note one consequence of this proof, combined with Lemma 4.3.

**COROLLARY 4.7.** *Let  $R$  be an algebra which satisfies  $(\dagger)$ , and let  $T$  denote the support of the gap divisor  $\Omega$ . Then, for  $i, j \gg 0$ , we have  $\mathcal{L}_i \mathcal{L}_j^{\tau^i} = \mathcal{L}_{i+j}$  at all points  $p \in X \setminus \sigma^{-i}T$ . ■*

**PROPOSITION 4.8.** *Let  $R$  be an algebra which satisfies  $(\dagger)$ . For  $n \gg 0$ , the order  $E_n = E(\mathcal{L}_n)$  is independent of  $n$ .*

*Proof.* Since  $E_n$  is a subring of  $A$ , it is a local problem to show independence of  $n$ . By Theorem 4.2,  $\mathcal{L}_n$  is invertible at all points, and is a normal lattice at all points not on  $F$ . Moreover, Lemma 4.3 shows that  $\mathcal{L}_n$ , and hence  $E_n$  is independent of  $n \gg 0$  at  $p \in F$ .

Now suppose that  $p$  belongs to an infinite orbit of  $\sigma$ . For  $i, j \gg 0$ , Proposition 1.14(iii) and Corollary 4.7 show that  $E_{i+j} = E_i$  except on  $\sigma^{-i}T$ . Since  $i$  is arbitrary,  $E_n$  is independent of  $n$  everywhere. ■

## 5. THE STABLE GEOMETRIC MODEL

Throughout this section we assume that  $R$  is an order in  $Q = A[z, z^{-1}; \tau]$  which satisfies  $(\dagger)$ , and we take  $z \in R_1$ . We retain the assumption that  $k$  is infinite. Let  $n$  be a large integer. If  $u \in \bar{R}_n$  is a regular element, let  $\Lambda_u = k\langle \bar{R}_n u^{-1} \rangle$  denote the subring of  $A$  generated by  $\bar{R}_n u^{-1}$  over  $k$ . Let  $C_u = Z(\Lambda_u)$  be the center of  $\Lambda_u$ , let  $\tilde{C}_u$  be the integral closure of  $C_u$  in  $K$ , and let  $\tilde{\Lambda}_u = \tilde{C}_u \langle \Lambda_u \rangle$ . Since  $K$  is a product of function fields,  $\text{GK-dim}(\Lambda_u) = 1$ , and so  $\Lambda_u$  is a finitely generated  $C_u$ -module, and  $C_u$  is a finitely generated  $k$ -algebra [SSW]. These objects (and related ones defined later in the section) also depend upon the integer  $n$ , but we have suppressed that dependence for notational simplicity.

The aim of this section is to show that the  $\Lambda_u$  patch together to define a sheaf of algebras  $\mathcal{E}$  over a complete curve  $Y$  that is birational to  $X$ , and such that  $\mathcal{E}$  and  $Y$  are independent of  $n \gg 0$ . The main results of this paper will be defined in terms of  $\mathcal{E}$  and  $Y$ . As already observed in Remark 1.12, the existence of  $\mathcal{E}$  and  $Y$  is not automatic: It depends on the condition  $(\dagger)$ .

**LEMMA 5.1.** *For  $n \gg 0$ , the total ring of fractions of  $C_u$  is  $K$ , and  $\Lambda_u$  is a  $C_u$ -order in  $R$ . ■*

Assume that  $n$  is chosen so that this is so. By Theorem 4.2 and the assumption that  $R$  satisfies  $(\dagger)$ , we may also assume that  $\mathcal{L}_n = \mathcal{O}_X \bar{R}_n$  is an invertible lattice. Set  $X_u = \text{Spec}(\tilde{C}_u)$  and  $Y_u = \text{Spec}(C_u)$ . Thus each  $X_u$  is an open subset of the smooth projective model  $X$  for  $K$ , and  $\tilde{\Lambda}_u$  is an order on  $X_u$ .

LEMMA 5.2. *Let  $P$  be a finite subset of  $X$ . Then for  $n \gg 0$ ,*

- (i)  $X_u$  is the set of points at which  $u \in \bar{R}_n$  has maximal pole;
- (ii)  $\tilde{\Lambda}_u = E(\mathcal{L}_n) = \mathcal{L}_n u^{-1}$  on  $X_u$ ;
- (iii) there is an element of  $\bar{R}_n$  which has maximal pole at every point of  $P$ .

(iv) *Let  $\mathcal{S}$  denote the set of regular elements  $u \in \bar{R}_n$  that have a maximal pole at all points  $p \in P$ . Then the open sets  $\{X_u : u \in \mathcal{S}\}$  cover  $X$ .*

*Proof.* (i) Since  $\bar{R}_n u^{-1}$  generates an order over  $\tilde{C}_u$ , it generates an order over  $X_u$ . This implies that  $u$  has maximal pole at the points of  $X_u$ . Conversely, suppose that  $u$  has maximal pole at  $p \in X$ , and let  $S = \mathcal{O}_{X,p}$ . Then since  $\mathcal{L}_n$  is invertible,  $\mathcal{L}_n u^{-1} = E(\mathcal{L}_n)$  locally at  $p$  (see Lemma 1.11). Since  $\bar{R}_n u^{-1}$  generates  $\mathcal{L}_n u^{-1}$  as an  $S$ -lattice,  $\tilde{\Lambda}_u \subset E(\mathcal{L}_n)_p$  and  $\tilde{C}_u \subset S$ . Therefore  $p \in X_u$ .

Assertion (ii) follows from the fact that both  $\tilde{\Lambda}_u$  and  $E(\mathcal{L}_n) = \mathcal{L}_n u^{-1}$  are generated over  $\tilde{C}_u$  by  $\bar{R}_n u^{-1}$ . Assertion (iii) is true because the condition that  $v \in V$  have maximal pole at  $p$  is open in  $V$  and  $k$  is infinite. Part (iv) follows immediately from (i) and (iii). ■

The rings  $\Lambda_u, \Lambda_v$  determined by two regular elements  $u, v \in \bar{R}_n$  are related by adjunction of inverses. The ring generated by  $\bar{R}_n u^{-1} \cup \bar{R}_n v^{-1}$  is

$$\Lambda_{u,v} = \Lambda_u \langle \alpha^{-1} \rangle = \Lambda_v \langle \alpha \rangle = \Lambda_{v,u}, \quad (5.3)$$

where  $\alpha = vu^{-1}$ . However, these inverse adjunctions need not be localizations, so some care is needed in analyzing them.

PROPOSITION 5.4. *Let  $n \gg 0$ . Then there exists a finite set  $Z = Z(n) \subset X$  such that, for all regular elements  $u \in \bar{R}_n$ , the map  $\pi_u: X_u \rightarrow Y_u$  is an isomorphism except on  $Z_u := X_u \cap Z$ .*

*Proof.* Choose a particular regular element  $u \in \bar{R}_n$ , and let  $Z$  denote the union of the following finite sets:

- (a) the points of  $X$  which are not in  $X_u$ ,
- (b) the points  $p \in X_u$  at which  $\pi_u$  is not an isomorphism,
- (c) the points  $p \in X_u$  such that the PI degree of  $\tilde{\Lambda}_u \otimes k(p)$  is less than the generic PI degree on the component of  $X_u$  which contains  $p$ .



To see that this set  $Z$  does the job, let  $v$  be another regular element of  $\bar{R}_n$ , and let  $p \in X_v - Z$ . Then  $p \in X_u$  and the map  $X_u \rightarrow Y_u$  is an isomorphism locally at  $p$ . Moreover, condition (c) shows that  $\tilde{\Lambda}_u \cong \Lambda_u$  is Azumaya, locally at  $p$ . Thus, locally at  $p$ , Lemma 1.4 implies that  $\Lambda_{u,v} = \Lambda_u[N(\alpha)^{-1}]$  is a central localization of  $\Lambda_u$ . Since  $p \in X_u \cap X_v$ ,  $N(\alpha)$  is a unit in  $(C_u)_p$ , and so  $\Lambda_{u,v} = \Lambda_u$ , locally at  $p$ . The following lemma shows that  $\Lambda_v$  is also Azumaya at  $p$  and so the same argument shows that the map  $\Lambda_v \rightarrow \Lambda_{u,v}$  is also a local isomorphism. Thus,  $X_v \rightarrow Y_v$  is an isomorphism at  $p$ . ■

**LEMMA 5.5.** *Let  $\Lambda \subseteq \Lambda'$  be Goldie  $k$ -algebras such that  $\Lambda' = \Lambda\langle\beta^{-1}\rangle$  for some regular element  $\beta \in \Lambda$ . Let  $\mathfrak{p}' \in \text{Spec } \Lambda'$  be a maximal ideal such that  $\Lambda'/\mathfrak{p}'$  is a finite-dimensional  $k$ -module. Then  $\mathfrak{p} = \Lambda \cap \mathfrak{p}'$  is a maximal ideal of  $\Lambda$ , and  $\Lambda/\mathfrak{p} \cong \Lambda'/\mathfrak{p}'$ .*

*Proof.* Since  $\beta$  is invertible in  $\Lambda'$ , its residue  $\bar{\beta} \in \Lambda/\mathfrak{p}$  is invertible in the finite-dimensional  $k$ -algebra  $\Lambda'/\mathfrak{p}'$ . So  $\bar{\beta}$  is regular, hence invertible, in  $\Lambda/\mathfrak{p}$  and  $\Lambda'/\mathfrak{p}' = (\Lambda/\mathfrak{p})\langle\bar{\beta}^{-1}\rangle = \Lambda/\mathfrak{p}$ . ■

**NOTATION 5.6.** For  $n \gg 0$ , let  $Z = Z(n)$  denote the smallest subset of  $X$  such that the maps  $\pi_u: X_u \rightarrow Y_u$  are isomorphisms except on  $Z \cap X_u$ . (We will see below that  $Z$  is contained in the fixed locus of  $\sigma$  on  $X$ .) Let  $\mathcal{S} = \mathcal{S}(n)$  denote the set of regular elements  $u \in \bar{R}_n$  which have maximal pole at all points  $p \in Z$ .

**PROPOSITION 5.7.** *Fix  $u, v \in \mathcal{S}$ . Let  $\alpha = vu^{-1}$ , and write  $\Lambda_{u,v} = \Lambda_u\langle\alpha^{-1}\rangle = \Lambda_v\langle\alpha\rangle$ , as before. Let  $C_{u,v}$  be the center of  $\Lambda_{u,v}$ . Then for  $n \gg 0$ ,*

- (i) *the map  $C_u \rightarrow C_{u,v}$  is a flat epimorphism, and  $\Lambda_{u,v} \cong C_{u,v} \otimes_{C_u} \Lambda_u$ ;*
- (ii) *the canonical map  $C_u \otimes_k C_v \rightarrow C_{u,v}$  is surjective.*

*Proof.* (i) The analogous assertion is true when  $\Lambda_u$  is replaced by  $\tilde{\Lambda}_u$  and  $C_u$  by  $\tilde{C}_u$ . This is because  $\tilde{C}_u$  is normal and so Lemma 1.4 implies, locally and hence globally, that  $\alpha^{-1}$  and  $N(\alpha)^{-1}$  generate the same extension of  $\tilde{\Lambda}_u$ . So  $\tilde{\Lambda}_u \rightarrow \tilde{\Lambda}_{u,v}$  is a central localization.

It suffices to prove part (i) locally on  $Y_u$ . Let  $q \in Y_u$ , and let  $\tilde{q} = \pi_u^{-1}(q)$ . By the last paragraph, the only case in question is when  $\pi_u$  is not an isomorphism locally at  $q$ ; that is, when  $\tilde{q} \subset Z$ . Let  $(\tilde{C}_u)_{\tilde{q}}$  denote the semilocal ring of  $\tilde{C}_u$  at the finite set  $\tilde{q}$ . The choices of  $u$  and  $v$  ensure that  $N(\alpha)$  is a unit in the semilocal ring  $(\tilde{C}_u)_{\tilde{q}}$ . Therefore  $\alpha^{-1} \in (\tilde{\Lambda}_u)_{\tilde{q}}$  and  $\alpha^{-1}$  is integral over  $(C_u)_q$ . Since  $\alpha \in (\Lambda_u)_q$ , this implies that  $\alpha^{-1} \in (\Lambda_u)_q$ . So the extension is locally trivial at such a point.

(ii) Let  $C'$  denote the image of  $C_u \otimes C_v$  in  $C_{u,v}$ . It suffices to prove that  $C' = C_{u,v}$  locally at each point  $q \in Y_u$ . Here, the only nontrivial case is when the map  $C_u \rightarrow C_{u,v}$  is not an isomorphism, locally at  $q$ . By the last

paragraph, this implies that  $\pi_u$  is a local isomorphism at  $q$ . In this case, the first paragraph implies that  $(C_{u,v})_q = (C_u)_q[N(\alpha)^{-1}]$ . However, since  $\alpha^{-1} \in \Lambda_v$ ,  $N(\alpha)^{-1} \in \tilde{C}_v$  and so  $N(\alpha)^{-1}$  is at least integral over  $C'$ . Since  $N(\alpha) \in C'$ , this forces  $N(\alpha)^{-1} \in C'$ . ■

**COROLLARY-DEFINITION 5.8.** (i) *For  $n \gg 0$ , the affine schemes  $Y_u = \text{Spec } C_u$  with  $u \in \mathcal{S}$  glue together to define a complete curve  $Y = Y^n$ , and the maps  $\pi_u$  glue together to define a finite, birational map  $\pi: X \rightarrow Y$  which is an isomorphism except on  $Z$ .*

(ii) *Setting  $\mathcal{E}(Y_u) = \Lambda_u$  defines a coherent sheaf of algebras  $\mathcal{E} = \mathcal{E}^n$  on  $Y$ .*

(iii) *Both  $Y$  and  $\mathcal{E}$  are defined independently of the choice of the regular element  $z \in R_1$ .*

*Proof.* The gluing data for  $Y$  is defined by the flat epimorphisms  $C_u \rightarrow C_{u,v}$ , which induce open immersions  $Y_{u,v} \rightarrow Y_u$ . The first two assertions follow by standard arguments from Proposition 5.7 and Lemmas 5.1 and 5.2(iv). To prove (iii), let  $w$  be another regular element in  $A_1$ . Then, as in Lemma 2.13,  $R = \bigoplus \bar{B}_i w^i$ , where  $\bar{R}_i = \bar{B}_i \alpha_i$ , for some unit  $\alpha_i \in A$ . It follows that, for any regular element  $u \in \bar{R}_i$ , there exists  $v \in \bar{B}_i$  such that  $\bar{R}_i u^{-1} = \bar{B}_i v^{-1}$ . In other words, the  $\Lambda_u$  do not depend upon  $z$ . ■

Our next step is to show that the curves  $Y^n$  and orders  $\mathcal{E}^n$  are independent of  $n \gg 0$ .

**THEOREM 5.9.** *Let  $R$  be a two-dimensional algebra which satisfies  $(\dagger)$ , and let  $\mathcal{R}_n = \mathcal{O}_Y \bar{R}_n = \bar{R}_n \mathcal{O}_Y$ . For  $n \gg 0$ ,*

(i)  *$Y = Y^n$  and  $\mathcal{E} = \mathcal{E}^n$  are independent of  $n$  and, up to canonical isomorphism,  $Y$  is determined by  $R$ ;*

(ii)  *$\mathcal{R}_n$  is an invertible  $\mathcal{O}_Y$ -lattice, and  $E(\mathcal{R}_n) = \mathcal{E}$ ;*

(iii) *the map  $\pi^n: X \rightarrow Y$  is an isomorphism except at the fixed points of  $\sigma$ ;*

(iv) *on every infinite orbit of  $\sigma$ ,  $\mathcal{E}$  is a maximal order and so, on such an orbit,  $\mathcal{R}_n$  is a normal lattice;*

(v) *the automorphism  $\sigma$  of  $K$  induces an automorphism of the curve  $Y$ .*

*Proof.* Let  $\tilde{\mathcal{E}}^n = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}$ . By Lemma 5.2,  $\tilde{\mathcal{E}}^n = E(\mathcal{L}_n)$  is generated by  $\bar{R}_n u_n^{-1}$  and, by Proposition 4.8,  $\tilde{\mathcal{E}}^n$  is independent of  $n \gg 0$ . Also,  $\mathcal{L}_n = \mathcal{O}_X \bar{R}_n$  is an invertible  $\mathcal{O}_X$ -lattice by Theorem 4.2. Let  $T$  denote the support of the gap divisor on  $X$ , as defined in (2.17), and let  $Z_i \subset X$  denote the union of the fixed locus of  $\sigma$  and of the set of points at which  $\pi^i: X \rightarrow Y^i$  is not an isomorphism. Suppose that  $p \notin \sigma^{-i} T \cup Z_i$ . We work locally at  $p$  and use the reasoning of Sect. 4, with  $i, j \gg 0$  and  $n = i + j$ . Thus, by

Corollary 4.7,  $\mathcal{L}_i \mathcal{L}_j^{\tau^i} = \mathcal{L}_{i+j}$ . Choose  $u_i \in \bar{R}_i$  and  $u_j \in \bar{R}_j$  with maximal poles at  $p$ . The definition of the gap divisor shows that  $u_n = u_i u_j^{\tau^i} \in \bar{R}_n$  has maximal pole at  $p$ . With  $\phi = \phi_i$  defined as in Lemma 4.3, we have

$$(\bar{R}_i u_i^{-1})(\bar{R}_j u_j^{-1})^\phi \subseteq \bar{R}_n u_n^{-1} \quad (5.10)$$

and

$$\bar{R}_i u_i^{-1} \subseteq \bar{R}_n u_n^{-1}. \quad (5.11)$$

Hence  $\mathcal{E}^i \subseteq \mathcal{E}^n$ . Also, because  $p \notin Z_i$ ,  $\mathcal{E}^i \cong E(\mathcal{L}_i)$ , and we know that  $E(\mathcal{L}_i)$  is a maximal order independent of  $i$ , by Theorem 4.2. Hence  $\mathcal{E}^i$  is maximal, and  $\mathcal{E}^i = \mathcal{E}^n$ . Since  $\mathcal{O}_Y$  is defined to be the center of  $\mathcal{E}^n$ ,  $X \approx Y^n$  for all  $p \notin \sigma^{-i}T \cup Z_i$ .

Similarly, if  $p \notin \sigma^{-i}T \cup \sigma^{-i}Z_j$ , then (5.10) shows that  $(\bar{R}_j u_j^{-1})^\phi \subseteq \bar{R}_n u_n^{-1}$ . Hence  $(\mathcal{E}^j)^\phi \subseteq \mathcal{E}^n$ , and since  $(\mathcal{E}^j)^\phi$  is a maximal order, the two orders are equal. We conclude that  $X \approx Y^n$  at such points. Interchanging the roles of  $i, j$  shows that  $X \approx Y^n$  if  $p \notin \sigma^{-j}T \cup \sigma^{-j}Z_i$ . If  $j \gg i$ , the sets  $\sigma^{-i}T \cup Z_i$  and  $\sigma^{-j}T \cup \sigma^{-j}Z_i$  have only fixed points of  $\sigma$  in common. Therefore  $X \approx Y^n$  at all points which are not fixed points. Thus, at the infinite orbits of  $\sigma$ , the theorem follows from Theorem 4.2 and Proposition 4.8.

Fix  $i \gg 0$ , let  $p \in X$  be a fixed point of  $\sigma$  with image  $q$  in  $Y^i$ , and write  $S^i$  for the finite set of points  $p' \in X$  such that  $\pi^i(p') = q$ . By the last paragraph,  $S^i$  consists entirely of fixed points. Let  $n = i + j$  with  $i, j \gg 0$ , and let  $S = S^i \cup S^j \cup S^n$ . Choosing  $u_i \in \bar{R}_i$  and  $u_j \in \bar{R}_j$  appropriately, we may assume that they have maximal pole at every point of  $S$ . By  $(\dagger)$ ,  $u_n = u_i u_j^{\tau^i}$  has maximal pole too. Then (5.11) holds and so  $\mathcal{E}^i \subseteq \mathcal{E}^n$ . Hence  $Y^n$  dominates  $Y^i$ , above  $S$ . This shows that  $S^n \subseteq S^i$ . Assertion 5.9(i) at the fixed points follows from the ascending chain condition.

Let  $Y = Y^i$  and  $\mathcal{E} = \mathcal{E}^i$  denote the stable curve and order at  $p$ , and let  $S = S^i$ . We work in the semilocal ring  $\tilde{\mathcal{O}} = \mathcal{O}_{X,S}$  of  $X$  at  $S$ , with Jacobson radical  $\mathfrak{m}$ . For large  $N$ ,  $\mathcal{O} = k + \mathfrak{m}^N$  is a  $\sigma$ -invariant local ring contained in  $\mathcal{O}_{Y,q}$  and which has  $\mathcal{O}$  as its normalization. Note that the localization  $\mathcal{E}_q$  of  $\mathcal{E}$  at  $q$  is the finitely generated  $\mathcal{O}$ -module generated by  $k\langle \bar{R}_n u_n^{-1} \rangle$  and  $\mathcal{O}$ . On the other hand, we may apply Lemma 4.3 and we change notation to be in accordance with that result; in particular, we now define  $\mathcal{L}_n = \mathcal{O} \bar{R}_n$ , which satisfies the hypotheses of Lemma 4.3. Then,  $\mathcal{L}_n$  is invertible and so, by Lemma 1.11,  $E(\mathcal{L}_n) = \mathcal{L}_n u_n^{-1} = \mathcal{O} \bar{R}_n u_n^{-1}$  is an  $\mathcal{O}$ -order. Hence  $E(\mathcal{L}_n) = \mathcal{E}_q$ . The assertions of the theorem at fixed points of  $\sigma$  now follow from Lemma 4.3. ■

**DEFINITION 5.12.** Let  $S$  be a two-dimensional algebra such that some Veronese  $R = S^{(n)}$ , for  $n \gg 0$ , satisfies  $(\dagger)$ . The curve  $Y$  and order  $\mathcal{E}$  described by Theorem 5.9 using the nice algebra  $R$  will be called the *stable*

*model* and *stable order* for  $S$  (and  $R$ ). By that theorem, they are independent of  $n$ .

Let us illustrate these constructions on two examples.

EXAMPLE 5.13 (Continuation of Example 2.7). Keep the notation of Example 2.7. From the description of  $\mathcal{P}(\overline{R}_n; p)$  in that example, it is clear that we need to use three choices of  $u$  to cover  $X = \mathbb{P}^1$ ; specifically we use the diagonal matrices  $u = \text{diag}\{w \cdots w^{\sigma^i}, w \cdots w^{\sigma^i}\}$  in the neighborhood of the point  $p_i$  for  $0 \leq i \leq n-2$ , but  $u = \text{diag}\{w \cdots w^{\sigma^{n-1}}, 1\}$  if  $p = p_{n-1}$  and  $u = \text{diag}\{1, 1\}$  otherwise. In doing so, one finds that  $E(\mathcal{L}_n)$  is the full  $2 \times 2$  matrix ring over the appropriate subring of  $k(w)$  and so  $\mathcal{E} = M_2(\mathcal{O}_{\mathbb{P}^1})$ . Thus,  $Y^n = X = \mathbb{P}^1$  and this is independent of  $n \geq 1$ . The sheaf  $\mathcal{R}_n$  is defined locally by

$$(\mathcal{R}_n)_p = \begin{cases} M_2(\mathcal{O}_{X,p}) \mathfrak{m}_p^{-1} & \text{if } p = p_i \text{ where } 0 \leq i \leq n-2 \\ \begin{pmatrix} \mathfrak{m}_p^{-1} & \mathcal{O}_{X,p} \\ \mathfrak{m}_p^{-1} & \mathcal{O}_{X,p} \end{pmatrix} & \text{if } p = p_{n-1} \\ M_2(\mathcal{O}_{X,p}) & \text{otherwise,} \end{cases}$$

where  $\mathfrak{m}_p$  denotes the maximal ideal of  $\mathcal{O}_{X,p}$ . One should note that  $\mathcal{R}_n$  is not a (twisted or untwisted) bimodule over  $\mathcal{E}$ . Instead,  $E'(\mathcal{R}_n) = \mathcal{E}'^{\tau^n}$ , where  $\mathcal{E}'$  is defined locally by  $\mathcal{E}'_p \cong M_2(\mathcal{O}_{X,p})$  for all points  $p$ , except  $p = p_{-1}$ , where

$$\mathcal{E}'_p = \begin{pmatrix} \mathcal{O}_{X,p} & \mathfrak{m}_p^{-1} \\ \mathfrak{m}_p & \mathcal{O}_{X,p} \end{pmatrix}.$$

EXAMPLE 5.14. One can repeat these computations for the ring defined in (0.2); that is for the algebra  $R = (k + yM_2(k[x, y]))\langle \hat{\alpha} \rangle$ , where  $\hat{\alpha} = \begin{pmatrix} x & x \\ 0 & x \end{pmatrix}$ . If one takes  $z = \hat{\alpha}$  and  $w = yx^{-1}$ , then  $Q(R) = M_2(k(w))[z, z^{-1}; \tau]$  where  $\tau$  denotes conjugation by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Thus,  $X = \mathbb{P}^1$ . In this case,

$$\mathcal{P}(\overline{R}_n; p) = \begin{cases} 2n & \text{for } p = p_0 \text{ (the pole of } w) \\ 0 & \text{otherwise.} \end{cases}$$

A simple computation shows that  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} + M_2(\mathcal{O}(1))$ , as in Example (0.2). In this case,  $\mathcal{R}_n$  is the left  $\mathcal{E}$ -module generated by 1,  $w^n$  while  $E'(\mathcal{R}_n) = \mathcal{E}^{\tau^n}$ . The details are left to the reader.

The next result, the proof of which will take up essentially the remainder of this section, defines the geometric data that is determined by a nice algebra  $R$ . We recall from Section 1 that  $\bullet$  denotes the tensor product of lattices, and that the inverse of an invertible lattice  $\mathcal{R}$  is denoted by  $\mathcal{R}^*$ .

**THEOREM 5.15.** *Let  $R$  be a two-dimensional algebra which satisfies  $(\dagger)$ , and let  $Y$  be the associated stable model. There is a unique set of data consisting of invertible lattices  $\mathcal{S}_n$  defined for all  $n \geq 1$ , and an invertible “gap lattice”  $\mathcal{G}$  such that*

- (i) *for  $n \gg 0$ ,  $\mathcal{S}_n = \mathcal{R}_n$ , where  $\mathcal{R}_n = \mathcal{O}_Y \bar{R}_n$ ;*
- (ii)  *$E(\mathcal{S}_1) = E'(\mathcal{G})$ ; this algebra is equal to the ring  $\mathcal{E}$  of Corollary 5.8;*
- (iii)  *$E(\mathcal{G}) = E'(\mathcal{S}_1)^{\tau^{-1}}$ , which will be denoted by  $\mathcal{E}'$ ;*
- (iv)  *$\mathcal{G} = \mathcal{E} = \mathcal{E}'$  locally at the fixed points of  $\sigma$ ;*
- (v) *for all  $i, j \geq 1$ , one has  $\mathcal{S}_i \cdot \mathcal{G}^i \cdot \mathcal{S}_j^{\tau^i} = \mathcal{S}_{i+j}$ .*

*Proof.* To begin the proof, we work with indices  $i, j, n \gg 0$ , since this enables us to analyze the  $\mathcal{R}_n$  in detail before defining the  $\mathcal{S}_m$  for small  $m$ . Theorem 5.9 tells us that  $\mathcal{R}_n = \mathcal{O}_Y R_n$  is invertible, and that  $\mathcal{E} = E(\mathcal{R}_n)$  is independent of  $n \gg 0$ . Moreover, (2.1) implies that  $\mathcal{R}_i \mathcal{R}_j^{\tau^i} \subseteq \mathcal{R}_{i+j}$  for  $i, j \gg 0$ . Since  $R$  satisfies  $(\dagger)$ , so does  $R^{\text{op}}$ , by Lemma 2.13(ii). Switching left and right, we may therefore define an order  $\mathcal{E}'$  analogous to  $\mathcal{E}$  by writing  $\bar{R} = \oplus z^n \bar{R}'_n$  and using the rings  $\Lambda'_u = k\langle u'^{-1} \bar{R}'_n \rangle$ . The earlier results of this paper, applied to the opposite ring  $R^{\text{op}}$ , imply that all the results phrased in terms of  $\bar{R}_n$  have left-right analogues, defined in terms of  $\bar{R}'_n$ . We will denote the corresponding objects with a “dash.” In particular, we obtain a sheaf of orders  $\mathcal{E}'$  defined over a complete curve  $Y'$ .

**LEMMA 5.16.** (i) *With the above notation,  $Y' = Y$ .*

(ii) *For  $n \gg 0$ ,  $E'(\mathcal{R}_n) = \mathcal{E}'^{\tau^n}$ .*

*Proof.* (i) The relation  $z^n \bar{R}'_n = \bar{R}_n z^n$  implies that  $\bar{R}'_n{}^{\tau^n} = \bar{R}_n$  for all  $n \geq 1$ . If  $u' \in \bar{R}'_n$  has a maximal pole at a point  $p \in X$ , then  $u = u'^{\tau^n} \in \bar{R}_n$  has maximal pole at  $\sigma^{-n}(p)$ . Hence,

$$\Lambda_u = k\langle \bar{R}_n u^{-1} \rangle = k\langle (\bar{R}'_n u'^{-1})^{\tau^n} \rangle = [(\Lambda'_{u'})^{\tau^n}]^{\text{op}}$$

and the respective centers  $C_u$  and  $C'_{u'}$  are related by  $C_u = \sigma^n(C'_{u'})$ . Since  $Y$  is  $\sigma$ -invariant, by Theorem 5.9, this implies that  $Y' = Y$ .

(ii) If  $\mathcal{R}'_n = \mathcal{O}_Y \bar{R}'_n$ , then Theorem 5.9 implies that  $E'(\mathcal{R}'_n) = \mathcal{E}'$ . By part (i),  $(\mathcal{R}'_n)^{\tau^n} = \mathcal{R}_n$ . So, in the notation of part (i),  $\mathcal{E}'^{\tau^n} = (u'^{-1} \mathcal{R}'_n)^{\tau^n} = u^{-1} \mathcal{R}_n = u^{-1} \mathcal{E} u = E'(\mathcal{R}_n)$ , locally at  $p$ . ■

In general,  $\mathcal{E}$  is not  $\tau$ -invariant and  $\mathcal{E} \neq \mathcal{E}'$ , as was shown in Example 5.13. Indeed, as will be shown in Corollary 5.24,  $\mathcal{E} \neq \mathcal{E}'$  whenever  $(\dagger\dagger)$  fails.

**LEMMA 5.17.** *Let  $T = \text{supp}(\Omega)$  denote the support in  $X$  of the gap divisor  $\Omega$ , as defined in Definition 2.17. Pick  $i, j \gg 0$ . Then,  $\mathcal{R}_i \cdot \mathcal{R}_j^{\tau^i} = \mathcal{R}_{i+j}$  except on  $\sigma^{-i}T$ .*

*Proof.* At the fixed points, the lemma follows from Lemma 4.3(iii). At the other points, we have the inclusion  $\mathcal{R}_i \mathcal{R}_j^{\tau^i} \subseteq \mathcal{R}_{i+j}$ , and both sides are normal lattices. Since  $D_i + \sigma^{-i} D_j + \sigma^{-i} \Omega = D_{i+j}$ , Corollary 4.7 and Proposition 1.14(iv) show that the inclusion is an equality at all points not in  $\sigma^{-i} T$ . It is a tensor product by Proposition 1.14(v). ■

Let  $n_0$  be an integer so that  $\mathcal{R}_n$  is invertible for  $n \geq n_0$ , and that the last two lemmas hold for  $n, i, j \geq n_0$ . Pick  $i, j \geq n_0$ . We define a *gap lattice*  $\mathcal{G}_{i,j}$  by the formula

$$\mathcal{G}_{i,j} = (\mathcal{R}_i^{\tau^{-i}})^* \cdot \mathcal{R}_{i+j}^{\tau^{-i}} \cdot \mathcal{R}_j^*. \quad (5.18)$$

It follows from Theorem 5.9(ii) and Lemma 5.16(ii) that the products appearing in this formula are, indeed, tensor products. By Proposition 1.14,

$$E(\mathcal{G}_{i,j}) = \mathcal{E}' \quad \text{and} \quad E'(\mathcal{G}_{i,j}) = \mathcal{E}. \quad (5.19)$$

Bringing terms to the other side and shifting by  $\tau^i$  in (5.18) gives us the desired tensor product formula:

$$\mathcal{R}_i \cdot \mathcal{G}_{i,j}^{\tau^i} \cdot \mathcal{R}_j^{\tau^i} = \mathcal{R}_{i+j}. \quad (5.20)$$

LEMMA 5.21. (i)  $\mathcal{E}' = \mathcal{G}_{i,j} = \mathcal{E}$  at all points not in  $T = \text{supp}(\Omega)$ .

(ii) The gap lattice  $\mathcal{G}_{i,j}$  is independent of  $i, j \gg 0$ .

*Proof.* (i) This is immediate from Lemma 5.17.

(ii) We compute  $\mathcal{R}_{i+j+k}$  in two ways:

$$(\mathcal{R}_i \cdot \mathcal{G}_{i,j}^{\tau^i} \cdot \mathcal{R}_j^{\tau^i}) \cdot \mathcal{G}_{i+j,k}^{\tau^{i+j}} \cdot \mathcal{R}_k^{\tau^{i+j}} = \mathcal{R}_{i+j+k} = \mathcal{R}_i \cdot \mathcal{G}_{i,j+k}^{\tau^i} \cdot (\mathcal{R}_j \cdot \mathcal{G}_{j,k}^{\tau^j} \cdot \mathcal{R}_k^{\tau^j})^{\tau^i}.$$

Cancelling  $\mathcal{R}_i$  and  $\mathcal{R}_k^{\tau^{i+j}}$  on left and right and shifting by  $\tau^{-i}$  gives the tensor product formula

$$\mathcal{G}_{i,j} \cdot \mathcal{R}_j \cdot \mathcal{G}_{i+j,k}^{\tau^j} = \mathcal{G}_{i,j+k} \cdot \mathcal{R}_j \cdot \mathcal{G}_{j,k}^{\tau^j}.$$

Because  $T \cap \sigma^{-j} T = \emptyset$  if  $j \gg 0$ , we conclude from part (i) and the next lemma that  $\mathcal{G}_{i,j} = \mathcal{G}_{i,j+k}$  and that  $\mathcal{G}_{i+j,k} = \mathcal{G}_{j,k}$  if  $j \gg 0$ . The assertion follows. ■

LEMMA 5.22. Let  $\mathcal{L}_i, \mathcal{M}_i$  be invertible lattices such that  $\mathcal{L}_i \mathcal{M}_i$  is a tensor product for  $i = 1, 2$ , and that  $\mathcal{L}_1 \cdot \mathcal{M}_1 = \mathcal{L}_2 \cdot \mathcal{M}_2$ . Let  $U, V$  be open subsets of  $Y$  such that  $\mathcal{L}_1 = \mathcal{L}_2$  on  $U$  and that  $\mathcal{M}_1 = \mathcal{M}_2$  on  $V$ . If  $U \cup V = Y$ , then  $\mathcal{L}_1 = \mathcal{L}_2$  and  $\mathcal{M}_1 = \mathcal{M}_2$ .

*Proof.* It suffices to show that  $\mathcal{L}_1 = \mathcal{L}_2$  on  $V$ . However, on that set,  $\mathcal{L}_1 = \mathcal{L}_1 \cdot \mathcal{M}_1 \cdot \mathcal{M}_1^* = \mathcal{L}_2 \cdot \mathcal{M}_2 \cdot \mathcal{M}_2^* = \mathcal{L}_2$ . ■

DEFINITION 5.23. Henceforth, we drop the subscripts and define the *gap lattice* to be  $\mathcal{G} = \mathcal{G}_{i,j}$  for any  $i, j \gg 0$ .

*Proof of Theorem 5.15.* We have the orders  $\mathcal{E}$  and  $\mathcal{E}'$ , the lattice  $\mathcal{G}$ , and the lattices  $\mathcal{S}_m = \mathcal{R}_m$  for  $m \gg 0$ . Once we have defined the remaining  $\mathcal{S}_i$  in such a way that part (v) holds, the theorem will follow from (5.16), (5.19), and (5.21).

Thus, it remains to define  $\mathcal{S}_i$  for small  $i$ , and we may try to define it by the tensor product formula  $\mathcal{S}_{i,m} \cdot \mathcal{G}^{\tau^i} \cdot \mathcal{R}_m^{\tau^i} = \mathcal{R}_{i+m}$ , with  $m \gg 0$ . Since  $\mathcal{G}$ ,  $\mathcal{R}_m$  and  $\mathcal{R}_{i+m}$  are invertible, this equation has a unique solution  $\mathcal{S}_{i,m} = \mathcal{R}_{i+m} \cdot (\mathcal{R}_m^{\tau^i})^* \cdot (\mathcal{G}^{\tau^i})^*$  and  $\mathcal{S}_{i,m} = \mathcal{R}_i$  for large  $i$ . We must show that  $\mathcal{S}_{i,m}$  is independent of  $m \gg 0$ . If  $m, n \gg 0$ , then  $\mathcal{R}_m \cdot \mathcal{G}^{\tau^m} \cdot \mathcal{R}_n^{\tau^m} = \mathcal{R}_{m+n}$  by Lemma 5.20. Thus,

$$\begin{aligned} \mathcal{S}_{i,m} \cdot \mathcal{G}^{\tau^i} \cdot \mathcal{R}_{m+n}^{\tau^i} &= \mathcal{S}_{i,m} \cdot \mathcal{G}^{\tau^i} \cdot (\mathcal{R}_m \cdot \mathcal{G}^{\tau^m} \cdot \mathcal{R}_n^{\tau^m})^{\tau^i} = \mathcal{S}_{i,m} \cdot \mathcal{G}^{\tau^i} \cdot \mathcal{R}_m^{\tau^i} \cdot \mathcal{G}^{\tau^{i+m}} \cdot \mathcal{R}_n^{\tau^{i+m}} \\ &= \mathcal{R}_{i+m} \cdot \mathcal{G}^{\tau^{i+m}} \cdot \mathcal{R}_n^{\tau^{i+m}} = \mathcal{R}_{i+m+n}. \end{aligned}$$

Thus  $\mathcal{S}_{i,m} = \mathcal{S}_{i,m+n}$ . Since this holds for all  $m, n \gg 0$ , the independence of  $m$  follows and we may define  $\mathcal{S}_i = \mathcal{S}_{i,m}$ . The verification of the relation  $\mathcal{S}_i \cdot \mathcal{G}^{\tau^i} \cdot \mathcal{S}_j^{\tau^i} = \mathcal{S}_{i+j}$  for small  $j$  is similar and left to the reader. This completes the proof of the theorem. ■

COROLLARY 5.24. *Let  $R$  be a two-dimensional algebra that satisfies  $(\dagger)$ . Then,  $\text{div } \mathcal{G} = \Omega$  by (1.13) and (2.18). Thus  $\mathcal{E} = \mathcal{E}'$  if and only if  $R$  satisfies  $(\dagger\dagger)$ . In this case,  $E'(\mathcal{S}_n) = \mathcal{E}^{\tau^n}$ , and  $\mathcal{S}_i \cdot \mathcal{S}_j^{\tau^i} \cong \mathcal{S}_{i+j}$ . ■*

Using the notation of Theorem 5.15, we define two further objects,

$$\mathcal{B}_n = \mathcal{S}_n \cdot \mathcal{G}^{\tau^n} \quad \text{and} \quad \mathcal{B}'_n = \mathcal{G} \cdot \mathcal{S}_n \text{ for } n \geq 0, \quad (5.25)$$

where we have adopted the convention that  $\mathcal{S}_0 = \mathcal{G}^*$ , so that  $\mathcal{B}_0 = \mathcal{E}$  and  $\mathcal{B}'_0 = \mathcal{E}'$ . Thus  $\mathcal{B}_n$  is an invertible  $(\mathcal{E}, \mathcal{E}^{\tau^n})$ -bimodule, and  $\mathcal{B}'_n$  is an invertible  $(\mathcal{E}', \mathcal{E}'^{\tau^n})$ -bimodule. Then we have the following tensor product formulas, valid for all  $n, i, j > 0$ . We emphasize that they also hold for  $\mathcal{R}_n = \mathcal{S}_n$ , when  $n$  is large:

$$\begin{aligned} \mathcal{B}_i \cdot \mathcal{S}_j^{\tau^i} &= \mathcal{S}_{i+j} = \mathcal{S}_i \cdot \mathcal{B}_j^{\tau^i}, & \mathcal{B}_i \cdot \mathcal{B}_j^{\tau^i} &= \mathcal{B}_{i+j} \\ \mathcal{B}_n &= \mathcal{B}_1 \cdot \mathcal{B}_1^{\tau} \cdots \mathcal{B}_1^{\tau^{n-1}}, & \mathcal{B}'_n &= \mathcal{B}'_1 \cdot \mathcal{B}'_1^{\tau} \cdots \mathcal{B}'_1^{\tau^{n-1}} \\ \mathcal{S}_n &= \mathcal{B}_1 \cdot \mathcal{B}_1^{\tau} \cdots \mathcal{B}_1^{\tau^{n-2}} \cdot \mathcal{S}_1^{\tau^{n-1}}. \end{aligned} \quad (5.26)$$

COROLLARY 5.27. *Let  $R$  be a two-dimensional algebra that satisfies  $(\dagger)$ . Then, for all  $i, j \geq 1$  and  $n \geq 0$  we have*

$$(i) \quad E(\mathcal{S}_i) = \mathcal{E} \text{ and } E'(\mathcal{S}_i) = \mathcal{E}'^{\tau^i},$$

- (ii)  $\mathcal{S}_i \mathcal{S}_j^{\tau^i} = \mathcal{S}_i \cdot (\mathcal{E}'^{\tau^i} \mathcal{E}^{\tau^i}) \cdot \mathcal{S}_j^{\tau^i} \subseteq \mathcal{S}_{i+j},$
- (iii)  $\mathcal{E}' \mathcal{E} \subseteq \mathcal{G}$  and  $1 \in \mathcal{G},$
- (iv)  $\mathcal{S}_n \subseteq \mathcal{B}_n$  and  $\mathcal{S}_n \subseteq \mathcal{B}'_n.$  Similarly,  $\mathcal{R}_n \subseteq \mathcal{B}_n$  and  $\mathcal{R}_n \subseteq \mathcal{B}'_n.$

*Remark.* In general, the products  $\mathcal{E}' \mathcal{E}$  and  $\mathcal{R}_i \mathcal{R}_j^{\tau^i}$  are not tensor products.

*Proof.* By Lemma 5.21, the orders  $\mathcal{E}'^{\tau^i}$  and  $\mathcal{E}^{\tau^i}$  are equal at the fixed points of  $\tau$  and Theorem 5.9(iv) implies that both orders are maximal elsewhere. Thus, Proposition 1.14(i), (iii) implies that  $\mathcal{E}'^{\tau^i} \mathcal{E}^{\tau^i}$  is an invertible  $(\mathcal{E}'^{\tau^i}, \mathcal{E}^{\tau^i})$ -bimodule. Therefore, the  $\cdot$  notation is legitimate in the statement of part (ii).

Part (i) follows from Theorem 5.15(ii), (iii) and the equality in (ii) follows directly from (i). By (5.26), the inclusion of (ii) is true for  $i, j \gg 0$ . Then (iii) follows by applying part (ii) and Theorem 5.15(v) with  $i, j \gg 0$ . In turn, part (iii) and Theorem 5.15(v) imply part (ii) for all  $i, j$ . The first sentence of part (iv) follows from part (iii) combined with (5.25). Finally, the inclusion  $\mathcal{R}_n \subseteq \mathcal{B}_n$  follows from the fact that, by Theorem 5.9(ii) and Theorem 5.15(v),

$$\mathcal{R}_n \subseteq \mathcal{R}_n \mathcal{R}_m^{\tau^n} (\mathcal{R}_m^{\tau^n})^* \subseteq \mathcal{R}_{n+m} (\mathcal{R}_m^{\tau^n})^* = \mathcal{R}_{n+m} \cdot (\mathcal{R}_m^{\tau^n})^* = \mathcal{S}_n \cdot \mathcal{G}^{\tau^n} = \mathcal{B}_n,$$

for any  $m \gg 0$ . ■

We check these results on the examples from (5.13) and (5.14). In the latter case there is nothing further to say, since  $\mathcal{B}_n = \mathcal{R}_n$  (as one should expect for a twisting ring). For the ring from (Example 5.13),  $\mathcal{B}_1$  is the sheaf  $\mathcal{E} + \mathcal{E}u = \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(1)$ . Equivalently,

$$(\mathcal{B}_1)_p = \begin{cases} M_2(\mathcal{O}_{X,p}) \mathfrak{m}_p^{-1} & \text{if } p = p_0 \\ M_2(\mathcal{O}_{X,p}) & \text{otherwise} \end{cases}$$

and so this module satisfies  $\mathcal{B}_1 \otimes_{\mathcal{E}^\tau} \mathcal{R}_{n-1}^\tau = \mathcal{R}_n$ , for all  $n$ .

## 6. TWISTED HOMOGENEOUS COORDINATE RINGS

Once again,  $k$  is assumed to be infinite in this section. As in [VdB], a sequence  $\{\mathcal{L}_n\}$  of coherent sheaves on a projective curve  $Y$  will be called *ample* if for all coherent sheaves  $\mathcal{G}$  and all  $n \gg 0$ , the sheaf  $\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L}_n$  is generated by global sections, and  $H^1(Y, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L}_n) = 0$ . We are primarily interested ample sequences  $\{\mathcal{L}_n\}$  of invertible lattices.

**LEMMA 6.1.** *Let  $\mathcal{E}$  be a coherent  $\mathcal{O}_Y$ -algebra over a projective curve  $Y$ , and let  $\{\mathcal{L}_n\}$  be a sequence of coherent left  $\mathcal{E}$ -modules. The following assertions are equivalent.*



(i) *The sequence  $\{\mathcal{L}_n\}$  is ample.*

(ii) *For all coherent right  $\mathcal{E}$ -modules  $\mathcal{F}$  and all  $n \gg 0$ ,  $\mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n$  is generated by its global sections, and  $H^1(Y, \mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n) = 0$ .*

*Proof.* To derive (i) from (ii), it suffices to note that  $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{L}_n \cong (\mathcal{G} \otimes_{\mathcal{O}} \mathcal{E}) \otimes_{\mathcal{E}} \mathcal{L}_n$  and that  $\mathcal{G} \otimes_{\mathcal{O}} \mathcal{E}$  is a coherent right  $\mathcal{E}$ -module. The other implication results from the surjective  $\mathcal{O}_Y$ -linear map  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}_n \rightarrow \mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n$ . If  $\mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n$  is generated by its sections, so is  $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}_n$ . The vanishing of cohomology also follows because, since  $Y$  is a curve,  $H^1$  is a right exact functor. ■

LEMMA 6.2. *Let  $Y$  be a projective curve.*

(i) *A sequence  $\{\mathcal{L}_n\}$  is ample if  $\mathcal{L}_n$  has finite length for  $n \gg 0$ .*

(ii) *Let  $\mathcal{E}$  be a coherent  $\mathcal{O}_Y$ -algebra, let  $\{\mathcal{L}_n\}$  be an ample sequence of left  $\mathcal{E}$ -modules, and let  $\mathcal{N}$  be coherent right  $\mathcal{E}$ -module. Then the sequence  $\{\mathcal{N} \otimes_{\mathcal{E}} \mathcal{L}_n\}$  is ample.*

(iii) *Let  $\{\mathcal{L}_n \rightarrow \mathcal{M}_n \rightarrow \mathcal{N}_n \rightarrow 0\}$  be a sequence of exact sequences of coherent sheaves on  $Y$ . If  $\{\mathcal{L}_n\}$  and  $\{\mathcal{N}_n\}$  are ample, then  $\{\mathcal{M}_n\}$  is ample.*

(iv) *Let  $\{\mathcal{L}_n\}$  be an ample sequence of coherent sheaves on  $Y$ , and let  $\{\sigma_n\}$  be a sequence of automorphisms of  $Y$ . Then the sequence  $\{\mathcal{L}_n^{\sigma_n}\}$  is ample.*

*Proof.* The first assertion is clear and (ii) follows directly from Lemma 6.1.

(iii) Suppose that  $\{\mathcal{L}_n\}$  and  $\{\mathcal{N}_n\}$  are ample, and let  $\mathcal{G}$  be a coherent sheaf on  $Y$ . Then

$$\mathcal{G} \otimes \mathcal{L}_n \xrightarrow{\phi_n} \mathcal{G} \otimes \mathcal{M}_n \rightarrow \mathcal{G} \otimes \mathcal{N}_n \rightarrow 0$$

is exact for every  $n$ , and as  $H^1$  is right exact  $H^1(\mathcal{G} \otimes \mathcal{M}_n) = 0$  for  $n \gg 0$ . Let  $\mathcal{D}_n$  denote the image of  $\phi_n$ . Since  $\mathcal{G} \otimes \mathcal{L}_n$  is generated by its global sections and has vanishing  $H^1$  for large  $n$ , the same is true for  $\mathcal{D}_n$ . Then, for large  $n$ , the exact sequence

$$0 \rightarrow \mathcal{D}_n \rightarrow \mathcal{G} \otimes \mathcal{M}_n \rightarrow \mathcal{G} \otimes \mathcal{N}_n \rightarrow 0$$

shows that the global sections of  $\mathcal{G} \otimes \mathcal{N}_n$  lift to global sections of  $\mathcal{G} \otimes \mathcal{M}_n$ . This, together with the fact that  $\mathcal{D}_n$  and  $\mathcal{G} \otimes \mathcal{N}_n$  are generated by global sections, shows that  $\mathcal{G} \otimes \mathcal{M}_n$  is also generated by its sections, as required.

(iv) Let  $\mathcal{G}$  be a coherent sheaf on  $Y$ . We must show for  $n \gg 0$ , that  $\mathcal{G} \otimes \mathcal{L}_n^{\sigma_n}$  is generated by its sections, and  $H^1(\mathcal{G} \otimes \mathcal{L}_n^{\sigma_n}) = 0$ . We pull back the tensor product using  $\sigma_n^{-1}$ : It suffices to show that for  $n \gg 0$ ,  $\mathcal{G}^{\sigma_n^{-1}} \otimes \mathcal{L}_n$  is generated by global sections and has vanishing  $H^1$ . We write  $\mathcal{G}$  as quotient of a sum of invertible sheaves  $\mathcal{O}(-Z)$ , where  $Z$  is a divisor supported

on the nonsingular locus of  $Y$ . Since tensor products and  $H^1$  are right exact, it suffices to prove the lemma in the case when  $\mathcal{G} = \mathcal{O}(-Z)$ . In this case,  $\mathcal{G}^{\sigma_n^{-1}} = \mathcal{O}(-\sigma_n Z)$ . We note that the degree of the divisor  $\sigma_n Z$  on a component  $Y_0$  of  $Y$  takes on only finitely many values. It follows from the Riemann–Roch theorem that, if  $\Delta$  is a divisor on  $Y$  whose degree on each component is sufficiently large, then  $\mathcal{O}(\Delta - \sigma_n Z)$  is generated by global sections for every  $n$ . We choose such a divisor and obtain, for every  $n$ , a surjective map from a sum of copies of  $\mathcal{O}(-\Delta)$  to  $\mathcal{G}^{\sigma_n^{-1}}$ . Again because tensor products and  $H^1$  are right exact, we are reduced to showing that  $\mathcal{O}(-\Delta) \otimes \mathcal{L}_n$  is generated by its sections and has vanishing  $H^1$ . This is true because the sequence  $\{\mathcal{L}_n\}$  is ample. ■

**COROLLARY 6.3.** *Let  $\mathcal{E}, \mathcal{E}'$  be orders in  $A$ , and let  $\{\mathcal{L}_n\}$  be an ample sequence of invertible lattices, with  $E(\mathcal{L}_n) = \mathcal{E}$  and  $E'(\mathcal{L}_n) = \mathcal{E}'^{\tau^n}$ . Then*

- (i) *for all coherent right  $\mathcal{E}$ -modules  $\mathcal{F}$  and all  $n \gg 0$ , the sheaf  $\mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n$  is generated by global sections as an  $\mathcal{O}_Y$ -module, and  $H^1(Y, \mathcal{F} \otimes_{\mathcal{E}} \mathcal{L}_n) = 0$ ;*
- (ii) *for all coherent left  $\mathcal{E}'$ -modules  $\mathcal{F}$ , the sheaf  $\mathcal{L}_n \otimes_{\mathcal{E}'^{\tau^n}} \mathcal{F}^{\tau^n}$  is generated by global sections as an  $\mathcal{O}_Y$ -module, and  $H^1(Y, \mathcal{L}_n \otimes_{\mathcal{E}'^{\tau^n}} \mathcal{F}^{\tau^n}) = 0$ , for all  $n \gg 0$ .*

*Proof.* The first assertion follows directly from Lemma 6.1. To prove (ii), we pull back via the automorphism  $\tau^n$ . It suffices to show that  $\mathcal{L}_n^{\tau^{-n}} \otimes_{\mathcal{E}'} \mathcal{F}$  is generated by its sections and has vanishing  $H^1$ . By Lemma 6.1 and left–right symmetry, it suffices to show that  $\{\mathcal{L}_n^{\tau^{-n}}\}$  is an ample sequence. Now as  $\mathcal{O}_Y$ -modules,  $\mathcal{L}_n^{\tau^{-n}} \cong \mathcal{L}_n^{\sigma^{-n}}$ . So Lemma 6.2(iv) applies. ■

**PROPOSITION 6.4.** *Let  $R$  be an algebra which satisfies  $(\dagger)$ . The sequences of invertible lattices  $\{\mathcal{R}_n\}$ ,  $\{\mathcal{B}_n\}$  and  $\{\mathcal{B}'_n\}$  defined in Theorem 5.15 and (5.25) are ample.*

*Proof.* Since  $\mathcal{R}_n \subseteq \mathcal{B}_n$  and  $\mathcal{R}_n \subseteq \mathcal{B}'_n$  for large  $n$ , it suffices by Lemma 6.2(i), (iii) to show that  $\mathcal{R}_n$  is ample. In order to prove this, it suffices to prove the following assertion: For every simple point  $p$  of  $Y$ , there is an integer  $n$  such that  $\mathcal{R}_n$  contains a subsheaf isomorphic to  $\mathcal{E}(p) = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(p)$ . Assume that this has been proved. The inclusions  $\mathcal{R}_i \mathcal{R}_j^{\tau^i} \subset \mathcal{R}_{i+j}$  show that, for any positive divisor  $\Delta$  supported at simple points of  $Y$  and for  $n \gg 0$ ,  $\mathcal{R}_n$  contains a subsheaf isomorphic to  $\mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(\Delta)$ . Now, if  $\Delta_k$  is a sequence of Cartier divisors on  $Y$  whose degrees on each irreducible component of  $Y$  tend to  $\infty$  with  $k$ , then the sequence of invertible sheaves  $\mathcal{O}(\Delta_k)$  on  $Y$  is ample [Ha, Exercise III.5.7 and Corollary IV.3.2]. By Lemma 6.1(ii), the sequence  $\mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(\Delta_k)$  is also ample, and Lemma 6.2(i), (iii) show that  $\mathcal{R}_n$  is an ample sequence.

It remains to prove the assertion. Let  $p$  be a simple point of  $Y$  and  $t \in K$  a local parameter at  $p$ . Then the left  $\mathcal{E}$ -module spanned by  $\{1, t^{-1}\}$  contains  $\mathcal{E}(p)$ . Because  $Q$  is the graded ring of fractions of  $R$ , we can write  $t = \beta\alpha^{-1}$  for some  $\alpha, \beta \in \overline{R}_n$  whenever  $n$  is sufficiently large. Then

$$\mathcal{R}_n \supseteq \mathcal{E}\alpha + \mathcal{E}\beta = (\mathcal{E} + \mathcal{E}t^{-1})\alpha \supset \mathcal{E}(p)\alpha,$$

as required. ■

We now move to a discussion of twisted homogeneous coordinate rings. Let  $\mathcal{E}$  be an order in  $\mathcal{A}$  on the curve  $Y$ , and let  $\mathcal{B}_1$  be an invertible lattice with  $E(\mathcal{B}_1) = \mathcal{E}$  and  $E'(\mathcal{B}_1) = \mathcal{E}^\tau$ . Set

$$\mathcal{B}_0 = \mathcal{E} \quad \text{and} \quad \mathcal{B}_n = \mathcal{B}_1 \cdot \mathcal{B}_1^\tau \cdot \dots \cdot \mathcal{B}_1^{\tau^{n-1}}. \quad (6.5)$$

Following Van den Bergh [VdB], we call  $\mathbb{B} = \mathbb{B}(\mathcal{E}, \mathcal{B}_1; \tau) = \bigoplus_{n \geq 0} \mathcal{B}_n$  a *sheaf of bimodule algebras*, or simply a *bimodule algebra*. Multiplication on  $\mathbb{B}$  is defined by the tensor product formula  $\mathcal{B}_i \cdot \mathcal{B}_j^{\tau^i} = \mathcal{B}_{i+j}$  for  $i, j \geq 1$ .

Van den Bergh [VdB] defines an  $(\mathcal{E}, \mathcal{E})$ -bimodule differently, as a coherent sheaf  $\widetilde{\mathcal{B}}_1$  on  $Y \times Y$  whose support is finite over each factor, and with the structure of a left module over  $pr_1^*(\mathcal{E}) \otimes pr_2^*(\mathcal{E}^{\text{op}})$ . This definition is nicer because the automorphism  $\tau$  can be absorbed into the bimodule structure. However we will not use it here.

A *graded right module*  $\mathcal{N}$  over a bimodule algebra  $\mathbb{B}$  is a sequence of quasi-coherent  $\mathcal{E}^{\tau^n}$ -modules  $\mathcal{N}_n$  with compatible maps

$$\mathcal{N}_i \otimes_{\mathcal{E}^{\tau^i}} \mathcal{B}_j^{\tau^i} \longrightarrow \mathcal{N}_{i+j}. \quad (6.6)$$

The module  $\mathcal{N}$  is called *coherent* if  $\mathcal{N}_i$  is coherent for all  $i$  and if the maps (6.6) are surjective when  $i$  is sufficiently large. A *graded left*  $\mathbb{B}$ -module is defined analogously, as a sequence of quasi-coherent  $\mathcal{E}$ -modules with compatible maps

$$\mathcal{B}_i \otimes_{\mathcal{E}^{\tau^i}} \mathcal{N}_j^{\tau^i} \longrightarrow \mathcal{N}_{i+j}. \quad (6.7)$$

In order to simplify our notation in the rest of the paper, we will frequently write  $\otimes$  for the tensor product  $\otimes_{\mathcal{E}^{\tau^n}}$  over the appropriate shift of  $\mathcal{E}$ . The ring over which we are tensoring will be clear from the context. The  $\cdot$  notation is reserved for tensor products of invertible lattices, as before.

**PROPOSITION 6.8.** *Let  $\mathbb{B} = \mathbb{B}(\mathcal{E}, \mathcal{B}_1; \tau)$  be a sheaf of bimodule algebras. Then*

(i) *a graded  $\mathbb{B}$ -module is noetherian if and only if it is coherent, hence  $\mathbb{B}$  is graded noetherian;*

(ii) *the category of coherent right  $\mathbb{B}$ -modules is abelian.*

*Proof.* (i) By symmetry, and by standard arguments [VdB, Propositions 3.5 and 3.6], it is enough to prove that every graded right ideal  $\mathcal{I} = \oplus \mathcal{I}_i$  of  $\mathbb{B}$  is generated by finitely many of its graded parts  $\mathcal{I}_i$ . Let  $\mathcal{B}_i^*$  denote the dual lattice of  $\mathcal{B}_i$ , as in (1.13). Then  $\mathcal{I}_i \mathcal{B}_i^*$  and  $\mathcal{I} = \sum_{i=1}^{\infty} \mathcal{I}_i \mathcal{B}_i^*$  are right ideals of  $\mathcal{E}$ . Since  $\mathcal{E}$  is noetherian,  $\mathcal{I} = \sum_{i=1}^t \mathcal{I}_i \mathcal{B}_i^*$ , for some  $t$ . Then for  $m \geq t$ ,  $\mathcal{I}_m = \sum_{i=1}^t \mathcal{I}_i \mathcal{B}_i^* \cdot \mathcal{B}_m = \sum_{i=1}^t \mathcal{I}_i \mathcal{B}_{m-i}^*$ , and  $\mathcal{I}$  is finitely generated, as required.

(ii) If  $\mathbb{B}$  were flat over  $Y$ , then this would follow from [VdB, Proposition 3.6(iii) and Corollary 3.8]. Since  $\mathbb{B}$  is flat over  $\mathcal{E}$  the proof carries over. ■

Define the *tail*  $M_{\gg 0}$  of a graded module (or sheaf of modules)  $M$  to be the sum  $\oplus_{i \geq n} M_i$ , for  $n$  sufficiently large. A noetherian graded right  $\mathbb{B}$ -module  $\mathcal{N}$  will be called a *torsion module* if  $\mathcal{N}_{\gg 0} = 0$  for  $n \gg 0$ . We denote by  $\overline{\mathbf{gr}}\text{-}\mathbb{B}$  the quotient category of noetherian graded right  $\mathbb{B}$ -modules, modulo torsion. As usual, this quotient category can also be interpreted as the category of *tails*  $\mathcal{N}_{\gg 0}$  of noetherian graded  $\mathbb{B}$ -modules.

PROPOSITION 6.9. *The tensor product  $-\otimes_{\mathcal{E}} \mathbb{B}$  defines an equivalence of categories*

$$\text{mod-}\mathcal{E} \longrightarrow \overline{\mathbf{gr}}\text{-}\mathbb{B},$$

where  $\text{mod-}\mathcal{E}$  denotes the category of coherent sheaves of right  $\mathcal{E}$ -modules.

Similarly, the map sending a coherent left  $\mathcal{E}$  module  $\mathcal{F}$  to  $\oplus_{i \geq 0} \mathcal{B}_i \otimes \mathcal{F}^i$  defines an equivalence of categories  $\mathcal{E}\text{-mod} \rightarrow \mathbb{B}\text{-}\overline{\mathbf{gr}}$ .

*Proof.* By symmetry, and taking the differences between (6.6) and (6.7) into account, it suffices to prove the assertion for right modules. The equivalence we wish to establish is induced by the map  $\theta$  that sends a coherent right  $\mathcal{E}$ -module  $\mathcal{F}$  to the graded  $\mathbb{B}$ -module  $\theta(\mathcal{F}) = \mathcal{F} \otimes \mathbb{B}$ . In order to define a functor  $\phi$  in the other direction, let  $\mathcal{N} = \oplus \mathcal{N}_i$  be a graded, coherent  $\mathbb{B}$ -module, and set  $\mathcal{P}_i = \mathcal{N}_i \otimes \mathcal{B}_i^*$ . There are canonical maps  $\mathcal{P}_i \rightarrow \mathcal{P}_{i+j}$  defined by

$$\mathcal{P}_i = \mathcal{N}_i \otimes \mathcal{B}_i^* = \mathcal{N}_i \otimes \mathcal{B}_j^{\tau^i} \cdot \mathcal{B}_{i+j}^* \rightarrow \mathcal{N}_{i+j} \otimes \mathcal{B}_{i+j}^* = \mathcal{P}_{i+j},$$

and we define  $\phi(\mathcal{N})$  to be the class of  $\lim_{\substack{\longrightarrow \\ i \gg 0}} \mathcal{P}_i$ . Because  $\mathcal{N}$  is coherent and  $\mathcal{B}_i$  are invertible,  $\mathcal{P}_i = \mathcal{P}_{i+1}$ , for  $i \gg 0$ . Thus,  $\phi(\mathcal{N})$  is a coherent  $\mathcal{E}$ -module with  $\theta\phi(\mathcal{N})_{\gg 0} \approx \mathcal{N}_{\gg 0}$ , for large  $i$ . Conversely,  $\phi\theta(\mathcal{F}) \approx \mathcal{F}$ . ■

As above, let  $\mathcal{E}$  be an order in  $A$  on the curve  $Y$ , and let  $\mathcal{B}_1$  be an invertible lattice with  $E(\mathcal{B}_1) = \mathcal{E}$  and  $E'(\mathcal{B}_1) = \mathcal{E}^{\tau}$ . If  $\{\mathcal{B}_n = \mathcal{B}_1 \cdot \mathcal{B}_1^{\tau} \cdot \dots \cdot \mathcal{B}_1^{\tau^{n-1}}\}$  is an ample sequence of sheaves, we say that  $\mathcal{B}_1$  is an *ample lattice*. If  $\mathcal{B}_1$  is an ample lattice, we define the *twisted homogeneous coordinate ring*, or

twisting ring for short, to be the ring  $B = B(\mathcal{E}, \mathcal{B}_1; \tau) = \bigoplus_{i \geq 0} B_n$ , where  $\mathcal{B}_0 = \mathcal{E}$ ,

$$\overline{B}_n = H^0(Y, \mathcal{B}_n) \quad \text{for } n \geq 0, \quad (6.10)$$

and  $B_n = \overline{B}_n z^n$ . Multiplication in  $B$  is induced by multiplication in the bimodule algebra  $\mathcal{B}$ , with the convention that  $z\beta = \beta^\tau z$ .

**COROLLARY 6.11.** *Let  $\mathcal{B}_1$  be an ample lattice, with associated twisting ring  $B = B(\mathcal{E}, \mathcal{B}_1; \tau)$  and bimodule algebra  $\mathbb{B} = \mathbb{B}(\mathcal{E}, \mathcal{B}_1; \tau)$ . Define a functor  $\Lambda : \overline{\mathbf{gr}}\text{-}\mathbb{B} \rightarrow \overline{\mathbf{gr}}\text{-}B$  as follows: If  $\mathcal{M} = \bigoplus_{i \geq 0} \mathcal{M}_i$  be a coherent right  $\mathbb{B}$ -module, set  $\Lambda(\mathcal{M}) = \bigoplus_{i \geq 0} H^0(Y, \mathcal{M}_i) z^i$ . Then*

(i) *The functor  $\Lambda$  is an equivalence of categories, its quasi-inverse being  $\mathbb{B} \otimes_B \_$ .*

(ii) *The categories  $\overline{\mathbf{gr}}\text{-}B$  and  $\text{mod-}\mathcal{E}$  are equivalent via the functor that sends  $\mathcal{F} \in \text{mod-}\mathcal{E}$  to the tail of*

$$\bigoplus_{n \geq 0} H^0(Y, \mathcal{F} \otimes_{\mathcal{E}} \mathcal{B}_n) z^n.$$

(iii) *The categories  $B\text{-}\overline{\mathbf{gr}}$  and  $\mathcal{E}\text{-mod}$  are equivalent via the functor that sends  $\mathcal{F} \in \mathcal{E}\text{-mod}$  to the tail of*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \geq 0} H^0(Y, \mathcal{B}_n \otimes_{\mathcal{E}^{\tau^n}} \mathcal{F}^{\tau^n}) z^n.$$

(iv)  *$B$  is a noetherian two-dimensional algebra.*

*Proof.* Parts (i) and (iv) follow from Proposition 6.8 and [VdB, Theorem 5.2]. (A couple of comments about Van den Bergh's result are in order. First, [VdB] contains the hypothesis that  $k$  is algebraically closed, but this is not used in the proof of [VdB, Theorem 5.2]. Secondly, [VdB] defines  $\mathcal{B}_0 = X$  rather than  $\mathcal{B}_0 = \mathcal{E}$ . Clearly, this does not affect the validity of part (i) while, by [AS, Lemma 1.14], the truth of part (iv) is also unaffected.) Parts (ii) and (iii) then follow from Proposition 6.9. ■

## 7. SEMIPRIME NOETHERIAN RINGS OF DIMENSION TWO

Assume in this section that  $k$  is infinite. The object of this section is to prove the following two theorems.

**THEOREM 7.1.** *A two-dimensional algebra  $R$  is right noetherian if and only if it is left noetherian, if and only if  $(\dagger)$  holds for some Veronese  $R^{(n)}$ .*

In particular, this theorem implies that, if  $R^{(n)}$  satisfies  $(\dagger)$ , then  $R$  is a finitely generated algebra, justifying a comment made in Remark 2.12.

**THEOREM 7.2.** (i) *Let  $R = \bigoplus_{i \geq 0} R_n$  be a two-dimensional algebra such that some Veronese  $R^{(m)}$  satisfies  $(\dagger)$ , let  $Y$  denote the stable model for  $R$ , in the sense of Definition 5.12, and let  $\mathcal{R}'_n = \mathcal{O}_Y R_n$ . Then, for  $n \gg 0$ , the natural embedding  $R_n \subset \mathcal{R}'_n$  induces an equality  $R_n = H^0(Y, \mathcal{R}'_n)$ .*

(ii) *Assume that  $R = \bigoplus_{i \geq 0} \bar{R}_n z^n$  is a nice algebra in part (i) and set  $\mathcal{R}_n = \mathcal{O}_Y \bar{R}_n$ . Then,  $R$  has finite codimension in the algebra  $\tilde{R} = \bigoplus H^0(Y, \mathcal{R}_n) z^n$ . The multiplication in  $\tilde{R}$  is the natural one induced from that of  $Q(R) = A[z, z^{-1}; \tau]$ .*

The only reason for the distinction between parts (i) and (ii) of the theorem is that the multiplication in  $\bigoplus H^0(Y, \tilde{\mathcal{R}}_n)$  is more awkward to describe. The next result provides some easy consequences of the theorem.

**COROLLARY 7.3.** *Let  $R = \bigoplus_{i \geq 0} \bar{R}_n z^n$  be a nice algebra. Then*

(i)  *$R$  has finite codimension in a twisting ring  $B = B(\mathcal{E}, \mathcal{B}_1, \tau)$  if and only if  $R$  satisfies  $(\dagger\dagger)$ .*

(ii) *Suppose that  $R$  satisfies  $(\dagger)$ . Then, possibly after replacing  $R$  by some Veronese ring, there exists a twisting ring  $B = B(\mathcal{E}, \mathcal{B}_1, \tau)$  such that  $R_{\geq 1} = \bigoplus_{i \geq 1} R_i$  is a left ideal of  $B$ .*

(iii) *Let  $R$ ,  $B$  and  $\mathcal{E}$  be defined as in part (ii). Then, there exists an essential left  $\mathcal{E}$ -submodule  $\mathcal{R}_1$  of  $\mathcal{B}_1$  such that*

(a)  *$1 \in H^0(Y, \mathcal{R}_1)$  and  $\mathcal{R}_1 = \mathcal{B}_1$  locally on every finite orbit of  $\sigma$  on  $Y$ ,*

(b) *If  $\mathcal{R}_n = \mathcal{B}_1 \otimes \mathcal{B}_1^\tau \otimes \cdots \otimes \mathcal{B}_1^{\tau^{(n-2)}} \otimes \mathcal{R}_1^{\tau^{(n-1)}}$ , then  $\bar{R}_n = H^0(Y, \mathcal{R}_n)$ , for all  $n \gg 0$ .*

(iv) *If  $R$ ,  $B$  and  $\mathcal{E}$  are as in part (ii), then the categories  $\overline{\mathbf{gr}}\text{-}R$ ,  $\overline{\mathbf{gr}}\text{-}B$  and  $\text{mod-}\mathcal{E}$  are all equivalent. This equivalence sends a sheaf of right  $\mathcal{E}$ -modules  $\mathcal{F}$  to the  $R$ -module  $\bigoplus_{i \gg 0} H^0(Y, \mathcal{F} \otimes_{\mathcal{E}} \mathcal{R}_i) z^i$ .*

*Proof.* (i) Corollary 5.27(iv) implies that  $R \subseteq B$ , for any nice algebra  $R$ . By Corollary 5.24 and (5.25),  $(\dagger\dagger)$  holds for  $R$  if and only if  $\mathcal{R}_n = \mathcal{B}_n$  for all  $n \gg 0$ . Thus, the result follows immediately from Theorem 7.2.

(ii) Replacing  $R$  by some Veronese ring, we may assume that  $\bar{R}_n = H^0(Y, \mathcal{R}_n)$  for all  $n \geq 1$ . The sheaf  $\mathcal{B}_1$  is defined by (5.25). Then, as in (6.10),  $B = \bigoplus_{i \geq 0} \bar{B}_n z^n$ , where  $\bar{B}_n = H^0(Y, \mathcal{B}_n)$ . By taking global sections of the equation  $\mathcal{B}_n \cdot \mathcal{R}_m^{\tau^n} = \mathcal{R}_{n+m}$  from (5.26), it follows that  $\bar{B}_n \bar{R}_m^{\tau^n} \subseteq \bar{R}_{n+m}$  for all  $n \geq 1$  and all  $m \gg 0$ .

(iii) This follows from part (ii) and Theorem 5.15.

(iv) By [SZ, Proposition 2.7] and its proof,  $\overline{\mathbf{gr}}\text{-}B$  is equivalent to  $\overline{\mathbf{gr}}\text{-}R$  via the map that sends a right  $B$ -module  $M$  to  $M \otimes_B R_{\gg 0}$ . The result therefore follows from Corollary 6.11(ii). ■

Part (ii) of this corollary is illustrated by Examples 5.13 and 5.14. The latter example is just the twisting ring  $B(\mathcal{E}, \mathcal{B}_1, \tau)$ , where  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} + M_2(\mathcal{O}_{\mathbb{P}^1}(1))$ ,  $\mathcal{B}_1 = \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(1)$  and  $\tau$  is given by conjugation by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , as predicted by (0.2). However, the former example is more interesting:

EXAMPLE 7.4 (Continuation of Example 5.13, and Example 2.7). Keep the notation of those two examples. Thus,  $\mathcal{E} = M_2(\mathcal{O}_{\mathbb{P}^1})$ . By the observations from the end of Sect. 5,  $\mathcal{B}_1 = \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(1)$  and so  $B = B(\mathcal{E}, \mathcal{B}_1; \tau) = M_2(U)$ , where  $U = k\{x, y\}/(xy - yx - x^2)$ , as in Example 2.7. By the computations of Example 5.13 it is easy to check that  $\mathcal{R}_n = \mathcal{B}_{n-1}\mathcal{R}_1^{n-1}$ , for all  $n \geq 1$ , that the conclusion of Theorem 7.2 holds for all  $n \geq 1$  and that  $R_{\geq 1}$  is a left ideal of  $B$ . It follows from [SZ], or directly, that  $B$  is a finitely generated left  $R$ -module, but that  $B_R$  is infinitely generated.

The proof of the theorems will distinguish several cases. We first consider an intermediate case defined by the next lemma, then deal with rings satisfying  $(\dagger)$  and, finally, deal with the general case.

LEMMA 7.5. *Let  $R$  be a two-dimensional algebra which satisfies  $(\dagger)$  and adopt the notation of Theorem 5.15 and (5.25). Then, the following are equivalent.*

- (i) *The gap lattice  $\mathcal{G}$ , as defined in (5.23), is the product lattice  $\mathcal{E}'\mathcal{E}$ .*
- (ii) *For  $n \gg 0$ ,  $\overline{R}_n$  generates  $\mathcal{B}_n$  as right  $\mathcal{E}^{\tau^n}$ -module.*
- (iii) *For  $i, j \gg 0$ ,  $\mathcal{R}_i\mathcal{R}_j^{\tau^i} = \mathcal{R}_{i+j}$ .*

*These conditions hold if  $R$  is generated in degree 1.*

*Proof.* The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i) follow from (5.25), (5.26), and (5.18). ■

DEFINITION 7.6. We say that a nice algebra  $R$  satisfies  $(\dagger')$  if the equivalent conditions of Lemma 7.5 are satisfied and if, in addition,  $\mathcal{S}_i = \mathcal{R}_i$  for all  $i \geq 1$ , in the notation of Theorem 5.15.

This final condition in this definition is harmless in the sense that, by Theorem 5.15, it can always be achieved by replacing  $R$  by some Veronese subring. It is easy to find rings satisfying  $(\dagger')$ , and here is a typical method that will be used in this paper. Let  $R$  be a order satisfying  $(\dagger)$  and take  $S$  to be the  $k$ -subalgebra generated by  $\overline{R}_n z^n$ , with the grading adjusted so that  $S_1 = \overline{R}_n z^n$ . If  $n$  is large enough so that  $\mathcal{A}$  is generated by  $\overline{R}_n$ , then it is routine to check that  $S$  satisfies  $(\dagger)$  and so, modulo passing to a further Veronese, Lemma 7.5 says that  $S$  does satisfy  $(\dagger')$ .

We begin with a preliminary result.

LEMMA 7.7. *Let  $R$  be a two-dimensional algebra which satisfies  $(\dagger)$ , let  $\mathcal{I}$  be a non-zero,  $\sigma$ -invariant ideal of  $\mathcal{O} = \mathcal{O}_Y$ , and set  $\tilde{\mathcal{O}} = \mathcal{O}/\mathcal{I}$ ,  $\tilde{\mathcal{E}} = \mathcal{E}/\mathcal{I}\mathcal{E}$  and  $\tilde{\mathcal{R}}_n = \mathcal{R}_n \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ . Let  $S$  denote the set of points of  $X$  lying over the support of  $\tilde{\mathcal{O}}$ , and let  $u_n \in \bar{R}_n$  be an element which has maximal pole at all points of  $S$ . If  $n \gg 0$ , the canonical maps  $f_n: \bar{R}_n u_n^{-1} \rightarrow H^0(Y, \tilde{\mathcal{E}})$  and  $g_n: \bar{R}_n \rightarrow H^0(Y, \tilde{\mathcal{R}}_n)$  are surjective.*

*Proof.* Recall that  $\mathcal{R}_n$  is invertible, for large  $n$ . Since  $u_n$  has maximal pole on  $S$ , its residue generates  $\tilde{\mathcal{R}}_n$ , i.e.,  $\tilde{\mathcal{R}}_n = \tilde{\mathcal{E}} u_n$  for  $n \gg 0$ . Therefore  $f_n$  is surjective if and only if  $g_n$  is surjective. Let  $V_n$  denote the image of the map  $f_n$ . For given  $n$ , the validity of the lemma is independent of the choice of  $u_n$ , because if  $u, v \in \bar{R}_n$  have maximal poles on  $S$ , then  $uv^{-1}$  is invertible on  $Y_u$ , hence it is invertible in  $\tilde{\mathcal{E}}$ . Let  $i, j \gg 0$  and  $i + j = n$ . By (2.10), we may choose  $u_n = u_i u_j^{\tau^i}$ . This being done, the equation  $(\bar{R}_i u_i^{-1})(\bar{R}_j u_j^{-1})^{\mu \circ \tau^i} \subseteq \bar{R}_n u_n$ , where  $\mu$  denotes conjugation by  $u_i$ , holds at all points in the support of  $\mathcal{O}/\mathcal{I}$ . Hence

$$V_i(u_i V_j^{\tau^i} u_i^{-1}) \subseteq V_{i+j},$$

and  $V_i \subseteq V_n$ . Therefore, the sequence  $V_i$  is essentially constant. We choose  $i, j$  large enough so that  $V_i = V_n = V_j = V$ . Then the last displayed equation shows that  $V = u_i V^{\tau^i} u_i^{-1}$  and hence that  $V$  is closed under multiplication. So  $V$  is a subring of  $\tilde{\mathcal{E}}$ .

The ring  $k\langle \bar{R}_n u_n^{-1} \rangle$  generated by  $\bar{R}_n u_n^{-1}$  is the affine ring  $\mathcal{E}(Y_{u_n})$ , by the definition of  $Y$ , and so it maps surjectively onto  $H^0(Y, \tilde{\mathcal{E}})$ . So  $V$  generates  $\tilde{\mathcal{E}}$ , and, as it is a subring, it is equal to  $\tilde{\mathcal{E}}$ . ■

We will now begin the proof of Theorem 7.2 in the case that  $R$  satisfies  $(\dagger')$ ; so suppose we are given such an order. As in (6.10), we define two graded rings by setting  $G = \oplus_n \bar{G}_n z^n$  and  $B = \oplus_n \bar{B}_n z^n$ , where

$$\bar{G}_n = H^0(Y, \mathcal{R}_n) \quad \text{and} \quad \bar{B}_n = H^0(Y, \mathcal{B}_n).$$

Multiplication is induced by the usual tensor product formulae (5.26), and with the usual commutation relation for  $z$ . As elsewhere, we write  $G_n = \bar{G}_n z^n$ , etc. Thus  $B = B(Y, \mathcal{B}; \tau)$  is a twisting ring, and so  $B$  is noetherian by Corollary 6.11. Since  $\mathcal{R}_n = \mathcal{S}_n$  for all  $n \geq 1$ , Corollary 5.27(iii) implies that  $R \subseteq G \subseteq B$ , and (5.26) implies that  $G_{\geq 1}$  is a left ideal of  $B$ . We must show that  $\bar{R}_n = \bar{G}_n$  for  $n \gg 0$ .

If  $\mathcal{F}$  is a left  $\mathcal{E}$ -module, we use the notation

$$\Gamma_*(\mathcal{F}) = \bigoplus_n H^0(Y, \mathcal{B}_n \otimes_{\mathcal{E}^{\tau^n}} \mathcal{F}^{\tau^n}) z^n, \quad (7.8)$$

as in Corollary 6.11(iii). So  $\Gamma_*(\mathcal{F})$  is a left  $B$ -module and  $\Gamma_*(\mathcal{E}) = B$ .



LEMMA 7.9. *If  $R$  satisfies  $(\dagger')$ , then  ${}_R B$ ,  ${}_R G$ , and  $G_R$  are finitely generated modules.*

*Remark.* In general  $B_R$  will not be finitely generated, as is illustrated by Example 7.4.

*Proof.* It suffices to show that  ${}_R B$  is finitely generated. Indeed, if this is the case, then  ${}_R G$  is finitely generated, since  $G_{\geq 1}$  is a left ideal of the noetherian ring  $B$ . By left–right symmetry  $G_R$  is finitely generated.

By Lemma 7.5(ii), we can choose  $i_0$  large enough so that  $\bar{R}_i$  generates  $\mathcal{B}_i$  as a right  $\mathcal{E}^i$ -module for all  $i \geq i_0$ . For such  $i$  we obtain a surjective map  $\bar{R}_i \otimes_k \mathcal{E}^i \rightarrow \mathcal{B}_i \rightarrow 0$  of right  $\mathcal{E}^i$ -modules. We tensor on the right with  $\mathcal{B}_j^{\tau^i}$ , obtaining a surjection

$$\bar{R}_i \otimes_k \mathcal{B}_j^{\tau^i} \rightarrow \mathcal{B}_{i+j} \rightarrow 0.$$

Applying the functor  $H^0(Y, \_)$  yields a map  $\psi: \bar{R}_i \otimes_k \bar{B}_j^{\tau^i} \rightarrow \bar{B}_{i+j}$ . By Proposition 6.4,  $\{\mathcal{B}_j^{\tau^i}\}$  is an ample sequence and so  $\psi$  is surjective if  $j$  is sufficiently large. We may choose  $j_0$  large enough so that  $\psi$  is surjective for each of the indices  $i = i_0, \dots, 2i_0 - 1$  and for all  $j \geq j_0$ . Then  $B_1 + \dots + B_{i_0+j_0}$  spans  $B$  as a left  $R$ -module. ■

LEMMA 7.10. *Suppose that  $(\dagger')$  holds for the two-dimensional algebra  $R$ . Then there exists a nonzero,  $\sigma$ -invariant sheaf of ideals  $\mathcal{I} \subseteq \mathcal{O}_Y$  such that the  $(B, G)$ -bimodule  $\Gamma_*(\mathcal{I}\mathcal{R}_1)_{\gg 0}$  is contained in  $R$ .*

*Proof.* Recall that  $(\dagger')$  and (5.26) together imply that  $\mathcal{R}_n = \mathcal{S}_n = \mathcal{B}_{n-1} \cdot \mathcal{R}_1^{\tau^{n-1}}$  for all  $n$ . The rings  $R$  and  $B$  are two-dimensional algebras with the same Goldie quotient ring. Therefore,  $(B/R)_R$  is a torsion module while, by Lemma 7.9,  ${}_R(B/R)$  is finitely generated. By [GW, Lemma 7.3] it follows that right annihilator  $U = r\text{-ann}_R(B/R)$  contains a homogeneous, regular element  $u$ ; one simply picks  $u$  to annihilate each homogeneous generator of  ${}_R(B/R)$ . Similarly, the left annihilator  $V = \ell\text{-ann}_R(G/R)$  contains a regular element, hence so does  $W = UV \subseteq R$ . Note that  $W$  is an  $(B, G)$ -bimodule. Since  $W$  is a left ideal of  $B$ , Corollary 6.11(iii) shows that there is a sheaf of left ideals  $\mathcal{W}$  of  $\mathcal{E}$  such that  $W_{\gg 0} = \Gamma_*(\mathcal{W})_{\gg 0}$ . Similarly, (5.26) implies that  $G_{\gg 0} = \Gamma_*(\mathcal{R}_1)_{\gg 0}$ . Since  $W \subseteq G$ , it follows that

$$W_n = H^0(Y, \mathcal{B}_{n-1} \otimes \mathcal{W}^{\tau^{n-1}})z^n \subseteq G_n = H^0(Y, \mathcal{B}_{n-1} \otimes \mathcal{R}_1^{\tau^{n-1}})z^n.$$

If  $n$  is large enough so that both  $\mathcal{B}_{n-1} \otimes \mathcal{W}^{\tau^{n-1}}$  and  $\mathcal{B}_{n-1} \otimes \mathcal{R}_1^{\tau^{n-1}}$  are generated by their sections, this implies that  $\mathcal{B}_{n-1} \otimes \mathcal{W}^{\tau^{n-1}} \subseteq \mathcal{B}_{n-1} \otimes \mathcal{R}_1^{\tau^{n-1}}$  and hence that  $\mathcal{W} \subseteq \mathcal{R}_1$ .

Thus  $\mathcal{N} = \mathcal{R}_1/\mathcal{W}$  is a coherent sheaf of torsion modules over  $\mathcal{O} = \mathcal{O}_Y$ , and we claim that  $\mathcal{N}$  is supported on the fixed locus  $F$  of  $\sigma$ . Assume this to be

true. Then, all powers of  $\mathfrak{m} = \bigcap_{p \in F} \mathfrak{m}_{X,p}$  are  $\sigma$ -invariant and some power  $I = \mathfrak{m}^r$  will annihilate  $\mathcal{N}$ . Therefore, if  $\mathcal{I}$  is the sheaf of ideals defined by  $I$ , then  $\mathcal{W}$  will contain  $\mathcal{I}\mathcal{R}_1$ . Thus,  $W \supseteq \Gamma_*(\mathcal{I}\mathcal{R}_1)$  and the lemma follows.

It remains to prove the claim. As usual, we write  $W = \bigoplus_{i \geq 0} \overline{W}_i z^i$ . Let  $p \in Y$  be a point of infinite order. In order to prove the claim, we need to show that  $p \notin \text{Supp}(\mathcal{R}_1/\mathcal{W})$  and to do this it suffices to find, for all  $n \gg 0$ , an element  $w_n \in \overline{W}_n$  that generates the  $\mathcal{E}$ -module  $\mathcal{R}_n$  locally at  $\sigma^{-n}(p)$  (see Corollary 6.11(iii)). Fix a regular element  $\alpha_i \in \overline{W}_i$ . Then,  $\alpha_i$  will span  $\mathcal{B}_i$  locally at all but finitely points  $q \in Y$ . Thus, it certainly spans  $\mathcal{B}_i$  locally at  $\sigma^{-n}(p)$  provided that  $j = n - i \gg 0$ . Next, as  $\mathcal{R}_j$  is invertible and spanned by  $R_j$ , we can pick  $\beta_j \in R_j$  that spans  $\mathcal{R}_j$  locally at  $\sigma^{-j}(p)$ . Thus,  $\beta_j^{\tau^i}$  spans  $\mathcal{R}_j^{\tau^i}$  locally at  $\sigma^{-n}(p)$ . Finally, as  $W$  is a right  $G$ -module,  $\alpha_i \beta_j^{\tau^i} \in \overline{W}_n$  and, as  $\mathcal{R}_n = \mathcal{B}_i \cdot \mathcal{R}_j^{\tau^i}$ , the element  $\alpha_i \beta_j^{\tau^i}$  spans  $\mathcal{R}_n$  locally at  $\sigma^{-n}(p)$ . ■

LEMMA 7.11. *Theorem 7.2 is true if  $R$  satisfies  $(\dagger')$ .*

*Proof.* Let  $\mathcal{I}$  be defined by Lemma 7.10 and note that  $\mathcal{R}_n \mathcal{I} = \mathcal{I} \mathcal{R}_n \cong \mathcal{I} \mathcal{E} \otimes_{\mathcal{E}} \mathcal{R}_n$ , since  $\mathcal{I}$  is central and  $\mathcal{R}_n$  is invertible (Theorem 5.9). We set  $\overline{J}_n = H^0(Y, \mathcal{I} \mathcal{R}_n)$  and  $J = \bigoplus_{n \geq 1} \overline{J}_n z^n$ . So  $J$  is a right ideal of  $G$ . Similarly, (5.26) implies that  $J$  is a left ideal of  $B$  and hence of  $G$ . But Lemma 7.10 ensures that  $J_{\gg 0} \subseteq R$ . Consequently,  $J_{\gg 0}$  is an ideal of  $R$  that kills  $G/R$  on both sides.

Set  $\widetilde{\mathcal{R}}_n = \mathcal{R}_n / \mathcal{I} \mathcal{R}_n$  as in Lemma 7.7. Because  $\mathcal{R}_n$  is an  $(\mathcal{E}, \mathcal{E}')$ -ample sequence, applying  $H^0$  to the exact sequence

$$0 \longrightarrow \mathcal{I} \mathcal{R}_n \longrightarrow \mathcal{R}_n \longrightarrow \widetilde{\mathcal{R}}_n \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \overline{J}_n \longrightarrow \overline{G}_n \longrightarrow H^0(Y, \widetilde{\mathcal{R}}_n) \longrightarrow 0$$

for all  $n \gg 0$ . If  $n$  is large, then  $\overline{J}_n \subseteq \overline{R}_n \subseteq \overline{G}_n$  and by Lemma 7.7 the map  $\overline{R}_n \rightarrow H^0(Y, \widetilde{\mathcal{R}}_n)$  is surjective, by Lemma 7.7. It follows that  $\overline{R}_n = \overline{G}_n$  for  $n \gg 0$ , as required. ■

We now turn to the proofs of Theorems 7.1 and 7.2 for the case when  $(\dagger)$  holds for  $R$ . We fix  $n \gg 0$ , and we let  $C = \bigoplus_{i \geq 0} \overline{C}_i z^{ni}$  denote the  $k$ -subalgebra of  $R$  generated by  $R_n = C_1$ . Then  $C$  satisfies  $(\dagger')$ , and the lattice spanned by  $\mathcal{C}_i$  is

$$\mathcal{C}_i = \mathcal{R}_n \mathcal{R}_n^{\tau^n} \cdots \mathcal{R}_n^{\tau^{(i-1)n}}.$$

This lattice is invertible by Theorem 5.9(iv) and Proposition 1.14(iii). Also, the stable model  $Y$  for  $C$  and for  $R$  will be the same, provided that  $n$  is sufficiently large. To see this, we note that, for  $n \gg 0$ , both models are smooth at points of infinite order 5.9(ii), and that  $\mathcal{C}_k = \mathcal{R}_{nk}$  at the fixed points (5.17).

LEMMA 7.12.  $E(\mathcal{C}_i) = E(\mathcal{R}_i)$  for all  $i, j \gg 0$ . Thus, in the notation of Corollary 5.8,  $E(\mathcal{C}_i) = \mathcal{C}$ .

*Proof.* This follows from Lemma 5.17 at the fixed points, and from Proposition 1.14(iii) at the points of infinite order. ■

Since  $C$  satisfies  $(\dagger')$ , Lemma 7.11 implies that

$$\overline{R}_n \overline{R}_n^{\tau^n} \cdots \overline{R}_n^{\tau^{(i-1)n}} = \overline{C}_i = H^0(Y, \mathcal{C}_i) \quad \text{for } i \gg 0. \quad (7.13)$$

If  $\{D_n\}$  is the divisor sequence for  $R$ , as in (2.3), then the divisor sequence for the ring  $C$  is

$$E_i = D_n + \sigma^{-n} D_n + \cdots + \sigma^{-(i-1)n} D_n.$$

We choose an integer  $1 < s < n$  such that the support  $T$  of the gap divisor  $\Omega$  of  $R$  (see Definition 2.17) does not meet its translate by  $\sigma^{-s}$ . This is possible if  $n$  is sufficiently large.

LEMMA 7.14. (i) *If  $n \gg 0$  and  $s$  is suitably chosen, the natural inclusions  $\mathcal{C}_i \subseteq \mathcal{R}_{ni}$  and  $\mathcal{C}_{i-1}^{\tau^s} \subseteq \mathcal{R}_{ni}$  determine an exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{H}_i \longrightarrow \mathcal{C}_i \oplus \mathcal{C}_{i-1}^{\tau^s} \xrightarrow{f} \mathcal{R}_{ni} \longrightarrow 0. \quad (7.15)$$

(ii) *For  $i \gg 0$ ,  $\mathcal{H}_i$  is an invertible lattice, and the sequence  $\{\mathcal{H}_i\}$  is ample.*

*Proof.* (i) This is similar to the proof of [AS, Lemma 5.5]. The inclusion  $\mathcal{C}_i \subseteq \mathcal{R}_{ni}$  is induced by the inclusion  $\overline{C}_i \subseteq \overline{R}_{ni}$ , while the inclusion  $\mathcal{C}_{i-1}^{\tau^s} \subseteq \mathcal{R}_{ni}$  is induced by  $\overline{C}_{i-1}^{\tau^s} \subseteq \overline{R}_{n(i-1)}^{\tau^s} \subseteq \overline{R}_{ni}$ . In order to prove that  $f$  is surjective, it suffices to prove it locally at each point  $p \in Y$ . At the fixed points,  $\mathcal{R}_{in} = \mathcal{C}_i$ , by  $(\dagger)$ , so consider the points of infinite order. By Theorem 5.9(iv) and Proposition 1.14(iv), it suffices to prove the corresponding statement for divisors. Thus we need to prove that  $E_i \cup E_{i-1}^{\sigma^s} \geq D_{ni}$ .

We prove this by induction on  $i$ , with the case  $i = 1$  being trivial. Now,  $E_i = D_n + \sigma^{-n}(E_{i-1})$ . Thus, by induction,

$$\begin{aligned} E_i \cup \sigma^{-s} E_{i-1} &= [D_n + \sigma^{-n} E_{i-1}] \cup [\sigma^{-s} D_n + \sigma^{-s-n} E_{i-2}] \\ &\geq F := \sigma^{-n} D_{(i-1)n} \cup D_n \cup \sigma^{-s} D_n. \end{aligned}$$

The choice of  $s$  and Corollary 2.18(ii) ensure that  $\sigma^{-s} D_n \geq \sigma^{-n} \Omega$  on each infinite orbit. Thus, Corollary 2.18(i) implies that  $F = D_{in}$ , as is required to prove that  $f$  is surjective. The sheaf  $\mathcal{H}_i$  is then defined so that the sequence (7.15) is exact.

(ii) The sheaf  $\mathcal{H}_i$  is an invertible lattice because the other three terms are invertible. Locally at any point  $p \in Y$ , one of the two sheaves  $\mathcal{C}_{i-1}^{\tau^s}$  or  $\mathcal{C}_i$  maps surjectively to  $\mathcal{R}_{ni}$ , hence  $\mathcal{H}_i$  is isomorphic to the other one at that point. To show that the sequence  $\{\mathcal{H}_i\}$  is ample, we exhibit an ample sequence of sublattices. We choose a large integer  $m$  so that  $m + s \leq n$ . Then by (2.1),  $\bar{R}_m^{\tau^s} \subset \bar{R}_n$  and  $\bar{R}_m^{\tau^s} \subset \bar{R}_n^{\tau^s}$ , hence  $\mathcal{R}_m^{\tau^{m+s}} \subset \mathcal{R}_n^{\tau^{jn}}$ , and also  $\mathcal{R}_m^{\tau^{m+s}} \subset \mathcal{R}_n^{\tau^{jn+s}}$ . Set

$$\mathcal{H}_i = \mathcal{R}_m^{\tau^s} \mathcal{R}_m^{\tau^{n+s}} \cdots \mathcal{R}_m^{\tau^{(i-1)n+s}}.$$

Then  $\mathcal{H}_{i-1} \subset \mathcal{C}_i$  and also  $\mathcal{H}_{i-1} \subset \mathcal{C}_{i-1}^{\tau^s}$ . The map  $(+, -) : \mathcal{H}_{i-1} \rightarrow \mathcal{C}_i \oplus \mathcal{C}_{i-1}^{\tau^s}$  identifies  $\mathcal{H}_{i-1}$  as a sublattice of  $\mathcal{H}_i$ . Moreover,  $\{\mathcal{H}_i\}$  is the sequence of invertible lattices defined by the graded ring  $B(Y, \mathcal{R}_m^{\tau^s}; \tau^n)$ , and so Proposition 6.4 implies that it is ample. Finally, for any  $i \gg 0$ ,  $\mathcal{H}_{i-1} = \mathcal{H}_i$  at all but a finite number of points  $p \in Y$ , simply because  $\mathcal{H}_{i-1}$  and  $\mathcal{H}_i$  are both lattices. Thus, by Lemma 6.2(i), (iii), the amplitude of  $\{\mathcal{H}_i\}$  follows from that of  $\{\mathcal{H}_i\}$ . ■

LEMMA 7.16. *If  $R$  satisfies  $(\dagger)$ , then Theorem 7.2 holds for some Veronese of  $R$ .*

*Proof.* By (7.13),  $H^0(Y, \mathcal{C}_i \oplus \mathcal{C}_{i-1}^{\tau^s}) = \bar{C}_i \oplus \bar{C}_{i-1}^{\tau^s}$  and so, by Lemma 7.14(i), we have an exact sequence

$$\bar{C}_i \oplus \bar{C}_{i-1}^{\tau^s} \xrightarrow{\psi} H^0(Y, \mathcal{R}_{ni}) \rightarrow H^1(Y, \mathcal{H}_i).$$

By Lemma 7.14(ii), the final term is zero and so  $\psi$  is surjective for large  $i$ . Since  $\bar{C}_i + \bar{C}_{i-1}^{\tau^s} \subseteq \bar{R}_{ni} \subseteq H^0(Y, \mathcal{R}_{ni})$ , it follows that  $\bar{R}_{ni} = H^0(Y, \mathcal{R}_{ni})$  for all  $n, i \gg 0$ . ■

We next wish to prove Theorem 7.1. Since  $R$  is right noetherian if and only if  $R^{(m)}$  is right noetherian (Proposition 3.2) we may freely replace  $R$  by a Veronese ring. Thus, by Lemma 7.16, we may suppose that the conclusion of Theorem 7.2 holds for  $R$ . Given a nonzero graded right ideal  $M = \oplus_{i \geq 0} \bar{M}_n z^n$  of  $R$ , we write  $\mathcal{M}_n = \mathcal{O}_Y \bar{M}_n = \bar{M}_n \mathcal{O}_Y$ . Thus  $\mathcal{M}_n \subseteq \mathcal{R}_n$  and  $\mathcal{M}_i \mathcal{R}_j^{\tau^i} \subseteq \mathcal{M}_{i+j}$ . We set  $\mathcal{J}_i = \mathcal{M}_i \mathcal{R}_i^*$ .

LEMMA 7.17. *Let  $M$  be a graded right ideal of the right noetherian, two-dimensional algebra  $R$  and keep the notation and assumptions of the last paragraph. Then, for  $i, j \gg 0$ ,*

- (i)  $\mathcal{J}_i$  is a right ideal of  $\mathcal{E}$  which is independent of  $i$ ;
- (ii) in the notation of (5.25),  $\mathcal{M}_i \mathcal{B}'^{\tau^i}_j = \mathcal{M}_{i+j}$ ;
- (iii)  $\mathcal{M}_i$  is a right  $\mathcal{E}^{\tau^i}$ -module.

*Proof.* Since  $\mathcal{R}_i^*$  is a right  $\mathcal{E}$ -module, so is  $\mathcal{F}_i = \mathcal{M}_i \mathcal{R}_i^*$ . Since  $\mathcal{M}_i \mathcal{R}_j^i \subseteq \mathcal{M}_{i+j}$ , we have

$$\mathcal{F}_i = \mathcal{M}_i \mathcal{R}_j^i \cdot \mathcal{R}_j^{\tau^i} \cdot \mathcal{R}_i^* \subseteq \mathcal{M}_{i+j} \mathcal{R}_{i+j}^* = \mathcal{F}_{i+j}.$$

Hence  $\mathcal{F}_i$  is an increasing family of right ideals of  $\mathcal{E}$ , which stabilizes, say to  $\mathcal{F}$ , for  $i \gg 0$ . This proves (i). Then for  $i, j \gg 0$ , (5.26) implies that

$$\mathcal{M}_i \mathcal{B}_j^{\tau^i} = \mathcal{F} \mathcal{R}_i \cdot \mathcal{B}_j^{\tau^i} = \mathcal{F} \mathcal{R}_{i+j} = \mathcal{M}_{i+j},$$

which proves (ii). Finally,  $\mathcal{M}_{i+j} = \mathcal{M}_i \mathcal{B}_j^{\tau^i}$  is a  $\mathcal{E}'^{\tau^{i+j}}$ -module, simply because  $\mathcal{B}_j'$  is a right  $\mathcal{E}'^{\tau^j}$ -module. ■

LEMMA 7.18. *Keep the notation of Lemma 7.17. For large  $i$ , and all  $j \gg i$ , the canonical map*

$$\overline{M}_i \otimes_k \overline{R}_j^i \longrightarrow H^0(Y, \mathcal{M}_i \mathcal{R}_j^i)$$

*is surjective.*

*Proof.* Recall that we are assuming that the conclusion of Theorem 7.2 holds for  $R$ . We fix a large integer  $i$ , and we choose  $j_0$  large enough so that  $\overline{R}_j = H^0(Y, \mathcal{R}_j)$  if  $j \geq j_0$ . For such  $j$ , we form the exact sequence of right  $\mathcal{E}'^{\tau^{i+j}}$ -modules

$$0 \longrightarrow \mathcal{K}_j \longrightarrow \overline{M}_i \otimes_k \mathcal{R}_j^i \xrightarrow{f_j} \mathcal{M}_i \mathcal{R}_j^i \longrightarrow 0.$$

If we tensor the map  $f_j$  on the right by  $\mathcal{B}_k'^{\tau^{i+j}}$ , (5.26) implies that we obtain the map  $f_{j+k}$ . Therefore,  $\mathcal{K}_{j+k} = \mathcal{K}_j \otimes_{\mathcal{E}'^{\tau^{i+j}}} \mathcal{B}_k'^{\tau^{i+j}}$ . Since  $\mathcal{B}_k'$  is an ample sequence, it follows that  $H^1(Y, \mathcal{K}_j) = 0$  if  $j \gg 0$ . For such  $j$ , the map  $H^0(Y, \overline{M}_i \otimes_k \mathcal{R}_j^i) \longrightarrow H^0(Y, \mathcal{M}_i \mathcal{R}_j^i)$  is surjective, as required. ■

LEMMA 7.19. *Keep the notation of Lemma 7.17. Let  $k$  be large enough so that the supports of  $\Omega$  and  $\sigma^{-k}\Omega$  are disjoint. Then, for large  $i$  and all  $j \gg i$  one has*

$$(i) \quad \mathcal{M}_i \mathcal{R}_j^{\tau^i} + \mathcal{M}_{i+k} \mathcal{R}_{j-k}^{\tau^{i+k}} = \mathcal{M}_{i+j},$$

(ii) *The map  $H^0(Y, \mathcal{M}_i \mathcal{R}_j^{\tau^i}) \oplus H^0(Y, \mathcal{M}_{i+k} \mathcal{R}_{j-k}^{\tau^{i+k}}) \longrightarrow H^0(Y, \mathcal{M}_{i+j})$  is surjective.*

*Proof.* (i) By Lemma 7.17(ii),  $\mathcal{M}_i \mathcal{R}_j^{\tau^i} = \mathcal{M}_{i+j}$  at all points  $q \in Y$  at which  $\mathcal{R}_j^{\tau^i} = \mathcal{B}_j'^{\tau^i}$ ; that is, at all points  $q \notin \text{Supp}(\sigma^{-i}\Omega)$  (see Lemma 5.21). Similarly,  $\mathcal{M}_{i+k} \mathcal{R}_{j-k}^{\tau^{i+k}} = \mathcal{M}_{i+j}$  holds at all points at which  $\mathcal{R}_{j-k}^{\tau^{i+k}} = \mathcal{B}_{j-k}'^{\tau^{i+k}}$ , and these are the points  $q \notin \text{Supp}(\sigma^{-(i+k)}\Omega)$ . By the choice of  $k$ , these two sets cover  $Y$ .

(ii) We form an exact sequence of right  $\mathcal{E}'^{\tau^{i+j}}$ -modules

$$0 \longrightarrow \mathcal{L}_j \longrightarrow \mathcal{M}_i \mathcal{R}_j^{\tau^i} \oplus \mathcal{M}_{i+k} \mathcal{R}_{j-k}^{\tau^{i+k}} \longrightarrow \mathcal{M}_{i+j} \longrightarrow 0.$$

As in the proof of Lemma 7.18, one finds that  $\mathcal{L}_{j+k} \cong \mathcal{L}_j \otimes \mathcal{B}^{\tau^{i+j}}$ , hence that  $H^1(Y, \mathcal{L}_j) = 0$  if  $j \gg 0$ . The lemma follows by taking global sections. ■

LEMMA 7.20. *Keep the notation of Lemma 7.17. Then*

(i) *The map  $(\overline{M}_i \otimes_k \overline{R}_j^{\tau^i}) \oplus (\overline{M}_{i+k} \otimes_k \overline{R}_{j-k}^{\tau^{i+k}}) \longrightarrow \overline{M}_{i+j}$  is surjective for  $i, j, k$  as in Lemma 7.19.*

(ii) *For large  $n$ ,  $\overline{M}_n = H^0(Y, \mathcal{M}_n)$ .*

(iii)  *$M$  is a finitely generated right ideal of  $R$ .*

*Proof.* Combining Lemmas 7.18 and 7.19(ii) we find that, for  $i, j, k$  as above, the map

$$(\overline{M}_i \otimes_k \overline{R}_j^{\tau^i}) \oplus (\overline{M}_{i+k} \otimes_k \overline{R}_{j-k}^{\tau^{i+k}}) \xrightarrow{\psi} H^0(Y, \mathcal{M}_{i+j})$$

is surjective. Since  $\text{Im}(\psi) \subseteq \overline{M}_{i+j} \subseteq H^0(Y, \mathcal{M}_{i+j})$ , the first two assertions of the lemma follow. Part (iii) follows from (i). ■

*Proof of Theorem 7.1.* If some Veronese ring  $R^{(m)}$  of  $R$  satisfies  $(\dagger)$ , then Lemma 7.20 implies that  $R^{(m)}$  is right noetherian and so, by Proposition 3.2,  $R$  is also right noetherian. By Lemma 2.13(ii), the opposite ring  $(R^{(m)})^{\text{op}}$  satisfies  $(\dagger)$ , and so  $R^{\text{op}}$  is right noetherian. Equivalently,  $R$  is left noetherian. Conversely, if  $R$  is right noetherian, then Theorem 3.5 implies that some Veronese ring  $R^{(m)}$  satisfies  $(\dagger)$ . ■

*Proof of Theorem 7.2.* By Lemma 7.16, the theorem holds for some Veronese ring  $S = R^{(m)}$ . Set  $M^i = \oplus_{j \geq 0} R_{mj+i}$ , for  $0 \leq i \leq m-1$ . Up to a shift of degree, this is a graded right  $S$ -module and so, applied to this  $S$ -module, Lemma 7.20 implies that  $R_{mj+i} \cong H^0(Y, \mathcal{O}_Y R_{mj+i})$ , for all  $0 \leq i \leq m-1$  and all  $j \gg 0$ . ■

*Remark 7.21.* Let  $R$  be a two-dimensional algebra such that some Veronese ring  $R^{(m)}$  satisfies  $(\dagger)$ . Although Theorem 7.2 appears to determine  $R$  in terms of infinitely many pieces of geometric data, in the form of the  $\mathcal{R}_n$ , in fact only finitely many pieces of geometric data is required. Indeed, for  $m$  sufficiently large, Corollary 7.3 shows that  $R^{(m)}$  is determined by  $Y, \mathcal{E}, \mathcal{B}_1, \tau$  and the left ideal  $(R^{(m)})_{\geq 1}$  of  $B(\mathcal{E}, \mathcal{B}_1, \tau)$ . Then, as in the proof of Theorem 7.2, the algebra  $R$  is determined (up to a finite dimensional vector space) by  $R^{(m)}$  together with the modules  $M^i = \oplus_{k \geq 0} R_{ki+m}$ , for  $1 \leq i \leq m-1$ . By the equivalence of Corollary 7.3(iv), one may also regard these modules as objects in  $\text{mod-}\mathcal{E}$ .

By [AS, Proposition 6.1], the  $i$ -critical  $R$ -modules generated in degrees  $m\mathbb{Z}$  are in bijection with the  $i$ -critical  $R^{(m)}$ -modules, for any integer  $i$ . In this way one may also relate  $\overline{\mathbf{gr}}\text{-}R$  to  $\text{mod-}\mathcal{E}$ , as was done for domains in [AS, Sect. 6]. For example, combined with Theorem 7.2, [AS, Proposition 6.1] implies.

**COROLLARY 7.22.** *Let  $R$  be a two-dimensional algebra such that some Veronese ring  $R^{(m)}$  satisfies  $(\dagger)$ , and let  $\mathcal{E}$  denote the stable order for  $R$ . Then, the simple objects in  $\overline{\mathbf{gr}}\text{-}R$  generated in degrees  $m\mathbb{Z}$  are in one-to-one correspondence with the simple  $\mathcal{E}$ -modules. ■*

The converse to Theorem 7.2 (and Theorem 5.15) is given by the next result.

**THEOREM 7.23.** *With the notation of (0.11), suppose given: a  $\sigma$ -stable complete model  $Y$  of  $K$ , an ample lattice  $\mathcal{B}_1$  in  $A$ , and an essential left  $E(\mathcal{B}_1)$ -submodule  $\mathcal{R}_1$  of  $\mathcal{B}_1$ . Assume that  $1 \in H^0(Y, \mathcal{R}_1)$  and that  $\mathcal{R}_1 = \mathcal{B}_1$  locally on every finite orbit of  $\sigma$  on  $Y$ . Set  $\mathcal{B}_{n-1} = \mathcal{B}_1 \cdot \mathcal{B}_1^\tau \cdots \mathcal{B}_1^{\tau^{(n-2)}}$  and  $\mathcal{R}_n = \mathcal{B}_{n-1} \cdot \mathcal{R}_1^{\tau^{(n-1)}}$ , for all  $n \geq 1$ . Then  $R = k + \bigoplus_{n \geq 1} H^0(Y, \mathcal{R}_n)$  is a noetherian nice algebra such that some Veronese  $R^{(m)}$  satisfies  $(\dagger)$ .*

*Proof.* In order for the theorem to make sense, we first need to check that  $\mathcal{R}_1$  is invertible, with  $E(\mathcal{R}_1) = \mathcal{E}$ . As  $\mathcal{R}_1$  is essential in  $\mathcal{B}_1$ , it is a lattice. By hypothesis, invertibility is trivial on the finite orbits of  $\sigma$ , so consider what happens locally at point  $p \in Y$  on an infinite orbit. Then, Theorem 5.9 applied to the twisting ring  $B = B(\mathcal{E}, \mathcal{B}_1, \tau)$ , shows that  $\mathcal{E}$  is a maximal order and so  $\mathcal{E} = E(\mathcal{R}_1)$  at  $p$ . Thus, by Proposition 1.15,  $\mathcal{R}_1$  is invertible.

By Theorems 7.1 and 7.2, it suffices to prove that some Veronese ring of  $R$  is a nice algebra satisfying  $(\dagger)$ . We replace  $R$  by some such Veronese ring  $R^{(m)}$  so that the finite orbits of  $\sigma$  are singletons. Thus, (2.9c) holds. If  $\mathcal{G} = \mathcal{B}_1^* \cdot \mathcal{R}_1$ , then  $\mathcal{R}_n = \mathcal{B}_n \cdot \mathcal{G}^{\tau^{n-1}}$  and so Lemma 6.2 implies that  $\mathcal{R}_n$  is generated by its sections for all  $n \gg 0$ . Since  $\mathcal{R}_n$  is invertible and  $\mathcal{E}$  is an order in  $A$ , it follows that  $A$  is generated (as a semisimple ring) by those sections for all large  $n$ . Consequently, if we set  $z = 1 \in H^0(Y, \mathcal{R}_1)$ , it follows that  $R$  is an order in  $Q = A[z, z^{-1}; \tau]$  and hence that  $R$  is a nice algebra. Conditions (2.9a), (2.9b), and (2.9c) follow from Lemma 1.13(iii) combined with the fact that  $\mathcal{R}_n = \mathcal{B}_n$  on all finite orbits of  $\sigma$ . ■

In this theorem, it is the fact that  $\mathcal{R}_1 \neq \mathcal{B}_1$  that highlights the differences between the commutative and noncommutative situations. If one allows  $\mathcal{R}_1 \neq \mathcal{B}_1$  to hold on a finite orbit of  $\sigma$  one can, for example, obtain the non-noetherian, commutative ring  $k + xk[x, y]$  by this construction. In contrast, algebras like Example 5.13 arise when  $\mathcal{R}_1 \neq \mathcal{B}_1$  on some infinite orbit.

We end this section by illustrating how these results can be used to describe rings that do not contain regular elements in degree one. In this case, since the element  $z \in R_1$  does not exist, there is no reason to suppose that our twisting is achieved by some automorphism  $\tau$  (see Example 7.25). Thus, we need a more general notion of twisted coordinate ring. This is defined as follows: Let  $\mathcal{B}_1$  be a  $\mathcal{E}$ -bimodule (not necessarily contained in  $A$ ) and assume that  $\mathcal{B}_1$  is locally projective on both sides with  $\mathcal{B}_1 \otimes_{\mathcal{E}} \mathcal{B}_1^* \cong \mathcal{E}$  and  $\mathcal{B}_1^* \otimes_{\mathcal{E}} \mathcal{B}_1 \cong \mathcal{E}$ . If the sequence  $\mathcal{B}_n = \mathcal{B}_1 \cdots \mathcal{B}_1^{\tau^{n-1}}$  is ample, then the *bimodule algebra*  $\mathbb{B}(\mathcal{E}, \mathcal{B}_1) = \bigoplus \mathcal{B}_n$  and the *twisting ring*  $B = B(\mathcal{E}, \mathcal{B}_1) = \bigoplus H^0(Y, \mathcal{B}_n)z^n$  can be defined just as was done for ample sequences of lattices in Sect. 6. However, as the twist  $\tau$  has been absorbed into the bimodule structure,  $z$  is now a dummy, commuting variable. Corollary 6.11 still holds in this generality, with the same proof.

**PROPOSITION 7.24.** *Let  $R = \bigoplus_{i \geq 0} R_i$  be a two-dimensional algebra that satisfies a polynomial identity and such that  $R_i R_j = R_{i+j}$ , for all  $i, j \gg 0$ . Then,  $R$  is noetherian if and only if, up to a finite dimensional vector space,  $R$  is the twisting ring  $B(\mathcal{E}, \mathcal{S}_1)$  of an ample bimodule  $\mathcal{S}_1$ .*

*Proof.*  $\Leftarrow$  This is Corollary 6.11.

$\Rightarrow$  If  $R$  is noetherian, then so is each  $R^{(n)}$ , by Proposition 3.2. Thus, by Theorem 3.5 we may choose  $m$  so that  $B = R^{(m)}$  is a nice algebra satisfying  $(\dagger)$ . Since  $B$  is PI,  $|\sigma| < \infty$  and so  $(\dagger\dagger)$  also holds. Thus, by Corollary 7.3,  $B = B(\mathcal{E}, \mathcal{B}_1^\tau)$  is a twisting ring. As usual, we write  $B = \bigoplus B_i = \bigoplus B_i z^i$ . Let  $M^i = \bigoplus_{j \geq 1} R_{mj+i}$ , as in the proof of Theorem 7.2, which we regard as a graded  $B$ -bimodule, by shifting degrees; thus  $R_{mj+i}$  has degree  $j$ . Set  $\mathcal{M}_{mj+i} = {}_{\mathcal{O}_Y} R_{mj+i}$ . Proposition 3.2 implies that  $M^i$  is a finitely generated right  $B$ -module and so, as  $B$  is generated in degree one,  $R_{m(j-1)+i} B_1 = R_{mj+i}$ , for all  $j \gg 0$ . Equivalently,  $\mathcal{M}_{m(j-1)+i} \mathcal{B}_1^{\tau^{j-1}} = \mathcal{M}_{mj+i}$  and therefore, since  $\mathcal{B}_1$  is a right  $\mathcal{E}^\tau$ -module, one sees that  $\mathcal{M}_{mj+i}$  is a right  $\mathcal{E}^{\tau^j}$ -module. Similarly, it is a left  $\mathcal{E}$ -module.

Fix  $j \gg 0$  and set  $\mathcal{S}_{mj+i} = \mathcal{M}_{mj+i}$ . Then define  $\mathcal{S}_{mk+i} = \mathcal{M}_{mj+i} \cdot (\mathcal{B}_{j-k}^{\tau^k})^*$ , for any  $0 \leq i \leq n-1$  and all  $k \geq 0$ . As in the proof of Theorem 5.15, this is an  $(\mathcal{E}, \mathcal{E}^{\tau^k})$ -bimodule that is independent of the choice of  $j$ . By transport of the right structure we regard these modules  $\mathcal{S}_\ell$  as  $(\mathcal{E}, \mathcal{E})$ -bimodules. The hypothesis that  $R_u R_v = R_{u+v}$ , for all  $u, v \gg 0$ , translates into the equation:

$$\mathcal{S}_{i+km} \otimes_{\mathcal{E}} \mathcal{S}_1 = \begin{cases} \mathcal{S}_{i+1+km} & \text{if } i < m-1 \\ \mathcal{S}_{(k+1)m} & \text{if } i = m-1. \end{cases}$$

(The two cases are needed because of the way in which the modules  $M^i$  were shifted to make them into  $B$ -modules.) Equivalently,  $\mathcal{S}_a \mathcal{S}_b = \mathcal{S}_{a+b}$ , for all  $a, b \geq 1$ . Finally,  $R_\ell = H^0(Y, \mathcal{S}_\ell)$ , for all large  $\ell$ , by Lemma 7.20 applied to the  $B$ -modules  $M^i$ . Thus, we have shown that, up to a finite dimensional



factor,  $R$  is a twisting ring  $B(\mathcal{E}, \mathcal{S}_1)$ . The amplitude of  $\mathcal{S}_1$  follows from the amplitude of  $\mathcal{B}_1 = \mathcal{S}_1^{\otimes m}$ . ■

Here is a typical example of the sort of ring that come up in the construction of this proposition.

EXAMPLE 7.25. Let  $C = k[x, y]$ , graded by  $\deg(x) = \deg(y) = 1$ , write  $C^0$ , respectively  $C^1$ , for the elements of even and odd degree. As usual, we grade matrix rings by defining the matrix units to have degree zero. Then, set

$$R = \begin{pmatrix} C^0 & C^0 & C^1 \\ C^0 & C^0 & C^1 \\ C^1 & C^1 & C^0 \end{pmatrix} \supset R^{(2)} = \begin{pmatrix} C^0 & C^0 & 0 \\ C^0 & C^0 & 0 \\ 0 & 0 & C^0 \end{pmatrix}.$$

Clearly,  $R$  is generated in degree one and  $B = R^{(2)}$  satisfies  $(\dagger\dagger)$ . In this case, we find that  $R = B(\mathcal{E}, \mathcal{S}_1)$ , where

$$\mathcal{E} = \begin{pmatrix} \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1} & 0 \\ \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1} & 0 \\ 0 & 0 & \mathcal{O}_{\mathbb{P}^1} \end{pmatrix} \quad \text{and} \quad \mathcal{S}_1 = \begin{pmatrix} 0 & 0 & \mathcal{O}(1) \\ 0 & 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) & 0 \end{pmatrix}.$$

This example also shows that one really does need the more general form of a twisting ring in Proposition 7.24, since  $\mathcal{S}_1$  does not embed in the quotient ring  $\mathcal{A}$  of  $\mathcal{E}$  and the twisting given by tensoring with  $\mathcal{S}_1$  cannot be replaced by an automorphism of  $\mathcal{A}$ . Indeed, since  $R$  has no regular elements in odd degrees, there is not even an element  $z \in R_1$  that one can use to define the automorphism  $\tau$ .

## 8. NON-NOETHERIAN RINGS

In this section,  $k$  can be any field. It is easy to write down finitely generated, non-noetherian, graded rings, standard examples being

$$\tilde{R} = \begin{pmatrix} k[x, y] & k[x, y] \\ yk[x, y] & k + yk[x, y] \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} k[x] & k[x] \\ 0 & k \end{pmatrix}. \quad (8.1)$$

Here,  $\tilde{R}$  is a prime ring of GK-dimension 2, while its factor ring  $\tilde{S}$  has GK-dimension 1. The prime ideals  $P = \begin{pmatrix} 0 & k[x] \\ 0 & k \end{pmatrix}$  and  $M = \begin{pmatrix} k[x] & k[x] \\ 0 & 0 \end{pmatrix}$  of  $S = \tilde{S}$  have the following abstract properties.

(8.2)  $S$  is a locally finite, left noetherian, but not right noetherian, graded  $k$ -algebra of GK-dimension 1. Moreover, there exists a graded maximal ideal  $M \supseteq S_{\geq 1} = \bigoplus_{i \geq 0} S_i$  and a graded prime ideal  $P$  with  $\text{GK-dim}(S/M) = 0$  and  $\text{GK-dim}(S/P) = 1$ , such that  $PM = 0$  but  $P \cap M$  is a non-zero, torsion-free (and hence infinite dimensional) left  $S/P$ -module.

One consequence of (8.2) is that there are bad linkages  $P \rightsquigarrow M$  between prime ideals of  $S$ , as summarized in the next result.

**LEMMA 8.3.** *Let  $S$  be a ring satisfying the properties of (8.2). Then  $P$  and  $M$  are prime ideals of  $S$  with  $\text{GK dim } S/P > \text{GK dim } S/M$ , such that  $P \rightsquigarrow M$ , in the sense that  $P \cap M/PM$  is neither torsion as a left  $S/P$ -module nor as a right  $S/M$ -module. Also, there are infinitely many ungraded maximal ideals  $Q$  of  $S$  such that  $Q \rightsquigarrow M$ .*

*Proof.* Only the final assertion needs proof. By [SW],  $S/Q$  is a noetherian PI ring, for every prime ideal  $Q$  of  $S$ . If  $Q$  is such an ideal, and  $L$  is a left  $S$ -module, let  $\rho(L, Q)$  denote the torsion-free rank of  $L/QL$ ; that is, if  $D$  is the (ungraded) Goldie quotient ring of  $S/Q$ , then  $\rho(L, Q)$  is the length of  $D \otimes_{S/Q} L/QL$  divided by the length of  $D$ . As  $S/P$  has Gelfand–Kirillov dimension one, it has infinitely many maximal ideals  $Q$  [SW] and  $\rho(M, Q) = \rho(M, P)$  for all but finitely many of them (see [Wa, Theorem 4]). Since  $\rho(M, P) = \rho(M/M \cap P, P) + \rho(M \cap P, P) > 1$ , this implies that  $\rho(M, Q) > 1$  for infinitely many  $Q$ . In particular,  $QM \subsetneq Q \cap M$  and  $Q \rightsquigarrow M$  for any such  $Q$ . ■

**Remark 8.4.** Let  $S$  be a ring of Gelfand–Kirillov dimension one for which one of the conclusions of Lemma 8.3 holds; in other words, assume that either  $P \rightsquigarrow M$ , for prime ideals  $P$  and  $M$  with  $\text{GK dim } S/P > \text{GK dim } S/M$ , or that there exist infinitely many prime ideals  $Q$  such that  $Q \rightsquigarrow M$ , for one prime ideal  $M$ . Then  $S$  is not noetherian. Indeed, suppose that  $S$  is noetherian. Then  $S$  is PI, by [SSW], and the two cases contradict [KL, Corollary 5.4], respectively [Wa, Theorem 4].

The aim of this section is to prove the converse to this observation.

**LEMMA 8.5.** *Let  $R$  be a graded, semiprime Goldie ring and let  $P$  be an essential, graded ideal of the Veronese  $R^{(n)}$ . There exists an essential, graded ideal  $I$  of  $R$  such that  $I^{(n)} \subseteq P$ .*

*Proof.* As in the proof of Proposition 3.2, set  $L_i = L_i(R) = \bigoplus_k R_{i+nk}$ , for  $0 \leq i \leq n-1$ . We will regard the  $L_i$  as indexed by  $\mathbb{Z}_n$ . Thus, the  $L_i$  are  $R^{(n)}$ -bimodules with  $L_i L_j \subseteq L_{i+j}$ . Set  $K_i = L_{-i} P L_i + r\text{-ann}_{R^{(n)}} L_{-i} L_i$ . We first show that each  $K_i$  is an essential ideal of  $R^{(n)}$ . Let  $Q$  denote the graded quotient ring of  $R$ ; thus, by Lemma 3.1,  $Q^{(n)} = Q(R^{(n)})$ . Moreover,

$RQ^{(n)} = Q$  and  $L_i(R) = R \cap L_i(Q)$ . Thus,  $L_iQ^{(n)} = L_i(Q) = Q^{(n)}L_i$ , while, as  $P$  is essential,  $PQ^{(n)} = Q^{(n)}$ . Hence,

$$L_{-i}PL_iQ^{(n)} = L_{-i}PQ^{(n)}L_i = L_{-i}L_iQ^{(n)}$$

and so  $r\text{-ann}_{R^{(n)}}(L_{-i}PL_i) = R \cap r\text{-ann}_{Q^{(n)}}(L_{-i}PL_iQ^{(n)}) = r\text{-ann}_{R^{(n)}}(L_{-i}L_i)$ . Therefore,  $K_i = V + r\text{-ann}(V)$ , where  $V = L_{-i}PL_i$ . Any such ideal is essential.

The ideal we want is  $I = RJR$ , where  $J = \bigcap_0^{n-1} K_i$ . First,  $J$  is essential, so it contains a regular element  $\alpha$  of  $R^{(n)}$ . Then,  $\alpha$  is a regular element of  $R$  contained in  $I$  and so  $I$  is essential. Next, since  $I = RJR$ ,  $I^{(n)} = \sum_{i=0}^{n-1} L_iJL_{-i}$ . However,  $W = L_i(r\text{-ann}(L_{-i}L_i))L_{-i}$  is a nilpotent ideal of the semiprime ring  $R^{(n)}$  and hence is zero. Thus,

$$L_iJL_{-i} \subseteq L_iK_iL_{-i} \subseteq L_iL_{-i}PL_iL_{-i} \subseteq P.$$

Hence,  $I^{(n)} \subseteq P$ , as required. ■

**THEOREM 8.6.** *Let  $R$  be a finitely generated, semiprime graded Goldie ring with  $\text{GK-dim}(R) = 2$ . Suppose that  $R$  is not right noetherian. Then,  $R$  has a factor ring  $S$  that satisfies the properties of (8.2) and (8.3).*

*Proof.* As  $R$  is not noetherian, neither is  $R/Q$ , for some minimal prime ideal  $Q$ . Since  $Q$  is automatically graded [NV, Theorem A.II.7.3], we may replace  $R$  by  $R/Q$  and assume that  $R$  is prime. By [SW],  $R$  still has Gelfand–Kirillov dimension two and so  $R$  is a two-dimensional algebra, by [AS, Theorem 1.1].

By Theorem 7.1, no Veronese ring  $R^{(n)}$  satisfies (†). There are two possible cases: First, some  $R^{(n)}$  may not be finitely generated. In this case,  $[R^{(n)}]_{\geq 1}$  will not be finitely generated as a right ideal and so, by the graded version of Nakayama's lemma,  $R^{(n)}/[R^{(n)}]_{\geq 1}^2$  will be a proper, non-noetherian factor ring of  $R^{(n)}$ . Alternatively, suppose that  $R^{(n)}$  is always finitely generated. Then, some such Veronese ring is a finitely generated nice algebra. Consequently, by Lemma 3.7, there exists a Veronese ring  $R^{(n)}$  with a non-zero ideal  $F$  such that  $R^{(n)}/F$  is not right noetherian. In either case, Lemma 8.5 implies that  $R$  has a proper factor ring  $\Lambda$  that is not right noetherian either. By [KL, Proposition 3.15],  $\text{GK-dim}(\Lambda) \leq 1$ .

Let  $J$  denote the nilradical of  $\Lambda$ . By [SSW],  $\Lambda/J$  is a finite module over its noetherian center and  $J$  is nilpotent. Since  $\Lambda$  is not right noetherian, some  $J^i/J^{i-1}$  is not finitely generated as a right  $\Lambda$ -module. Then  $J^i$  is not finitely generated either. By Zorn's lemma, there is a graded ideal  $I \supseteq J^i$  which is maximal with the property that  $I_\Lambda$  is not finitely generated, and by the graded Nakayama lemma,  $I/I_{\geq 1}$  is not finitely generated. Then there exists a maximal ideal  $L \supseteq \Lambda_{\geq 1}$  such that  $I/IL$  is not finitely generated. Set  $S = \Lambda/IL$ ,  $P = I/IL$  and  $M = L/IL$ . Thus,  $P$  is not finitely generated as a

right  $S$ -module. We claim that  $P$  is a prime ideal of  $S$ . If not, then [SSW] implies that there exists a graded ideal  $J \supsetneq P$  such that  $JQ \subseteq P$ , for some (graded) prime ideal  $Q \supsetneq P$ . The maximality of  $P$  ensures that both  $J$  and  $Q$  are finitely generated right  $S$ -modules. Hence, so is  $JQ$ . So  $X = P/JQ$  is still infinitely generated as a right  $S/M$ -module. This contradicts the fact that  $X$  is a submodule of  $J/JQ$ , which is a finitely generated module over the noetherian ring  $S/Q$ . Thus  $P$  must be prime.

In summary,  $P$  and  $M$  are prime ideals of  $S$  and the nilradical of  $S$  is  $N = P \cap M$ , with  $N^2 = 0$ . Also,  $N$  is infinite dimensional and  $S/P$  is noetherian, and  $\text{GK-dim}(S) \leq 1$ . Since  $S$  is finitely generated, but infinite dimensional,  $\text{GK-dim}(S) = 1$ .

**SUBLEMMA 8.7.** (i) *The nilradical  $N$  is finitely generated as a left ideal.*

(ii)  *$S$  is left noetherian.*

*Proof.* (i) Since  $PM = 0$ , we have  $PN = 0$  and  $NM = 0$ . So  $N$  is an  $(A, B)$ -bimodule, where  $A = S/P$  and  $B = S/M$ . It suffices to show that  $N$  is finitely generated as a two-sided ideal, i.e., as a left  $A \otimes B^{\text{op}}$ -module, because  $B$  has finite dimension over  $k$ . As before, [SSW] implies that  $S/N$  is a finite module over its finitely presented center  $Z$ . Thus,  $S/N$  is finitely presented, and this implies that  $N$  is finitely generated as a two-sided ideal. To see this, suppose that  $S/N = F/J$ , where  $F$  is a finitely generated free  $k$ -algebra and  $J$  is a finitely generated two-sided ideal. The projection  $F \rightarrow S/N$  lifts to a surjective map  $F \rightarrow S$ , and the image of  $J$  in  $S$  is the ideal  $N$ , as required.

(ii) Since  $S/N$  is noetherian, this follows from part (i). ■

We return to the proof of the theorem. The sum  $t(N)$  of the finite dimensional, left  $S$ -submodules of  $N$  is finite dimensional and hence  $N/t(N)$  is infinite dimensional. This also implies that  $\text{GK-dim}(S/P) = 1$ . Thus, if one replaces  $S$  by  $S/t(N)$  then  $S$  satisfies all the properties of (8.2) and hence (8.3). ■

Finally, we remark that any ring  $S$  satisfying (8.2) is very close to the ring  $\tilde{S}$  of (8.1). More precisely, that there is a factor  $S'$  of  $S$  such that  $\tilde{S} \hookrightarrow S'$  with  $S'$  finitely generated as both a left and a right  $\tilde{S}$ -module. The proof is left to the interested reader.

## 9. GENERALIZATIONS

We retain the notation of (0.11), but we do not assume that the ground field  $k$  is infinite. Indeed, the first aim of this section is to extend the main results of the paper to two-dimensional algebras over an arbitrary field.

PROPOSITION 9.1. *Let  $R$  be a noetherian two-dimensional algebra over a field  $k$ , and let  $k'$  be a separable algebraic field extension of  $k$ . Then  $R' = R \otimes k'$  is noetherian if and only if  $R$  is noetherian.*

*Proof.* It is clear that  $R$  is noetherian if  $R'$  is. Conversely, suppose that  $R'$  is not noetherian. The data (0.11) carry over to  $k'$  by base extension. Since  $R$  is semiprime, so is  $R'$ . Also,  $R'$  is a Goldie ring because the regular elements  $S$  of  $R$  form an Ore set in  $R'$ , and  $S^{-1}R' = Q(A) \otimes k' = Q'$ . In other words,  $R'$  is a two-dimensional algebra over  $k'$ . It is finitely generated because  $R$ , being noetherian, is finitely generated. The assertion of the proposition is trivial when  $[k' : k] < \infty$ , so we may assume that  $k'$  is an infinite field, hence that the results of the previous sections are true for  $R'$ , and we are free to replace  $k$  by a finite field extension. We may also assume that  $R$  is prime. Then every proper quotient of  $R$  has GK-dimension  $\leq 1$ . (See [AS, Theorem 1.2].)

If  $R'$  is not noetherian then, by Theorem 8.6, it has a non-noetherian quotient  $\Lambda'$  of GK-dimension 1. The image  $\Lambda$  of  $R$  in  $\Lambda'$  is also of GK-dimension 1. Then, since  $\Lambda'$  is a quotient of  $\Lambda \otimes k'$ , it follows that  $\Lambda \otimes k'$  is not noetherian. Since  $\text{GK dim}_k \Lambda \leq 1$ , [SSW] implies that  $\Lambda$  satisfies a polynomial identity and so, by [Rw, Ex. 7, p.180],  $\Lambda \otimes k'$  is also noetherian. This contradiction proves the result. ■

Let  $k$  be a finite field and let  $k'$  be an infinite separable extension of  $k$ . Let  $R$  be a nice algebra over  $k$  and let  $'$  denote base change to  $k'$ . Since the data in Theorems 5.9 and 5.15 are canonical, the next two corollaries follow by descent.

COROLLARY 9.2. *With the above notation, assume that the nice algebra  $R' = R \otimes_k k'$  satisfies  $(\dagger)$ , and let  $Y', \mathcal{R}'_n, \mathcal{G}'$ , be the stable model and the lattices described in Theorems 5.9, 5.15 for  $R'$ . There exist canonical data consisting of a stable model  $Y$  of  $K$ , and lattices  $\mathcal{R}_n, \mathcal{G}$  such that conditions (i)–(v) of Theorem 5.9 and (i)–(v) of Theorem 5.15 hold for  $R$ . Moreover,  $R$  is noetherian. ■*

COROLLARY 9.3. *With the above notation, assume that  $R$  is noetherian, and let  $m$  be an integer such that the Veronese ring  $R^{(m)}$  satisfies  $(\dagger)$ . Let  $Y$  denote the stable model for  $R^{(m)}$ , in the sense of Corollary 9.2, and let  $\mathcal{R}_n = \mathcal{O}_Y \bar{R}_n$ . Then  $\bar{R}_n = H^0(Y, \mathcal{R}_n)$  for  $n \gg 0$ . The analogues of Corollary 7.3 and Theorem 7.23 also hold for  $R$ . ■*

The second generalization that can be made to the results of this paper concerns the assumption, in the definition of a two-dimensional algebra, that  $\text{GK-dim}(R/P) \leq 2$  for all minimal prime ideals  $P$ . Throughout, this can be weakened to the assumption that  $\text{GK-dim}(R) \leq 2$ . To make this precise, assume that  $R$  is a finitely generated, semiprime Goldie, locally finite graded algebra with  $\text{GK-dim}(R) \leq 2$ . (We have returned to the case of finitely generated algebras since this is the case that interests us and, as Theorem 7.1 shows, finite generation is automatic for the algebras we have

been considering.) Let  $I_2 = I_2(R)$  denote the intersection of the minimal prime ideals  $P$  of  $R$  such that  $\text{GK-dim}(R/P) = 2$  and let  $I_1 = I_1(R)$  denote the intersection of the remaining minimal prime ideals of  $R$ . Then, as  $I_1$  and  $I_2$  are incomparable,  $R$  has finite codimension in  $R/I_1 \oplus R/I_2$ . By construction,  $R/I_2$  is a two-dimensional algebra (providing that it is non-zero) and so it remains to consider  $R/I_1$ . We may, in turn, write  $R/I_1 = S \oplus S'$ , where  $S'$  is finite dimensional and  $S$  has no finite dimensional algebra summands. Obviously, it suffices to study  $S$ . By [SSW],  $S$  is a noetherian ring, finitely generated as a module over its center and so its structure is easy to describe. The following lemma gives the required properties.

**LEMMA 9.4.** *Let  $S$  be a finitely generated, semiprime Goldie, locally finite graded algebra such that  $\text{GK-dim}(S) = 1$  and  $S$  has no finite dimensional algebra summands. Then*

(i) *Pick a regular element  $z \in S_n$  and set  $T = S^{(n)}$ . Then, there exists a semisimple artinian, finite dimensional  $k$ -algebra  $D$ , with an automorphism  $\theta$ , such that  $T$  has finite codimension in the Ore extension  $D[z; \theta]$ . In turn,  $D[z; \theta]$  may be identified with the twisting ring  $B = B(D, L, \theta)$ , for  $L = D$ .*

(ii)  $\overline{\text{gr}}\text{-}T \approx \text{mod-}D$ .

*Proof.* (i) By [NV, Theorem C.I.1.6(2)],  $S$  does contain homogeneous regular elements in positive degree. Since  $z \in T_1$ ,  $\dim_k T_m \leq \dim_k T_{m+1}$  for all  $m \geq 1$ . However, since  $\text{GK-dim}(T) = 1$ ,  $\dim_k T_m$  is bounded above. Thus, there exists  $m_0$  such that  $\dim T_m$  is constant and  $T_{m+1} = T_m z = z T_m$ , for all  $m \geq m_0$ . Let  $Q(T)$  denote the graded quotient ring of  $T$ . For any other regular, homogeneous element  $r \in T$ , one has  $T_m r = T_{m+d}$ , for  $d = \deg(r)$  and  $m \gg 0$ . Thus,  $z^{m+d} = tr$ , for some  $t \in T$ , and so  $Q(T) = T[z^{-1}]$ . Write  $Q(T) = D[z, z^{-1}; \theta]$ . It follows easily that  $Q(T)_m = T_m$  for  $m \gg 0$  and so  $T$  has finite codimension in  $B = D[z; \theta]$ . Since  $\text{GK-dim}(D) \leq \text{GK-dim}(D[z; \theta]) - 1 = \text{GK-dim}(T) - 1 = 0$ , the algebra  $D$  is finite dimensional over  $k$ . Thus, the associated projective model  $W(S)$  of  $Z(D)/k$  consists of a finite number of points and  $W(S)$ -modules are equal to their global sections. In particular,  $B$  can be identified with the twisting ring  $B = B(D, L, \theta)$ , for  $L = D$ .

(ii) Use the proof of [AZ, Examples 5.4 and 5.5]. ■

One can view the lemma as saying that:  $S$  satisfies all the conclusions of (7.1)–(7.3), provided that one replaces  $Y$  by the variety  $W(S)$ . In this statement, we again define  $S^{(m)}$  to be nice if there exists a regular element  $z \in S_1^{(m)}$  and we ignore all references to  $(\dagger)$  or  $(\dagger\dagger)$ . In fact, the analogue of  $(\dagger\dagger)$  automatically holds for  $B$  since, as the reader may check,  $B = \bigoplus \overline{B}_i z^i$  with  $\overline{B}_i = D$  for all  $i$  and so  $\mathcal{P}(\overline{B}_i; p) = 0$ , for all  $p \in W(S)$ .

Combining these observations with the results from Sect. 7, we obtain

**COROLLARY 9.5.** *Let  $R$  is a finitely generated, semiprime Goldie, locally finite graded algebra with  $\text{GK-dim}(R) = 2$ . Let  $I_2$  denote the intersection of*

the minimal prime ideals  $P$  of  $R$  such that  $\text{GK-dim}(R/P) = 2$  and let  $I_1$  denote the intersection of the remaining minimal primes. Set  $R_v = R/I_v$ . Then

(i)  $R_2$  is a two-dimensional algebra,  $R_1$  has Gelfand–Kirillov dimension at most one, and  $R$  has finite codimension in  $R_2 \oplus R_1$ .

(ii) Define  $R$  to satisfy  $(\dagger)$  or  $(\dagger\dagger)$  if and only if  $R_2$  satisfies that condition. Then, up to a finite dimensional vector space, Theorems 7.1 and 7.2 and Corollary 7.3 all hold for  $R$ , provided that one replaces  $Y$  by the projective variety  $Y \cup W(R_1)$  of dimension one.

## ACKNOWLEDGMENT

This work was supported in part by NSF grants.

## GLOSSARY

Definitions are given in, or just above, the display whose number is indicated. Notation is also established in (0.11).

2.9	$(\dagger)$
7.6	$(\dagger')$
2.10	$(\dagger\dagger)$
6.10	ample lattice
6.1	ample sequence
5.25	$\mathcal{B}_n$
6.6	$\mathbb{B}$
6.5	bimodule algebra
6.6	coherent $\mathbb{B}$ -module
1.7	divisor $\text{div}(V)$
2.3	divisor sequence $D_n$
5.12	$\mathcal{E}$
5.15	$\mathcal{E}'$
2.17	gap divisor $\Omega$
5.15, 5.23	gap lattice $\mathcal{G}$
6.9	$\overline{\mathbf{gr}}\text{-}R, R\text{-}\overline{\mathbf{gr}}$
1.10	invertible lattice
1.6	lattice
1.8	left order $E(\mathcal{L})$ , right order $E'(\mathcal{L})$
8.3	linked prime ideals
0.1	nice algebra
1.7	norm $N$
1.14	normal lattice

1.1	$\nu$ , the valuation
1.2	pole, $\mathcal{P}(V)$
1.7	$\mathcal{P}(V, p)$
5.9	$\mathcal{R}_n$
5.15	$\mathcal{S}_n$
5.12	stable model $Y$ , stable order $\mathcal{E}$
6.9	tail $M_{\gg 0}$ of a module $M$
1.13	tensor product of lattices, $\mathcal{L} \cdot \mathcal{M}$
6.10	twisting ring
0.1	two-dimensional algebra
0.1	Veronese subring

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