

On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves

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Abstract

In this paper, we shall study finite generation of symbolic Rees rings of the defining ideal of the space monomial curves (t^a, t^b, t^c) for pairwise coprime integers a, b, c such that $(a, b, c) \neq (1, 1, 1)$. If such a ring is not finitely generated over a base field, then it is a counterexample to the Hilbert's fourteenth problem. Finite generation of such rings is deeply related to existence of negative curves on certain normal projective surfaces. We study a sufficient condition (Definition 3.6) for existence of a negative curve. Using it, we prove that, in the case of $(a + b + c)^2 > abc$, a negative curve exists. Using a computer, we shall show that there exist examples in which this sufficient condition is not satisfied.

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1. Introduction

Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k -algebra or not. It is called the *Hilbert's fourteenth problem*.

The first counterexample to this problem was discovered by Nagata [13] in 1958. An easier counterexample was found by Paul C. Roberts [15] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

The Hilbert's fourteenth problem is deeply related to the following question of Cowsik [2]. Let R be a regular local ring (or a polynomial ring over a field). Let P be a prime ideal of R . Cowsik asked whether the symbolic Rees ring

$$R_s(P) = \bigoplus_{r \geq 0} P^{(r)} T^r$$

of P is a Noetherian ring or not. His aim is to give a new approach to the Kronecker's problem, that asks whether affine algebraic curves are set theoretic complete intersection or not. Kronecker's problem is still open, however, Roberts [14] gave a counterexample to Cowsik's question in 1985. Roberts constructed a regular local ring and a prime ideal such that the completion coincides with Nagata's counterexample to the Hilbert's fourteenth problem. In Roberts' example, the regular local ring contains a field of characteristic zero, and the prime ideal splits after completion. Later, Roberts [15] gave a new easier counterexample to both Hilbert's fourteenth problem and Cowsik's question. In his new example, the prime ideal does not split after completion, however, the ring still contains a field of characteristic zero. It was proved that analogous rings of characteristic positive are finitely generated [9,10].

On the other hand, let $\mathfrak{p}_k(a, b, c)$ be the defining ideal of the space monomial curves (t^a, t^b, t^c) in k^3 . Then, $\mathfrak{p}_k(a, b, c)$ is generated by at most three binomials in $k[x, y, z]$. The symbolic Rees rings are deeply studied by many authors. Huneke [7] and Cutkosky [3] developed criteria for finite generation of such rings. In 1994, Goto, Nishida and Watanabe [4] proved that $R_s(\mathfrak{p}_k(7n-3, (5n-2)n, 8n-3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0 \pmod{3}$. In their proof of infinite generation, they proved the finite generation of $R_s(\mathfrak{p}_k(7n-3, (5n-2)n, 8n-3))$ in the case where k is of characteristic positive. Goto and Watanabe conjectured that, for any a, b and c , $R_s(\mathfrak{p}_k(a, b, c))$ is always finitely generated over k if the characteristic of k is positive.

On the other hand, Cutkosky [3] gave a geometric meaning to the symbolic Rees ring $R_s(\mathfrak{p}_k(a, b, c))$. Let X be the blow-up of the weighted projective space $\text{Proj}(k[x, y, z])$ at the smooth point $V_+(\mathfrak{p}_k(a, b, c))$. Let E be the exceptional curve of the blow-up. Finite generation of $R_s(\mathfrak{p}_k(a, b, c))$ is equivalent to that of the total coordinate ring

$$TC(X) = \bigoplus_{D \in \text{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

of X . If $-K_X$ is ample, one can prove that $TC(X)$ is finitely generated using the cone theorem (cf. [8]) as in [6]. Cutkosky proved that $TC(X)$ is finitely generated if $(-K_X)^2 > 0$, or equivalently $(a+b+c)^2 > abc$. Finite generation of $TC(X)$ is deeply related to existence of a negative curve C , i.e., a curve C on X satisfying $C^2 < 0$ and $C \neq E$. In fact, in the case where $\sqrt{abc} \notin \mathbb{Z}$, a negative exists if $TC(X)$ is finitely generated. If a negative exists in the case where the characteristic of k is positive, then $TC(X)$ is finitely generated by a result of M. Artin [1].

By a standard method (mod p reduction), if there exists a negative curve in the case of characteristic zero, then one can prove that a negative curve exists in the case of characteristic positive, therefore, $R_s(\mathfrak{p}_k(a, b, c))$ is finitely generated in the case of characteristic positive (cf. Lemma 3.4). In the examples of Goto, Nishida and Watanabe [4], a negative curve exists, however, $R_s(\mathfrak{p}_k(a, b, c))$ is not finitely generated over k in the case where k is of characteristic zero (cf. Remark 3.5 below).

In Section 2, we shall prove that if $R_s(\mathfrak{p}_k(a, b, c))$ is not finitely generated, then it is a counterexample to the Hilbert's fourteenth problem (cf. Theorem 2.1 and Remark 2.2).

In Section 3, we review some basic facts on finite generation of $R_s(\mathfrak{p}_k(a, b, c))$. We define sufficient conditions for X to have a negative curve (cf. Definition 3.6).

In Section 4, we shall prove that there exists a negative curve in the case where $(a + b + c)^2 > abc$ (cf. Theorem 4.3). We should mention that if $(a + b + c)^2 > abc$, then Cutkosky [3] proved that $R_s(\mathfrak{p}_k(a, b, c))$ is finitely generated. Moreover if we assume $\sqrt{abc} \notin \mathbb{Z}$, existence of a negative curve follows from finite generation. Existence of negative curves in these cases is an immediate conclusion of the cone theorem. Our proof of existence of a negative curve is very simple, purely algebraic, and do not need the cone theorem as Cutkosky's proof.

In Section 5, we discuss the degree of a negative curve (cf. Theorem 5.4). It is used in a computer programming in Section 6.1.

In Section 6.1, we prove that there exist examples in which a sufficient condition ((C2) in Definition 3.6) is not satisfied using a computer. In Section 6.2, we give a computer programming to check whether a negative curve exists or not.

2. Symbolic Rees rings of monomial curves and Hilbert's fourteenth problem

Throughout of this paper, we assume that rings are commutative with unit.

For a prime ideal P of a ring A , $P^{(r)}$ denotes the r th symbolic power of P , i.e.,

$$P^{(r)} = P^r A_P \cap A.$$

By definition, it is easily seen that $P^{(r)} P^{(r')} \subset P^{(r+r')}$ for any $r, r' \geq 0$, therefore,

$$\bigoplus_{r \geq 0} P^{(r)} T^r$$

is a subring of the polynomial ring $A[T]$. This subring is called the *symbolic Rees ring* of P , and denoted by $R_s(P)$.

Let k be a field and m be a positive integer. Let a_1, \dots, a_m be positive integers. Consider the k -algebra homomorphism

$$\phi_k : k[x_1, \dots, x_m] \rightarrow k[t]$$

given by $\phi_k(x_i) = t^{a_i}$ for $i = 1, \dots, m$, where x_1, \dots, x_m, t are indeterminates over k . Let $\mathfrak{p}_k(a_1, \dots, a_m)$ be the kernel of ϕ_k . We sometimes denote $\mathfrak{p}_k(a_1, \dots, a_m)$ simply by \mathfrak{p} or \mathfrak{p}_k if no confusion is possible.

Theorem 2.1. *Let k be a field and m be a positive integer. Let a_1, \dots, a_m be positive integers. Consider the prime ideal $\mathfrak{p}_k(a_1, \dots, a_m)$ of the polynomial ring $k[x_1, \dots, x_m]$.*

Let $\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t, T$ be indeterminates over k . Consider the following injective k -homomorphism

$$\xi : k[x_1, \dots, x_m, T] \rightarrow k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t)$$

given by $\xi(T) = \alpha_2/\alpha_1$ and $\xi(x_i) = \alpha_1\beta_i + t^{a_i}$ for $i = 1, \dots, m$.

Then,

$$\begin{aligned} & k(\alpha_1\beta_1 + t^{a_1}, \alpha_1\beta_2 + t^{a_2}, \dots, \alpha_1\beta_m + t^{a_m}, \alpha_2/\alpha_1) \cap k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t] \\ &= \xi(R_s(\mathfrak{p}_k(a_1, \dots, a_m))) \end{aligned}$$

holds true.

Proof. Set $L = k(\alpha_1\beta_1 + t^{a_1}, \dots, \alpha_1\beta_m + t^{a_m}, \alpha_2/\alpha_1)$. Set $d = \text{GCD}(a_1, \dots, a_m)$. Then, L is included in $k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d)$. Since

$$k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t] \cap k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d) = k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d],$$

we obtain the equality

$$L \cap k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t] = L \cap k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d].$$

By the commutativity of the diagram

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow & & \\ k[x_1, \dots, x_m, T] & \longrightarrow & k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d) & \supset & k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t^d] \\ & \searrow \xi & \downarrow & & \downarrow \\ & & k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t) & \supset & k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t] \end{array}$$

it is enough to prove this theorem in the case where $\text{GCD}(a_1, \dots, a_m) = 1$.

In the rest of this proof, we assume $\text{GCD}(a_1, \dots, a_m) = 1$.

Consider the following injective k -homomorphism

$$\tilde{\xi} : k[x_1, \dots, x_m, T, t] \rightarrow k(\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t)$$

given by $\tilde{\xi}(T) = \alpha_2/\alpha_1$, $\tilde{\xi}(t) = t$ and $\tilde{\xi}(x_i) = \alpha_1\beta_i + t^{a_i}$ for $i = 1, \dots, m$. Here, remark that $\alpha_2/\alpha_1, \alpha_1\beta_1 + t^{a_1}, \alpha_1\beta_2 + t^{a_2}, \dots, \alpha_1\beta_m + t^{a_m}, t$ are algebraically independent over k . By definition, the map ξ is the restriction of $\tilde{\xi}$ to $k[x_1, \dots, x_m, T]$.

We set $S = k[x_1, \dots, x_m]$ and $A = k[x_1, \dots, x_m, t]$. Let \mathfrak{q} be the ideal of A generated by $x_1 - t^{a_1}, \dots, x_m - t^{a_m}$. Then \mathfrak{q} is the kernel of the map $\tilde{\phi}_k : A \rightarrow k[t]$ given by $\tilde{\phi}_k(t) = t$ and $\tilde{\phi}_k(x_i) = t^{a_i}$ for each i . Since ϕ_k is the restriction of $\tilde{\phi}_k$ to S , $\mathfrak{q} \cap S = \mathfrak{p}$ holds.

Now we shall prove $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$ for each $r > 0$. Since \mathfrak{q} is a complete intersection, $\mathfrak{q}^{(r)}$ coincides with \mathfrak{q}^r for any $r > 0$. Therefore, it is easy to see $\mathfrak{q}^r \cap S \supset \mathfrak{p}^{(r)}$.

Since $\text{GCD}(a_1, \dots, a_m) = 1$, there exists a monomial M in S such that $\phi_k(x_1^u)t = \phi_k(M)$ for some $u > 0$. Let

$$\tilde{\phi}_k \otimes 1 : k[x_1, \dots, x_m, x_1^{-1}, t] \rightarrow k[t, t^{-1}]$$

be the localization of $\tilde{\phi}_k$. Then, the kernel of $\tilde{\phi}_k \otimes 1$ is equal to

$$\mathfrak{q}k[x_1, \dots, x_m, x_1^{-1}, t] = \left(\mathfrak{p}, t - \frac{M}{x_1^u} \right) k[x_1, \dots, x_m, x_1^{-1}, t].$$

Setting $t' = t - \frac{M}{x_1^u}$,

$$\mathfrak{q}A[x_1^{-1}] = (\mathfrak{p}, t')k[x_1, \dots, x_m, x_1^{-1}, t']$$

holds. Since x_1, \dots, x_m, t' are algebraically independent over k ,

$$\mathfrak{q}^r A[x_1^{-1}] \cap S[x_1^{-1}] = (\mathfrak{p}, t')^r k[x_1, \dots, x_m, x_1^{-1}, t'] \cap k[x_1, \dots, x_m, x_1^{-1}] = \mathfrak{p}^r S[x_1^{-1}]$$

holds. Therefore,

$$\mathfrak{q}^r \cap S \subset \mathfrak{q}^r A[x_1^{-1}] \cap S = \mathfrak{p}^r S[x_1^{-1}] \cap S \subset \mathfrak{p}^{(r)}.$$

We have completed the proof of $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$.

Let $R(\mathfrak{q})$ be the Rees ring of the ideal \mathfrak{q} , i.e.,

$$R(\mathfrak{q}) = \bigoplus_{r \geq 0} \mathfrak{q}^r T^r \subset A[T].$$

Then, since $\mathfrak{q}^r \cap S = \mathfrak{p}^{(r)}$ for $r \geq 0$,

$$R(\mathfrak{q}) \cap S[T] = R_s(\mathfrak{p})$$

holds. It is easy to verify

$$R(\mathfrak{q}) \cap Q(S[T]) = R_s(\mathfrak{p})$$

because $Q(S[T]) \cap A[T] = S[T]$, where $Q(\)$ means the field of fractions. Here remark that $S[T] = k[x_1, \dots, x_m, T]$ and $A[T] = k[x_1, \dots, x_m, T, t]$. Therefore, we obtain the equality

$$\tilde{\xi}(R(\mathfrak{q})) \cap L = \xi(R_s(\mathfrak{p})). \quad (1)$$

Here, remember that L is the field of fractions of $\text{Im}(\xi)$.

On the other hand, setting $x'_i = x_i - t^{a_i}$ for $i = 1, \dots, m$, we obtain the following:

$$\begin{aligned} R(\mathfrak{q}) &= k[x_1, \dots, x_m, x'_1 T, \dots, x'_m T, t] \\ &= k[x'_1, \dots, x'_m, x'_1 T, \dots, x'_m T, t]. \end{aligned}$$

Here, remark that x'_1, \dots, x'_m, T, t are algebraically independent over k .

By definition, $\tilde{\xi}(x'_i) = \alpha_1 \beta_i$, and $\tilde{\xi}(x'_i T) = \alpha_2 \beta_i$ for each i .

We set

$$B = \tilde{\xi}(R(\mathbf{q})) \quad (2)$$

and $C = k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t]$. Here,

$$B = (k[\alpha_i \beta_j \mid i = 1, 2; j = 1, \dots, m])[t] \subset C.$$

Since B is a direct summand of C as a B -module, the equality

$$C \cap Q(B) = B \quad (3)$$

holds in $Q(C)$.

Then, since $L \subset Q(B)$, we obtain

$$C \cap L = (C \cap Q(B)) \cap L = B \cap L = \xi(R_s(\mathbf{p}))$$

by Eqs. (1)–(3). \square

Remark 2.2. Let k be a field. Let R be a polynomial ring over k with finitely many variables. For a field L satisfying $k \subset L \subset Q(R)$, Hilbert asked in 1900 whether the ring $L \cap R$ is finitely generated as a k -algebra or not. It is called the *Hilbert's fourteenth problem*.

The first counterexample to this problem was discovered by Nagata [13] in 1958. An easier counterexample was found by Paul C. Roberts [15] in 1990. Further counterexamples were given by Kuroda, Mukai, etc.

On the other hand, Goto, Nishida and Watanabe [4] proved that $R_s(\mathbf{p}_k(7n - 3, (5n - 2)n, 8n - 3))$ is not finitely generated over k if the characteristic of k is zero, $n \geq 4$ and $n \not\equiv 0 \pmod{3}$. By Theorem 2.1, we know that they are new counterexamples to the Hilbert's fourteenth problem.

Remark 2.3. With notation as in Theorem 2.1, we set

$$D_1 = \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2} - \beta_1 \frac{\partial}{\partial \beta_1} - \dots - \beta_m \frac{\partial}{\partial \beta_m},$$

$$D_2 = a_1 t^{a_1-1} \frac{\partial}{\partial \beta_1} + \dots + a_m t^{a_m-1} \frac{\partial}{\partial \beta_m} - \alpha_1 \frac{\partial}{\partial t}.$$

Assume that the characteristic of k is zero.

Then, one can prove that $\xi(R_s(\mathbf{p}_k(a_1, \dots, a_m)))$ is equal to the kernel of the derivations D_1 and D_2 , i.e.,

$$\xi(R_s(\mathbf{p}_k(a_1, \dots, a_m))) = \{f \in k[\alpha_1, \alpha_2, \beta_1, \dots, \beta_m, t] \mid D_1(f) = D_2(f) = 0\}.$$

3. Symbolic Rees rings of space monomial curves

In the rest of this paper, we restrict ourselves to the case $m = 3$. For the simplicity of notation, we write x, y, z, a, b, c for $x_1, x_2, x_3, a_1, a_2, a_3$, respectively. We regard the polynomial ring $k[x, y, z]$ as a \mathbb{Z} -graded ring by $\deg(x) = a, \deg(y) = b$ and $\deg(z) = c$.

$\mathfrak{p}_k(a, b, c)$ is the kernel of the k -algebra homomorphism

$$\phi_k : k[x, y, z] \rightarrow k[t]$$

given by $\phi_k(x) = t^a, \phi_k(y) = t^b, \phi_k(z) = t^c$.

By a result of Herzog [5], we know that $\mathfrak{p}_k(a, b, c)$ is generated by at most three elements. For example, $\mathfrak{p}_k(3, 4, 5)$ is minimally generated by $x^3 - yz, y^2 - zx$ and $z^2 - x^2y$. On the other hand, $\mathfrak{p}_k(3, 5, 8)$ is minimally generated by $x^5 - y^3$ and $z - xy$. We can choose a generating system of $\mathfrak{p}_k(a, b, c)$ which is independent of k .

We are interested in the symbolic powers of $\mathfrak{p}_k(a, b, c)$. If $\mathfrak{p}_k(a, b, c)$ is generated by two elements, then the symbolic powers always coincide with ordinary powers because $\mathfrak{p}_k(a, b, c)$ is a complete intersection. However, it is known that, if $\mathfrak{p}_k(a, b, c)$ is minimally generated by three elements, the second symbolic power is strictly bigger than the second ordinary power. For example, the element

$$h = (x^3 - yz)^2 - (y^2 - zx)(z^2 - x^2y)$$

is contained in $\mathfrak{p}_k(3, 4, 5)^2$, and is divisible by x . Therefore, h/x is an element in $\mathfrak{p}_k(3, 4, 5)^{(2)}$ of degree 15. Since $[\mathfrak{p}_k(3, 4, 5)^2]_{15} = 0$, h/x is not contained in $\mathfrak{p}_k(3, 4, 5)^2$.

We are interested in finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k(a, b, c))$. It is known that this problem is reduced to the case where a, b and c are pairwise coprime, i.e.,

$$(a, b) = (b, c) = (c, a) = 1.$$

In the rest of this paper, we always assume that a, b and c are pairwise coprime.

Let $\mathbb{P}_k(a, b, c)$ be the weighted projective space $\text{Proj}(k[x, y, z])$. Then

$$\mathbb{P}_k(a, b, c) \setminus \{V_+(x, y), V_+(y, z), V_+(z, x)\}$$

is a regular scheme. In particular, $\mathbb{P}_k(a, b, c)$ is smooth at the point $V_+(\mathfrak{p}_k(a, b, c))$. Let $\pi : X_k(a, b, c) \rightarrow \mathbb{P}_k(a, b, c)$ be the blow-up at $V_+(\mathfrak{p}_k(a, b, c))$. Let E be the exceptional divisor, i.e.,

$$E = \pi^{-1}(V_+(\mathfrak{p}_k(a, b, c))).$$

We sometimes denote $\mathfrak{p}_k(a, b, c)$ (respectively $\mathbb{P}_k(a, b, c), X_k(a, b, c)$) simply by \mathfrak{p} or \mathbb{P}_k (respectively \mathbb{P} or \mathbb{P}_k, X or X_k) if no confusion is possible.

It is easy to see that

$$\text{Cl}(\mathbb{P}) = \mathbb{Z}H \simeq \mathbb{Z},$$

where H is a Weil divisor corresponding to the reflexive sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Set $H = \sum_i m_i D_i$, where D_i 's are subvarieties of \mathbb{P} of codimension one. We may choose D_i 's such that $D_i \not\supset V_+(\mathfrak{p})$ for any i . Then, set $A = \sum_i m_i \pi^{-1}(D_i)$.

One can prove that

$$\mathrm{Cl}(X) = \mathbb{Z}A + \mathbb{Z}E \simeq \mathbb{Z}^2.$$

Since all Weil divisors on X are \mathbb{Q} -Cartier, we have the intersection pairing

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \rightarrow \mathbb{Q},$$

that satisfies

$$A^2 = \frac{1}{abc}, \quad E^2 = -1, \quad A \cdot E = 0.$$

Therefore, we have

$$(n_1 A - r_1 E) \cdot (n_2 A - r_2 E) = \frac{n_1 n_2}{abc} - r_1 r_2.$$

Here, we have the following natural identification:

$$H^0(X, \mathcal{O}_X(nA - rE)) = \begin{cases} [\mathfrak{p}^{(r)}]_n & (r \geq 0), \\ S_n & (r < 0). \end{cases}$$

Therefore, the *total coordinate ring* (or *Cox ring*)

$$TC(X) = \bigoplus_{n, r \in \mathbb{Z}} H^0(X, \mathcal{O}_X(nA - rE))$$

is isomorphic to the extended symbolic Rees ring

$$R_s(\mathfrak{p})[T^{-1}] = \cdots \oplus ST^{-2} \oplus ST^{-1} \oplus S \oplus \mathfrak{p}T \oplus \mathfrak{p}^{(2)}T^2 \oplus \cdots.$$

We refer the reader to Hu and Keel [6] for finite generation of total coordinate rings. It is well known that $R_s(\mathfrak{p})[T^{-1}]$ is Noetherian if and only if so is $R_s(\mathfrak{p})$.

Remark 3.1. By Huneke's criterion [7] and a result of Cutkosky [3], the following four conditions are equivalent:

- (1) $R_s(\mathfrak{p})$ is a Noetherian ring, or equivalently, finitely generated over k .
- (2) $TC(X)$ is a Noetherian ring, or equivalently, finitely generated over k .
- (3) There exist positive integers $r, s, f \in \mathfrak{p}^{(r)}, g \in \mathfrak{p}^{(s)}$, and $h \in (x, y, z) \setminus \mathfrak{p}$ such that

$$\ell_{S_{(x, y, z)}}(S_{(x, y, z)}/(f, g, h)) = rs \cdot \ell_{S_{(x, y, z)}}(S_{(x, y, z)}/(\mathfrak{p}, h)),$$

where $\ell_{S_{(x, y, z)}}$ is the length as an $S_{(x, y, z)}$ -module.

(4) There exist curves C and D on X such that

$$C \neq D, \quad C \neq E, \quad D \neq E, \quad C.D = 0.$$

Here, a curve means a closed irreducible reduced subvariety of dimension one.

The condition (4) as above is equivalent to that just one of the following two conditions is satisfied:

(4-1) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C^2 < 0, \quad D^2 > 0, \quad C.D = 0.$$

(4-2) There exist curves C and D on X such that

$$C \neq E, \quad D \neq E, \quad C \neq D, \quad C^2 = D^2 = 0.$$

Definition 3.2. A curve C on X is called a *negative curve* if

$$C \neq E \quad \text{and} \quad C^2 < 0.$$

Remark 3.3. Suppose that a divisor F is linearly equivalent to $nA - rE$. Then, we have

$$F^2 = \frac{n^2}{abc} - r^2.$$

If (4-2) in Remark 3.1 is satisfied, then all of a , b and c must be squares of integers because a , b , c are pairwise coprime. In the case where one of a , b and c is not square, the condition (4) is equivalent to (4-1). Therefore, in this case, a negative curve exists if $R_s(\mathfrak{p})$ is finitely generated over k .

Suppose $(a, b, c) = (1, 1, 1)$. Then \mathfrak{p} coincides with $(x - y, y - z)$. Of course, $R_s(\mathfrak{p})$ is a Noetherian ring since the symbolic powers coincide with the ordinary powers. By definition it is easy to see that there is no negative curve in this case, therefore, (4-2) in Remark 3.1 happens.

The authors know no other examples in which (4-2) happens.

In the case of $(a, b, c) = (3, 4, 5)$, the proper transform of

$$V_+ \left(\frac{(x^3 - yz)^2 - (y^2 - zx)(z^2 - x^2y)}{x} \right)$$

is the negative curve on X , that is linearly equivalent to $15A - 2E$.

It is proved that two distinct negative curves never exist.

In the case where the characteristic of k is positive, Cutkosky [3] proved that $R_s(\mathfrak{p})$ is finitely generated over k if there exists a negative curve on X .

We remark that there exists a negative curve on X if and only if there exists positive integers n and r such that

$$\frac{n}{r} < \sqrt{abc} \quad \text{and} \quad [\mathfrak{p}^{(r)}]_n \neq 0.$$

We are interested in existence of a negative curve. Let a, b and c be pairwise coprime positive integers. By the following lemma, if there exists a negative curve on $X_{k_0}(a, b, c)$ for a field k_0 of characteristic 0, then there exists a negative curve on $X_k(a, b, c)$ for any field k .

Lemma 3.4. *Let a, b and c be pairwise coprime positive integers.*

1. *Let K/k be a field extension. Then, for any integers n and r ,*

$$[\mathfrak{p}_k(a, b, c)^{(r)}]_n \otimes_k K = [\mathfrak{p}_K(a, b, c)^{(r)}]_n.$$

2. *For any integers n, r and any prime number p ,*

$$\dim_{\mathbb{F}_p} [\mathfrak{p}_{\mathbb{F}_p}(a, b, c)^{(r)}]_n \geq \dim_{\mathbb{Q}} [\mathfrak{p}_{\mathbb{Q}}(a, b, c)^{(r)}]_n$$

holds, where \mathbb{Q} is the field of rational numbers, and \mathbb{F}_p is the prime field of characteristic p . Here, $\dim_{\mathbb{F}_p}$ (respectively $\dim_{\mathbb{Q}}$) denotes the dimension as an \mathbb{F}_p -vector space (respectively \mathbb{Q} -vector space).

Proof. Since $S \rightarrow S \otimes_k K$ is flat, it is easy to prove the assertion (1).

We shall prove the assertion (2). Let \mathbb{Z} be the ring of rational integers. Set $S_{\mathbb{Z}} = \mathbb{Z}[x, y, z]$. Let $\mathfrak{p}_{\mathbb{Z}}$ be the kernel of the ring homomorphism

$$\phi_{\mathbb{Z}} : S_{\mathbb{Z}} \rightarrow \mathbb{Z}[t]$$

given by $\phi_{\mathbb{Z}}(x) = t^a$, $\phi_{\mathbb{Z}}(y) = t^b$ and $\phi_{\mathbb{Z}}(z) = t^c$. Since the cokernel of $\phi_{\mathbb{Z}}$ is \mathbb{Z} -free module, we know

$$\mathfrak{p}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k = \text{Ker}(\phi_{\mathbb{Z}}) \otimes_{\mathbb{Z}} k = \text{Ker}(\phi_k) = \mathfrak{p}_k$$

for any field k .

Consider the following exact sequence of \mathbb{Z} -free modules:

$$0 \rightarrow \mathfrak{p}_{\mathbb{Z}}^{(r)} \rightarrow S_{\mathbb{Z}} \rightarrow S_{\mathbb{Z}}/\mathfrak{p}_{\mathbb{Z}}^{(r)} \rightarrow 0.$$

For any field k , the following sequence is exact:

$$0 \rightarrow \mathfrak{p}_{\mathbb{Z}}^{(r)} \otimes_{\mathbb{Z}} k \rightarrow S \rightarrow S/\mathfrak{p}_{\mathbb{Z}}^{(r)} \otimes_{\mathbb{Z}} k \rightarrow 0.$$

Since $\mathfrak{p}_{\mathbb{Z}} S_{\mathbb{Z}}[x^{-1}]$ is generated by a regular sequence, we know

$$\mathfrak{p}_{\mathbb{Z}}^{(r)} S_{\mathbb{Z}}[x^{-1}] = \mathfrak{p}_{\mathbb{Z}}^r S_{\mathbb{Z}}[x^{-1}]$$

for any $r \geq 0$. Therefore, for any $f \in \mathfrak{p}_{\mathbb{Z}}^{(r)}$, there is a positive integer u such that

$$x^u f \in \mathfrak{p}_{\mathbb{Z}}^r.$$

Let p be a prime number. Consider the natural surjective ring homomorphism

$$\eta : S_{\mathbb{Z}} \rightarrow S_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Suppose $f \in \mathfrak{p}_{\mathbb{Z}}^{(r)}$. Since $x^u f \in \mathfrak{p}_{\mathbb{Z}}^r$ for some positive integer u , we obtain

$$x^u \eta(f) \in \eta(\mathfrak{p}_{\mathbb{Z}}^r) = \mathfrak{p}_{\mathbb{F}_p}^r.$$

Hence we know

$$\mathfrak{p}_{\mathbb{Z}}^{(r)} \otimes_{\mathbb{Z}} \mathbb{F}_p = \eta(\mathfrak{p}_{\mathbb{Z}}^{(r)}) \subset \mathfrak{p}_{\mathbb{F}_p}^{(r)}.$$

We obtain

$$\text{rank}_{\mathbb{Z}}[\mathfrak{p}_{\mathbb{Z}}^{(r)}]_n = \dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{Z}}^{(r)}]_n \otimes_{\mathbb{Z}} \mathbb{F}_p \leq \dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}^{(r)}]_n$$

for any $r \geq 0$ and $n \geq 0$. Here, $\text{rank}_{\mathbb{Z}}$ denotes the rank as a \mathbb{Z} -module.

On the other hand, it is easy to see that

$$\mathfrak{p}_{\mathbb{Z}}^{(r)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{p}_{\mathbb{Q}}^{(r)}$$

for any $r \geq 0$. Therefore, we have

$$\text{rank}_{\mathbb{Z}}[\mathfrak{p}_{\mathbb{Z}}^{(r)}]_n = \dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}^{(r)}]_n$$

for any $r \geq 0$ and $n \geq 0$.

Hence, we obtain

$$\dim_{\mathbb{Q}}[\mathfrak{p}_{\mathbb{Q}}^{(r)}]_n \leq \dim_{\mathbb{F}_p}[\mathfrak{p}_{\mathbb{F}_p}^{(r)}]_n$$

for any $r \geq 0$, $n \geq 0$, and any prime number p . \square

Remark 3.5. Let a, b, c be pairwise coprime positive integers. Assume that there exists a negative curve on $X_{k_0}(a, b, c)$ for a field k_0 of characteristic zero.

By Lemma 3.4, we know that there exists a negative curve on $X_k(a, b, c)$ for any field k . Therefore, if k is a field of characteristic positive, then the symbolic Rees ring $R_s(\mathfrak{p}_k)$ is finitely generated over k by a result of Cutkosky [3]. However, if k is a field of characteristic zero, then $R_s(\mathfrak{p}_k)$ is not necessary Noetherian. In fact, assume that k is of characteristic zero and $(a, b, c) = (7n - 3, (5n - 2)n, 8n - 3)$ with $n \not\equiv 0 \pmod{3}$ and $n \geq 4$ as in Goto, Nishida and Watanabe [4]. Then there exists a negative curve, but $R_s(\mathfrak{p}_k)$ is not Noetherian.

Definition 3.6. Let a, b, c be pairwise coprime positive integers. Let k be a field.

We define the following three conditions:

- (C1) There exists a negative curve on $X_k(a, b, c)$, i.e., $[\mathfrak{p}_k(a, b, c)^{(r)}]_n \neq 0$ for some positive integers n, r satisfying $n/r < \sqrt{abc}$.
- (C2) There exist positive integers n, r satisfying $n/r < \sqrt{abc}$ and $\dim_k S_n > r(r + 1)/2$.
- (C3) There exist positive integers q, r satisfying $abcq/r < \sqrt{abc}$ and $\dim_k S_{abcq} > r(r + 1)/2$.

Here, \dim_k denotes the dimension as a k -vector space.

By the following lemma, we know the implications

$$(C3) \implies (C2) \implies (C1)$$

since $\dim_k [\mathfrak{p}^{(r)}]_n = \dim_k S_n - \dim_k [S/\mathfrak{p}^{(r)}]_n$.

Lemma 3.7. *Let a, b, c be pairwise coprime positive integers. Let r and n be non-negative integers. Then,*

$$\dim_k [S/\mathfrak{p}^{(r)}]_n \leq r(r+1)/2$$

holds true for any field k .

Proof. Since x, y, z are non-zero divisors on $S/\mathfrak{p}^{(r)}$, we have only to prove that

$$\dim_k [S/\mathfrak{p}^{(r)}]_{abcq} = r(r+1)/2$$

for $q \gg 0$.

The left-hand side is the multiplicity of the abc th Veronese subring

$$[S/\mathfrak{p}^{(r)}]^{(abc)} = \bigoplus_{q \geq 0} [S/\mathfrak{p}^{(r)}]_{abcq}.$$

Therefore, for $q \gg 0$, we have

$$\begin{aligned} \dim_k [S/\mathfrak{p}^{(r)}]_{abcq} &= \ell([S/\mathfrak{p}^{(r)} + (x^{bc})]^{(abc)}) \\ &= e((x^{bc}), [S/\mathfrak{p}^{(r)}]^{(abc)}) \\ &= \frac{1}{abc} e((x^{bc}), S/\mathfrak{p}^{(r)}) \\ &= \frac{1}{a} e((x), S/\mathfrak{p}^{(r)}) \\ &= \frac{1}{a} e((x), S/\mathfrak{p}) \ell_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}^r S_{\mathfrak{p}}) \\ &= \frac{r(r+1)}{2}. \quad \square \end{aligned}$$

Remark 3.8. It is easy to see that $[\mathfrak{p}_k(a, b, c)]_n \neq 0$ if and only if $\dim_k S_n \geq 2$. Therefore, if we restrict ourselves to $r = 1$, then (C1) and (C2) are equivalent.

However, even if $[\mathfrak{p}_k(a, b, c)^{(2)}]_n \neq 0$, $\dim_k S_n$ is not necessary bigger than 3. In fact, since $\mathfrak{p}_k(5, 6, 7)$ contains $y^2 - zx$, we know $[\mathfrak{p}_k(5, 6, 7)^2]_{24} \neq 0$. In this case, $\dim_k S_{24}$ is equal to three.

Here assume that (C1) is satisfied for $r = 2$. Furthermore, we assume that the characteristic of k is zero. Then, there exists $f \neq 0$ in $[\mathfrak{p}_k(a, b, c)^{(2)}]_n$ such that $n < 2\sqrt{abc}$ for some $n > 0$. Let $f = f_1 \cdots f_s$ be the irreducible decomposition. Then, at least one of f_i 's satisfies the condition (C1). If it satisfies (C1) with $r = 1$, then (C2) is satisfied as above. Suppose that the irreducible component satisfies (C2) with $r = 2$. For the simplicity of notation, we assume that f itself is

irreducible. We want to show $\dim_k S_n \geq 4$. Assume the contrary. By Lemma 3.4(1), we may assume that f is a polynomial with rational coefficients. Set

$$f = k_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} - k_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} + k_3 x^{\alpha_3} y^{\beta_3} z^{\gamma_3}.$$

Furthermore, we may assume that k_1, k_2, k_3 are non-negative integers such that $\text{GCD}(k_1, k_2, k_3) = 1$. Since

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}_k(a, b, c)$$

as in Remark 5.1, we have

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we have

$$(x^{\alpha_1} y^{\beta_1} z^{\gamma_1})^{k_1} (x^{\alpha_3} y^{\beta_3} z^{\gamma_3})^{k_3} = (x^{\alpha_2} y^{\beta_2} z^{\gamma_2})^{k_2}.$$

Since f is irreducible, $x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$ and $x^{\alpha_3} y^{\beta_3} z^{\gamma_3}$ have no common divisor. Note that $k_2 = k_1 + k_3$ since $f \in \mathfrak{p}_k(a, b, c)$. Since k_1 and k_3 are relatively prime, there exist monomials N_1 and N_3 such that $x^{\alpha_1} y^{\beta_1} z^{\gamma_1} = N_1^{k_1+k_3}$, $x^{\alpha_3} y^{\beta_3} z^{\gamma_3} = N_3^{k_1+k_3}$ and $x^{\alpha_2} y^{\beta_2} z^{\gamma_2} = N_1^{k_1} N_3^{k_3}$. Then

$$f = k_1 N_1^{k_1+k_3} - (k_1 + k_3) N_1^{k_1} N_3^{k_3} + k_3 N_3^{k_1+k_3}.$$

Then, f is divisible by $N_1 - N_3$. Since f is irreducible, f is equal to $N_1 - N_3$. It contradicts to

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}_k(a, b, c).$$

Consequently, if (C1) is satisfied with $r \leq 2$ for a field k of characteristic zero, then (C2) is satisfied.

We shall discuss the difference between (C1) and (C2) in Section 6.1.

Remark 3.9. Let a, b and c be pairwise coprime positive integers. Assume that $\mathfrak{p}_k(a, b, c)$ is a complete intersection, i.e., generated by two elements.

Permuting a, b and c , we may assume that

$$\mathfrak{p}_k(a, b, c) = (x^b - y^a, z - x^\alpha y^\beta)$$

for some $\alpha, \beta \geq 0$ satisfying $\alpha a + \beta b = c$. If $ab < c$, then

$$\deg(x^b - y^a) = ab < \sqrt{abc}.$$

If $ab > c$, then

$$\deg(z - x^\alpha y^\beta) = c < \sqrt{abc}.$$

If $ab = c$, then (a, b, c) must be equal to $(1, 1, 1)$. Ultimately, there exists a negative curve if $(a, b, c) \neq (1, 1, 1)$.

4. The case where $(a + b + c)^2 > abc$

In the rest of this paper, we set $\xi = abc$ and $\eta = a + b + c$ for pairwise coprime positive integers a, b and c .

For $v = 0, 1, \dots, \xi - 1$, we set

$$S^{(\xi, v)} = \bigoplus_{q \geq 0} S_{\xi q + v}.$$

This is a module over $S^{(\xi)} = \bigoplus_{q \geq 0} S_{\xi q}$.

Lemma 4.1.

$$\dim_k [S^{(\xi, v)}]_q = \dim_k S_{\xi q + v} = \frac{1}{2} \{ \xi q^2 + (\eta + 2v)q + 2 \dim_k S_v \}$$

holds for any $q \geq 0$.

The following simple proof is due to Professor Kei-ichi Watanabe. We appreciate him very much.

Proof of Lemma 4.1. We set $a_n = \dim_k S_n$ for each integer n . Set

$$f(t) = \sum_{n \in \mathbb{Z}} a_n t^n.$$

Here we put $a_n = 0$ for $n < 0$. Then, the equality

$$f(t) = \frac{1}{(1 - t^a)(1 - t^b)(1 - t^c)}$$

holds.

Set $b_n = a_n - a_{n-\xi}$. Then, b_n is equal to the coefficient of t^n in $(1 - t^\xi)f(t)$ for each n . Furthermore, $b_n - b_{n-1}$ is equal to the coefficient of t^n in $(1 - t)(1 - t^\xi)f(t)$ for each n .

On the other hand, we have the equality

$$(1 - t)(1 - t^\xi)f(t) = g(t) \times \frac{1}{1 - t} = g(t) \times (1 + t + t^2 + \dots), \quad (4)$$

where

$$g(t) = \frac{1 + t + \dots + t^{\xi-1}}{(1 + t + \dots + t^{a-1})(1 + t + \dots + t^{b-1})(1 + t + \dots + t^{c-1})}.$$

Since a, b and c are pairwise coprime, $g(t)$ is a polynomial of degree $\xi - \eta + 2$. Therefore, the coefficient of t^n in $(1 - t)(1 - t^\xi)f(t)$ is equal to $g(1)$ for $n \geq \xi - \eta + 2$ by Eq. (4). It is easy to see $g(1) = 1$.

Since $b_n - b_{n-1} = 1$ for $n \geq \xi + 1$,

$$b_n = b_\xi + (n - \xi)$$

holds for any $n \geq \xi$. Then,

$$\begin{aligned} a_{\xi q+v} - a_v &= \sum_{i=1}^q (a_{\xi i+v} - a_{\xi(i-1)+v}) \\ &= \sum_{i=1}^q b_{\xi i+v} \\ &= \sum_{i=1}^q (b_\xi + \xi(i-1) + v) \\ &= b_\xi q + \xi \frac{(q-1)q}{2} + vq \\ &= \frac{\xi}{2} q^2 + \left(b_\xi - \frac{\xi}{2} + v \right) q. \end{aligned}$$

Recall that b_ξ is the coefficient of t^ξ in

$$(1 - t^\xi) f(t) = \frac{g(t)}{(1-t)^2} = g(t) \times (1 + 2t + \cdots + (n+1)t^n + \cdots). \quad (5)$$

Setting

$$g(t) = c_0 + c_1 t + \cdots + c_{\xi-\eta+2} t^{\xi-\eta+2},$$

it is easy to see

$$c_i = c_{\xi-\eta+2-i} \quad (6)$$

for each i . Therefore, by Eqs. (5) and (6), we have

$$b_\xi = c_0(\xi + 1) + c_1 \xi + \cdots + c_{\xi-\eta+2}(\eta - 1) = (c_0 + c_1 + \cdots + c_{\xi-\eta+2}) \times \frac{\xi + \eta}{2}.$$

Since $g(1) = 1$, we have $b_\xi = \frac{\xi+\eta}{2}$. Thus,

$$a_{\xi q+v} = \frac{\xi}{2} q^2 + \left(\frac{\xi + \eta}{2} - \frac{\xi}{2} + v \right) q + a_v. \quad \square$$

Before proving Theorem 4.3, we need the following lemma:

Lemma 4.2. Assume that a , b and c are pairwise coprime positive integers such that $(a, b, c) \neq (1, 1, 1)$. Then, $\eta - \sqrt{\xi} \neq 0, 1, 2$.

Proof. We may assume that all of a , b and c are squares of integers. It is sufficient to show

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma \neq 0, 1, 2$$

for pairwise coprime positive integers α , β , γ such that $(\alpha, \beta, \gamma) \neq (1, 1, 1)$.

Assume the contrary. Suppose that $(\alpha_0, \beta_0, \gamma_0)$ is a counterexample such that $\alpha_0 + \beta_0 + \gamma_0$ is minimum. We may assume $1 \leq \alpha_0 \leq \beta_0 \leq \gamma_0$.

Set

$$f(x) = x^2 - \alpha_0\beta_0x + \alpha_0^2 + \beta_0^2.$$

First suppose $\alpha_0\beta_0 \leq \gamma_0$. Then,

$$f(\gamma_0) \geq f(\alpha_0\beta_0) = \alpha_0^2 + \beta_0^2 \geq 2.$$

Since $f(\gamma_0) = 0, 1$, or 2 , we have

$$\gamma_0 = \alpha_0\beta_0 \quad \text{and} \quad \alpha_0^2 + \beta_0^2 = 2.$$

Then, we obtain the equality $\alpha_0 = \beta_0 = \gamma_0 = 1$ immediately. It is a contradiction.

Next, suppose $\frac{\alpha_0\beta_0}{2} < \gamma_0 < \alpha_0\beta_0$. Then, $0 < \alpha_0\beta_0 - \gamma_0 < \gamma_0$ and

$$f(\alpha_0\beta_0 - \gamma_0) = f(\gamma_0) = 0, 1, \text{ or } 2.$$

It is easy to see that α_0 , β_0 , $\alpha_0\beta_0 - \gamma_0$ are pairwise coprime positive integers. By the minimality of $\alpha_0 + \beta_0 + \gamma_0$, we have $\alpha_0 = \beta_0 = \alpha_0\beta_0 - \gamma_0 = 1$. Then, γ_0 must be zero. It is a contradiction.

Finally, suppose $0 < \gamma_0 \leq \frac{\alpha_0\beta_0}{2}$. Since $\beta_0 \leq \gamma_0 \leq \frac{\alpha_0\beta_0}{2}$, we have $\alpha_0 \geq 2$. If $\alpha_0 = 2$, then $2 \leq \beta_0 = \gamma_0$. It contradicts to $(\beta_0, \gamma_0) = 1$. Assume $\alpha_0 \geq 3$. Since $\beta_0 < \gamma_0$,

$$f(\gamma_0) < f(\beta_0) = (2 - \alpha_0)\beta_0^2 + \alpha_0^2 \leq 0.$$

It is a contradiction. \square

Theorem 4.3. Let a , b and c be pairwise coprime integers such that $(a, b, c) \neq (1, 1, 1)$.

Then, we have the following:

1. Assume that $\sqrt{abc} \notin \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > abc$.
2. Assume that $\sqrt{abc} \in \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > 9abc$.
3. If $(a + b + c)^2 > abc$, then, (C2) holds. In particular, a negative curve exists in this case.

Proof. Remember that, by Lemma 4.1, we obtain

$$\dim_k S_{\xi q} = \frac{1}{2}(\xi q^2 + \eta q + 2)$$

for any $q \geq 0$.

First we shall prove the assertion (1). Assume that (C3) is satisfied. Then,

$$\begin{cases} \sqrt{\xi} > \frac{\xi q}{r}, \\ \frac{\xi q^2 + \eta q + 2}{2} > \frac{r(r+1)}{2} \end{cases}$$

is satisfied for some positive integers r and q . The second inequality is equivalent to $\xi q^2 + \eta q \geq r(r+1)$ since both integers are even. Since

$$\xi q^2 + \eta q \geq r^2 + r > \xi q^2 + \sqrt{\xi} q,$$

we have $\eta > \sqrt{\xi}$ immediately.

Assume $\eta > \sqrt{\xi}$ and $\sqrt{\xi} \notin \mathbb{Z}$. Let ϵ be a real number satisfying $0 < \epsilon < 1$ and

$$2\epsilon\sqrt{\xi} < \frac{\eta - \sqrt{\xi}}{2}. \quad (7)$$

Since $\sqrt{\xi} \notin \mathbb{Q}$, there exist positive integers r and q such that

$$\epsilon > r - \sqrt{\xi}q > 0.$$

Then,

$$\frac{r}{q} < \sqrt{\xi} + \frac{\epsilon}{q} \leq \sqrt{\xi} + \epsilon < \sqrt{\xi} + \frac{\eta - \sqrt{\xi}}{2} = \frac{\eta + \sqrt{\xi}}{2}.$$

Since $\sqrt{\xi}q + \epsilon > r$, we have

$$\xi q^2 + 2\epsilon\sqrt{\xi}q + \epsilon^2 > r^2.$$

Therefore

$$r^2 + r < \xi q^2 + 2\epsilon\sqrt{\xi}q + \epsilon^2 + \frac{\eta + \sqrt{\xi}}{2}q < \xi q^2 + \eta q + \epsilon^2 < \xi q^2 + \eta q + 2$$

by Eq. (7).

Next we shall prove the assertion (2). Suppose $\sqrt{\xi} \in \mathbb{Z}$. Since $r > \sqrt{\xi}q$, we may assume that $r = \sqrt{\xi}q + 1$. Then,

$$(\xi q^2 + \eta q + 2) - (r^2 + r) = (\eta - 3\sqrt{\xi})q.$$

Therefore, the assertion (2) immediately follows from this.

Now, we shall prove the assertion (3). Assume $\eta > \sqrt{\xi}$. Since $(a, b, c) \neq (1, 1, 1)$, we know $\xi > 1$. If $\sqrt{\xi} \notin \mathbb{Z}$, then the assertion immediately follows from assertion (1). Therefore, we may assume $\sqrt{\xi} \in \mathbb{Z}$.

Let n, q and v be integers such that

$$n = \xi q + v, \quad v = \sqrt{\xi} - 1.$$

We set

$$r = \sqrt{\xi} q + 1.$$

Then,

$$\sqrt{\xi} r = \xi q + \sqrt{\xi} > n.$$

Furthermore, by Lemma 4.1,

$$\begin{aligned} 2 \dim_k S_n - (r^2 + r) &= (\xi q^2 + (\eta + 2v)q + 2 \dim_k S_v) - (\xi q^2 + 3\sqrt{\xi}q + 2) \\ &= (\eta - \sqrt{\xi} - 2)q + (2 \dim_k S_v - 2). \end{aligned}$$

Since $\eta - \sqrt{\xi}$ is a non-negative integer, we know $\eta - \sqrt{\xi} \geq 3$ by Lemma 4.2. Consequently, we have $2 \dim_k S_n - (r^2 + r) > 0$ for $q \gg 0$. \square

Remark 4.4. If $(a + b + c)^2 > abc$, then $R_s(\mathfrak{p})$ is Noetherian by a result of Cutkosky [3].

If $(a + b + c)^2 > abc$ and $\sqrt{abc} \notin \mathbb{Q}$, then the existence of a negative curve follows from Nakai's criterion for ampleness, Kleimann's theorem and the cone theorem (e.g. [11, Theorems 1.2.23 and 1.4.23], [8, Theorem 4-2-1]).

The condition $(a + b + c)^2 > abc$ is equivalent to $(-K_X)^2 > 0$. If $-K_X$ is ample, then the finite generation of the total coordinate ring follows from Proposition 2.9 and Corollary 2.16 in Hu and Keel [6].

If $(a, b, c) = (5, 6, 7)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $12A - E$. Since $(a + b + c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X \cdot C = (18A - E) \cdot (12A - E) = 0.028 \dots > 0,$$

$-K_X$ is ample by Nakai's criterion.

If $(a, b, c) = (7, 8, 9)$, then the negative curve C is the proper transform of the curve defined by $y^2 - zx$. Therefore, C is linearly equivalent to $16A - E$. Since $(a + b + c)^2 > abc$, $(-K_X)^2 > 0$. Since

$$-K_X \cdot C = (24A - E) \cdot (16A - E) = -0.23 \dots < 0,$$

$-K_X$ is not ample by Nakai's criterion.

5. Degree of a negative curve

Remark 5.1. Let k be a field of characteristic zero, and R be a polynomial ring over k with variables x_1, x_2, \dots, x_m . Suppose that P is a prime ideal of R . By [12], we have

$$P^{(r)} = \left\{ h \in R \mid 0 \leq \alpha_1 + \dots + \alpha_m < r \Rightarrow \frac{\partial^{\alpha_1 + \dots + \alpha_m} h}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}} \in P \right\}.$$

In particular, if $f \in P^{(r)}$, then

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \in P^{(r-1)}.$$

Proposition 5.2. Let a, b and c be pairwise coprime integers, and k be a field of characteristic zero. Suppose that a negative curve exists, i.e., there exist positive integers n and r satisfying $[\mathfrak{p}_k(a, b, c)^{(r)}]_n \neq 0$ and $n/r < \sqrt{abc}$.

Set n_0 and r_0 to be

$$n_0 = \min \{ n \in \mathbb{N} \mid \exists r > 0 \text{ such that } n/r < \sqrt{\xi} \text{ and } [\mathfrak{p}^{(r)}]_n \neq 0 \},$$

$$r_0 = \left\lfloor \frac{n_0}{\sqrt{\xi}} \right\rfloor + 1,$$

where $\lfloor \frac{n_0}{\sqrt{\xi}} \rfloor$ is the maximum integer which is less than or equal to $\frac{n_0}{\sqrt{\xi}}$.

Then, the negative curve C is linearly equivalent to $n_0 A - r_0 E$.

Proof. Suppose that the negative curve C is linearly equivalent to $n_1 A - r_1 E$. Since $n_1/r_1 < \sqrt{\xi}$ and $[\mathfrak{p}^{(r_1)}]_{n_1} \neq 0$, we have $n_1 \geq n_0$. Since $H^0(X, \mathcal{O}(n_0 A - r_0 E)) \neq 0$ with $n_0/r_0 < \sqrt{abc}$, $n_0 A - r_0 E - C$ is linearly equivalent to an effective divisor. Therefore, $n_0 \geq n_1$. Hence, $n_0 = n_1$.

Since $n_0/r_1 < \sqrt{\xi}$, $r_0 \leq r_1$ holds. Now, suppose $r_0 < r_1$. Let f be the defining equation of $\pi(C)$, where $\pi : X \rightarrow \mathbb{P}$ is the blow-up at $V_+(\mathfrak{p})$. Then, we have

$$[\mathfrak{p}^{(r_1-1)}]_{n_0} = [\mathfrak{p}^{(r_1)}]_{n_0} = k \cdot f.$$

If n is an integer less than n_0 , then $[\mathfrak{p}^{(r_1-1)}]_n = 0$ because

$$\frac{n}{r_1 - 1} < \frac{n_0}{r_1 - 1} \leq \frac{n_0}{r_0} < \sqrt{\xi}.$$

By Remark 5.1, we have

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \in \mathfrak{p}^{(r_1-1)}.$$

Since their degrees are strictly less than n_0 , we know

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

On the other hand, the equality

$$ax \frac{\partial f}{\partial x} + by \frac{\partial f}{\partial y} + cz \frac{\partial f}{\partial z} = n_0 f$$

holds. Remember that k is a field of characteristic zero. It is a contradiction. \square

Remark 5.3. Let a , b and c be pairwise coprime integers, and k be a field of characteristic zero. Assume that the negative curve C exists, and C is linearly equivalent to $n_0 A - r_0 E$.

Then, by Proposition 5.2, we obtain

$$n_0 = \min \{ n \in \mathbb{N} \mid [p^{(\lfloor \frac{n}{\sqrt{\xi}} \rfloor + 1)}]_n \neq 0 \},$$

$$r_0 = \left\lfloor \frac{n_0}{\sqrt{\xi}} \right\rfloor + 1.$$

Theorem 5.4. Let a , b and c be pairwise coprime positive integers such that $\sqrt{\xi} > \eta$. Assume that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1 + 1)/2$. Suppose $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$.

Then, $q_1 < \frac{2 \dim_k S_{v_1}}{\sqrt{\xi} - \eta}$ holds.

In particular,

$$n_1 = \xi q_1 + v_1 < \frac{2\xi \max\{\dim_k S_t \mid 0 \leq t < \xi\}}{\sqrt{\xi} - \eta} + \xi.$$

Proof. We have

$$r_1 > \frac{n_1}{\sqrt{\xi}} = \sqrt{\xi} q_1 + \frac{v_1}{\sqrt{\xi}}.$$

Therefore,

$$2 \dim_k S_{n_1} > r_1^2 + r_1 > \xi q_1^2 + 2v_1 q_1 + \frac{v_1^2}{\xi} + \sqrt{\xi} q_1 + \frac{v_1}{\sqrt{\xi}}.$$

By Lemma 4.1, we have

$$(\sqrt{\xi} - \eta) q_1 < 2 \dim_k S_{v_1} - \frac{v_1^2}{\xi} - \frac{v_1}{\sqrt{\xi}} \leq 2 \dim_k S_{v_1}. \quad \square$$

Remember that, if $\sqrt{\xi} < \eta$, then (C2) is always satisfied by Theorem 4.3(3).

6. Calculation by computer

In this section, we assume that the characteristic of k is zero.

6.1. Examples that do not satisfy (C2)

Suppose that (C2) is satisfied, i.e., there exist positive integers n_1 and r_1 such that $n_1/r_1 < \sqrt{\xi}$ and $\dim_k S_{n_1} > r_1(r_1 + 1)/2$. Put $n_1 = \xi q_1 + v_1$, where q_1 and v_1 are integers such that $0 \leq v_1 < \xi$. If $\sqrt{\xi} > \eta$, then $q_1 < \frac{2\dim_k S_{v_1}}{\sqrt{\xi}-\eta}$ holds by Theorem 5.4.

By the following programming on MATHEMATICA, we can check whether (C2) is satisfied or not in the case where $\sqrt{\xi} > \eta$.

```
c2[a_, b_, c_] :=
Do[
  If[(a + b + c)^2 > a b c, Print["-K: self-int positive"]; Goto[fin]];
  s = Series[(1 - t^a)(1 - t^b)(1 - t^c))^(1), {t, 0, a b c}];
  Do[ h = SeriesCoefficient[s, k];
    m = IntegerPart[2 h/(Sqrt[a b c] - a - b - c)];
    Do[ r = IntegerPart[(a b c q + k)(Sqrt[a b c]^(1))]] + 1;
      If[2 h + q(a + b + c) + a b c q^2 + 2q k > r (r + 1),
        Print[StringForm["n=", r=" ", a b c q + k, r]];
        Goto[fin]],
      {q, 0, m}],
    {k, 0, a b c - 1}];
Print["c2 is not satisfied"];
Label[fin];
Print["finished"]]
```

Calculations by a computer show that (C2) is not satisfied in some cases, for example, $(a, b, c) = (5, 33, 49), (7, 11, 20), (9, 10, 13), \dots$

The examples due to Goto, Nishida and Watanabe [4] have negative curves with $r = 1$. Therefore, by Remark 3.8, they satisfy the condition (C2).

In the case where $(a, b, c) = (5, 33, 49), (7, 11, 20), (9, 10, 13), \dots$, the authors do not know whether $R_s(p_k)$ is Noetherian or not.

Remark 6.1. Set

$$A = \{(a, b, c) \mid 0 < a \leq b \leq c \leq 50, a, b, c \text{ are pairwise coprime}\},$$

$$B = \{(a, b, c) \in A \mid a + b + c > \sqrt{abc}\},$$

$$C = \{(a, b, c) \in A \mid (a, b, c) \text{ does not satisfy (C2)}\}.$$

$\sharp A = 6156$, $\sharp B = 1950$, $\sharp C = 457$. By Theorem 4.3, we know $B \cap C = \emptyset$.

6.2. Does a negative curve exist?

By the following simple computer programming on MATHEMATICA, it is possible to know whether a negative curve exists or not.

```
n[a1_, b1_, c1_, r1_, d1_] := (V = 0;
Do[
  mono = {};
  Do[ e1 = d1 - i*a1;
    Do[ h1 = e1 - j*b1; k1 = Floor[h1/c1];
      If[ h1 / c1 == k1,
```

```

        mono = Join[mono, {x^i y^j z^(k1)}}],
        {j, 0, Floor[e1/b1]}
    ], {i, 0, Floor[d1/a1]}
];
w = Length[mono];
If[w > N[r1*(r1 + 1)/2],
    V = 1; W = w; J = d1; R = r1; H = N[r1*(r1 + 1)/2],
    If[ w > 0,
        f[x_, y_, z_] := mono;
        mat = {};
        Do[
            Do[
                mat = Join[mat, { D[f[x, y, z], {x, j}, {y, i - j}] }], {j, 0, i}
            ], {i, 0, r1 - 1}
        ];
        mat = mat /. x -> 1 /. y -> 1 /. z -> 1;
        q = MatrixRank[mat];
        If[ q < w, V = 1; W = w; J = d1; R = r1; H = N[r1*(r1 + 1)/2] ]
    ]
];
t[a_, b_, c_, rmade_] := (
    Do[
        W = 0;
        p = Ceiling[r*Sqrt[a*b*c]] - 1;
        Do[
            n[a, b, c, r, p - u];
            If[V == 1,
                J1 = J; Break[], {u, 0, a - 1}
            ];
        If[
            V == 1,
            Do[
                n[a, b, c, r, J1 - a*u];
                If[V == 0,
                    J1 = J; Break[], {u, 1, b*c}
                ];
            ];
            Do[
                n[a, b, c, r, J1 - b*u];
                If[V == 0,
                    J1 = J; Break[], {u, 1, a*c}
                ];
            ];
            Do[
                n[a, b, c, r, J1 - c*u];
                If[V == 0,
                    J1 = J; Break[], {u, 1, c*a}
                ];
            ];
        ];
    If[W > 0, Break[]];

    Print["r th symbolic power does not contain a negative curve if r <= ",
        r], {r, 1, rmade}
];

```

```

If[W == 0,
  Print["finished"],
  Print["There exists a negative curve. Degree = ", J, ", r = ", R,
    ", Dimension of homog. comp. = ", W, ", # of equations = ", H]
]
)

```

By the command $t[a, b, c, r]$, we can check whether $p(a, b, c)^{(m)}$ contains an equation of a negative curve for $m = 1, 2, \dots, r$.

$p(9, 10, 13)^{(m)}$ does not contain an equation of a negative curve if $m \leq 24$. Remember that $(9, 10, 13)$ does not satisfy (C2). Our computer gave up computation of $p(9, 10, 13)^{(25)}$ for scarcity of memories. We don't know whether there exists a negative curve in the case $(9, 10, 13)$.

On the other hand, there are examples that (C2) is not satisfied but there exists a negative curve.

- Suppose $(a, b, c) = (5, 33, 49)$. Then (C2) is not satisfied, but $[p(5, 33, 49)^{(18)}]_{1617}$ contains a negative curve.
- Suppose $(a, b, c) = (8, 15, 43)$. Then (C2) is not satisfied, but $[p(8, 15, 43)^{(9)}]_{645}$ contains a negative curve.

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