

Commutation relations on the covariant derivative<sup>☆</sup>

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## ARTICLE INFO

## Article history:

Received 8 June 2009

Available online 11 August 2009

Communicated by Michel Broué

## Keywords:

Non-associative algebra

Affine connection

## ABSTRACT

The  $n$ -th order covariant derivative on a smooth manifold with affine connection is a differential operator which maps a function to a tensor field of type  $(0, n)$ . In the paper the properties of this operator related to the permutations of indices are investigated by means of non-associative algebra. The general formula for commutation relations of this kind is obtained.

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Let  $\mathfrak{g}$  be a non-associative algebra over a field  $\mathbb{k}$ . The operation in  $\mathfrak{g}$  will be denoted by the diamond sign  $\diamond$ , for example,  $(x \diamond y) \diamond z \in \mathfrak{g}$ ,  $x, y, z \in \mathfrak{g}$ . Let  $T(\mathfrak{g}) = T_{\mathbb{k}}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$  denote the tensor algebra of  $\mathfrak{g}$  and let  $L_u : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ ,  $u \in T(\mathfrak{g})$ , denote the operator of left multiplication,  $L_u : v \mapsto u \otimes v$ . For typographical reasons we shall write sometimes  $L(u)$  instead of  $L_u$ . Denote by  $\tau_x \in \text{Der}_{\mathbb{k}}(T(\mathfrak{g}))$ ,  $x \in \mathfrak{g}$ , the derivation of the tensor algebra defined by the condition  $\tau_x : y \mapsto x \diamond y$ ,  $y \in \mathfrak{g}$ . There exists a unique linear map  $K : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$  such that  $K(1) = 1$  and  $KL_x = L_x K - K\tau_x$ ,  $x \in \mathfrak{g}$ . For example,  $K(x) = K(L_x 1) = L_x(K(1)) - K(\tau_x 1) = x$ ,  $K(x \otimes y) = K(L_x y) = L_x(K(y)) - K(\tau_x y) = x \otimes y - x \diamond y$ , etc. The author introduced this map in [1,2] (it is unlikely that such a map has never been considered before, however, I have not found an appropriate reference).

The problem we are interested in appears when  $\mathfrak{g}$  is a non-associative algebra and a Lie algebra simultaneously (a “framed Lie algebra” in terms of [1]). In other words, it is an algebra with two operations, one of which is antisymmetric and satisfies the Jacobi identity. As  $\mathfrak{g}$  is a Lie algebra, there exists the exact sequence  $0 \rightarrow I(\mathfrak{g}) \rightarrow T(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \rightarrow 0$ , where  $U(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$  and  $I(\mathfrak{g})$  is the two-sided ideal of  $T(\mathfrak{g})$  generated by the elements of the form  $x \otimes y - y \otimes x - [x, y]$ . Let  $\Sigma(\mathfrak{g}) \subset T(\mathfrak{g})$  denote the linear space of symmetric tensors (i.e.  $\Sigma_n(\mathfrak{g}) = \Sigma(\mathfrak{g}) \cap \mathfrak{g}^{\otimes n} = (\mathfrak{g}^{\otimes n})^{S_n}$ ). The map  $K$  has the property  $\deg(Ku - u) < \deg(u)$ ,  $u \in T(\mathfrak{g})$ , hence it is invertible. Moreover, if  $\text{char } \mathbb{k} = 0$ , then the restriction of the natural map to  $K(\Sigma(\mathfrak{g})) \rightarrow U(\mathfrak{g})$  is a linear isomorphism by the Poincaré–Birkhoff–Witt theorem. Thus we have the decomposition  $T(\mathfrak{g}) = \Sigma(\mathfrak{g}) \oplus K^{-1}(I(\mathfrak{g}))$ . The problem is to describe the corresponding projection  $\pi : T(\mathfrak{g}) \rightarrow \Sigma(\mathfrak{g})$  explicitly.

<sup>☆</sup> Supported in part by the grant RFFI-08-01-92001.

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The origin of the problem lies in differential geometry; it will be discussed below. Even in the case  $K = \text{id}$  (i.e.  $\mathfrak{g} \diamond \mathfrak{g} = \{0\}$ ) it is not trivial [5]. In the general case it looks like a formidable task. This paper concerned with a much more simple related problem. Denote  $\Omega(\mathfrak{g}) = T(\mathfrak{g}) \otimes \bigwedge^2 \mathfrak{g}$ , where  $\bigwedge^2 \mathfrak{g}$  is the exterior square of  $\mathfrak{g}$ . Let  $\iota: \Omega(\mathfrak{g}) \otimes T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$  be the natural map,  $\iota: a \otimes x \wedge y \otimes b \mapsto a \otimes (x \otimes y - y \otimes x) \otimes b$ . The aim is to find in an explicit form a map  $R: \Omega(\mathfrak{g}) \otimes T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$  satisfying for any  $Q \in \Omega(\mathfrak{g}) \otimes T(\mathfrak{g})$  the following two properties:  $R(Q) + \iota(Q) \in K^{-1}(I(\mathfrak{g}))$  and  $\deg R(Q) < \deg \iota(Q)$ . These properties do not determine the map  $R$  uniquely; however, the solution proposed below is probably the simplest one.

The projection  $\pi$  may be expressed via  $R$  as follows. Let  $s: T(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  be the natural algebra homomorphism and  $J(\mathfrak{g}) = \ker(s, T(\mathfrak{g}))$ . The restriction  $s: \Sigma(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  is a linear isomorphism, hence  $T(\mathfrak{g}) = \Sigma(\mathfrak{g}) \oplus J(\mathfrak{g})$ . Denote by  $\pi_0: T(\mathfrak{g}) \rightarrow \Sigma(\mathfrak{g})$  the corresponding projection. The map  $\iota: \Omega(\mathfrak{g}) \otimes T(\mathfrak{g}) \rightarrow J(\mathfrak{g})$  is surjective, hence there exists a (non-unique) right inverse  $\iota^{-1}: J(\mathfrak{g}) \rightarrow \Omega(\mathfrak{g}) \otimes T(\mathfrak{g})$ . Denote  $\rho = R \circ \iota^{-1} \circ (\text{id} - \pi_0): T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ . One can choose an inverse map  $\iota^{-1}$  satisfying the natural condition  $\deg \iota^{-1}(u) < \deg u$ ,  $u \in T(\mathfrak{g})$ . Then we have  $\deg \rho(u) < \deg u$ , hence the map  $\text{id} + \rho$  is invertible. Let  $\pi = \pi_0(\text{id} + \rho)^{-1}$ . By the definitions,  $\rho: \Sigma(\mathfrak{g}) \rightarrow \{0\}$  and  $\text{id} + \rho: J(\mathfrak{g}) \rightarrow K^{-1}(I(\mathfrak{g}))$ . Hence  $\pi$  is exactly the projection we need (though not in an explicit form). We shall see below that the map  $R$  may be interpreted as a collection of commutation relations on the covariant derivative.

## 1. Relations

In this section,  $\mathfrak{g}$  is an algebra with two operations: the first one denoted by  $\diamond$  and the anti-symmetric second one denoted by the brackets  $[\cdot, \cdot]$ . In the applications it is a Lie algebra, i.e. the second operation satisfies the Jacobi identity, but we shall not actually need this identity here. The base field  $\mathbb{k}$  is an arbitrary one.

Let  $I(\mathfrak{g}) \subset T(\mathfrak{g})$  be the two-sided ideal generated by the elements of the form  $x \otimes y - y \otimes x - [x, y]$ ,  $x, y \in \mathfrak{g}$ . Note that if  $\mathfrak{g}$  is not a Lie algebra then  $T(\mathfrak{g})/I(\mathfrak{g})$  is no more the universal enveloping algebra. Let  $\Omega(\mathfrak{g})$  and  $\tau_x$ ,  $x \in \mathfrak{g}$ , be defined as above. The derivation  $\tau_x: T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$  can be naturally lifted to the map  $\tau_x: \Omega(\mathfrak{g}) \rightarrow \Omega(\mathfrak{g})$  by the condition  $\tau_x \circ \iota = \iota \circ \tau_x$ ; namely  $\tau_x: a \otimes y \wedge z \mapsto \tau_x(a) \otimes y \wedge z + a \otimes x \diamond y \wedge z + a \otimes y \wedge x \diamond z$ . The linear maps  $t: \Omega(\mathfrak{g}) \rightarrow \mathfrak{g}$ ,  $r: \Omega(\mathfrak{g}) \rightarrow \text{Der}_{\mathbb{k}}(T(\mathfrak{g}))$  and  $e: \Omega(\mathfrak{g}) \rightarrow I(\mathfrak{g})$  are defined as follows. If  $x, y, z \in \mathfrak{g}$  and  $Q \in \Omega(\mathfrak{g})$ , then

$$\begin{aligned} t(x \wedge y) &= x \diamond y - y \diamond x - [x, y], & t(x \otimes Q + \tau_x Q) &= x \diamond t(Q), \\ r(x \wedge y): z &\mapsto x \diamond (y \diamond z) - y \diamond (x \diamond z) - [x, y] \diamond z, & r(x \otimes Q + \tau_x Q) &= [\tau_x, r(Q)], \\ e(x \wedge y) &= x \otimes y - y \otimes x - [x, y], & e(x \otimes Q + \tau_x Q) &= x \otimes e(Q) - e(Q) \otimes x. \end{aligned}$$

It is well known that tensor algebra is a bialgebra (actually it is a Hopf algebra but we make no use of antipode). The comultiplication  $\Delta: T(\mathfrak{g}) \rightarrow T(\mathfrak{g}) \otimes T(\mathfrak{g})$  is an algebra homomorphism defined by the condition  $\Delta: x \mapsto 1 \hat{\otimes} x + x \hat{\otimes} 1$ ,  $x \in \mathfrak{g}$  (to avoid misunderstanding the “exterior” tensor product is denoted by the hatted sign, i.e.  $1 \hat{\otimes} x$  and  $x \hat{\otimes} 1$  are the elements of  $T(\mathfrak{g}) \otimes T(\mathfrak{g})$ , not of  $T(\mathfrak{g})$ ). We shall use the common Sweedler notation, e.g.  $\Delta(u) = \sum_{(u)} u_{(1)} \hat{\otimes} u_{(2)}$ .

**Theorem.** Let  $\mathfrak{g}$  be as above,  $u, v \in T(\mathfrak{g})$  and  $\omega \in \bigwedge^2 \mathfrak{g}$ . Then

$$K\left(u \otimes \iota(\omega) \otimes v + \sum_{(u)} u_{(1)} \otimes (t(u_{(2)} \otimes \omega) \otimes v + r(u_{(2)} \otimes \omega)v)\right) = \sum_{(u)} e(u_{(1)} \otimes \omega) \otimes K(u_{(2)} \otimes v).$$

As an easy consequence, the map

$$R: \Omega(\mathfrak{g}) \otimes T(\mathfrak{g}) \rightarrow T(\mathfrak{g}), \quad R: u \otimes \omega \otimes v \mapsto \sum_{(u)} u_{(1)} \otimes (t(u_{(2)} \otimes \omega) \otimes v + r(u_{(2)} \otimes \omega)v)$$

has the required properties: if  $Q \in \Omega(\mathfrak{g}) \otimes T(\mathfrak{g})$ , then  $R(Q) + \iota(Q) \in K^{-1}(I(\mathfrak{g}))$  and  $\deg R(Q) < \deg \iota(Q)$ .

It is convenient to introduce the linear maps  $\lambda_x = L_x + \tau_x$ ,  $x \in \mathfrak{g}$ , and  $q(Q) = L_{t(Q)} + r(Q) : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ ,  $Q \in \Omega(\mathfrak{g})$ .

Denote by  $Z(u, \omega) : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})$ ,  $u \in T(\mathfrak{g})$ ,  $\omega \in \bigwedge^2 \mathfrak{g}$  the linear map defined by the equality

$$Z(u, \omega) = KL(u)L(\iota(\omega)) + \sum_{(u)} KL(u_{(1)})q(u_{(2)} \otimes \omega) - L(e(u_{(1)} \otimes \omega))KL(u_{(2)}).$$

The statement of the theorem may be written in the form  $Z(u, \omega)v = 0$ , so it remains to prove that  $Z(u, \omega)$  is zero. By the definitions,

$$Z(1, x \wedge y) = K[L_x, L_y] + Kq(x \wedge y) - ([L_x, L_y] - L_{[x, y]})K.$$

Substituting

$$q(x \wedge y) = [\tau_x, L_y] - [\tau_y, L_x] + [\tau_x, \tau_y] - \lambda_{[x, y]}$$

and taking into account the equality  $K\lambda_x = L_x K$ ,  $x \in \mathfrak{g}$ , we get

$$Z(1, x \wedge y) = K[\lambda_x, \lambda_y] - [L_x, L_y]K - K\lambda_{[x, y]} + L_{[x, y]}K = 0.$$

An easy computation shows that  $L(\lambda_x u) = \lambda_x L(u) - L(u)\tau_x$ ,  $q(\lambda_x Q) = [\tau_x, q(Q)]$  and  $\Delta(\lambda_x u) = \sum_{(u)} \lambda_x u_{(1)} \hat{\otimes} u_{(2)} + u_{(1)} \hat{\otimes} \lambda_x u_{(2)}$  for any  $x \in \mathfrak{g}$ . Applying all these equalities we have

$$\begin{aligned} Z(\lambda_x u, \omega) + Z(u, \tau_x \omega) &= KL(\lambda_x(u \otimes \iota(\omega))) + KL(\lambda_x u_{(1)})q(u_{(2)} \otimes \omega) + KL(u_{(1)})q(\lambda_x(u_{(2)} \otimes \omega)) \\ &\quad - L(e(\lambda_x(u_{(1)} \otimes \omega)))KL(u_{(2)}) - L(e(u_{(1)} \otimes \omega))KL(\lambda_x u_{(2)}) \\ &= L_x KL(u \otimes \iota(\omega)) - KL(u \otimes \iota(\omega))\tau_x + L_x KL(u_{(1)})q(u_{(2)} \otimes \omega) \\ &\quad - KL(u_{(1)})\tau_x q(u_{(2)} \otimes \omega) + KL(u_{(1)})[\tau_x, q(u_{(2)} \otimes \omega)] \\ &\quad - L(x \otimes e(u_{(1)} \otimes \omega) - e(u_{(1)} \otimes \omega) \otimes x)KL(u_{(2)}) \\ &\quad - L(e(u_{(1)} \otimes \omega))L_x KL(u_{(2)}) + L(e(u_{(1)} \otimes \omega))KL(u_{(2)})\tau_x \\ &= L_x Z(u, \omega) - Z(u, \omega)\tau_x. \end{aligned}$$

One can write this as

$$Z(x \otimes u, \omega) = L_x Z(u, \omega) - Z(u, \omega)\tau_x - Z(\tau_x u, \omega) - Z(u, \tau_x \omega).$$

By the induction on the degree of  $u$ , we have  $Z(u, \omega) = 0$ .

## 2. Geometric interpretation

Let  $\mathcal{M}$  be a smooth manifold, let  $\mathfrak{F}(\mathcal{M})$  be the algebra of smooth functions on  $\mathcal{M}$  and let  $\mathcal{V}(\mathcal{M}) = \text{Der}_{\mathbb{R}}(\mathfrak{F}(\mathcal{M}))$  be the Lie algebra of smooth vector fields. Let us denote  $\mathcal{V} = \mathcal{V}(\mathcal{M})$  and  $T(\mathcal{V}) = T_{\mathbb{R}}(\mathcal{V})$ .

Denote by  $\Gamma(\mathcal{M}, T^n \mathcal{M})$  the space of smooth global sections of the rank  $n$  tensor bundle  $T^n \mathcal{M} = \otimes^n T\mathcal{M}$ . For example,  $\Gamma(\mathcal{M}, T^0 \mathcal{M}) = \mathfrak{F}(\mathcal{M})$  and  $\Gamma(\mathcal{M}, T^1 \mathcal{M}) = \mathcal{V}(\mathcal{M})$ . Denote  $T(\mathcal{M}) = \bigoplus_{n=0}^{\infty} \Gamma(\mathcal{M}, T^n \mathcal{M})$ . It is well known that  $T(\mathcal{M}) = T_{\mathfrak{F}(\mathcal{M})}(\mathcal{V})$  (e.g. [3, Chapter I, Proposition 3.1]). Then there is a natural map  $\mathfrak{t}: T(\mathcal{V}) \rightarrow T(\mathcal{M})$ .

Denote by  $\mathcal{D}(\mathcal{M})$  the algebra of scalar differential operators with smooth coefficients on  $\mathcal{M}$ . Any vector field is a first-order differential operator. By the definition of  $T(\mathcal{V})$ , the natural inclusion map  $\tau: \mathcal{V} \rightarrow \mathcal{D}(\mathcal{M})$  can be extended to the algebra homomorphism  $\tau: T(\mathcal{V}) \rightarrow \mathcal{D}(\mathcal{M})$ . By the definition of the Lie bracket,  $\tau: x \otimes y - y \otimes x - [x, y] \mapsto 0$ ,  $x, y \in \mathcal{V}$ , hence  $\tau: I(\mathcal{V}) \rightarrow \{0\}$ .

Let us suppose the manifold to be endowed with a smooth affine connection. Let  $\mu: T(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{M})$  denote the  $\mathfrak{F}(\mathcal{M})$  – linear map defined by  $\mu: v_1 \otimes \cdots \otimes v_n \mapsto \nabla_{v_1, \dots, v_n}^n$ ,  $v_1, \dots, v_n \in \mathcal{V}$ . Here  $\nabla^n$  is the  $n$ -th order covariant derivative (in the notation of [3, Chapter III, §2]  $\nabla_{v_1, \dots, v_n}^n: f \mapsto f(v_n; \dots; v_1)$ ). For example, if  $f, g \in \mathfrak{F}(\mathcal{M})$  and  $v \in \mathcal{V}(\mathcal{M})$ , then  $\mu(f): g \mapsto fg$  and  $\mu(v): g \mapsto v(g)$ . The operator  $\mu(v \otimes w) = \nabla_{v, w}^2 = vw - (\nabla_v w)$  depends on the connection, as well as the images of the higher degree tensor fields.

Let  $\Sigma(\mathcal{M}) = \mathfrak{t}(\Sigma(\mathcal{V})) \subset T(\mathcal{M})$  be the space of (formal sums of) symmetric tensor fields. By the methods of geometry it may be shown that there exists a unique map  $\sigma: \mathcal{D}(\mathcal{M}) \rightarrow \Sigma(\mathcal{M})$ , such that  $\mu \circ \sigma = \text{id}_{\mathcal{D}(\mathcal{M})}$ . This map is a surjective  $\mathfrak{F}(\mathcal{M})$  – module homomorphism. The image  $\sigma(A)$  is called a symbol of the differential operator  $A$ . In almost the same form the symbol map was introduced in [6, §2] but actually it has been known long before (see the references in [6]). A proper investigation of the symbol map leads inevitably to the following natural question: what is the projection  $\Sigma = \sigma \circ \mu: T(\mathcal{M}) \rightarrow \Sigma(\mathcal{M})$ ? For example, this map is of importance when the symbol of a composition of two (pseudo)differential operators is considered. In some simple cases, Sharafutdinov has computed it in the unpublished supplements to [6].

The aforementioned projection  $\pi$  is closely related to  $\Sigma$ . The space  $\mathcal{V}$  may be considered as an algebra with two operations: the Lie bracket and the covariant derivative (probably Nomizu was the first who take this view [4, Chapter III, §6]). Put  $v \diamond w = \nabla_v w$ ,  $v, w \in \mathcal{V}$ . The corresponding map  $K: T(\mathcal{V}) \rightarrow T(\mathcal{V})$  is then connected to  $\mu$  by the relation  $\mu \circ \mathfrak{t} = \tau \circ K$  [1, Proposition 1], [2, Lemma 2]. This relation is actually a simple consequence of the well-known covariant derivation rules [3, Chapter III, Proposition 2.10]. Let  $\pi: T(\mathcal{V}) \rightarrow \Sigma(\mathcal{V})$  be the projection defined above. By the definition,  $\text{id} - \pi: T(\mathcal{V}) \rightarrow K^{-1}(I(\mathcal{V}))$ , hence  $\mu \circ \mathfrak{t} = \mu \circ \mathfrak{t} \circ \pi$ . If  $\mathfrak{t}^{-1}: T(\mathcal{M}) \rightarrow T(\mathcal{V})$  is any right inverse of  $\mathfrak{t}^{-1}$ , then  $\mu = \mu \circ \mathfrak{t} \circ \pi \circ \mathfrak{t}^{-1}$ . The map  $\Sigma$  is determined uniquely by the property  $\mu = \mu \circ \Sigma$ , hence  $\Sigma = \mathfrak{t} \circ \pi \circ \mathfrak{t}^{-1}$ . Note that the equality does not depend on the choice of  $\mathfrak{t}^{-1}$ , which means  $\pi: \ker(\mathfrak{t}, T(\mathcal{V})) \rightarrow \ker(\mathfrak{t}, T(\mathcal{V}))$ . We shall call a linear map with this property a special one. In other words, the map  $T(\mathcal{V}) \rightarrow T(\mathcal{V})$  is special if it may be lifted to an  $\mathfrak{F}(\mathcal{M})$  – linear map  $T(\mathcal{M}) \rightarrow T(\mathcal{M})$ .

In the expression  $\pi = \pi_0(\text{id} + \rho)^{-1}$  the map  $\pi_0$  is special, so it is natural to ask for speciality of  $\rho$ . Under natural assumptions on  $\iota^{-1}$  this is indeed the case, because the functions  $t$  and  $r$  are nothing but the (derivatives of) the torsion tensor and the curvature tensor respectively:

$$\begin{aligned} t(v \wedge w) &= T(v, w), & t(u \otimes v \wedge w) &= (\nabla_u T)(v, w), \\ r(v \wedge w)h &= R(v, w)h, & r(u \otimes v \wedge w)h &= (\nabla_u R)(v, w)h, \end{aligned}$$

etc., where  $u, v, w, h \in \mathcal{V}(\mathcal{M})$  [4, Chapter III, §5], [1, §7]. The statement  $R(Q) + \iota(Q) \in K^{-1}(I(\mathcal{V}))$ ,  $Q \in \Omega(\mathcal{V}) \otimes T(\mathcal{V})$  may then be considered as a series of commutation relations on the covariant derivative. For example, if  $Q = u \otimes v \wedge w \otimes h$ , it takes the form  $u \otimes v \otimes w \otimes h - u \otimes w \otimes v \otimes h + u \otimes t(v \wedge w) \otimes h + t(u \otimes v \wedge w) \otimes h + u \otimes r(v \wedge w)h + r(u \otimes v \wedge w)h \in K^{-1}(I(\mathcal{V}))$ . Note that  $\mu \circ \mathfrak{t} = \tau \circ K: K^{-1}(I(\mathcal{V})) \rightarrow \{0\}$ . Applying this map, we get

$$\nabla_{u, v, w, h}^4 - \nabla_{u, v, v, h}^4 + \nabla_{u, T(v, w), h}^3 + \nabla_{(\nabla_u T)(v, w), h}^2 + \nabla_{u, R(v, w)h}^2 + \nabla_{(\nabla_u R)(v, w)h} = 0.$$

Calculating relations of this kind manually is not an easy task even for relatively small degrees.

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