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Tropical matrices and group representations

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ABSTRACT

The paper gives a complete description of the subgroups of the semigroup of tropical n-by-n matrices up to an isomorphism.

1. Introduction

The set $\mathbb{R}$ of reals extended by adding an infinite negative element $-\infty$ is called the *tropical semiring* and is also known as the *max-plus algebra*. The tropical arithmetic operations on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ are $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$. The main object of our research is the set $\overline{\mathbb{R}}^{n\times n}$ of tropical n-by-n matrices. We are interested in studying the multiplicative structure of tropical matrices. The multiplication of such matrices is defined as ordinary matrix multiplication with $+$ and $\cdot$ replaced by the tropical operations $\oplus$ and $\otimes$.

The study of linear algebra over the tropical semiring is important for many different applications (see [1,5]). There are a number of purely linear-algebraic important problems for tropical matrices, for example, the eigenvalues and eigenvectors problem, the problem of solving linear systems, computational problems for the rank functions, see [3,5]. Another important approach considers the set of tropical matrices from the point of view of the semigroup theory. The paper [4] is devoted to the solution of the Burnside-type problem for semigroups of tropical matrices. Johnson and Kambites in the recent paper [9] have developed the study of the semigroup-theoretic structure of tropical matrices. They consider Green's relations on the semigroup $(\overline{\mathbb{R}}^{n\times n}, \otimes)$, groups of tropical matrices, and

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idempotent tropical matrices. They give a complete description of the subgroups of $(\overline{\mathbb{R}}^{n\times n}, \otimes)$ in the case when $n = 2$. The study of Green's relations on the semigroup of tropical matrices has been developed in [6,8]. In [6], the complete description of the $\mathcal{D}$-relation has been provided. In [8], the important characterization of the $\mathcal{J}$-order has been given, and the connection of Green's relations with the rank functions of tropical matrices has been studied.

The aim of our paper is to solve the problem that has arisen from the paper [9]. Namely, we are interested in a complete characterization of the subgroups of the semigroup $(\overline{\mathbb{R}}^{n\times n}, \otimes)$. We show that every subgroup of the semigroup of tropical n-by-n matrices admits a faithful representation with tropical monomial n-by-n matrices. We prove that every subgroup $\mathcal{G}$ of $(\overline{\mathbb{R}}^{n\times n}, \otimes)$ is isomorphic to a subgroup of the wreath product $\mathbb{R} \wr \mathcal{S}_n$, and, conversely, every subgroup of $\mathbb{R} \wr \mathcal{S}_n$ can be realized with tropical n-by-n matrices. Our results confirm the conjecture proposed in [9] that every group admitting a faithful representation by $n \times n$ tropical matrices must have a torsion-free abelian subgroup of index at most $n!$. We also give an upper bound for the order of a periodic group of tropical n-by-n matrices, developing the result proven in [2].

We note that a result that is similar to the main result of our paper is contained in the paper [7] by Izhakian, Johnson, and Kambites. However, they consider the case of matrices without infinite elements, and the technique used in their proof seems to be considerably different from that used in ours.

Throughout our paper $\mathcal{S}_n$ will denote the symmetric group on $\{1, \ldots, n\}$. By $a_{i(\cdot)}$ we denote the ith row of a matrix A, by $A[r_1, \ldots, r_k]$ the submatrix of A formed by the rows with numbers $r_1, \ldots, r_k$. We say that a matrix $P \in \overline{\mathbb{R}}^{n\times n}$ is *monomial* if there exists $\sigma = \sigma(P) \in \mathcal{S}_n$ such that $p_{ij} \neq -\infty$ if and only if $i = \sigma(j)$. In this case, P is called *diagonal* if $\sigma(P)$ is an identity permutation. Note that the diagonal matrix with zeros on the diagonal is the identity element of the semigroup $(\overline{\mathbb{R}}^{n\times n}, \otimes)$.

2. Subgroups of the semigroup $(\overline{\mathbb{R}}^{n\times n}, \otimes)$

In order to prove our main results, we need to introduce some auxiliary notions. The concept of the row rank (see [1]) of a tropical matrix is useful for our considerations.

Definition 2.1. A tropical matrix $B \in \overline{\mathbb{R}}^{n\times m}$ is said to be of *full row rank* if no row of A can be expressed as a linear combination of other rows, that is, the condition

$$b_{i(\cdot)} = \bigoplus_{k\in\{1,\ldots,n\}\setminus\{i\}} \lambda_k \otimes b_{k(\cdot)} \tag{1}$$

fails to hold for every $i \in \{1, \ldots, n\}$ and $\lambda_1, \ldots, \lambda_n \in \overline{\mathbb{R}}$.

The following theorem gives a linear-algebraic reformulation of the fact that is well known in the semimodule theory, see [10, Theorem 5].

Theorem 2.2. *Let $A, B, C, D \in \overline{\mathbb{R}}^{n\times n}$ be such that $B = C \otimes A$, $A = D \otimes B$. If the row rank of B is full, then there exists a monomial matrix $P \in \overline{\mathbb{R}}^{n\times n}$ such that $B = P \otimes A$.*

Proof. Denote the set of all tropical linear combinations of the rows of A by $M(A)$. The row rank of B is full, so the rows of B form a *weak basis* of the *pseudomodule* $M(B)$, see [10, Definition 2.2.5]. Since $B = C \otimes A$, we see that $M(B) \subset M(A)$, and also $M(A) \subset M(B)$ because $A = D \otimes B$. Therefore, $M(A) = M(B)$, and the rows of B are a *weak basis* of $M(A)$ as well. By Theorem 5.1 from [10], the set of rows of B coincides with the set of rows of A up to scaling. □

The following lemma deals with matrices whose row rank is not full.

Lemma 2.3. *Let $\mathcal{G}$ be a subgroup of $(\overline{\mathbb{R}}^{n\times n}, \otimes)$, $n \geqslant 2$. If the row rank of some matrix A from $\mathcal{G}$ is not full, then $\mathcal{G}$ admits a faithful representation with tropical $(n-1)$-by-$(n-1)$ matrices.*

Proof. By Definition 2.1, some row of A is a linear combination of the other rows. So for some $i \in \{1,\dots,n\}$ we have $A = P \otimes \overline{A}$, where the matrix $\overline{A} \in \overline{\mathbb{R}}^{(n-1)\times n}$ is obtained from A by removing the ith row, and $P \in \overline{\mathbb{R}}^{n\times(n-1)}$ is such that the matrix $P[1,\dots,i-1,i+1,\dots,n]$ is the identity matrix of $(\overline{\mathbb{R}}^{(n-1)\times(n-1)},\otimes)$, and the ith row of P consists of the coefficients occurring in the linear combination identified above. Since $\mathcal{G}$ is a group, for every $G \in \mathcal{G}$ there exists $B \in \mathcal{G}$ such that $G = A \otimes B$.

Thus we see that $G = P \otimes \overline{A} \otimes B$. Moreover, since $P[1,\dots,i-1,i+1,\dots,n]$ is the identity matrix, $\overline{G} = \overline{A} \otimes B$ is the unique matrix satisfying $G = P \otimes \overline{G}$. The map φ sending $G \in \mathcal{G}$ to $\overline{G} \otimes P \in \overline{\mathbb{R}}^{(n-1)\times(n-1)}$ is therefore well defined.

Let $G, H \in \mathcal{G}$ and let $\overline{G}$ and $\overline{H}$ denote the unique matrices satisfying $G = P \otimes \overline{G}$ and $H = P \otimes \overline{H}$. Then $G \otimes H = P \otimes \overline{G} \otimes P \otimes \overline{H}$, so that $\overline{G \otimes H} = \overline{G} \otimes P \otimes \overline{H}$, giving

$$\varphi(G \otimes H) = \overline{G \otimes H} \otimes P = \overline{G} \otimes P \otimes \overline{H} \otimes P = \varphi(G) \otimes \varphi(H),$$

so φ is a homomorphism. Moreover, if $\varphi(G) = \varphi(H)$, then $\overline{G} \otimes P = \overline{H} \otimes P$, so in this case $P \otimes \overline{G} \otimes P \otimes \overline{G} = P \otimes \overline{H} \otimes P \otimes \overline{G}$, or $G \otimes G = H \otimes G$. Since $\mathcal{G}$ is a group, the condition $\varphi(G) = \varphi(H)$ therefore implies that $G = H$, proving that φ is injective. □

Now we are ready to prove the one of our main results.

Theorem 2.4. *Every subgroup of the semigroup $(\overline{\mathbb{R}}^{n\times n},\otimes)$ admits a faithful representation with tropical monomial n-by-n matrices.*

Proof. The case of $n = 1$ is trivial, and we proceed by the induction on n. Let $n \geqslant 2$, $\mathcal{G}$ be a subgroup of $(\overline{\mathbb{R}}^{n\times n},\otimes)$, E be a neutral element of $\mathcal{G}$. The two cases are then possible.

1. Let $\mathcal{G}$ contain a matrix whose row rank is not full. Lemma 2.3 shows that in this case $\mathcal{G}$ admits a faithful representation with tropical $(n-1)$-by-$(n-1)$ matrices. The inductive hypothesis then shows that $\mathcal{G}$ has a faithful representation with tropical monomial $(n-1)$-by-$(n-1)$ matrices, so the result follows.

2. Now let the matrices from $\mathcal{G}$ be of full row rank. In this case, from Theorem 2.2 it follows that for every $G \in \mathcal{G}$ there exists a monomial matrix $\mathcal{P}_G \in \overline{\mathbb{R}}^{n\times n}$ such that $G = \mathcal{P}_G \otimes E$. Since the row rank of G is full, we see that the matrix $\mathcal{P}_G$ is uniquely determined. So we can define the map ψ sending $G \in \mathcal{G}$ to the monomial matrix $\mathcal{P}_G$. Clearly, ψ is injective. We also see that for every $G, H \in \mathcal{G}$ it holds that

$$\psi(G \otimes H) = \psi(\mathcal{P}_G \otimes E \otimes H) = \psi(\mathcal{P}_G \otimes H) = \psi(\mathcal{P}_G \otimes \mathcal{P}_H \otimes E) = \mathcal{P}_G \otimes \mathcal{P}_H,$$

so ψ is a homomorphism. □

Johnson and Kambites in [9, Section 4] conjectured that every group admitting a faithful representation by $n \times n$ tropical matrices has a torsion-free abelian subgroup of index at most $n!$. Now we are ready to prove this conjecture.

Theorem 2.5. *Let a group $\mathcal{G}$ admit a faithful representation by $n \times n$ tropical matrices. Then $\mathcal{G}$ has a torsion-free abelian subgroup of index at most $n!$.*

Proof. By Theorem 2.4, we assume without a loss of generality that $\mathcal{G}$ consists of tropical monomial n-by-n matrices. Consider the subgroup D of all diagonal matrices from $\mathcal{G}$. Clearly, D is normal in $\mathcal{G}$, abelian and torsion-free. It remains to note that matrices $A, B \in \mathcal{G}$ belong to the same coset of D in $\mathcal{G}$ if and only if $\sigma(A) = \sigma(B)$. □

D'Alessandro and Pasku have shown that every periodic finitely generated subgroup of $(\overline{\mathbb{R}}^{n\times n},\otimes)$ is finite, see [2, Proposition 5]. Theorem 2.4 allows us to derive a more precise characterization. Recall that a group H is *periodic* if each element of H has finite order.

Corollary 2.6. *The order of any periodic subgroup of the semigroup* $(\overline{\mathbb{R}}^{n\times n}, \otimes)$ *is at most* $n!$.

Proof. By definition, any torsion-free subgroup of a periodic group is trivial. So the result follows from Theorem 2.5. □

Finally, we note that the group of all tropical monomial n-by-n matrices is isomorphic to the wreath product $\mathbb{R} \wr \mathcal{S}_n$. This gives the following group-theoretic description of the subgroups of $(\overline{\mathbb{R}}^{n\times n}, \otimes)$.

Theorem 2.7. *A group* $\mathcal{G}$ *admits a faithful representation with tropical n-by-n matrices if and only if* $\mathcal{G}$ *is isomorphic to a subgroup of the wreath product* $\mathbb{R} \wr \mathcal{S}_n$.

Proof. Follows from Theorem 2.4. □

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