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Integral domains of finite t-character

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ABSTRACT

An integral domain D is said to be of finite t-character if each nonzero nonunit of D is contained in only finitely many maximal t-ideals of D. For example, Noetherian domains and Krull domains are of finite t -character. In this paper, we study several properties of integral domains of finite t-character. We also show when the ring $D^{(S)} = D + XD_S[X]$ is of finite t-character, where X is an indeterminate over D and S is a multiplicative subset of D.

Introduction

An integral domain D is said to be of *finite character* (resp., *finite t-character*) if every nonzero nonunit of D belongs to at most a finite number of maximal ideals (resp., maximal t-ideals). It is well known that integral domains in which each t-ideal is a v-ideal (e.g., Noetherian, Mori, or Krull domains) are of finite t-character [27, Theorem 1.3]. Also, if D is of finite t-character, then D is a w-LPI domain (i.e., each nonzero t-locally principal ideal is t-invertible), and hence D is an LPI domain that is an integral domain in which every nonzero locally principal ideal is invertible. In particular,

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if D is a Prüfer domain (resp., Prüfer v-multiplication domain (PvMD)), then D is of finite character (resp., finite t-character) if and only if D is an LPI domain (resp., a w-LPI domain) [26, Theorem 10], [23, Theorem 6.1], [40, Proposition 5]. The properties of LPI domains (resp., w-LPI domains) are further studied in [9] (resp., [12]).

Let S be a multiplicative subset of an integral domain D. It is known that if A is a t-ideal of D_S, then $A \cap D$ is a t-ideal of D [30, Lemma 3.17]. However, I being a t-ideal of D does not imply that ID_S is a t-ideal of D_S. As in [38], we say that D is *conditionally well behaved* if for each maximal t-ideal M of D, the prime ideal MD_M is a t-ideal. In Section 1 of this paper, we study the finite t-character property of integral domains. We first show that integral domains of finite t-character are conditionally well behaved. As a corollary, we have that if D is of finite t-character, then D is t-locally (resp., locally) a GCD-domain if and only if D is a PvMD (resp., generalized GCD-domain). We shall also give some examples of situations where the requirements/properties yield the property of being of finite t-character. In Section 2, we study when the ring $D^{(S)} = D + XD_S[X]$ is a PvMD of finite t-character, where X is an indeterminate over D. Precisely, we show that $D^{(S)}$ is a PvMD of finite t-character if and only if D is a PvMD of finite t-character, S is a t-splitting set, and $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| < \infty$. In particular, if D is a Krull domain, then $D^{(S)}$ is of finite t-character if and only if $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| < \infty$. Finally, in Section 3, we give a kind of Nagata-like theorem. We then use this result to prove some sufficient conditions for $D^{(S)}$ to be of finite t-character even when $D^{(S)}$ is not a PvMD.

It is apparent that this paper will be steeped in the so-called star-operations. So let us start with a set of working definitions. Most of the information given below can be found in [39] and [19]. Let D denote an integral domain with quotient field K and let $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) *fractional* ideals of D. A fractional ideal that is contained in D will be called an *integral ideal*.

A *star operation* $*$ on D is a function $*: F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(i) $(x)^* = (x)$ and $(xA)^* = xA^*$,
(ii) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,
(iii) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ for some $B \in f(D)$. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid B \subseteq A \text{ and } B \in f(D)\}$. For $A \in F(D)$, define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly, every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid B \in f(D) \text{ and } B \subseteq A\}$, the w-operation defined by $A \mapsto A_w = \{x \in K \mid xJ \subseteq A \text{ for some } J \in f(D) \text{ with } J^{-1} = D\}$, and the d-operation that is the identity function of $F(D)$ onto itself. Given two star operations $*_1$ and $*_2$, we say that $*_1 \leqslant *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leqslant *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. For any star operation $*$, we have $* \leqslant v$. For $A \in F(D)$, A^{-1} is a v-ideal. If $\{D_\alpha\}$ is a family of overrings of D such that $D = \bigcap D_\alpha$, then the operation $*$ defined on $F(D)$ by $A \mapsto \bigcap AD_\alpha$ is a star operation induced by $\{D_\alpha\}$. Thus if $*$ is a star operation induced by $\{D_\alpha\}$, then $A^{-1} = (A^{-1})^* = \bigcap A^{-1}D_\alpha$. By definition t is of finite character, $t \leqslant v$ while $* \leqslant t$ for every star operation $*$ of finite character. If $*$ is a star operation of finite character, then using Zorn's lemma we can show that an integral ideal maximal among proper integral $*$-ideals is a prime ideal and that every integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $*$-Max(D). It can also be easily established that if $*$ is a star operation of finite character on D, then $D = \bigcap_{M \in *\text{-Max}(D)} D_M$, and an $A \in F(D)$ is $*$-invertible if and only if $AA^{-1} \nsubseteq P$ for any maximal $*$-ideal P of D.

A v-ideal A of finite type is t-invertible if and only if A is t-locally principal, i.e., for every $M \in t\text{-Max}(D)$, we have that AD_M is principal. We say that an ideal A is t-locally t-invertible if AD_M is t-invertible for every maximal t-ideal M. Recall from [5, Corollary 2.17] that $t\text{-Max}(D) = w\text{-Max}(D)$; so $A \in F(D)$ is t-locally principal (resp., t-invertible) if and only if A is w-locally principal (resp., w-invertible). An integral domain D is called a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal of D is t-invertible. It is well known that D is a PvMD if and only if D_M is a valuation domain for each $M \in t$-Max(D) [21, Theorem 5]. An integral domain D is called an *essential* domain if $D = \bigcap D_P$ where P ranges over prime ideals of D such that D_P is a valuation domain. By definition, PvMDs are essential. Any pair of elements $a, b \in D$ is said to be *v-coprime* if $(a, b)_v = D$. Obviously, two elements a, b in D are v-coprime if and only if a, b do not share a maximal t-ideal. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$ and let $Prin(D)$ be its subgroup of principal fractional ideals. Then $Cl(D) = T(D)/Prin(D)$, called the (t-)class group of D, is an abelian group. Clearly, $Cl(D) = (0)$ means that every t-invertible fractional t-ideal of D is principal. It is well know that D is a GCD-domain if and only if D is a PvMD and $Cl(D) = (0)$.

Let $\mathcal{F}$ be a family of prime ideals of D. Then $\mathcal{F}$ is called a *defining family* for D if $D = \bigcap_{P \in \mathcal{F}} D_P$. For example, t-Max(D) and Max(D) are defining families for D, where Max(D) represents the set of maximal ideals of D. If $\mathcal{F}$ is a defining family for D, then we denote by $*_{\mathcal{F}}$ the star operation on D induced by $\{D_P \mid P \in \mathcal{F}\}$. We call $\mathcal{F}$ *independent* if no two distinct members of $\mathcal{F}$ contain a nonzero prime ideal. An integral domain D is called an *h-local domain* (resp., *weakly Matlis domain* or WM-domain) if D is of finite character (resp., finite t-character) and Max(D) (resp., t-Max(D)) is independent. Clearly, a weakly Matlis domain is a t-operation version of the h-local domains. The h-local domains were introduced and studied by Eben Matlis (see [8]). Recall that D is an h-local domain (resp. WM-domain) if and only if for every maximal ideal (resp., maximal t-ideal) M and for every nonzero $x \in M$, the ideal $xD_M \cap D$ is invertible (resp., t-invertible) [8, Corollaries 3.4 and 4.4].

The prime t-ideals have this annoying property that if P is a prime t-ideal of D then PD_S may not be a prime t-ideal for some multiplicative set S disjoint with P. The authors of [31] were led to this conclusion seeing an example in [25] of an essential domain that is not a PvMD. In any case, in [38], a prime ideal P in D was called *well behaved* if PD_P is a prime t-ideal of D_P. We say that D is *well behaved* if every prime t-ideal of D is well behaved. In [38], the last named author characterized well behaved domains and showed that most of the known domains, including PvMDs, are well behaved. Clearly, well behaved domains are conditionally well behaved. In [38], there was also an example of a conditionally well behaved domain that is not well behaved (or see Example 1.4).

1. Integral domains of finite t-character

Let D be an integral domain with quotient field K. We will say that a maximal t-ideal is *potent* if it contains a nonzero finitely generated ideal that is not contained in any other maximal t-ideal. Clearly, if P is finitely generated, the radical of a finitely generated ideal, or a v-ideal of finite type, then P is automatically potent. Hence Mori domains and Noetherian domains all have potent maximal t-ideals.

Theorem 1.1. *Let D be an integral domain.*

(1) *Let P be a maximal t-ideal of D that is potent, then P is well behaved.*
(2) *If D is of finite t-character, then every maximal t-ideal of D is potent, and hence D is conditionally well behaved.*

Proof. (1) Let A be a nonzero finitely generated ideal such that $A \subseteq P$ but $A \nsubseteq Q$ for all $Q \in t$-Max(D) with $Q \neq P$. We first show that $(AD_P)_v \subseteq PD_P$. Deny. Then $(AD_P)_v = D_P$ which gives $(AD_P)^{-1} = D_P$. Since A is finitely generated, we have $A^{-1}D_P = D_P$ [35, Lemma 4]. Next, for any maximal t-ideal Q of D with $Q \neq P$ we have $AD_Q = D_Q$ and so $A^{-1}D_Q = D_Q$. Thus, $A^{-1} = \bigcap_{M \in t\text{-Max}(D)} A^{-1}D_M = \bigcap_{M \in t\text{-Max}(D)} D_M = D$. But, this means $A_v = D$, a contradiction to the fact that A is contained in P a maximal t-ideal of D. Hence our denial of $(AD_P)_v \subseteq PD_P$ is refuted. Now take any nonzero finitely generated ideal $B \subseteq P$ and note that $B + A$ is contained in P and in no maximal t-ideal other than P, because of A. By the above, we conclude that $((B + A)D_P)_v \subseteq PD_P$. But as $BD_P \subseteq (B + A)D_P \subseteq PD_P$ we have $(BD_P)_v \subseteq ((B + A)D_P)_v \subseteq PD_P$. Thus PD_P is a t-ideal of D_P.

(2) To prove this, let x be a nonzero element in a maximal t-ideal P of D. If x belongs to no other maximal t-ideal, we have nothing to prove. So let us assume that x belongs also to other maximal t-ideals. Since D is of finite t-character, there can be only finitely many maximal t-ideals

$M_1, M_2, \ldots, M_n$, in all, besides P. Now construct $A = (x, x_1, \ldots, x_n)$ where $x_i \in P \backslash M_i$. Clearly, $A \subseteq P$ and A is in no other maximal t-ideal. Thus, P is potent, and hence P is well behaved by (1). Now as P was arbitrary, D is conditionally well behaved. $\square$

The next result is similar to [17, Proposition 2.8 and Corollary 2.8] where it was studied when t-locally a PvMD or a GCD-domain is of finite t-character.

Corollary 1.2. *If D is of finite t-character, then the following are equivalent.*

(1) *D is t-locally a GCD-domain.*
(2) *D is t-locally a PvMD.*
(3) *D is a PvMD.*
(4) *D is locally a PvMD.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) Let M be a maximal t-ideal of D. Then D_M is a PvMD, and since MD_M is a t-ideal by Theorem 1.1(2), $D_M = (D_M)_{MD_M}$ is a valuation domain. Thus D is a PvMD. (3) $\Rightarrow$ (1) and (4) Clear. (4) $\Rightarrow$ (3) Let P be a maximal t-ideal of D. Then PD_P is a t-ideal of D_P by Theorem 1.1(2), and so if M is a maximal ideal of D containing P, then PD_M is a t-ideal of D_M because $PD_P \cap D_M = PD_M$. Thus, by (4), $D_P = (D_M)_{PD_M}$ is a valuation domain. $\square$

An integral domain D is called a *generalized GCD-domain* (GGCD-domain) if the intersection of two invertible ideals of D is invertible. It is known that D is a GGCD-domain if and only if I_v is invertible for each nonzero finitely generated ideal I of D, if and only if $aD \cap bD$ is invertible for all $0 \neq a, b \in D$ [2, Theorem 1].

Corollary 1.3. *If D is of finite t-character, then D is locally a GCD-domain if and only if D is a GGCD-domain.*

Proof. ($\Rightarrow$) Note that locally a GCD-domain is t-locally a GCD-domain; so D is a PvMD by Corollary 1.2. Hence if $0 \neq a, b \in D$, then $aD \cap bD$ is of finite type because $aD \cap bD$ is t-invertible. Also, $(aD \cap bD)_M = aD_M \cap bD_M$ is principal for all maximal ideals M of D by assumption. Thus $aD \cap bD$ is invertible.

($\Leftarrow$) This is well known, but we give the proof. Let $0 \neq x, y \in D$. Then $xD \cap yD$ is invertible by assumption, and hence $xD_M \cap yD_M = (xD \cap yD)_M$ is invertible (so principal) for all maximal ideals M of D. $\square$

We next give an example of integral domains of finite t-character that is not well-behaved.

Example 1.4. Let $R = \mathbb{R}[[X, Y, Z]]$ be the power series ring over the field $\mathbb{R}$ of real numbers, $M = (X, Y, Z)\mathbb{R}[[X, Y, Z]]$, and $D = \mathbb{Q} + M$, where $\mathbb{Q}$ is the field of rational numbers. Then R is a 3-dimensional local Noetherian Krull domain with maximal ideal M, and D is a quasi-local domain with maximal ideal M such that $\operatorname{Spec}(R) = \operatorname{Spec}(D)$ and M is a v-ideal of D. Hence D is of finite t-character. But, if P is a prime ideal of D with $\operatorname{ht}P = 2$, then P is a prime ideal of R such that $D_P = R_P$. Clearly, R_P is a 2-dimensional Krull domain and $\operatorname{ht}PR_P = 2$, and thus $PD_P = PR_P$ is not a t-ideal. Thus D is not well-behaved.

Theorem 1.1(2) shows that if D is of finite t-character, then every maximal t-ideal of D is potent. However, if $D = \mathbb{Z} + X\mathbb{Q}[[X]]$, then every maximal t-ideal of D is potent, but D is not of finite t-character. It may be noted that while a domain with potent maximal t-ideals is conditionally well behaved by Theorem 1.1(1), even a well behaved domain may not have potent maximal t-ideals.

Example 1.5. Let $S = \{X^\alpha \mid \alpha \in \mathbb{Q}^+\}$ where $\mathbb{Q}^+$ denotes the set of nonnegative rational numbers and let K be an algebraically closed field with characteristic zero. Also, let R be the semigroup ring $K[S] = \{\sum c_i X^{\alpha_i} \mid c_i \in K \text{ and } \alpha_i \in \mathbb{Q}^+\}$.

(1) R is a one-dimensional Bezout domain, and hence a well-behaved domain.
(2) No maximal ideal containing $(X-1)R$ is potent.

Proof. (1) Note that R can be regarded as an ascending union of the PIDs $R_{n!} = K[X^{\frac{1}{n!}}]$ where $n!$ denotes the factorial of the natural number n. That is, $R = \bigcup R_{n!}$, where obviously $R_{n!} \subseteq R_{(n+1)!}$ for all natural numbers n. Being an ascending union of PIDs, R is a one-dimensional Bezout domain.

(2) Note that every finitely generated ideal of R is principal by (1), and so every nonzero ideal is a t-ideal. Now by [6, Theorem 1], R is an antimatter domain, i.e., every nonzero nonunit element of R is expressible as a product of at least two nonunits. Since R is a Bezout domain, a maximal ideal P of R is potent if and only if there is an element $r \in P$ such that r belongs to no other maximal ideal. Now it is easy to show that in a Bezout domain R, a nonzero element r belongs to a unique maximal ideal if and only if r is such that for all $x, y \mid r$ we have $x \mid y$ or $y \mid x$, i.e. r is rigid. Clearly, as no element of R is a prime, nor a prime power, we have a factorization $r = xy$ where x and y are nonunits. Now as r is rigid, $x \mid y$ or $y \mid x$ and so $x^2 \mid r$ or $y^2 \mid r$. Now let P be a maximal ideal containing $(X-1)R$ such that P is potent containing a rigid element s. But then there is a rigid element r dividing $X-1$ and so there is a nonunit factor x such that $x^2 \mid r$ and so $x^2 \mid (X-1)$, contradicting the fact that $(X-1)R$ is a radical ideal as shown in [34, Example 3.6 and Lemma 3.7]. □

Let $*$ be a star operation of finite character on D. We say that D is of *finite $*$-character* if each nonzero nonunit of D is contained in only finitely many maximal $*$-ideals of D. We shall call an ideal A of D $*$-locally principal if AD_P is principal for each maximal $*$-ideal P of D and we shall call D a *$*$-LPI domain* if every nonzero $*$-locally principal ideal of D is $*$-invertible. Obviously, finite d-character $\Leftrightarrow$ finite character; finite t-character $\Leftrightarrow$ finite w-character; d-LPI domain $\Leftrightarrow$ LPI domain; and w-LPI domain $\Leftrightarrow$ t-LPI domain.

It is now well-established that a Prüfer domain (resp. PvMD) D is of finite character (resp., finite t-character) if and only if D is an LPI (resp., w-LPI) domain. Look up [40] for the relevant results and history. As it was shown in [14], most cases where LPI (resp., w-LPI) implies finite character (resp., finite t-character) fall under the cases where every finitely generated ideal is contained in at least one ideal of a fixed type, e.g., invertible ideal (resp., t-invertible t-ideal). Finocchiaro et al. in [17], took a direction that could avoid using the approach used in [14]. Here we show that there are some situations, mostly involving conditionally well behaved prime t-ideals such that $*$-LPI implies finite $*$-character, where $*$ is a star operation of finite character.

Theorem 1.6. *Let $*$ be a star operation of finite character on D such that $\mathcal{F} = *\text{-Max}(D)$ is independent. Then D is of finite $*$-character if and only if D is $*$-LPI.*

Proof. ($\Rightarrow$) This is an immediate consequence of [7, Lemma 2.2].

($\Leftarrow$) Suppose that D is $*$-LPI. Let $x \in P \backslash \{0\}$ where $P \in *\text{-Max}(D)$ and consider $xD_P \cap D$. Then $xD_P \cap D \subseteq P$ and to no other member of $*\text{-Max}(D)$ [8, Lemma 2.3]. Now $(xD_P \cap D)D_P = xD_P$ and $(xD_P \cap D)D_Q = D_Q$ for all $Q \in *\text{-Max}(D)$ with $Q \neq P$, and so $xD_P \cap D$ is $*$-locally principal. Hence by the assumption, $xD_P \cap D$ is $*$-invertible. Next, note that $*_{\mathcal{F}} = *_w$ because the members of $\mathcal{F}$ are maximal $*$-ideals and $*_w$ is of finite character and that $*$-invertible is $*_w$-invertible [5]. Thus $*_{\mathcal{F}}$ is of finite character, for each $P \in \mathcal{F}$ and $0 \neq x \in P$, $xD_P \cap D$ is $*_{\mathcal{F}}$-invertible and unidirectional (is contained in P and in no other member of $\mathcal{F}$); hence all requirements of (4) of [8, Theorem 3.3] are met, and so by [8, Theorem 3.3] $\mathcal{F}$ is independent of finite character which translates to "D is of finite $*$-character". □

We say that D is of *t-dimension one* if every member of $t\text{-Max}(D)$ is of height-one. Obviously, if D is of t-dimension one, then $t\text{-Max}(D)$ is independent and D is well behaved (and hence conditionally well behaved). An integral domain of t-dimension one that is also of finite t-character is called a *weakly Krull domain*. (These domains were studied in [7, Theorem 3.1], called weakly Krull domains in [3].) We recall from [7, Theorem 3.1] that D is a weakly Krull domain if and only if every nonzero prime ideal of D contains a nonzero t-invertible primary t-ideal, if and only if P being minimal over a proper principal ideal (x) implies $xD_P \cap D$ is t-invertible.

Corollary 1.7. *Let D be of t-dimension one. Then the following are equivalent.*

(1) *D is of finite t-character.*
(2) *D is a weakly Krull domain.*
(3) *Every maximal t-ideal of D is potent.*
(4) *D is a w-LPI domain.*

Proof. (1) $\Leftrightarrow$ (2) This follows because D is of t-dimension one.

(1) $\Rightarrow$ (3) Theorem 1.1(2).

(3) $\Rightarrow$ (1) Assume that every maximal t-ideal of D is potent. If P is a maximal t-ideal of D, then there is a nonzero finitely generated ideal A such that $A \subseteq P$ but $A \nsubseteq Q$ for all $Q \in t\text{-Max}(D) \setminus \{P\}$. Hence $P = \sqrt{A}$ because D is of t-dimension one. Let x be a nonzero nonunit of D. Then each minimal prime ideal of xD is a maximal t-ideal, and hence the radical of a finitely generated ideal. Thus, x is contained in only finitely many maximal t-ideals [20, Theorem 2.1]. Hence D is of finite t-character.

(1) $\Leftrightarrow$ (4) Since D is of t-dimension one, $t\text{-Max}(D)$ is independent. Thus, the result follows directly from Theorem 1.6. $\square$

Remark 1.8. We had originally proved Theorem 1.6 for t-operation alone, thanks are due to the reviewer for pointing out to us the more general result that is Theorem 1.6 now. This result was first proved for $* = d$ in [33, Lemma 3.9].

Note that every height-one prime ideal is a t-ideal, because it is minimal over each of its nonzero principal subideals. Also, note that if every maximal ideal is a t-ideal, then the notions of "t-invertible" and "invertible" coincide. This gives the following corollary.

Corollary 1.9. *Let D be a one-dimensional integral domain. Then the following are equivalent.*

(1) *D is of finite character.*
(2) *D is an LPI domain.*
(3) *For every nonzero prime ideal P of D and for every nonzero $x \in P$, $xD_P \cap D$ is invertible.*

Let D be an integral domain and Δ be a set of prime ideals of D such that $D = \bigcap_{P \in \Delta} D_P$. In [17, Proposition 1.8], it was shown that if $D = \bigcap_{P \in \Delta} D_P$ is locally finite, then $I_t = \bigcap_{P \in \Delta} (ID_P)_t$ for all $I \in F(D)$.

Proposition 1.10. *Let $D = \bigcap_\alpha D_{S_\alpha}$, where $\{S_\alpha\}$ is a nonempty family of multiplicative subsets of D. If the intersection $D = \bigcap_\alpha D_{S_\alpha}$ is locally finite, then*

$$A_t = \bigcap_\alpha (AD_{S_\alpha})_t$$

for all $A \in F(D)$.

Proof. For each $A \in F(D)$, let $A^* = \bigcap_\alpha (AD_{S_\alpha})_t$. It is routine to check that $*$ is a star operation on D (for the property (iii) of star operations, note that $A^* \subseteq (AD_{S_\alpha})_t$, and hence $(A^*D_{S_\alpha})_t \subseteq ((AD_{S_\alpha})_t)_t = (AD_{S_\alpha})_t$ for all α). Also, since the intersection is locally finite, $*$ is of finite character on D [1, Theorem 2]. Note that if S is a multiplicative set of D, then $(ID_S)_t = (I_tD_S)_t$ for all $I \in F(D)$ [30, Lemma 3.4]. Hence $I_t \subseteq \bigcap_\alpha (ID_{S_\alpha})_t = I^*$. Thus $t \leqslant *$, and so $* = t$ since $*' \leqslant t$ for any star operation $*'$ of finite character on D. Therefore $A^* = A_t$ for all $A \in F(D)$. $\square$

Corollary 1.11. *Let $D = \bigcap_\alpha D_{S_\alpha}$, where $\{S_\alpha\}$ is a nonempty family of multiplicative subsets of D, and suppose that the intersection is locally finite. If P is a maximal t-ideal of D, then PD_{S_α} is a maximal t-ideal of D_{S_α} for some S_α.*

Proof. By Proposition 1.10, $P_t = \bigcap_\alpha (PD_{S_\alpha})_t$. Hence $PD_{S_\alpha} \subseteq (PD_{S_\alpha})_t \subsetneq D_{S_\alpha}$ for some S_α. Note that if Q is a maximal t-ideal of D_{S_α} with $(PD_{S_\alpha})_t \subseteq Q$, then $Q \cap D$ is a t-ideal of D, and since $P \subseteq Q \cap D$ and P is a maximal t-ideal, we have $P = Q \cap D$. Hence $Q = PD_{S_\alpha}$, and thus PD_{S_α} is a maximal t-ideal. □

We use Corollary 1.11 to give another proof of Theorem 1.1(2) that an integral domain of finite t-character is conditionally well-behaved.

Proof of Theorem 1.1(2). Let M be a maximal t-ideal of D. Then, by Corollary 1.11, MD_P is a maximal t-ideal of D_P for some maximal t-ideal P of D. But, note that if $P \neq M$, then $MD_P = D_P$, and so $(MD_P)_t = D_P$. Thus MD_M is a maximal t-ideal of D_M. □

Continuing with the theme of conditionally well behaved domains we note the following result.

Proposition 1.12. *The following hold for an integral domain D.*

(1) *If D is a quasi-local domain with maximal ideal M, with M a t-ideal, then every t-invertible ideal of D is principal.*
(2) *If D is conditionally well behaved, then "t-locally t-invertible" is equivalent to "t-locally principal".*

Proof. (1) Let A be a t-invertible ideal of D. Then $(AA^{-1})_t = D$ implies that AA^{-1} is in no maximal t-ideals of D. That means that $AA^{-1} = D$. This forces A to be invertible and hence principal.

(2) Note that t-locally principal is t-locally invertible, and hence t-locally t-invertible anyway. For the reverse implication, note that if A is t-locally t-invertible, then AD_M is t-invertible for each maximal t-ideal M of D. But as M is well behaved, MD_M is a t-ideal and so, by (1) above, AD_M is principal. □

Corollary 1.13. *If D is of finite t-character, then every t-locally t-invertible ideal of D is t-invertible.*

Proof. Let I be a t-locally t-invertible ideal of D. Then I is t-locally principal by Theorem 1.1(2) and Proposition 1.12, and thus I is t-invertible because integral domains of finite t-character are w-LPI domains [12, Corollary 2.2]. □

Let $\mathrm{GV}(D) = \{J \in F(D) \mid J$ is finitely generated and $J_v = D\}$, and let $2\text{-}\mathrm{GV}(D) = \{J \in \mathrm{GV}(D) \mid J$ is generated by two elements$\}$. For each $I \in F(D)$, let $I^{t_2} = \{x \in K \mid xJ \subseteq I$ for some $J \in 2\text{-}\mathrm{GV}(D)\}$. In [24, Proposition 3.3], it was shown that if D is Noetherian, then $t_2 = w$, and hence t_2 is a star operation on D. More generally, assume that D is of finite t-character. Let $I \in \mathrm{GV}(D)$, and choose a nonzero $a \in I$. Then there exist only finitely many maximal t-ideals of D containing a, say, $P_1, \ldots, P_n$. Note that $I_t = D$; so $I \nsubseteq P_i$ for $i = 1, \ldots, n$, and hence $I \nsubseteq \bigcup_{i=1}^n P_i$. Choose another $b \in I - \bigcup_{i=1}^n P_i$. Then $(a, b) \subseteq I$ and $(a, b)_v = D$. Thus $t_2 = w$ on D. We next give a necessary and sufficient condition for t_2 to be a star operation on D.

Theorem 1.14. *Let $I^{t_2} = \{x \in K \mid xJ \subseteq I$ for some $J \in 2\text{-}\mathrm{GV}(D)\}$ for all $I \in F(D)$. Then t_2 is a star operation on D if and only if for $J_1, J_2 \in 2\text{-}\mathrm{GV}(D)$, there exists a $J \in 2\text{-}\mathrm{GV}(D)$ with $J \subseteq J_1 J_2$.*

Proof. ($\Rightarrow$) Let $J_1, J_2 \in 2\text{-}\mathrm{GV}(D)$, and put $I = J_1 J_2$. Clearly, $J_1^{t_2} = J_2^{t_2} = D$, and since t_2 is a star operation on D, we have $I^{t_2} = (J_1^{t_2} J_2^{t_2})^{t_2} = D$. Hence $1 \in I^{t_2}$, and thus there is a $J \in 2\text{-}\mathrm{GV}(D)$ so that $J = 1 \cdot J \subseteq I$.

($\Leftarrow$) Let $0 \neq x \in K$ and $A, B \in F(D)$. It is easy to check that (i) $(xD)^{t_2} = xD$ and $(xA)^{t_2} = xA^{t_2}$ and (ii) $A \subseteq A^{t_2}$, and $A \subseteq B$ implies $A^{t_2} \subseteq B^{t_2}$. Hence it suffices to show that (iii) $(A^{t_2})^{t_2} = A^{t_2}$.

To do this, we first note that A^{t_2} is a D-module because $A^{t_2} = \bigcup\{(A : J) \mid J \in 2\text{-}\mathrm{GV}(D)\}$ is a direct union of D-modules by assumption. Moreover, if $zA \subseteq D$ for some $0 \neq z \in D$, then $zA^{t_2} = (zA)^{t_2} \subseteq D^{t_2} = D$. Therefore $A^{t_2} \in F(D)$.

Now, we prove that $(A^{t_2})^{t_2} \subseteq A^{t_2}$, and thus $(A^{t_2})^{t_2} = A^{t_2}$ by (ii) above. Let $x \in (A^{t_2})^{t_2}$. Then $x(\alpha, \beta) \subseteq A^{t_2}$ for some $(\alpha, \beta) \in 2\text{-GV}(D) \Rightarrow x\alpha J_1 + x\beta J_2 \subseteq A$ for some $J_1, J_2 \in 2\text{-GV}(D)$ because $A^{t_2} \in F(D) \Rightarrow x(\alpha, \beta)J \subseteq x\alpha J + x\beta J \subseteq A$, where $J \in 2\text{-GV}(D)$ with $J \subseteq J_1 J_2$, $\Rightarrow xJ' \subseteq A$, where $J' \in 2\text{-GV}(D)$ with $J' \subseteq (\alpha, \beta)J$, $\Rightarrow x \in A^{t_2}$. Thus $(A^{t_2})^{t_2} \subseteq A^{t_2}$. $\square$

We end this section with an example of integral domains (that need not be of finite t-character) on which $t_2 = w$.

Example 1.15. (See [24, Theorem 4.5].) If $D = R[y]$ is the polynomial ring over an integral domain R, then t_2 is a star operation on D with $t_2 = w$, but D need not be of finite t-character.

Proof. Let A be a nonzero finitely generated ideal of $R[y]$ such that $A^{-1} = R[y]$. Then (i) $A \cap R \neq (0)$, (ii) there is a nonzero $f \in A$ with $c(f)_v = R$, where $c(f)$ is the ideal of R generated by the coefficients of f, and (iii) $(a, f)_v = R[y]$ for all $0 \neq a \in A \cap R$ [24, Lemma 4.4]. Therefore, $t_2 = w$ on $R[y]$. Note that $R[y]$ is of finite t-character if and only if R is of finite t-character (see Corollary 3.4). Hence $R[y]$ need not be of finite t-character. $\square$

2. PvMDs of finite t-character

Let D be an integral domain with quotient field K, S be a multiplicative subset of D, X be an indeterminate over D, and $D^{(S)} = D + XD_S[X]$. Clearly, $D^{(S)}$ is an integral domain with $D[X] \subseteq D^{(S)} \subseteq D_S[X] \subseteq K[X]$. As in [22], we say that D is a *ring of Krull type* if D is a locally finite intersection of essential valuation overrings of D; equivalently, D is a PvMD of finite t-character. In this section, we study when $D^{(S)}$ is a ring of Krull type.

We first recall the prime t-ideal structure of $D^{(S)}$ intersecting S which is very useful in the sequel.

Lemma 2.1. *Let $\mathfrak{P}$ be the set of prime ideals of D intersecting S.*

(1) *$\{P + XD_S[X] \mid P \in \mathfrak{P}\}$ is the set of prime ideals of $D^{(S)}$ intersecting S.*
(2) *If A is an ideal of $D^{(S)}$ such that $A \cap S \neq \emptyset$, then $A = (A \cap D)D^{(S)} = (A \cap D) + XD_S[X]$. Moreover, $A_t = (A \cap D)_t + XD_S[X]$.*
(3) *If $P \in \mathfrak{P}$, then $P + XD_S[X]$ is a prime (resp., maximal) t-ideal of $D^{(S)}$ if and only if P is a prime (resp., maximal) t-ideal.*

Proof. (1) [13, Theorem 2.1]. (2) [16, Lemma 3.7]. (3) follows from (1) and (2). $\square$

It is known that $D^{(S)}$ is a PvMD if and only if D is a PvMD and S is a t-splitting set [4, Theorem 2.5], if and only if $D^{(S)}$ is well behaved [38, Proposition 3.3]. (The multiplicative set S is a *t-splitting set* of D if for each nonzero $d \in D$, we have $dD = (AB)_t$ for some integral ideals A and B of D with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S = \emptyset$; equivalently, $dD_S \cap D$ is t-invertible for all $0 \neq d \in D$ [4, Proposition 3.1].)

Lemma 2.2. *If $D^{(S)} = D + XD_S[X]$ is a PvMD, then*

$$\begin{aligned} t\text{-Max}\big(D^{(S)}\big) \subseteq \big\{A \cap D^{(S)} \bigm| A \in t\text{-Max}\big(D_S[X]\big) \text{ with } A \cap D_S = (0)\big\} \\ \cup \big\{PD_S[X] \cap D^{(S)} \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S = \emptyset\big\} \\ \cup \big\{P + XD_S[X] \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S \neq \emptyset\big\}. \end{aligned}$$

Proof. Let $R = D^{(S)}$ and $Q \in t\text{-Max}(R)$. Clearly, $R_S = D_S[X]$ and both R_S and D_S are PvMDs. If $Q \cap S \neq \emptyset$, then $Q = (Q \cap D) + XD_S[X]$ and $Q \cap D$ is a maximal t-ideal of D with $(Q \cap D) \cap S \neq \emptyset$ by Lemma 2.1. Hence we assume that $Q \cap S = \emptyset$. Then Q_S is a prime t-ideal of R_S because R is a

PvMD, and hence either $Q_S \cap D_S = (0)$ or $Q_S = PD_S[X]$ for some nonzero prime ideal P of D. If $Q_S \cap D_S = (0)$, then $Q_S \cap R = Q$ and Q_S is a maximal t-ideal of R_S because D_S is a PvMD. Next, assume $Q_S = PD_S[X]$. We claim that P is a maximal t-ideal of D. Since Q_S is a t-ideal, both PD_S and P are t-ideals. If P is not a maximal t-ideal, then there is a maximal t-ideal P' of D with $P \subsetneq P'$. Note that, since R is a PvMD, S is a t-splitting set of D, and hence if $P' \cap S \neq \emptyset$, then $P \cap S \neq \emptyset$ [4, Lemma 4.2], a contradiction. So $P' \cap S = \emptyset$, but, in this case, $Q_S \subsetneq P'D_S[X]$ and $P'D_S[X]$ is a t-ideal of R_S. So $Q \subsetneq P'D_S[X] \cap R$ and $P'D_S[X] \cap R$ is a t-ideal, a contradiction. Thus P is a maximal t-ideal. $\square$

Lemma 2.3. *If $D^{(S)} = D + XD_S[X]$ is a PvMD, then*

$$t\text{-Max}\big(D^{(S)}\big) \supseteq \big\{PD_S[X] \cap D^{(S)} \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S = \emptyset\big\} \\ \cup \big\{P + XD_S[X] \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S \neq \emptyset\big\}.$$

Proof. Let $R = D^{(S)}$ and P be a maximal t-ideal of D. If $P \cap S \neq \emptyset$, then $P + XD_S[X]$ is a maximal t-ideal of $D^{(S)}$ by Lemma 2.1. Next, assume $P \cap S = \emptyset$. Note that PD_S is a maximal t-ideal of D_S; so $PD_S[X]$ is a maximal t-ideal of $D_S[X]$, and hence $PD_S[X] \cap R$ is a t-ideal of R. Thus, by Lemma 2.2, $PD_S[X] \cap R$ is a maximal t-ideal of R. $\square$

Corollary 2.4. *Let $R = D^{(S)}$, i.e., $R = D + XD_S[X]$. If R is a PvMD, then*

$$\big\{Q \in t\text{-Max}(R) \bigm| Q \cap D \neq (0)\big\} = \big\{PD_S[X] \cap R \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S = \emptyset\big\} \\ \cup \big\{P + XD_S[X] \bigm| P \in t\text{-Max}(D) \text{ with } P \cap S \neq \emptyset\big\}.$$

Proof. This is an immediate consequence of Lemmas 2.2 and 2.3. $\square$

We are now ready to prove the main result of this section.

Theorem 2.5. *The following statements are equivalent for $D^{(S)} = D + XD_S[X]$.*

(1) *$D^{(S)}$ is a ring of Krull type.*
(2) *D is a ring of Krull type, S is a t-splitting set, and the set of maximal t-ideals of D that intersect S is finite.*

Proof. Let $R = D^{(S)}$. Then R is a PvMD because R is a PvMD if and only if D is a PvMD and S is t-splitting. Note that each nonzero nonunit of R is contained in finitely many maximal t-ideals Q of R with $Q \cap D = (0)$ because $R_{D\setminus\{0\}} = K[X]$ is a principal ideal domain (PID). Hence the t-finite characterness of $D^{(S)}$ is completely determined by $\{Q \in t\text{-Max}(R) \mid Q \cap D \neq (0)\}$. Clearly, $X \in P + XD_S[X]$ for each maximal t-ideal P of D intersecting S. Also, if $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| < \infty$, then D is of finite t-character if and only if each nonzero nonunit of $D^{(S)}$ is contained in finitely many maximal t-ideals of the form $PD_S[X] \cap R$, where $P \in t\text{-Max}(D)$ with $P \cap S = \emptyset$. Thus, by Corollary 2.4, R is of finite t-character if and only if D is of finite t-character and $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| < \infty$. $\square$

It is known that every multiplicative subset of D is a t-splitting set if and only if D is a weakly Krull domain [4, p. 8]. Thus, every multiplicative subset of a Krull domain is a t-splitting set. Let $X^1(D)$ be the set of height-one prime ideals of D. Obviously, if D is a Krull domain, then $X^1(D) = t\text{-Max}(D)$.

Corollary 2.6. *If D is a Krull domain, then $D^{(S)} = D + XD_S[X]$ is a ring of Krull type if and only if $|\{P \in X^1(D) \mid P \cap S \neq \emptyset\}| < \infty$.*

Proof. Note that every multiplicative subset of a Krull domain is t-splitting; hence $D^{(S)}$ is a PvMD. Thus the result follows from Theorem 2.5. □

A ring of Krull type D is called an *independent ring of Krull type* if t-Max(D) is independent. Hence D is an independent ring of Krull type if and only if D is a weakly Matlis PvMD. It is obvious that rings of Krull type that are of t-dimension one (e.g., Krull domains) are independent rings of Krull type.

Corollary 2.7. *$D^{(S)} = D + XD_S[X]$ is an independent ring of Krull type if and only if D is an independent ring of Krull type, S is a t-splitting set, and $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| \leqslant 1$.*

Proof. Let $R = D^{(S)}$.

($\Rightarrow$) First, note that $XD_S[X] \subseteq P + XD_S[X]$ for all $P \in t$-Max(D) with $P \cap S \neq \emptyset$; so $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| \leqslant 1$. Next, let P_0 be a prime t-ideal of D. If $P_0 \cap S = \emptyset$, then $Q := P_0 D_S[X] \cap R$ is a prime t-ideal of R, and hence Q is contained in a unique maximal t-ideal. Hence P_0 is contained in a unique maximal t-ideal of D by Corollary 2.4. Thus the proof is completed by Theorem 2.5.

($\Leftarrow$) By Theorem 2.5, R is a ring of Krull type. For the independence, let Q be a prime t-ideal of R. If $Q \cap D = (0)$, then $Q \cap S = \emptyset$, and hence Q_S must be a maximal t-ideal of $D_S[X]$ because D_S is a PvMD. Thus, either Q is a maximal t-ideal of R or Q is contained in a unique maximal t-ideal of the form $P + XD_S[X]$ for some $P \in t$-Max(D) with $P \cap S \neq \emptyset$. Next, assume $Q \cap D \neq (0)$. Then $Q \cap D[X]$ is a nonzero prime ideal of $D[X]$ such that $(Q \cap D[X]) \cap D \neq (0)$, and hence there is a unique maximal t-ideal P' of D so that $Q \cap D[X] \subseteq P'[X]$. Thus, by Corollary 2.4, Q is contained in a unique maximal t-ideal of R. □

Corollary 2.8. *If D is a Krull domain, then $D^{(S)} = D + XD_S[X]$ is an independent ring of Krull type if and only if $|\{P \in X^1(D) \mid P \cap S \neq \emptyset\}| \leqslant 1$.*

Proof. This follows from Corollaries 2.6 and 2.7. □

Corollary 2.9. *D is a ring (resp., an independent ring) of Krull type if and only if $D[X]$ is a ring (resp., an independent ring) of Krull type.*

Proof. Clearly, if S is the set of units in D, then $D^{(S)} = D[X]$ and S is a t-splitting set. Thus, the proof is completed by Theorem 2.5 and Corollary 2.7. □

An integral domain D is called an *almost GCD-domain* (AGCD-domain) if, for every pair of nonzero elements $a, b \in D$, there is an integer $n = n(a,b) \geqslant 1$ such that $a^n D \cap b^n D$ is principal. Clearly, a GCD-domain is an integrally closed AGCD-domain. We know that D is an integrally closed AGCD-domain if and only if D is a PvMD with $Cl(D)$ torsion [36, Theorem 3.9]. Also, $D^{(S)}$ is an integrally closed AGCD-domain if and only if D is an integrally closed AGCD-domain and S is an almost splitting set [15, Theorem 3.1(a)]. (Recall that S is said to be *almost splitting* if for each $d \in D \backslash \{0\}$, there is an $m \in \mathbb{N}$ such that $d^m = rs$ in D where $s \in S$ and r is v-coprime to each element of S. Clearly, almost splitting sets are t-splitting. It is known that if $Cl(D)$ is torsion, then t-splitting sets are almost splitting [10, Corollary 2.4].)

Let D be an AGCD domain. For a nonzero nonunit $x \in D$, let $S(x) = \{y \mid y$ is a nonunit factor of x^n for some $n \in \mathbb{N}\}$. If r is a nonzero nonunit of D such that no two members of $S(r)$ are v-coprime, we call r an *almost rigid* element. Clearly, a nonzero nonunit $r \in D$ is almost rigid if and only if, whenever x and y are two factors of some power of r, then $x^m \mid y^m$ or $y^m \mid x^m$ for some $m \in \mathbb{N}$. The notion of almost rigid element generalizes the notion of a rigid element in a GCD domain. By [15, Corollary 2.1], an AGCD domain D is of finite t-character if and only if for each nonzero nonunit $x \in D$, $S(x)$ contains at most a finite number of mutually v-coprime elements. Also, it follows from [15, Theorem 2.2] that in an AGCD domain of finite t-character, every maximal t-ideal P contains an almost rigid element r such that $P = \{x \in D \mid (x,r)_v \neq D\}$; in this case, we say that P is associated to r. Clearly, if P is associated to r, then P is a unique maximal t-ideal of D containing r, and thus two distinct maximal t-ideals of an AGCD domain of finite t-character are associated to a pair of v-coprime almost rigid elements.

Lemma 2.10. *Let D be an AGCD domain of finite t-character and let S be a saturated multiplicative set of D. Then S contains a sequence of mutually v-coprime almost rigid elements of infinite length if and only if $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| = \infty$.*

Proof. Let T be the set of all almost rigid elements of D. Then the relation "is non-v-coprime to" is an equivalence relation in T, and hence for each almost rigid element r, we have the unique equivalence class $[r]$ and correspondingly a unique maximal t-ideal $P(r)$. Also, note that as D is an AGCD domain of finite t-character, every maximal t-ideal of D is of the form $P(r) = \{x \in D \mid (x, r)_v \subsetneq D\}$ for an almost rigid element r [15, Corollary 2.1]. Now suppose that a maximal t-ideal $P(r)$ intersects S, say, $x \in P(r) \cap S$. Then we can find d dividing a power of x such that d is an almost rigid element that is non-v-coprime to r (see the second to the last paragraph on [15, p. 167]). Thus $P(r)$ intersecting S implies that there is an almost rigid element $d \in S$ such that $[r] = [d]$. Next as no two v-coprime elements can share a t-ideal, two distinct maximal t-ideals intersecting S would result in a pair of v-coprime elements in S. Thus, there are infinitely many distinct maximal t-ideals of D intersecting S if and only if there is an infinite set of mutually v-coprime elements in S. $\square$

Corollary 2.11. *(See [15, Theorem 3.1(b)].) Let D be an integrally closed AGCD domain of finite t-character and let S be an almost splitting set of D. Then $D^{(S)} = D + XD_S[X]$ is of finite t-character if and only if S contains no sequence of mutually v-coprime almost rigid elements of infinite length.*

Proof. This is an immediate consequence of Theorem 2.5 and Lemma 2.10, because an integrally closed AGCD-domain is a PvMD and almost splitting sets are t-splitting. $\square$

Next, we need to establish that Corollary 2.11 applies directly to the GCD domain case. For this, we start with the following simple lemma.

Lemma 2.12. *Let r be an almost rigid element in an integrally closed AGCD domain D. Then r, and every power of r, is rigid. In particular, every almost rigid element of a GCD domain is rigid.*

Proof. Let $x, y \in D$ be nonzero such that $x, y \mid r$ (resp., $x, y \mid r^n$ for any $n \in \mathbb{N}$). Then there is an $m \in \mathbb{N}$ such that $x^m \mid y^m$ or $y^m \mid x^m$, but this leads to $x \mid y$ or $y \mid x$ because D is integrally closed. The "in particular" part follows because GCD domains are integrally closed AGCD domains. $\square$

It is known that $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and $(d, X)_v$ is principal in $D^{(S)}$ for all $0 \neq d \in D$ [13, Theorem 1.1], if and only if D is a GCD domain and the saturation of S in D is a splitting set [37, Corollary 1.5]. (A saturated multiplicative set S is called a *splitting set* if each nonzero $d \in D$ can be written as $d = rs$ in D where $s \in S$ and $(r, s')_v = D$ for all $s' \in S$.)

Corollary 2.13. *Let D be a GCD domain of finite t-character and let S be a saturated multiplicative set of D such that $D^{(S)} = D + XD_S[X]$ is a GCD domain. Then R is of finite t-character if and only if S contains no sequence of mutually v-coprime rigid elements of infinite length.*

Proof. This follows directly from Corollary 2.11 and Lemma 2.12, because GCD domains are integrally closed AGCD domains. $\square$

3. Nagata-like theorems

As in Section 2, D denotes an integral domain, S is a saturated multiplicative set of D, X is an indeterminate over D, and $D^{(S)} = D + XD_S[X]$.

Nagata's theorem states that if S is generated by prime elements and D satisfies the ascending chain condition on principal ideals, then D_S is a factorial domain (if and) only if D is. (Nagata originally proved the result for a Noetherian domain and its multiplicative subset generated by a single prime element [32, Lemma 1].) In this section, we prove this kind of results for integral domains of finite t-character. We then use this result to give some sufficient conditions for $D^{(S)}$ to be of finite t-character even when $D^{(S)}$ is not a PvMD.

Proposition 3.1. *Let S be a saturated multiplicative set of D, and consider the following two conditions*: (i) *each nonzero nonunit $x \in D$ belongs to only finitely many maximal t-ideals intersecting S and* (ii) *every maximal t-ideal P of D with $P \cap S = \emptyset$ is contracted from a maximal t-ideal of D_S. If D_S is of finite t-character, then the following are equivalent.*

(1) *D is of finite t-character.*
(2) *The condition* (i) *holds and D is conditionally well behaved.*
(3) *The conditions* (i) *and* (ii) *hold.*

Proof. (1) $\Rightarrow$ (2) Clearly, the condition (i) holds, and by Theorem 1.1(2), D is conditionally well behaved.

(2) $\Rightarrow$ (3) The condition (ii) holds when PD_S is a prime t-ideal for each maximal t-ideal P of D that is disjoint from S. But, this follows because D is conditionally well behaved (note that PD_P is a t-ideal and $PD_S = PD_P \cap D_S$).

(3) $\Rightarrow$ (1) Let x be a nonzero nonunit of D. The maximal t-ideals of D containing x are of two types, ones that are disjoint from S and these are contractions from maximal t-ideals of D_S by (ii) and ones that intersect S and these are finite in number by (i). Thus, since D_S is of finite t-character, x is contained in a finite number of maximal t-ideals of D. $\square$

Remark 3.2. (1) The proof of Proposition 3.1 may leave one wondering about the maximal t-ideals of D_S that do not contract to maximal t-ideals in D. Let M be such a maximal t-ideal of D_S, then $M \cap D$ is a t-ideal. Suppose that $P = M \cap D$ is not a maximal t-ideal, and let Q be a maximal t-ideal of D containing P. We claim that $Q \cap S \neq \emptyset$. For if not, then $M \subsetneq QD_S \subsetneq D_S$ and by the condition (ii) of Proposition 3.1, QD_S is a maximal t-ideal of D_S. This is contrary to the assumption that M is a maximal t-ideal of D_S.

(2) Let $\mathbb{Z}$ (resp., $\mathbb{Q}$) be the ring (resp., field) of integers (resp., rational numbers), p be a nonzero prime number, X and Y be indeterminates over $\mathbb{Q}$, $R = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!]$, K be the quotient field of R, Z be an indeterminate over K and let $D = R + ZK[\![Z]\!]$. Then D is a quasi-local domain whose maximal ideal is a t-ideal, and hence D is of finite t-character. But if we set $S = \{p^n \mid n = 0, 1, 2, \ldots\}$, then D_S does not have finite t-character [12, Example 2.14].

Corollary 3.3. *Assume that D is conditionally well behaved and D_S is of finite t-character. Then D is of finite t-character if and only if each nonzero nonunit $x \in D$ belongs to only finitely many maximal t-ideals intersecting S.*

An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$. A domain D is called a UMT-domain if each upper to zero in $D[X]$ is a maximal t-ideal. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain [28, Proposition 3.2]. The next result is already known ([29, Proposition 4.2] and [18, Lemma 2.1]), but we include it here to indicate an application of Proposition 3.1.

Corollary 3.4. *D is of finite t-character if and only if $D[X]$ is of finite t-character.*

Proof. ($\Rightarrow$) Let $S = D \backslash \{0\}$. Recall that if M is a maximal t-ideal of $D[X]$ such that $M \cap S \neq \emptyset$, i.e., $M \cap D \neq (0)$, then $M = (M \cap D)[X]$ and $M \cap D$ is a maximal t-ideal of D (cf. [28, Proposition 1.1]); hence for any $0 \neq f \in D[X]$, $f \in M$ if and only if $c(f) \subseteq M \cap D$. Also, if Q is a maximal t-ideal of $D[X]$

with $Q \cap S = \emptyset$, then Q_S is a maximal t-ideal of $D[X]_S$ (note that $D[X]_S$ is a PID) and $Q_S \cap D[X] = Q$. So if D is of finite t-character, the conditions (i) and (ii) of Proposition 3.1 are satisfied. Clearly, $D[X]_S$ is of finite t-character. Thus, by Proposition 3.1, $D[X]$ is of finite t-character.

($\Leftarrow$) This follows directly from the fact that if P is a maximal t-ideal of D, then $P[X]$ is a maximal t-ideal of $D[X]$. $\square$

Let Γ be a numerical semigroup and $D[\Gamma]$ be the numerical semigroup ring of Γ over D. Then the map $\phi : t\text{-Spec}(D[X]) \to t\text{-Spec}(D[\Gamma])$, given by $Q \mapsto Q \cap D[\Gamma]$, is an order-preserving bijection, where t-Spec(A) is the set of prime t-ideals of an integral domain A [11, Theorem 1.4]. Hence $D[X]$ is of finite t-character (resp., weakly Matlis) if and only if $D[\Gamma]$ is of finite t-character (resp., weakly Matlis). By Corollary 3.4, D is of finite t-character if and only if $D[X]$ is. Also, it is known that $D[X]$ is a weakly Matlis domain if and only if D is weakly Matlis and each upper to zero in $D[X]$ is contained in a unique maximal t-ideal of $D[X]$ [18, Proposition 2.2]. Thus, we have

Corollary 3.5. *If Γ is a numerical semigroup, then*

(1) *D is of finite t-character if and only if $D[\Gamma]$ is of finite t-character.*
(2) *D is weakly Matlis and each upper to zero in $D[X]$ is contained in a unique maximal t-ideal of $D[X]$ if and only if $D[\Gamma]$ is weakly Matlis.*

Corollary 3.6. *If D is a UMT-domain, then D is weakly Matlis if and only if $D[\Gamma]$ is weakly Matlis.*

Proof. This follows directly from Corollary 3.5(2) because each upper to zero in $D[X]$ is a maximal t-ideal. $\square$

Corollary 3.7. *Let S be a saturated multiplicative set of D, and consider the condition (#): S meets at most a finite number of maximal t-ideals of D. If D_S is of finite t-character, then the following are equivalent.*

(1) *$D^{(S)}$ is of finite t-character.*
(2) *The condition (#) holds and $D^{(S)}$ is conditionally well behaved.*
(3) *The condition (#) holds and every maximal t-ideal of $D^{(S)}$ that is disjoint from S is contracted from a maximal t-ideal of $D_S[X]$.*

Proof. (1) $\Rightarrow$ (2) By Lemma 2.1, $\{P + XD_S[X] \mid P \in t\text{-Max}(D) \text{ and } P \cap S \neq \emptyset\}$ is the set of maximal t-ideals of $D^{(S)}$ intersecting S. Clearly, $X \in P + XD_S[X]$ for all $P \in t$-Max(D) with $P \cap S \neq \emptyset$. Thus, the condition (#) holds. Also, by Theorem 1.1(2), $D^{(S)}$ is conditionally well behaved.

(2) $\Rightarrow$ (3) The proof is similar to that of (2) $\Rightarrow$ (3) of Proposition 3.1.

(3) $\Rightarrow$ (1) Recall from Corollary 3.4 that $D_S[X]$ is of finite t-character if and only if D_S is; hence $D_S^{(S)}$ is of finite t-character because $D_S[X] = D_S^{(S)}$. Also, since every maximal t-ideal of $D^{(S)}$ disjoint from S is contracted from $D_S^{(S)} = D_S[X]$, we see that the conditions (i) and (ii) of Proposition 3.1 are satisfied. Thus, $D^{(S)}$ is of finite t-character. $\square$

Let D be a PID, L be a proper algebraic extension of the quotient field of D, and $R = D + XL[X]$. Recall from [40, Proposition 7] that R is of finite (t-)character if and only if D is a semilocal PID; and, in this case, R is not a PvMD.

Corollary 3.8. *Let K be the quotient field of D. Then $R = D + XK[X]$ is of finite t-character if and only if D is semi-quasi-local whose maximal ideals are t-ideals.*

Proof. Let $S = D \setminus \{0\}$. Then $R = D^{(S)}$, $D_S = K$ is of finite t-character, and the condition (#) of Corollary 3.7 means $|t\text{-Max}(D)| < \infty$. Note that $R_S = K[X]$ and $K[X]$ is a PID; so every nonzero prime ideal of R_S is a maximal t-ideal. Thus, every maximal t-ideal of R that is disjoint from S is contracted from a maximal t-ideal of R_S. Hence the result follows directly from Corollary 3.7. $\square$

The other possible use of Proposition 3.1 can be in shortening the proof of Theorem 2.5. However, we left the proof of Theorem 2.5 intact as it includes the structure of maximal t-ideals of the $D + XD_S[X]$ construction when it is a PvMD. The next application is an example of the $D + XD_S[X]$ domain of finite t-character that is not a PvMD.

Example 3.9. Let D be a valuation domain (hence a PvMD) of dimension $\geqslant 2$, Q be a nonzero non-maximal prime ideal of D, and $S = D \setminus Q$. Then $D^{(S)}$ is conditionally well behaved but not well behaved [38, Proposition 2.5], and hence $D^{(S)}$ is not a PvMD [38, Proposition 3.3]. However, note that there are only two types of maximal t-ideals: (i) principal rank one prime ideals P generated by discrete primes and (ii) the prime ideal M consisting of all non-discrete elements of $D^{(S)}$ [38, Lemmas 2.3 and 2.4]. (An $f \in D^{(S)}$ is said to be discrete if $f(0)$ is a unit in D.) Note that M is a unique maximal t-ideal of $D^{(S)}$ that intersects S; so the condition (i) of Proposition 3.1 is met and every maximal t-ideal different from M is principal. Now as principal prime ideals disjoint with S extend to principal primes in $D_S^{(S)} = D_Q[X]$ we conclude the condition (ii) of Proposition 3.1. Next as $D_Q[X]$ is of finite t-character by Corollary 3.4, the requirements of Proposition 3.1 are satisfied, and thus $D^{(S)}$ is of finite t-character.

An integral domain D is said to be *coherent* (resp., v-coherent) if I^{-1} is finitely generated (resp., of finite type) for all nonzero finitely generated ideals I of D. Clearly, Noetherian domain $\Rightarrow$ coherent domain $\Rightarrow$ v-coherent.

Corollary 3.10. *Let D be a Noetherian domain and let S be a saturated multiplicative set in D such that S meets at most a finite number of maximal t-ideals of D. Then $D^{(S)}$ is of finite t-character.*

Proof. D_S is Noetherian, and so D_S is of finite t-character. Hence, by Corollary 3.7, all we need to establish is that $D^{(S)}$ is conditionally well behaved. For this, we note that if D is Noetherian, then $D^{(S)}$ is coherent [13, Theorem 4.32] and hence $D^{(S)}$ is well behaved [38, Proposition 1.4]. $\square$

In general, if D is coherent, then $D^{(S)}$ may not be coherent. In Example 3.9, $D^{(S)}$ is not well behaved, and hence $D^{(S)}$ is not (v-)coherent because (v-)coherent domains are well behaved [38, pp. 201–202]. Thus, even when D is a valuation domain, $D^{(S)}$ need not be (v-)coherent.

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