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Limit  $T$ -subalgebras in free associative algebras

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## ABSTRACT

Let  $F\langle X \rangle$  be the free unitary associative algebra over a field  $F$  on a free generating set  $X$ . A unitary subalgebra  $R$  of  $F\langle X \rangle$  is called a  $T$ -subalgebra if  $R$  is closed under all endomorphisms of  $F\langle X \rangle$ . A  $T$ -subalgebra  $R^*$  in  $F\langle X \rangle$  is *limit* if every larger  $T$ -subalgebra  $W \supsetneq R^*$  is finitely generated (as a  $T$ -subalgebra) but  $R^*$  itself is not. It follows easily from Zorn's lemma that if a  $T$ -subalgebra  $R$  is not finitely generated then it is contained in some limit  $T$ -subalgebra  $R^*$ . In this sense limit  $T$ -subalgebras form a “border” between those  $T$ -subalgebras which are finitely generated and those which are not. In the present article we give the first example of a limit  $T$ -subalgebra in  $F\langle X \rangle$ , where  $F$  is an infinite field of characteristic  $p > 2$  and  $|X| \geq 4$ . Note that, by Shchigolev's result, over a field  $F$  of characteristic 0 every  $T$ -subalgebra in  $F\langle X \rangle$  is finitely

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Central polynomial  
 $T$ -ideal  
 $T$ -subspace

generated; hence, over such a field limit  $T$ -subalgebras in  $F\langle X \rangle$  do not exist.

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## 1. Introduction

Let  $F$  be a field,  $X$  a non-empty set and let  $F\langle X \rangle$  be the free unitary associative algebra over  $F$  on the free generating set  $X$ . Recall that a  $T$ -ideal of  $F\langle X \rangle$  is an ideal closed under all endomorphisms of  $F\langle X \rangle$ . Similarly, a  $T$ -subalgebra and a  $T$ -subspace (the latter is also called a  $T$ -space) are, respectively, an unitary subalgebra and a vector subspace in  $F\langle X \rangle$  closed under all endomorphisms of  $F\langle X \rangle$ .

Let  $A$  be an unitary associative algebra over  $F$ . Recall that a polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  is called a *polynomial identity* in  $A$  if  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in A$ . It can be easily checked that, for a given algebra  $A$ , its polynomial identities form a  $T$ -ideal  $T(A)$  in  $F\langle X \rangle$ . The converse also holds: for every  $T$ -ideal  $I$  in  $F\langle X \rangle$  there is an algebra  $A$  such that  $I = T(A)$ , that is,  $I$  is the ideal of all polynomial identities satisfied in  $A$ .

A polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  is a *central polynomial* of  $A$  if, for all  $a_1, \dots, a_n \in A$ ,  $f(a_1, \dots, a_n)$  is central in  $A$ . Clearly,  $f = f(x_1, \dots, x_n)$  is a central polynomial of  $A$  if and only if  $[f, x_{n+1}]$  is a polynomial identity of  $A$ . For a given algebra  $A$  its central polynomials form a  $T$ -subalgebra  $C(A)$  in  $F\langle X \rangle$ . However, not every  $T$ -subalgebra coincides with the  $T$ -subalgebra  $C(A)$  of all central polynomials of some algebra  $A$ .

Let  $I$  be a  $T$ -ideal in  $F\langle X \rangle$ . A subset  $S \subset I$  *generates*  $I$  as a  $T$ -ideal if  $I$  is the minimal  $T$ -ideal in  $F\langle X \rangle$  containing  $S$ . A  $T$ -subalgebra and a  $T$ -subspace of  $F\langle X \rangle$  generated by  $S$  (as a  $T$ -subalgebra and a  $T$ -subspace, respectively) are defined in a similar way. It is clear that the  $T$ -ideal ( $T$ -subalgebra,  $T$ -subspace) generated by  $S$  is the ideal (the subalgebra, the vector subspace) in  $F\langle X \rangle$  generated by all polynomials  $f(g_1, \dots, g_m)$ , where  $f = f(x_1, \dots, x_m) \in S$  and  $g_i \in F\langle X \rangle$  for all  $i$ .

We refer to [7,9,17,26] for further terminology and basic results concerning  $T$ -ideals and algebras with polynomial identities and to [2,6,14,15,17] for an account of results concerning  $T$ -subspaces.

If  $F$  is a field of characteristic 0 then every  $T$ -ideal in  $F\langle X \rangle$  is finitely generated (as a  $T$ -ideal); this is a celebrated result of Kemer [18,19] that solves the Specht problem. Moreover, over such a field  $F$  each  $T$ -subspace (and, therefore, each  $T$ -subalgebra) in  $F\langle X \rangle$  is finitely generated; this has been proved more recently by Shchigolev [29]. Very recently Belov [5] has proved that, for each Noetherian commutative and associative unitary ring  $K$  and each  $n \in \mathbb{N}$ , each  $T$ -ideal in  $K\langle x_1, \dots, x_n \rangle$  is finitely generated.

On the other hand, over a field  $F$  of characteristic  $p > 0$  there are  $T$ -ideals in  $F\langle x_1, x_2, \dots \rangle$  that are not finitely generated. This has been proved by Belov [3], Grishin [11] and Shchigolev [27] (see also [4,12,17,29]). The construction of such  $T$ -ideals

makes use of the non-finitely generated  $T$ -subspaces in  $F\langle x_1, x_2, \dots \rangle$  constructed by Grishin [11] for  $p = 2$  and by Shchigolev [28] for  $p > 2$  (see also [12]). Shchigolev [28] also constructed non-finitely generated  $T$ -subspaces and  $T$ -subalgebras in  $F\langle x_1, \dots, x_n \rangle$ , where  $n > 1$  and  $F$  is a field of characteristic  $p > 2$ .

A  $T$ -subalgebra  $R^*$  in  $F\langle X \rangle$  is called *limit* if every larger  $T$ -subalgebra  $W \not\supseteq R^*$  is finitely generated (as a  $T$ -subalgebra) but  $R^*$  itself is not. A *limit  $T$ -ideal* and a *limit  $T$ -subspace* are defined in a similar way. It follows easily from Zorn's lemma that if a  $T$ -subalgebra  $R$  is not finitely generated then it is contained in some limit  $T$ -subalgebra  $R^*$ . Similarly, each non-finitely generated  $T$ -ideal ( $T$ -subspace) is contained in a limit  $T$ -ideal ( $T$ -subspace). In this sense limit  $T$ -subalgebras ( $T$ -ideals,  $T$ -subspaces) form a “border” between those  $T$ -subalgebras ( $T$ -ideals,  $T$ -subspaces) which are finitely generated and those which are not.

By [3,11,27], over a field  $F$  of characteristic  $p > 0$  the algebra  $F\langle x_1, x_2, \dots \rangle$  contains non-finitely generated  $T$ -ideals; therefore, it contains at least one limit  $T$ -ideal. No example of a limit  $T$ -ideal is known so far. Even the cardinality of the set of limit  $T$ -ideals in  $F\langle x_1, x_2, \dots \rangle$  is unknown; it is possible that, for a given field  $F$  of characteristic  $p > 0$ , there is only one limit  $T$ -ideal. The non-finitely generated  $T$ -ideals constructed in [1,16] come closer to being limit than any other known non-finitely generated  $T$ -ideal. However, it is unlikely that the  $T$ -ideals found in [1,16] are limit. Note that a similar problem concerning limit verbal subgroups in a free group remains open for 40 years; there exist infinitely many such subgroups [24] but no example is known (see [22] for more details).

About limit  $T$ -subspaces in  $F\langle X \rangle$  we know more than about limit  $T$ -ideals. Brandão Jr., Koshlukov, Krasilnikov and Silva [6] have proved that the vector space  $C(G)$  of all central polynomials of the infinite dimensional Grassmann algebra  $G$  over an infinite field  $F$  of characteristic  $p > 2$  is a limit  $T$ -subspace in  $F\langle x_1, x_2, \dots \rangle$ . Very recently, Gonçalves, Krasilnikov and Sviridova [10] have found infinitely many other limit  $T$ -subspaces in  $F\langle x_1, x_2, \dots \rangle$ .

However, it can be easily verified that the limit  $T$ -subspaces described in [6,10] are not limit  $T$ -subalgebras:  $C(G)$  is finitely generated as a  $T$ -subalgebra and the limit  $T$ -subspaces of [10] are not subalgebras in  $F\langle x_1, x_2, \dots \rangle$ .

The aim of the present article is to construct the first example of a limit  $T$ -subalgebra in  $F\langle x_1, x_2, \dots \rangle$ . We write  $[a, b] = ab - ba$ ,  $[a, b, c] = [[a, b], c]$ . For each  $l \geq 0$ , let

$$q^{(l)}(x_1, x_2) = x_1^{p^l-1} [x_1, x_2] x_2^{p^l-1}.$$

Our main result is as follows.

**Theorem 1.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . Let  $\Gamma$  be the (unitary)  $T$ -subalgebra in  $F\langle x_1, x_2, \dots \rangle$  generated by the polynomials*

$$q^{(0)}, q^{(1)}, \dots, q^{(l)}, \dots \quad (1)$$

together with the polynomials  $x_1[x_2, x_3, x_4]$  and  $x_1[x_2, x_3][x_4, x_5]$ . Then  $\Gamma$  is a limit  $T$ -subalgebra in  $F\langle x_1, x_2, \dots \rangle$ .

Note that the  $T$ -subspace in  $F\langle x_1, x_2, \dots \rangle$  generated by the polynomials  $x_1[x_2, x_3, x_4]$  and  $x_1[x_2, x_3][x_4, x_5]$  coincides with the  $T$ -ideal  $T^{(3,2)}$  generated (as a  $T$ -ideal) by  $[x_2, x_3, x_4]$  and  $[x_2, x_3][x_4, x_5]$ . Thus,  $\Gamma$  is the  $T$ -subalgebra generated by the  $T$ -ideal  $T^{(3,2)}$  together with the polynomials (1).

We also prove the following.

**Theorem 2.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . Let  $\Gamma_n = \Gamma \cap F\langle x_1, \dots, x_n \rangle$ . Then  $\Gamma_n$  is a limit  $T$ -subalgebra in  $F\langle x_1, \dots, x_n \rangle$  for every  $n \geq 4$ .*

We make the following conjecture.

**Conjecture.** *Let  $F$  be an infinite field of characteristic  $p > 2$ . Then  $\Gamma$  is the only limit  $T$ -subalgebra in  $F\langle x_1, x_2, \dots \rangle$  and, for each  $n \geq 4$ ,  $\Gamma_n$  is the only limit  $T$ -subalgebra in  $F\langle x_1, \dots, x_n \rangle$ .*

Recall that it follows from [3,11,27] that over a field  $F$  of characteristic  $p > 0$  there exists at least one limit  $T$ -ideal in  $F\langle x_1, x_2, \dots \rangle$ .

**Problem.** For a field of characteristic  $p > 0$ , find an example of a limit  $T$ -ideal in  $F\langle x_1, x_2, \dots \rangle$ . How many such limit  $T$ -ideals exist for a given (infinite) field  $F$  of characteristic  $p > 0$ ?

**Remarks.** 1. As a  $T$ -subspace  $\Gamma$  is generated by 1 together with the polynomials (1) and the polynomials  $x_1[x_2, x_3, x_4]$ ,  $x_1[x_2, x_3][x_4, x_5]$ .  $\Gamma$  is not finitely generated as a  $T$ -subspace in  $F\langle x_1, x_2, \dots \rangle$  (otherwise it would be finitely generated as a  $T$ -subalgebra as well). However,  $\Gamma$  is not a limit  $T$ -subspace; it is contained in the (larger) limit  $T$ -subspace generated by  $x_1^p$  and  $\Gamma$  itself (see [10]).

2. The limit  $T$ -subalgebra  $\Gamma$  is not equal to the  $T$ -subalgebra  $C(A)$  of all central polynomials of any algebra  $A$ .

Indeed, suppose that  $\Gamma = C(A)$  for some  $A$ . Let  $T(A)$  be the  $T$ -ideal of all polynomial identities of  $A$ . Then, for each  $f \in C(A)$  and each  $g \in F\langle X \rangle$ , we have  $[f, g] \in T(A)$ . Since  $[x_1, x_2] \in \Gamma = C(A)$ , we have  $[x_1, x_2, x_3] \in T(A)$ .

Note that, over a field of characteristic  $p > 0$ ,  $[x_2, x_1^p] = [x_2, x_1, \dots, x_1]$ , where the commutator on the right hand side of the equality is of length  $p + 1$ . Since  $[x_1, x_2, x_3] \in T(A)$ , we have  $[x_2, x_1, \dots, x_1] \in T(A)$ , hence  $[x_2, x_1^p] \in T(A)$ . It follows that  $x_1^p \in C(A)$ ; on the other hand,  $x_1^p \notin \Gamma$ . This contradiction proves that  $\Gamma \neq C(A)$  for any algebra  $A$ , as claimed.

3. Recall that  $G$  is the infinite dimensional Grassmann algebra over a field  $F$  and  $C(G)$  is the vector space of all central polynomials of  $G$ . Let  $F$  be an infinite field of characteristic  $p > 2$ . Then, by [6],  $C(G)$  is a limit  $T$ -subspace in  $F\langle x_1, x_2, \dots \rangle$ , that is,

$C(G)$  is not a finitely generated  $T$ -subspace but every larger  $T$ -subspace  $W \supsetneq C(G)$  is finitely generated. We note that the former part of this assertion (stating that  $C(G)$  is not a finitely generated  $T$ -subspace) was also proved independently by Bekh-Ochir and Rankin [2] and by Grishin [13].

4. The first example of a limit  $T$ -subspace in the free non-unitary associative algebra of countable infinite rank over a field  $F$  of characteristic  $p > 0$  was found by Kireeva [20]. If  $p > 2$  then Kireeva's limit  $T$ -subspace coincides with the vector space  $C(H)$  of all central polynomials of the infinite dimensional non-unitary Grassmann algebra  $H$  (see [6]).

## 2. Preliminaries

Note that if  $I$  is a  $T$ -ideal in  $F\langle X \rangle$  then  $T$ -ideals,  $T$ -subalgebras and  $T$ -subspaces can be defined in the quotient algebra  $F\langle X \rangle/I$  in a natural way.

In the rest of the paper  $F$  will be an infinite field of characteristic  $p > 2$ . Let  $X = \{x_1, x_2, \dots\}$  be an infinite countable set, and  $F\langle X \rangle$  be the free unitary associative algebra over  $F$  generated by  $X$ .

Let  $T^{(3)}$  denote the  $T$ -ideal of  $F\langle X \rangle$  generated by the polynomial  $[x_1, x_2, x_3]$ . Then  $T^{(3)}$  is the ideal of  $F\langle X \rangle$  generated by all polynomials  $[g_1, g_2, g_3]$  ( $g_i \in F\langle X \rangle$ ). The next basic properties of  $T^{(3)}$  are well-known (see for example [2,6,8,14,25]).

**Lemma 3.** *Let  $F$  be a field of characteristic  $p > 2$ . Let  $g_1, g_2 \in F\langle X \rangle$ . Then*

1.  $g_1^p + T^{(3)}$  is central in  $F\langle X \rangle/T^{(3)}$ ;
2.  $(g_1 + g_2)^p + T^{(3)} = (g_1^p + g_2^p) + T^{(3)}$ ;
3.  $(g_1 g_2)^p + T^{(3)} = g_1^p g_2^p + T^{(3)}$ .

Recall that  $T^{(3,2)}$  is the  $T$ -ideal in  $F\langle X \rangle$  generated by the polynomials  $[x_1, x_2, x_3]$  and  $[x_1, x_2][x_3, x_4]$ . Since  $T^{(3)} \subset T^{(3,2)}$ , the statement of Lemma 3 remains valid if we replace  $T^{(3)}$  by  $T^{(3,2)}$  there.

It can be easily seen that if  $g_1, g_2, \dots, g_{n-1}, h_1, h_2 \in F\langle X \rangle$  and  $g_n = [h_1, h_2]$  then

$$g_1 g_2 \dots g_n + T^{(3,2)} = g_{\sigma(1)} g_{\sigma(2)} \dots g_{\sigma(n)} + T^{(3,2)} \quad (2)$$

for any permutation  $\sigma \in S_n$ .

For integers  $i_1, i_2$  such that  $1 \leq i_1 < i_2 \leq n$  and integers  $a_1, \dots, a_n \geq 0$  such that  $a_{i_1}, a_{i_2} \geq 1$ ,

$$\frac{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2}}$$

denotes the monomial  $x_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \in F\langle X \rangle$ , where  $b_j = a_j - 1$  if  $j \in \{i_1, i_2\}$  and  $b_j = a_j$  otherwise. The next lemma is a corollary of [7, Proposition 4.3.3].

**Lemma 4.** If  $f(x_1, \dots, x_n) \in F\langle X \rangle$  is a multihomogeneous polynomial of multidegree  $(a_1, \dots, a_n)$  then

$$f + T^{(3,2)} = \alpha x_1^{a_1} \dots x_n^{a_n} + \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_1^{a_1} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] + T^{(3,2)},$$

where  $\alpha, \alpha_{(i_1, i_2)} \in F$ .

Recall that  $\Gamma$  is the (unitary)  $T$ -subalgebra generated by polynomials

$$q^{(0)}, q^{(1)}, \dots, q^{(l)}, \dots, x_1[x_2, x_3, x_4], x_1[x_2, x_3][x_4, x_5]. \quad (3)$$

Note that  $T^{(3,2)} \subset \Gamma$ . Indeed,

$$x_1[x_2, x_3, x_4]x_5 = x_1[x_2, x_3, x_4x_5] - x_1x_4[x_2, x_3, x_5]$$

so  $T^{(3)} \subseteq \Gamma$ . Further,

$$\begin{aligned} x_1[x_2, x_3][x_4, x_5]x_6 + T^{(3)} &= x_1[x_2, x_3]x_6[x_4, x_5] + T^{(3)} \\ &= x_1x_6[x_2, x_3][x_4, x_5] + T^{(3)} \end{aligned}$$

so the  $T$ -ideal generated by  $[x_1, x_2][x_3, x_4]$  is also contained in  $\Gamma$ . Thus,  $T^{(3,2)} \subset \Gamma$ , as claimed.

For any  $g \in F\langle X \rangle$ , let  $\langle g \rangle^{TS}$  denote the  $T$ -subspace in  $F\langle X \rangle$  generated by  $g$ . The following lemma is a corollary of a result of Grishin and Tsybulya [14, Theorem 1.3, item 1]).

**Lemma 5.** (See [14].) Let  $F$  be an infinite field of characteristic  $p > 2$ . Let  $a_1, a_2 \geq 1$  and let  $l \geq 0$  be the largest integer such that  $p^l$  divides both  $a_1, a_2$ . Then

$$\langle x_1^{a_1-1} x_2^{a_2-1} [x_1, x_2] \rangle^{TS} + T^{(3)} = \langle q^{(l)} \rangle^{TS} + T^{(3)}.$$

Lemma 5 immediately implies the following.

**Lemma 6.** Let  $F$  be an infinite field of characteristic  $p > 2$ . For any integers  $a_1, a_2 \geq 1$ , the polynomial  $x_1^{a_1-1} x_2^{a_2-1} [x_1, x_2]$  belongs to  $\Gamma$ .

### 3. Proof of Theorem 1

Let  $f \in F\langle X \rangle$  be a polynomial. We denote by  $\Gamma_f$  the  $T$ -subalgebra of  $F\langle X \rangle$  generated by  $f$  and  $\Gamma$  and we write  $\Gamma(s)$  ( $s \geq 0$ ) for the  $T$ -subalgebra generated by  $\Gamma$  and by the polynomial

$$f = x_1^{p^s} q^{(s)}(x_2, x_3).$$

In particular,  $\Gamma(0)$  is the  $T$ -subalgebra generated by  $x_1[x_2, x_3]$  and  $\Gamma$ .

**Lemma 7.**  $\Gamma(s)$  is generated as a  $T$ -subalgebra by the polynomials

$$x_1^{p^s} q^{(s)}(x_2, x_3), \quad x_1[x_2, x_3, x_4], \quad x_1[x_2, x_3][x_4, x_5]. \quad (4)$$

In particular,  $\Gamma(0)$  is generated as a  $T$ -subalgebra by the polynomial  $x_1[x_2, x_3]$ .

**Proof.** Let  $\Gamma(s)^*$  be the  $T$ -subalgebra of  $F\langle X \rangle$  generated by the polynomials (4). It is clear that  $\Gamma(s)^* \subset \Gamma(s)$ .

Let  $s = 0$ . It can be easily seen that  $\Gamma(0)^*$  is generated as a  $T$ -subalgebra by the polynomial  $x_1[x_2, x_3]$ . Since  $T^{(3,2)} \subset \Gamma(0)^*$  and

$$q^{(l)}(x_1, x_2) + T^{(3,2)} = x_1^{p^l-1} x_2^{p^l-1} [x_1, x_2] + T^{(3,2)},$$

we obtain  $q^{(l)} \in \Gamma(0)^*$  for all  $l \geq 0$ . Hence,  $\Gamma(0)^* = \Gamma(0)$ .

Let  $s > 0$ . If  $l > s$ , then by (2) we have

$$q^{(l)}(x_1, x_2) + T^{(3,2)} = ((x_1 x_2)^{(p^{l-s}-1)})^{p^s} q^{(s)}(x_1, x_2) + T^{(3,2)}.$$

Hence  $q^{(l)} \in \Gamma(s)^*$ .

Suppose that  $l \leq s$ . Take  $f(x_1, x_2, x_3) = x_1^{p^s} q^{(s)}(x_2, x_3)$ . Note that the multihomogeneous component of multidegree  $(p^l, p^l)$  of the polynomial

$$f(1, x_1 + 1, x_2 + 1) = (x_1 + 1)^{p^s-1} [x_1 + 1, x_2 + 1] (x_2 + 1)^{p^s-1}$$

is equal to  $\alpha q^{(l)}(x_1, x_2)$  where  $\alpha = \binom{p^s-1}{p^l-1}^2$ . Since  $f(1, x_1 + 1, x_2 + 1) \in \Gamma(s)^*$  and  $\alpha \neq 0$  in  $F$ , we have  $q^{(l)} \in \Gamma(s)^*$ .

Thus,  $q^{(l)} \in \Gamma(s)^*$  for all  $l \geq 0$ . Since  $T^{(3,2)} \subset \Gamma(s)^*$ , we have  $\Gamma(s)^* = \Gamma(s)$ , as required.  $\square$

**Lemma 8.** Let  $f(x_1, \dots, x_n) \in F\langle X \rangle$  be a multihomogeneous polynomial of multidegree  $(a_1, \dots, a_n)$ . If  $a_i = 1$  for some  $i$ , then either

$$\Gamma_f = F\langle X \rangle \quad \text{or} \quad \Gamma_f = \Gamma \quad \text{or} \quad \Gamma_f = \Gamma(0).$$

**Proof.** Note that we can assume without loss of generality, permuting the generators  $x_1, \dots, x_n$  if necessary, that  $a_1 = 1$ . We also can suppose that

$$f = \alpha x_1^{a_1} \dots x_n^{a_n} + \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_1^{a_1} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}], \quad (5)$$

where  $\alpha, \alpha_{(i_1, i_2)} \in F$ . This follows easily from Lemma 4 and from the fact that  $\Gamma_{f+h} = \Gamma_f$  for any polynomial  $h \in T^{(3,2)} \subset \Gamma$ .

If  $\alpha \neq 0$  in (5) then  $\alpha^{-1} f(x_1, 1, \dots, 1) = x_1 \in \Gamma_f$ , and  $\Gamma_f = F\langle X \rangle$ .

Suppose that  $\alpha = 0$ . We claim that we may assume without loss of generality that  $f$  is of the form  $f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n)$ , where

$$g = \sum_{2 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}]. \quad (6)$$

Indeed, consider a term  $m = \frac{x_1^{a_1} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}]$  in (5). If  $i_1 > 1$  then

$$m = x_1 \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}]. \quad (7)$$

Suppose that  $i_1 = 1$ ; then  $m = m' [x_1, x_{i_2}]$ , where  $m' = \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_2}}$ . We have

$$m + T^{(3,2)} = m' [x_1, x_{i_2}] + T^{(3,2)} = [m' x_1, x_{i_2}] - x_1 [m', x_{i_2}] + T^{(3,2)}.$$

Note that  $[m' x_1, x_{i_2}] \in \Gamma$ . Using the equality  $[uv, x_{i_2}] = [u, x_{i_2}]v + u[v, x_{i_2}]$  and the property (2), we obtain that  $[m', x_{i_2}] = m'' + h''$ , where  $m''$  is of the form (6) and  $h'' \in T^{(3,2)}$ . Hence, we can write

$$m = x_1 g' + h' \quad (8)$$

where  $g'$  is of the form (6) and  $h' \in \Gamma$ .

It follows easily from (7) and (8) that there exists a multihomogeneous polynomial  $g = g(x_2, \dots, x_n) \in F\langle X \rangle$  of the form (6) such that  $f = x_1 g + h$ , where  $h \in \Gamma$ . Since  $\Gamma_f = \Gamma_{x_1 g}$  we can assume without loss of generality (replacing  $f$  with  $x_1 g$ ) that  $f = x_1 g(x_2, \dots, x_n)$ , where  $g$  is of the form (6), as claimed.

It is clear that if  $f = 0$  then  $\Gamma_f = \Gamma$ . Suppose that  $f \neq 0$ . Permuting the generators  $x_2, \dots, x_n$  if necessary we can assume that  $\alpha_{(2,3)} \neq 0$ ; then  $f$  is of the form

$$f = x_1 \left( \alpha_{(2,3)} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_2 x_3} [x_2, x_3] + \sum_{(i_1, i_2) \neq (2,3)} \alpha_{(i_1, i_2)} \frac{x_2^{a_2} \dots x_n^{a_n}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] \right).$$

Let  $f_1(x_1, x_2, x_3) = f(x_1, x_2, x_3, 1, \dots, 1) \in \Gamma_f$ ; then

$$f_1 = \alpha_{(2,3)} x_1 x_2^{a_2-1} x_3^{a_3-1} [x_2, x_3].$$

It can be easily seen that the multihomogeneous component of multidegree  $(1, 1, 1)$  of the polynomial  $f_1(x_1, x_2 + 1, x_3 + 1)$  is equal to  $\alpha_{(2,3)} x_1 [x_2, x_3]$ . It follows that  $x_1 [x_2, x_3] \in \Gamma_f$  and therefore  $\Gamma(0) \subseteq \Gamma_f$ .

Let  $T^{(2)}$  be the  $T$ -ideal in  $F\langle X \rangle$  generated by  $[x_1, x_2]$ . Clearly,  $f \in T^{(2)}$ . On the other hand, since

$$x_1 [x_2, x_3] x_4 = x_1 [x_2, x_3 x_4] - x_1 x_3 [x_2, x_4],$$



we have  $T^{(2)} \subset \Gamma(0)$ . Hence,  $f \in \Gamma(0)$  and  $\Gamma_f \subseteq \Gamma(0)$ . It follows that  $\Gamma_f = \Gamma(0)$ . The proof of Lemma 8 is completed.  $\square$

**Lemma 9.** Let  $f = f(x_1, \dots, x_n) \in F\langle X \rangle$  be a multihomogeneous polynomial of the form

$$f = \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}], \quad (9)$$

where  $\alpha_{(i_1, i_2)} \in F$ ,  $s_i \geq 1$  for all  $i$ . Then  $\Gamma_f = \Gamma$  or  $\Gamma_f = \Gamma(s)$  where  $s$  is the minimum of  $s_1, s_2, \dots, s_n$ .

**Proof.** We can assume without loss of generality (permuting the free generators  $x_1, \dots, x_n$  if necessary) that  $s = s_1 \leq s_i$  for all  $i = 1, \dots, n$ .

If  $n = 1$  then  $f = 0$ , and if  $n = 2$  then  $f = \alpha_{(1,2)} x_1^{p^{s_1}-1} x_2^{p^{s_2}-1} [x_1, x_2]$ . Thus, by Lemma 6, in both cases we have  $\Gamma_f = \Gamma$ .

Suppose that  $n > 2$ . We claim that we may assume without loss of generality that  $f$  is of the form

$$f(x_1, \dots, x_n) = x_1^{p^s} g(x_2, \dots, x_n), \quad (10)$$

where

$$g = \sum_{2 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}].$$

Indeed, consider a term  $m = \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}]$  in (9). If  $i_1 > 1$  then

$$m = x_1^{p^s} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}]. \quad (11)$$

Suppose that  $i_1 = 1$ . Let  $a_i = p^{s_i}$  for all  $i$ . Then

$$\begin{aligned} m + T^{(3,2)} &= x_1^{p^s-1} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_2}} [x_1, x_{i_2}] + T^{(3,2)} \\ &= x_1^{a_1-1} x_{j_1}^{a_{j_1}} \dots x_{j_l}^{a_{j_l}} [x_1, x_{i_2}] x_{i_2}^{a_{i_2}-1} + T^{(3,2)}, \end{aligned}$$

where  $\{j_1, \dots, j_l\} = \{1, \dots, n\} \setminus \{1, i_2\}$ ,  $l = n - 2 > 0$ . Suppose that

$$a_1 = a_{j_1} = \dots = a_{j_z} \quad \text{and} \quad a_{j_{z+1}}, a_{j_{z+2}}, \dots, a_{j_l} > a_1, \quad 0 \leq z \leq l.$$

Let

$$u = x_1 x_{j_1} \dots x_{j_z} x_{j_{z+1}}^{a'_{j_{z+1}}} \dots x_{j_l}^{a'_{j_l}},$$

where  $a'_i = a_i/p^s$  for all  $i$ . Let  $h = h(x_1, x_2) = x_1^{a_1-1}[x_1, x_2]x_2^{a_{i_2}-1}$ . By Lemma 6, we have  $h \in \Gamma$ . It follows that

$$h(u, x_{i_2}) = u^{p^s-1}[u, x_{i_2}]x_{i_2}^{a_{i_2}-1} \in \Gamma. \quad (12)$$

Since, by Lemma 3,  $[v_1^p, v_2] \in T^{(3)} \subset T^{(3,2)}$  for all  $v_1, v_2 \in F\langle X \rangle$ , we have

$$\begin{aligned} & h(u, x_{i_2}) + T^{(3,2)} \\ &= (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1 x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} + T^{(3,2)}. \end{aligned}$$

If  $z = 0$  then  $h(u, x_{i_2}) \equiv \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_1 x_{i_2}} [x_1, x_{i_2}] \pmod{T^{(3,2)}}$ . Thus, by (12),

$$m = \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_1 x_{i_2}} [x_1, x_{i_2}] \in \Gamma. \quad (13)$$

Suppose that  $z > 0$ . Then  $h(u, x_{i_2}) + T^{(3,2)} = h_1 + h_2 + T^{(3,2)}$ , where

$$\begin{aligned} h_1 &= (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1, x_{i_2}] x_{j_1} \cdots x_{j_z} x_{i_2}^{a_{i_2}-1}, \\ h_2 &= (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} x_1 [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1}. \end{aligned}$$

It can be easily checked using (2) that

$$\begin{aligned} h_1 + T^{(3,2)} &= \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_1 x_{i_2}} [x_1, x_{i_2}] + T^{(3,2)} = m + T^{(3,2)}, \\ h_2 + T^{(3,2)} &= \sum_{2 \leq i_1 < i_2 \leq n} \beta_{(i_1, i_2)} x_1^{p^s} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] + T^{(3,2)} \end{aligned}$$

for some  $\beta_{(i_1, i_2)} \in F$ . Since  $T^{(3,2)} \subset \Gamma$ , by (12) we obtain

$$m + \sum_{2 \leq i_1 < i_2 \leq n} \beta_{(i_1, i_2)} x_1^{p^s} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] \in \Gamma. \quad (14)$$

By (11), (13) and (14), we can write  $f = f_1 + f_2$ , where

$$f_1 = x_1^{p^s} \left( \sum_{2 \leq i_1 < i_2 \leq n} \gamma_{(i_1, i_2)} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] \right)$$

is of the form (10) and  $f_2 \in \Gamma$ . It is clear that  $\Gamma_f = \Gamma_{f_1}$ . Thus, we can assume (replacing  $f$  with  $f_1$ ) that the polynomial  $f$  is of the form (10), as claimed.

If  $f = 0$  then  $\Gamma_f = \Gamma$ . Suppose that  $f \neq 0$ . Then we can assume without loss of generality that  $\alpha_{(2,3)} \neq 0$ . It follows that the  $T$ -subalgebra  $\Gamma_f$  contains the polynomial

$$h(x_1, x_2, x_3) = \alpha_{(2,3)}^{-1} f(x_1, x_2, x_3, 1, 1, \dots, 1) = x_1^{p^s} x_2^{p^{s_2}-1} x_3^{p^{s_3}-1} [x_2, x_3].$$

Then  $\Gamma_f$  also contains the multihomogeneous component of the polynomial  $h(x_1, x_2 + 1, x_3 + 1)$  of degree  $p^s$  in each variable  $x_i$  ( $i = 1, 2, 3$ ), that is equal, modulo  $T^{(3,2)}$ , to

$$\gamma x_1^{p^s} x_2^{p^s-1} x_3^{p^s-1} [x_2, x_3],$$

where  $\gamma = \binom{p^{s_2}-1}{p^s-1} \binom{p^{s_3}-1}{p^s-1} \equiv 1 \pmod{p}$ . It follows that  $x_1^{p^s} q^{(s)}(x_2, x_3) \in \Gamma_f$  and hence  $\Gamma(s) \subseteq \Gamma_f$ .

Now let us prove that  $\Gamma_f \subseteq \Gamma(s)$ . We have

$$\begin{aligned} & x_1^{p^s} x_2^{p^{s_2}-1} x_3^{p^{s_3}-1} x_4^{p^{s_4}} \dots x_n^{p^{s_n}} [x_2, x_3] + T^{(3,2)} \\ &= x_1^{p^s} x_2^{p^{s_2}-p^s} x_3^{p^{s_3}-p^s} x_4^{p^{s_4}} \dots x_n^{p^{s_n}} x_2^{p^s-1} [x_2, x_3] x_3^{p^s-1} + T^{(3,2)} \\ &= (x_1 x_2^{p^{s_2}-s-1} x_3^{p^{s_3}-s-1} x_4^{p^{s_4}-s} \dots x_n^{p^{s_n}-s})^{p^s} q^{(s)}(x_2, x_3) + T^{(3,2)} \end{aligned}$$

so

$$x_1^{p^s} x_2^{p^{s_2}-1} x_3^{p^{s_3}-1} x_4^{p^{s_4}} \dots x_n^{p^{s_n}} [x_2, x_3] \in \Gamma(s).$$

Similarly we can prove that, for all  $i_1, i_2$  such that  $2 \leq i_1 < i_2 \leq n$ , we have

$$x_1^{p^s} \frac{x_2^{p^{s_2}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}] \in \Gamma(s).$$

Since  $f$  has the form (10), we get  $f \in \Gamma(s)$ . It follows that  $\Gamma_f \subseteq \Gamma(s)$  and therefore  $\Gamma_f = \Gamma(s)$ .

The proof of Lemma 9 is completed.  $\square$

**Lemma 10.** Let  $f = f(x_1, \dots, x_n)$  be a multihomogeneous polynomial of multidegree  $(p^{s_1}, \dots, p^{s_n})$ , where  $s_i \geq 0$  for all  $i$ . Denote by  $s$  the minimum of  $s_1, s_2, \dots, s_n$ . Then one of the following holds:

1.  $\Gamma_f = F\langle X \rangle$ ;
2.  $\Gamma_f = \Gamma$ ;
3.  $\Gamma_f = \Gamma(s)$ ;
4.  $\Gamma_f$  is the  $T$ -subalgebra generated by  $x_1^{p^s}$  and  $\Gamma(s)$ .

**Proof.** Since a  $T$ -subalgebra is invariant under automorphisms of  $F\langle X \rangle$ , we may assume without loss of generality that  $s = s_1$ .

If  $s = 0$  then, by Lemma 8, we have either  $\Gamma_f = F\langle X \rangle$  or  $\Gamma_f = \Gamma$  or  $\Gamma_f = \Gamma(0)$ .

Suppose that  $s \geq 1$ . By Lemma 4 we can assume that

$$f = \alpha x_1^{p^{s_1}} \dots x_n^{p^{s_n}} + \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}],$$

where  $\alpha, \alpha_{(i_1, i_2)} \in F$ . Denote by  $h(x_1, \dots, x_n)$  the following polynomial

$$h = \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{(i_1, i_2)} \frac{x_1^{p^{s_1}} \dots x_n^{p^{s_n}}}{x_{i_1} x_{i_2}} [x_{i_1}, x_{i_2}].$$

If  $\alpha = 0$  then  $f = h$ , and by Lemma 9 we have  $\Gamma_f = \Gamma$  or  $\Gamma_f = \Gamma(s)$ .

Suppose that  $\alpha \neq 0$ . Then  $\Gamma_f$  is the  $T$ -subalgebra generated by the polynomials  $x_1^{p^s}$ ,  $h$  and by  $\Gamma$ .

Indeed, since  $x_1^{p^{s_1}} x_2^{p^{s_2}} \dots x_n^{p^{s_n}} + T^{(3,2)} = (x_1 x_2^{p^{s_2-s}} \dots x_n^{p^{s_n-s}})^{p^s} + T^{(3,2)}$ , the monomial  $x_1^{p^{s_1}} x_2^{p^{s_2}} \dots x_n^{p^{s_n}}$  belongs to the  $T$ -subalgebra generated by  $x_1^{p^s}$  and  $\Gamma$ . Hence,  $f = \alpha x_1^{p^{s_1}} x_2^{p^{s_2}} \dots x_n^{p^{s_n}} + h$  belongs to the  $T$ -subalgebra generated by  $x_1^{p^s}$ ,  $h$  and by  $\Gamma$ .

On the other hand, it is clear that  $\alpha^{-1} f(x_1, 1, \dots, 1) = x_1^{p^s} \in \Gamma_f$  so  $x_1^{p^{s_1}} x_2^{p^{s_2}} \dots x_n^{p^{s_n}} \in \Gamma_f$  and therefore  $h = f - \alpha x_1^{p^{s_1}} x_2^{p^{s_2}} \dots x_n^{p^{s_n}} \in \Gamma_f$ . Thus,  $x_1^{p^s} \in \Gamma_f$  and  $h \in \Gamma_f$ . Since  $\Gamma \subset \Gamma_f$ ,  $\Gamma_f$  coincides with the  $T$ -subalgebra generated by the polynomials  $x_1^{p^s}$ ,  $h$  and by  $\Gamma$ , as claimed.

Thus,  $\Gamma_f$  is the  $T$ -subalgebra in  $F\langle X \rangle$  generated by  $x_1^{p^s}$  and by  $\Gamma_h$ . By Lemma 9, one of the following holds:

- (a)  $\Gamma_f$  coincides with the  $T$ -subalgebra  $\Gamma'$  generated by  $x_1^{p^s}$  and by  $\Gamma$ ;
- (b)  $\Gamma_f$  coincides with the  $T$ -subalgebra  $\Gamma''$  generated by  $x_1^{p^s}$  and  $\Gamma(s)$ .

Since  $x_1^{p^s}$  and  $q^{(s)}(x_2, x_3)$  belongs to  $\Gamma'$ , we have  $x_1^{p^s} q^{(s)}(x_2, x_3) \in \Gamma'$  so  $\Gamma' = \Gamma''$ . The proof of the lemma is completed.  $\square$

**Proposition 11.** Let  $\Omega$  be a  $T$ -subalgebra of  $F\langle X \rangle$  such that  $\Gamma \subsetneq \Omega$ . Then one of the following holds:

1.  $\Omega = F\langle X \rangle$ ;
2.  $\Omega = \Gamma(\mu)$  for some  $\mu \geq 0$ ;
3.  $\Omega$  is the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$  for some  $\lambda \geq \mu \geq 0$ .

**Proof.** The Vandermonde argument shows that over an infinite field of characteristic  $p > 0$  each  $T$ -ideal (see for example [7,9,17]), each  $T$ -subspace (see [10,14,15]) and each  $T$ -subalgebra in  $F\langle X \rangle$  is generated by its multihomogeneous polynomials  $f(x_1, \dots, x_n)$

of multidegrees  $(p^{s_1}, \dots, p^{s_n})$  for some  $s_i \geq 0$  ( $i = 1, \dots, n$ ). Let  $M$  be a subset of  $\Omega$  formed by such multihomogeneous polynomials that generates  $\Omega$  as a  $T$ -subalgebra.

Using [Lemma 10](#) we can describe the  $T$ -subalgebra  $\Gamma_f \subseteq \Omega$  for any  $f \in M$ . If  $\Gamma_f = F\langle X \rangle$  for some  $f \in M$  then  $\Omega = F\langle X \rangle$ .

Suppose that  $\Gamma_f \neq F\langle X \rangle$  for all  $f \in M$ . Since  $\Omega \supsetneq \Gamma$ , there exists  $f \in M$  such that  $\Gamma_f \neq \Gamma$ . In this case it follows from [Lemma 10](#) that  $\Omega \supseteq \Gamma(s)$  for some  $s \geq 0$ . Let  $\mu$  be the smallest non-negative integer such that  $\Gamma(\mu) \subseteq \Omega$ . Since

$$\begin{aligned} & x_1^{p^s} x_2^{p^s-1} [x_2, x_3] x_3^{p^s-1} + T^{(3,2)} \\ &= (x_1^{p^{s-\mu}} x_2^{p^{s-\mu}-1} x_3^{p^{s-\mu}-1})^{p^\mu} x_2^{p^\mu-1} [x_2, x_3] x_3^{p^\mu-1} + T^{(3,2)} \end{aligned}$$

then it is clear that for all  $s \geq \mu$  we have

$$\Gamma(s) \subseteq \Gamma(\mu). \quad (15)$$

Define the polynomial  $\omega(x_1)$  by  $\omega(x_1) = x_1^{p^\lambda}$ , where  $\lambda$  is the smallest integer  $r \geq 0$  such that  $x_1^{p^r} \in \Omega$ . If such  $\lambda$  does not exist (that is, if  $x_1^{p^r} \notin \Omega$  for any  $r \geq 0$ ), then we define  $\omega(x_1) = 0$ .

If  $\omega(x_1) = 0$  then, by [Lemma 10](#), we have  $\Gamma_f = \Gamma$  or  $\Gamma_f = \Gamma(s)$  ( $s \geq 0$ ) for any  $f \in M$ . By the definition of  $\mu$  we have  $\Gamma_f \subseteq \Gamma(\mu)$  for any  $f \in M$ . Hence,  $\Omega = \Gamma(\mu)$ .

Suppose that  $\omega(x_1) = x_1^{p^\lambda}$ . Let  $\Omega'$  be the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$ . It is clear that  $\Omega' \subseteq \Omega$ . We will prove that in this case  $\Omega = \Omega'$ .

Let  $f \in M$ . If  $\Gamma_f = \Gamma$  or  $\Gamma_f = \Gamma(s)$  ( $s \geq \mu$ ) then  $\Gamma_f \subseteq \Omega'$ . Suppose that  $\Gamma_f$  is generated by  $x_1^{p^s}$  and  $\Gamma(s)$  for some  $s \geq 0$ . Then it is clear that  $s \geq \mu$ , and  $\Gamma(s) \subseteq \Omega'$  by (15). Similarly,  $s \geq \lambda$  and  $x_1^{p^s} = (x_1^{p^{s-\lambda}})^{p^\lambda}$  belongs to  $\Omega'$ . Thus,  $\Gamma_f \subseteq \Omega'$  for all  $f \in M$ , and hence  $\Omega = \Omega'$  as desired.

Suppose now in order to get a contradiction that  $\lambda < \mu$ . Since  $x_1^{p^\lambda} \in \Omega$  and  $q^{(\lambda)}(x_2, x_3) \in \Gamma \subseteq \Omega$ , we have  $x_1^{p^\lambda} q^{(\lambda)}(x_2, x_3) \in \Omega$ . Therefore  $\Gamma(\lambda) \subseteq \Omega$ , a contradiction with the definition of  $\mu$ . Thus,  $\lambda \geq \mu$ . The proof of the proposition is completed.  $\square$

Now we are in a position to complete the proof of [Theorem 1](#).

First we prove that  $\Gamma$  is not finitely generated as a  $T$ -subalgebra. Suppose the contrary:  $\Gamma$  is generated as a  $T$ -subalgebra by some finite subset  $Q$ . Then  $Q$  lies in the  $T$ -subalgebra generated by a set

$$S = \{1, q^{(0)}, q^{(1)}, q^{(2)}, \dots, q^{(z)}, x_1[x_2, x_3, x_4], x_1[x_2, x_3][x_4, x_5]\}$$

for some integer  $z \geq 0$ . Hence,  $\Gamma$  is generated by  $S$  as a  $T$ -subalgebra. It follows that each element of  $\Gamma$  is a linear combination of 1 and of products of the form

$$f = f_1(g_{(1,1)}, \dots, g_{(1,5)}) f_2(g_{(2,1)}, \dots, g_{(2,5)}) \dots f_n(g_{(n,1)}, \dots, g_{(n,5)}),$$

where  $f_i \in S \setminus \{1\}$ , and  $g_{(i,j)} \in F\langle X \rangle$  for all  $i, j$ . Note that if  $n \geq 2$  then  $f \in T^{(3,2)}$  and if  $n = 1$  then  $f$  belongs to the  $T$ -subspace  $\langle S \rangle^{TS}$  generated by  $S$ . Since  $T^{(3,2)} \subseteq \langle S \rangle^{TS}$ , we have  $f \in \langle S \rangle^{TS}$  in either case. Since  $1 \in S$ ,  $\Gamma$  coincides with the  $T$ -subspace  $\langle S \rangle^{TS}$  generated by  $S$ . This contradicts to the fact that  $q^{(z+1)} \in \Gamma$  does not belong to the  $T$ -subspace of  $F\langle X \rangle$  generated by the polynomials

$$x_1^p, q^{(0)}, \dots, q^{(z)}, x_1[x_2, x_3, x_4], x_1[x_2, x_3][x_4, x_5]$$

(see, for example, [14, Theorem 3.1]). Hence,  $\Gamma$  is not finitely generated as a  $T$ -subalgebra.

Now, let  $\Omega$  be a  $T$ -subalgebra in  $F\langle X \rangle$  such that  $\Gamma \subsetneq \Omega$ . Then, by Proposition 11 and Lemma 7,  $\Omega$  is finitely generated.

Thus,  $\Gamma$  is a limit  $T$ -subalgebra in  $F\langle X \rangle$ . The proof of Theorem 1 is completed.

#### 4. Proof of Theorem 2

Recall that  $X = \{x_1, x_2, \dots\}$ . Define  $X_n = \{x_1, \dots, x_n\} \subset X$ . Let  $F\langle X_n \rangle$  be the free unitary associative  $F$ -algebra on the free generating set  $X_n$ ,  $F\langle X_n \rangle \subset F\langle X \rangle$ . If  $W \subset F\langle X \rangle$  then  $W_n$  denotes the intersection of  $W$  and  $F\langle X_n \rangle$ ,  $W_n = W \cap F\langle X_n \rangle$ . The following lemma can be easily verified.

**Lemma 12.** *If  $W$  is a  $T$ -subalgebra ( $T$ -ideal) in  $F\langle X \rangle$  then  $W_n$  is a  $T$ -subalgebra ( $T$ -ideal) in  $F\langle X_n \rangle$ . If  $W$  is generated as a  $T$ -subalgebra by a set  $H$ , then  $W_n$  is generated as an unitary subalgebra in  $F\langle X_n \rangle$  by the elements  $h(f_1, f_2, \dots, f_m)$ , where  $h(x_1, \dots, x_m) \in H$  and  $f_i \in F\langle X_n \rangle$  for all  $i$ .*

In particular,  $T_n^{(3,2)} = T^{(3,2)} \cap F\langle X_n \rangle$  is a  $T$ -ideal of the algebra  $F\langle X_n \rangle$ .

**Lemma 13.** *If  $U$  is a  $T$ -subalgebra in  $F\langle X_n \rangle/T_n^{(3,2)}$  then there is a  $T$ -subalgebra  $\Omega$  in  $F\langle X \rangle$  such that  $T^{(3,2)} \subset \Omega$  and  $U = \Omega_n/T_n^{(3,2)}$ .*

**Proof.** Consider the preimage  $V = \pi^{-1}(U)$  of  $U$  with respect to the canonical homomorphism  $\pi : F\langle X_n \rangle \rightarrow F\langle X_n \rangle/T_n^{(3,2)}$ . It is clear that  $V$  is a  $T$ -subalgebra in  $F\langle X_n \rangle$  such that  $U = V/T_n^{(3,2)}$ . Let  $\Omega$  be the  $T$ -subalgebra in  $F\langle X \rangle$  generated  $V$  and  $T^{(3,2)}$ . Since  $T_n^{(3,2)} \subset V$ , by Lemma 12 we have  $\Omega_n = V$ .  $\square$

**Theorem 14.** *If  $n \geq 3$  then  $\Gamma_n/T_n^{(3,2)}$  is a limit  $T$ -subalgebra in  $F\langle X_n \rangle/T_n^{(3,2)}$ .*

**Proof.** It is clear that  $\Gamma_n/T_n^{(3,2)}$  is a  $T$ -subalgebra of  $F\langle X_n \rangle/T_n^{(3,2)}$ . Let us prove that  $\Gamma_n/T_n^{(3,2)}$  is not finitely generated as a  $T$ -subalgebra. Denote by  $H$  the set of polynomials (3). By Lemma 12,  $\Gamma_n$  is generated as an unitary subalgebra in  $F\langle X_n \rangle$  by the elements  $h(f_1, f_2, \dots, f_m)$ , where  $h \in H$  and  $f_i \in F\langle X_n \rangle$  for all  $i$ . Hence,  $\Gamma_n/T_n^{(3,2)}$  is generated as an unitary algebra by the elements  $q^{(l)}(f_1, f_2) + T_n^{(3,2)}$ , where  $f_1, f_2 \in F\langle X_n \rangle$ ,

and  $l \geq 0$ . Thus,  $\Gamma_n/T_n^{(3,2)}$  is generated as a  $T$ -subalgebra in  $F\langle X_n \rangle/T_n^{(3,2)}$  by the elements

$$q^{(0)} + T_n^{(3,2)}, q^{(1)} + T_n^{(3,2)}, \dots, q^{(l)} + T_n^{(3,2)}, \dots$$

Using the argument similar to one used in the proof of [Theorem 1](#) we can prove that if  $\Gamma_n/T_n^{(3,2)}$  is finitely generated then it can be generated by

$$q^{(0)} + T_n^{(3,2)}, q^{(1)} + T_n^{(3,2)}, \dots, q^{(z)} + T_n^{(3,2)}$$

for some  $z \geq 0$ . Note that  $\Gamma$  is generated by  $T^{(3,2)}$  and by the polynomials  $q^{(l)}(x_1, x_2) \in F\langle X_2 \rangle$  ( $l \geq 0$ ). It follows that  $\Gamma$  is generated by

$$q^{(0)}, q^{(1)}, \dots, q^{(z)}, x_1[x_2, x_3, x_4], x_1[x_2, x_3][x_4, x_5].$$

This is a contradiction because, by [Theorem 1](#),  $\Gamma$  is not a finitely generated  $T$ -subalgebra in  $F\langle X \rangle$ . Hence,  $\Gamma_n/T_n^{(3,2)}$  is not finitely generated as a  $T$ -subalgebra.

Let  $U$  be a  $T$ -subalgebra in  $F\langle X_n \rangle/T_n^{(3,2)}$  such that  $\Gamma_n/T_n^{(3,2)} \subsetneq U$ . We will prove that  $U$  is finitely generated as a  $T$ -subalgebra. By [Lemma 13](#), there is a  $T$ -subalgebra  $\Omega$  in  $F\langle X \rangle$  such that  $T^{(3,2)} \subset \Omega$  and  $U = \Omega_n/T_n^{(3,2)}$ . Since  $\Gamma_n \subsetneq \Omega_n$  and  $T^{(3,2)} \subset \Omega$ , for  $n \geq 3$  we obtain  $\Gamma \subsetneq \Omega$ . Then, by [Proposition 11](#), one of the following holds:

1.  $\Omega = F\langle X \rangle$ ;
2.  $\Omega = \Gamma(\mu)$  for some  $\mu \geq 0$ ;
3.  $\Omega$  is the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$  for some  $\lambda \geq \mu \geq 0$ .

If  $\Omega = F\langle X \rangle$ , then  $U = F\langle X_n \rangle/T_n^{(3,2)}$ , and hence  $U$  is generated by  $x_1 + T_n^{(3,2)}$ .

Suppose  $\Omega = \Gamma(\mu)$  for some  $\mu \geq 0$ . By [Lemma 7](#),  $\Omega$  is generated by the polynomials  $x_1^{p^\mu} q^{(\mu)}(x_2, x_3), x_1[x_2, x_3, x_4], x_1[x_2, x_3][x_4, x_5]$ . Hence  $U = \Omega_n/T_n^{(3,2)}$  is generated as a unitary algebra by the elements  $(f_1)^{p^\mu} q^{(\mu)}(f_2, f_3) + T_n^{(3,2)}$ , where  $f_i \in F\langle X_n \rangle$  for all  $i$ . If  $n \geq 3$  then  $U$  is generated as a  $T$ -subalgebra by  $x_1^{p^\mu} q^{(\mu)}(x_2, x_3) + T_n^{(3,2)}$ .

Let us consider the case when  $\Omega$  is the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$  for some  $\lambda \geq \mu \geq 0$ . By the argument similar to one used above, we have that for  $n \geq 3$  the  $T$ -subalgebra  $U$  is generated (as a  $T$ -subalgebra) by the elements  $x_1^{p^\lambda} + T_n^{(3,2)}, x_1^{p^\mu} q^{(\mu)}(x_2, x_3) + T_n^{(3,2)}$ .

In all cases we have that  $U$  is finitely generated as a  $T$ -subalgebra. The proof is completed.  $\square$

**Corollary 15.** *If  $n \geq 3$  then the  $T$ -subalgebra  $\Gamma_n$  is not finitely generated.*

Let  $\Omega_n$  be a  $T$ -subalgebra in  $F\langle X_n \rangle$  such that  $\Gamma_n \subsetneq \Omega_n$ . To prove [Theorem 2](#) it suffices to check that  $\Omega_n$  is a finitely generated  $T$ -subalgebra in  $F\langle X_n \rangle$ .

Let  $\Omega$  be the  $T$ -subalgebra of  $F\langle X \rangle$  generated by  $\Omega_n$  and  $T^{(3,2)}$ . It's easy to see that  $\Omega_n = \Omega \cap F\langle X_n \rangle$  and  $\Gamma \subsetneq \Omega$  for  $n \geq 2$ . By [Proposition 11](#), one of the following holds:

1.  $\Omega = F\langle X \rangle$ ;
2.  $\Omega = \Gamma(\mu)$  for some  $\mu \geq 0$ ;
3.  $\Omega$  is the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$  for some  $\lambda \geq \mu \geq 0$ .

If  $\Omega = F\langle X \rangle$ , then  $\Omega_n = F\langle X_n \rangle$ , and hence  $\Omega_n$  is generated by  $x_1$ .

Suppose  $\Omega = \Gamma(\mu)$  for some  $\mu \geq 0$ . Then, by [Lemma 7](#),  $\Omega$  is generated by

$$x_1^{p^\mu} q^{(\mu)}(x_2, x_3), \quad x_1[x_2, x_3, x_4], \quad x_1[x_2, x_3][x_4, x_5]. \quad (16)$$

If  $n \geq 5$  then  $\Omega_n$  is generated as a  $T$ -subalgebra of  $F\langle X_n \rangle$  by the elements (16) as well. Suppose  $n = 4$ . Let us consider a polynomial of the form  $f = f_1[f_2, f_3][f_4, f_5]$ , where  $f_1, \dots, f_5 \in F\langle X_4 \rangle$ . We have that  $f + T^{(3)}$  is a linear combination of the elements  $m + T^{(3)}$ , where

$$m = x_1^{a_1-1}[x_1, x_2]x_2^{a_2-1}x_3^{a_3-1}[x_3, x_4]x_4^{a_4-1},$$

$T^{(3)}$  is the  $T$ -ideal generated by  $[x_1, x_2, x_3]$ , and  $a_i \geq 1$  for all  $i$  (see for example [\[21,23\]](#) or [\[6,10,14,15,25\]](#)). Let  $l_1$  and  $l_2$  the largest integers such that  $p^{l_1}$  divides both  $a_1, a_2$  and  $p^{l_2}$  divides both  $a_3, a_4$ . By [Lemma 5](#),  $m$  belongs to the  $T$ -subalgebra generated by  $q^{(l_1)}, q^{(l_2)}$  and  $T^{(3)}$ . Modifying the proof of [Lemma 7](#), one can prove that for any  $l$  the polynomial  $q^{(l)}$  belongs to the  $T$ -subalgebra generated by  $x_1^{p^\mu} q^{(\mu)}(x_2, x_3)$  and  $T^{(3)}$ .

We leave the details to the reader to check that  $f$  belongs to the  $T$ -subalgebra generated by  $x_1^{p^\mu} q^{(\mu)}(x_2, x_3)$  and  $x_1[x_2, x_3, x_4]$ . Thus,  $\Omega_4$  is generated as a  $T$ -subalgebra by polynomials

$$x_1^{p^\mu} q^{(\mu)}(x_2, x_3), \quad x_1[x_2, x_3, x_4].$$

To finish, suppose that  $\Omega$  is the  $T$ -subalgebra generated by  $x_1^{p^\lambda}$  and by  $\Gamma(\mu)$  for some  $\lambda \geq \mu \geq 0$ . By the argument similar to one used above, we have that if  $n \geq 5$  then  $\Omega_n$  is generated as a  $T$ -subalgebra in  $F\langle X_n \rangle$  by

$$x_1^{p^\lambda}, \quad x_1^{p^\mu} q^{(\mu)}(x_2, x_3), \quad x_1[x_2, x_3, x_4], \quad x_1[x_2, x_3][x_4, x_5],$$

and if  $n = 4$  then  $\Omega_4$  is generated as a  $T$ -subalgebra in  $F\langle X_4 \rangle$  by

$$x_1^{p^\lambda}, \quad x_1^{p^\mu} q^{(\mu)}(x_2, x_3), \quad x_1[x_2, x_3, x_4].$$

In all cases, we have that  $\Omega_n$  is finitely generated as a  $T$ -subalgebra. The proof of [Theorem 2](#) is completed.



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