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On subcategories closed under predecessors and the representation dimension

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A R T I C L E I N F O

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A B S T R A C T

Let Λ be an artin algebra, $\mathcal{A}$ and $\mathcal{C}$ be full subcategories of the category of finitely generated Λ-modules consisting of indecomposable modules and closed under predecessors and successors respectively. In this paper we relate, under various hypotheses, the representation dimension of Λ to those of the left support algebra of $\mathcal{A}$ and the right support algebra of $\mathcal{C}$. Our results are then applied to the classes of laura algebras, ada algebras and Nakayama oriented pullbacks.

Introduction

The aim of the representation theory of artin algebras is to characterize and to classify algebras using properties of module categories. The representation dimension of an artin algebra was introduced by Auslander [9] and he expected that this invariant would give a measure of how far an algebra is from being representation-finite. He proved that a non-semisimple algebra Λ is representation-finite if and only if its representation

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dimension rep.dim Λ is two. Iyama proved that the representation dimension of an artin algebra is always finite (see [19]) and Rouquier has constructed examples of algebras with rep.dim $\Lambda = r$ for any $r \geq 2$ (see [23]).

Igusa and Todorov gave in [18] an interesting connection with the finitistic dimension conjecture. They proved that if Λ has representation dimension at most three then its finitistic dimension is finite.

Auslander proved in [9] that if Λ is a hereditary algebra, then rep.dim Λ is at most three. Many other classes of algebras have representation dimension at most three, as for example, tilted and laura algebras [7], trivial extensions of hereditary algebras [14] and quasi-tilted algebras [22]. Other results can be found also in [15,26].

In order to calculate the representation dimension of an artin algebra Λ, one reasonable approach would be to split the module category mod Λ of the finitely generated modules into pieces and calculate the representation dimension of algebras associated to each piece. In this sense, we consider for a full subcategory $\mathcal{C}$ of ind Λ closed under successors its support algebra $\Lambda_{\mathcal{C}}$, in the sense of [2], and for a full subcategory $\mathcal{A}$ of ind Λ closed under predecessors its support algebra ${}_{\mathcal{A}}\Lambda$. Our two main theorems (Theorems 2.6 and 4.2) relate rep.dim Λ to rep.dim ${}_{\mathcal{A}}\Lambda$ or rep.dim $\Lambda_{\mathcal{C}}$ when $\mathcal{A}$ and $\mathcal{C}$ satisfy some additional hypotheses.

Before stating our first main theorem, we need to recall some definitions. Let Λ be an artin algebra and ind Λ be a full subcategory of mod Λ consisting of one representative from each isomorphism class of indecomposable modules. A trisection of ind Λ is a triple of disjoint full subcategories $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ such that ind $\Lambda = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $\mathrm{Hom}(\mathcal{C}, \mathcal{B}) = \mathrm{Hom}(\mathcal{C}, \mathcal{A}) = \mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$, see [1]. We say that $\mathcal{B}$ is finite if it contains only finitely many objects of ind Λ. We denote by $\mathcal{L}_\Lambda$ and $\mathcal{R}_\Lambda$, respectively, the left and the right parts of mod Λ in the sense of [16] (or see Section 1.2 below). For the definition of covariantly and contravariantly finite subcategories, we refer the reader to [12] (or see Section 1.3 below).

The first theorem is the following:

Theorem. *Let Λ be a representation-infinite artin algebra and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection of* ind Λ *with $\mathcal{B}$ finite.*

(a) *If $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ and* add $\mathcal{C}$ *is covariantly finite, then*

$$\mathrm{rep.dim}\,\Lambda = \max\{3, \mathrm{rep.dim}\,{}_{\mathcal{A}}\Lambda\}.$$

(b) *If $\mathcal{A} \subseteq \mathcal{L}_\Lambda$ and* add $\mathcal{A}$ *is contravariantly finite, then*

$$\mathrm{rep.dim}\,\Lambda = \max\{3, \mathrm{rep.dim}\,\Lambda_{\mathcal{C}}\}.$$

As consequences of this theorem we prove that the class of ada algebras, introduced and studied in [3], has representation dimension at most three (Corollary 5.3), and give another proof of Theorem 4.1 in [7] saying that strict laura algebras have representation dimension at most three.

If $\mathcal{C}$ is not necessarily contained in $\mathcal{R}_\Lambda$ and $\mathcal{A}$ is not necessarily contained in $\mathcal{L}_\Lambda$ the second theorem gives a relationship between the representation dimension of Λ and those of ${}_{\mathcal{A}}\Lambda$ and of $\Lambda_{\mathcal{C}}$. For this, however, we have to assume that $\operatorname{ind}\Lambda_{\mathcal{C}}$ is closed under successors or $\operatorname{ind}_{\mathcal{A}}\Lambda$ is closed under predecessors.

For a full subcategory $\mathcal{X}$ of $\operatorname{ind}\Lambda$, we denote by $\mathcal{X}^c = \operatorname{ind}\Lambda \setminus \mathcal{X}$ its complement.

Theorem. *Let Λ be an artin algebra with a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\operatorname{ind}\Lambda$. If*

(a) $(\mathcal{A} \cup \operatorname{ind}\Lambda_{\mathcal{C}})^c$ *is finite and* $\operatorname{ind}\Lambda_{\mathcal{C}}$ *is closed under successors*

or

(b) $(\operatorname{ind}{}_{\mathcal{A}}\Lambda \cup \mathcal{C})^c$ *is finite and* $\operatorname{ind}{}_{\mathcal{A}}\Lambda$ *is closed under predecessors,*
then,

$$\operatorname{rep.dim}\Lambda \leq \max\{\operatorname{rep.dim}{}_{\mathcal{A}}\Lambda, \operatorname{rep.dim}\Lambda_{\mathcal{C}}\}.$$

As a consequence of this second theorem we prove that if R is the Nakayama oriented pullback [20] of the morphisms $A \to B$ and $C \to B$, then we have $\operatorname{rep.dim} R \leq \max\{\operatorname{rep.dim} A, \operatorname{rep.dim} C\}$ (Corollary 5.8).

This paper is organized as follows. The first section is dedicated to preliminaries with some definitions and useful results. Section 2 and Section 4 are the proofs of the first and the second theorems, respectively. Section 3 studies the relation of the representation dimension of an algebra with the representation dimension of the support algebras of the complements of left and right parts; this study is useful for the proof of the result concerning ada algebras. Finally, Section 5 contains applications of the main results: laura algebras, ada algebras and Nakayama oriented pullbacks.

1. Preliminaries

In this first section, we recall some well-known definitions that we use in this text.

1.1. Notation

In this paper, all algebras are artin algebras. For an algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of all finitely generated right Λ-modules and by $\operatorname{ind}\Lambda$ a full subcategory of $\operatorname{mod}\Lambda$ consisting of exactly one representative from each isomorphism class of indecomposable modules. For a Λ-module M, we denote by ${}_\Lambda(-, M)$ the functor $\operatorname{Hom}_\Lambda(-, M)$. For a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ we write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$.

We denote by $\operatorname{add}\mathcal{C}$ the full subcategory of $\operatorname{mod}\Lambda$ with objects the finite direct sums of summands of modules in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$ as $\operatorname{add} M$. We denote the projective (or injective) dimension of a module M as $\operatorname{pd}_\Lambda M$ (or $\operatorname{id}_\Lambda M$, respectively). We say that $\mathcal{C}$ is **finite** if it has only finitely many isomorphism classes of indecomposable Λ-modules and we say that $\mathcal{C}$ is **cofinite** if $\mathcal{C}^c$ is finite. We say that Λ is a **representation-finite** algebra if $\operatorname{ind}\Lambda$ is finite. It is **representation-infinite** otherwise. We denote by $\operatorname{Gen} M$ (or $\operatorname{Cogen} M$) the full subcategory of $\operatorname{mod}\Lambda$ having as objects all modules generated (or cogenerated, respectively) by M. We denote by $\tau_\Lambda = \mathrm{DTr}$ and $\tau_\Lambda^{-1} = \mathrm{TrD}$ the Auslander–Reiten translations.

For an algebra that is determined by a quiver Q_Λ we denote by e_i the idempotent associated to the vertex $i \in (Q_\Lambda)_0$ and by $e_\Lambda = \sum_{i\in(Q_\Lambda)_0} e_i$ its identity. In this case, we denote by P_i, I_i and S_i the projective, injective and simple, respectively, associated to the vertex $i \in (Q_\Lambda)_0$.

For further definitions and facts on $\operatorname{mod}\Lambda$, we refer to [8,11].

1.2. Subcategories closed under predecessors

Given $M, N \in \operatorname{ind}\Lambda$, a **path** from M to N in $\operatorname{ind}\Lambda$ is a sequence of non-zero morphisms $M = X_1 \to X_2 \to \cdots \to X_t = N$ $(t \geq 1)$ where $X_i \in \operatorname{ind}\Lambda$ for all i. In this case, we say that M is a **predecessor** of N and that N is a **successor** of M.

We say that $\mathcal{A}$ is **closed under predecessors** if, whenever M is a predecessor of N with $N \in \mathcal{A}$, then $M \in \mathcal{A}$. Dually, we define subcategory **closed under successors**.

For a module M, we denote by $\operatorname{Succ} M$ the full subcategory of $\operatorname{ind}\Lambda$ consisting of all successors of any indecomposable summand of M. This category is, of course, closed under successors. Dually we denote by $\operatorname{Pred} M$ the full subcategory of $\operatorname{ind}\Lambda$ consisting of all predecessors of any indecomposable summand of M.

We recall from [16] that the **right part** $\mathcal{R}_\Lambda$ of $\operatorname{mod}\Lambda$ is the full subcategory of $\operatorname{ind}\Lambda$ defined by

$$\mathcal{R}_\Lambda = \{M \in \operatorname{ind}\Lambda \mid \operatorname{id}_\Lambda N \leq 1 \text{ for each successor } N \text{ of } M\}.$$

Clearly, $\mathcal{R}_\Lambda$ is closed under successors. Dually, the **left part**,

$$\mathcal{L}_\Lambda = \{M \in \operatorname{ind}\Lambda \mid \operatorname{pd}_\Lambda N \leq 1 \text{ for each predecessor } N \text{ of } M\}$$

is a full subcategory of $\operatorname{ind}\Lambda$ closed under predecessors.

Another way to produce subcategories closed under predecessors is by means of trisections [1]. A **trisection** of $\operatorname{ind}\Lambda$ is a triple of disjoint full subcategories $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\operatorname{ind}\Lambda$ such that $\operatorname{ind}\Lambda = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $\operatorname{Hom}(\mathcal{C}, \mathcal{B}) = \operatorname{Hom}(\mathcal{C}, \mathcal{A}) = \operatorname{Hom}(\mathcal{B}, \mathcal{A}) = 0$. If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a trisection of $\operatorname{ind}\Lambda$ then the subcategory $\mathcal{A}$ is closed under predecessors and $\mathcal{C}$ is closed under successors. Also, $\mathcal{B}$ is **convex** in $\operatorname{ind}\Lambda$, that is, if $M = M_1 \to M_2 \to \cdots \to M_{t-1} \to M_t = N$ is a path in $\operatorname{ind}\Lambda$ with $M, N \in \mathcal{B}$ then $M_i \in \mathcal{B}$ for all $i = 1, ..., t$.

1.3. Covariantly and contravariantly finite subcategories

The notions of contravariantly and covariantly finite subcategories were introduced in [12,13]. Let $\mathcal{X}$ be an additive full subcategory of $\operatorname{mod}\Lambda$. We say that $\mathcal{X}$ is **contravariantly finite** if for any Λ-module M, there is a morphism $f_M : X_M \to M$ with $X_M \in \mathcal{X}$ such that any morphism $f : X \to M$ with $X \in \mathcal{X}$ factors through f_M. Dually we define **covariantly finite** subcategories and $\mathcal{X}$ is called **functorially finite** if it is both contravariantly and covariantly finite. Finally, following [10], $\mathcal{X}$ is called **homologically finite** if it is contravariantly finite or covariantly finite. For instance, if $\mathcal{C}$ is a finite or cofinite subcategory of $\operatorname{ind}\Lambda$, then $\operatorname{add}\mathcal{C}$ is functorially finite in $\operatorname{mod}\Lambda$ (see [12]). In particular, for a module $M \in \operatorname{mod}\Lambda$, the category $\operatorname{add} M$ is functorially finite.

If $\mathcal{X}$ is an additive subcategory of $\operatorname{mod}\Lambda$, closed under extensions, then a module $M \in \mathcal{X}$ is called **Ext-projective** in $\mathcal{X}$ if $\operatorname{Ext}^1_\Lambda(M, -)|_{\mathcal{X}} = 0$. Dually, a module N to be **Ext-injective** in $\mathcal{X}$ if $\operatorname{Ext}^1_\Lambda(-, N)|_{\mathcal{X}} = 0$. If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair, then $M \in \mathcal{X}$ is Ext-projective in $\mathcal{X}$ if and only if $\tau_\Lambda M \in \mathcal{Y}$ and $N \in \mathcal{Y}$ is Ext-injective in $\mathcal{Y}$ if and only if $\tau_\Lambda^{-1} N \in \mathcal{X}$ (see [13]).

Let $\mathcal{A}$ be a full subcategory closed under predecessors of $\operatorname{ind}\Lambda$ then $\mathcal{C} = \mathcal{A}^c$ is closed under successors and in this case $(\operatorname{add}\mathcal{C}, \operatorname{add}\mathcal{A})$ is a split torsion pair. Denote by E the direct sum of a full set of representatives of the indecomposable Ext-injective modules in $\mathcal{A}$ and by F the direct sum of a full set of representatives of the indecomposable Ext-projective modules in $\mathcal{C}$. We need the following particular case of the main result of [24].

Lemma 1.1. *Let $\mathcal{A}$ be a full subcategory closed under predecessors of* $\operatorname{ind}\Lambda$ *and $\mathcal{C} = \mathcal{A}^c$. The following conditions are equivalent:*

(a) $\operatorname{add}\mathcal{A}$ *is contravariantly finite.*
(b) $\operatorname{add}\mathcal{A} = \operatorname{Cogen} N$ *for some* $N \in \operatorname{mod}\Lambda$.
(c) $\operatorname{add}\mathcal{A} = \operatorname{Cogen} E$.
(d) $\operatorname{add}\mathcal{C}$ *is covariantly finite.*
(e) $\operatorname{add}\mathcal{C} = \operatorname{Gen} M$ *for some* $M \in \operatorname{mod}\Lambda$.
(f) $\operatorname{add}\mathcal{C} = \operatorname{Gen} F$. □

Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind}\Lambda$ closed under successors such that $\operatorname{add}\mathcal{C}$ is covariantly finite. Denote by F the direct sum of all indecomposable Ext-projective modules in $\mathcal{C}$ and by N the direct sum of all indecomposable injective Λ-modules lying in $\mathcal{C}$.

Lemma 1.2. *(See [6] (5.3).) Let $\mathcal{C}$ be a full subcategory of* $\operatorname{ind}\Lambda$ *closed under successors. Assume that* $\operatorname{add}\mathcal{C}$ *is covariantly finite. Then:*

(a) *F is convex if and only if $\mathcal{C} \subseteq \mathcal{R}_\Lambda$.*
(b) *If, moreover,* $\operatorname{add}\mathcal{C}$ *contains all the injective Λ-modules, then $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ if and only if Λ is tilted having F as a slice module.* □

Note that, by [7] (2.1), the algebra Λ is tilted if and only if it has a convex tilting module. For properties of tilted algebras we refer to [8].

1.4. Support algebras

Let $\mathcal{A}$ be a full subcategory of $\operatorname{ind}\Lambda$ closed under predecessors. Following [2], we define its **support algebra** ${}_{\mathcal{A}}\Lambda$ to be the endomorphism algebra of the direct sum of a full set of representatives of the isomorphism classes of the indecomposable projectives lying in $\mathcal{A}$. Let $\mathcal{C}$ be a full subcategory of $\operatorname{ind}\Lambda$ closed under successors, we define dually the support algebra $\Lambda_{\mathcal{C}}$ of $\mathcal{C}$. Note that $\operatorname{mod}{}_{\mathcal{A}}\Lambda$ and $\operatorname{mod}\Lambda_{\mathcal{C}}$ are full subcategories of $\operatorname{mod}\Lambda$. We have the following properties from [6] (4.1).

Lemma 1.3. *Let $\mathcal{A}$ be a full subcategory of* $\operatorname{ind}\Lambda$ *closed under predecessors and $\mathcal{C}$ a full subcategory of* $\operatorname{ind}\Lambda$ *closed under successors.*

(a) *All indecomposable Λ-modules lying in $\mathcal{A}$ have a natural structure of indecomposable ${}_{\mathcal{A}}\Lambda$-modules;*
(b) *The indecomposable projective ${}_{\mathcal{A}}\Lambda$-modules are just the indecomposable projective Λ-modules lying in $\mathcal{A}$;*
(c) *For any indecomposable ${}_{\mathcal{A}}\Lambda$-module M we have* $\operatorname{pd}_{({}_{\mathcal{A}}\Lambda)}M = \operatorname{pd}_\Lambda M$ *and* $\operatorname{id}_{({}_{\mathcal{A}}\Lambda)}M \leq \operatorname{id}_\Lambda M$*;*
(a′) *All indecomposable Λ-modules lying in $\mathcal{C}$ have a natural structure of indecomposable $\Lambda_{\mathcal{C}}$-modules;*
(b′) *The indecomposable injective $\Lambda_{\mathcal{C}}$-modules are just the indecomposable injective Λ-modules lying in $\mathcal{C}$;*
(c′) *For any indecomposable $\Lambda_{\mathcal{C}}$-module M we have* $\operatorname{id}_{(\Lambda_{\mathcal{C}})}M = \operatorname{id}_\Lambda M$ *and* $\operatorname{pd}_{(\Lambda_{\mathcal{C}})}M \leq \operatorname{pd}_\Lambda M$. □

1.5. Representation dimension

A module M is called a **generator** of $\operatorname{mod}\Lambda$ if any projective Λ-module belongs to $\operatorname{add}M$, it is called a **cogenerator** of $\operatorname{mod}\Lambda$ if any injective Λ-module belongs to $\operatorname{add}M$ and it is called a **generator–cogenerator** of $\operatorname{mod}\Lambda$ if it is both a generator and a cogenerator of $\operatorname{mod}\Lambda$.

Definition 1.4. Let Λ be a non-semisimple artin algebra. The **representation dimension** $\operatorname{rep.dim}\Lambda$ of Λ is the infimum of the global dimensions of the algebras $\operatorname{End}M$ where M is a generator–cogenerator of $\operatorname{mod}\Lambda$.

For the original definition of representation dimension and further details, we refer to [9]. The characterization given above as definition appears in the same paper.

Recall that a morphism $f: M \to N$ is said to be **right minimal** if any morphism g such that $fg = f$ is an isomorphism. Let $\mathcal{X}$ be an additive full subcategory of $\operatorname{mod} \Lambda$. A **right $\mathcal{X}$-approximation of** M is a morphism $f: X \to M$ with $X \in \mathcal{X}$ such that the sequence of functors ${}_\Lambda(-, X) \to {}_\Lambda(-, M) \to 0$ is exact in $\mathcal{X}$. A morphism f is a **minimal right $\mathcal{X}$-approximation of** M if it is a right $\mathcal{X}$-approximation of M and also a right minimal morphism.

Remark 1.5. An additive full subcategory $\mathcal{X}$ of $\operatorname{mod} \Lambda$ is contravariantly finite if and only if any module $M \in \operatorname{mod} \Lambda$ has a right $\mathcal{X}$-approximation.

Consider $\bar{X} \in \operatorname{mod} \Lambda$ and $\mathcal{X} = \operatorname{add} \bar{X}$. Let $f: X \to M$ be a right $\mathcal{X}$-approximation of M. By [11] (I.2.2) there exists a decomposition $X = X' \oplus X''$ such that $f|_{X'}: X' \to M$ is right minimal and $f|_{X''} = 0$. Moreover f factors through $f|_{X'}$, that is, there exists $l: X \to X'$ such that $f = f_{X'} \circ l$. Therefore $f_{X'}$ is also a right $\mathcal{X}$-approximation of M and so it is a minimal right $\mathcal{X}$-approximation of M.

Definition 1.6. Let Λ be an artin algebra and $\mathcal{X}$ be an additive full subcategory of $\operatorname{mod} \Lambda$. An **$\mathcal{X}$-approximation resolution of length** r of a module M is an exact sequence $0 \to X_r \to X_{r-1} \to \cdots \to X_1 \to M \to 0$ such that $X_i \in \mathcal{X}$ for each i, and the induced sequence of functors

$$0 \to {}_\Lambda(-, X_r) \to {}_\Lambda(-, X_{r-1}) \to \cdots \to {}_\Lambda(-, X_1) \to {}_\Lambda(-, M) \to 0$$

is exact in $\mathcal{X}$.

Note that if $(*)$ $0 \to X_r \overset{\varphi_r}{\to} X_{r-1} \to \cdots \to X_2 \overset{\varphi_2}{\to} X_1 \overset{\varphi_1}{\to} M \to 0$ is an $\mathcal{X}$-approximation resolution of M then φ_1 and each restriction $\varphi_i: X_i \to \operatorname{Ker} \varphi_{i-1}$ are right $\mathcal{X}$-approximations. We are using, by abuse of notation, the same notation for the morphism φ_i and for the restriction of φ_i over its image. If each of these morphisms is right minimal, we say that $(*)$ is a **minimal $\mathcal{X}$-approximation resolution**.

For a Λ-module $\bar{X}$, each module has a minimal right $\operatorname{add} \bar{X}$-approximation. Then we can construct a minimal $\operatorname{add} \bar{X}$-approximation resolution for each module in $\operatorname{mod} \Lambda$.

Lemma 1.7. *Let $\bar{X}$ and M be Λ-modules in* $\operatorname{mod} \Lambda$. *If there exists an* $\operatorname{add} \bar{X}$*-approximation resolution of length r of M then there exists a minimal* $\operatorname{add} \bar{X}$*-approximation resolution of length at most r of M.*

Proof. Let $0 \to X_r \overset{\varphi_r}{\to} X_{r-1} \to \cdots \to X_2 \overset{\varphi_2}{\to} X_1 \overset{\varphi_1}{\to} M \to 0$ be an $\operatorname{add} \bar{X}$-approximation resolution of length r of M. We can construct an exact sequence $0 \to K \to X'_{r+1} \overset{\psi_{r+1}}{\to} X'_r \overset{\psi_r}{\to} X'_{r-1} \to \cdots \to X'_2 \overset{\psi_2}{\to} X'_1 \overset{\psi_1}{\to} M \to 0$ where each ψ_i (over its image) is a minimal right $\operatorname{add} \bar{X}$-approximation. Then, for $i \in \{1, 2, \cdots, r+1\}$, there exist $f_i: X_i \to X_i'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, "f_{r+1}"] & X_r \arrow[r, "\varphi_r"] \arrow[d, "f_r"] & \cdots \arrow[r] & X_2 \arrow[r, "\varphi_2"] \arrow[d, "f_2"] & X_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & X'_{r+1} \arrow[r, "\psi_{r+1}"] & X'_r \arrow[r, "\psi_r"] & \cdots \arrow[r] & X'_2 \arrow[r, "\psi_2"] & X'_1 \arrow[r, "\psi_1"] & M \arrow[r] & 0 \end{tikzcd}$$

is commutative. By minimality of each ψ_i we have that each f_i is a retraction and, in particular, we have $X'_{r+1} = 0$. This completes the proof. □

Remark 1.8. It follows from this lemma that, if there exists an $\operatorname{add} \bar{X}$-approximation resolution of length r of M, then we can assume that it is minimal.

Definition 1.9. A Λ-module $\bar{X}$ is said to have the **r-approximation property** if each indecomposable Λ-module has an $\operatorname{add} \bar{X}$-approximation resolution of length at most r.

Theorem 1.10. *(See [14,15,26].) For an artin algebra Λ, $\operatorname{rep.dim} \Lambda \leq r+1$ if and only if there exists a generator–cogenerator of $\operatorname{mod} \Lambda$ satisfying the r-approximation property.* □

Auslander proved in [9] that Λ is representation-finite if and only if $\operatorname{rep.dim} \Lambda \leq 2$. Thus, if Λ is representation-infinite, then $\operatorname{rep.dim} \Lambda \geq 3$.

An important class of algebras which has representation dimension at most 3 is the class of tilted algebras as demonstrated in [7]. There, it is proved that if T is a convex tilting module of a tilted algebra Λ then $\Lambda \oplus D\Lambda \oplus T$ is a generator–cogenerator having the 2-approximation property. Here, we use some arguments from this paper in Lemma 2.2 below.

Many other classes of algebras have been shown to have representation dimension at most 3, see, for instance [7,9,14,15,22,26].

2. Proof of the first theorem

The next trivial corollary of Lemma 1.1 will be useful in the sequel.

Corollary 2.1. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection of $\operatorname{ind} \Lambda$ such that $\mathcal{B}$ is finite. Then $\operatorname{add} \mathcal{C}$ is covariantly finite if and only if $\operatorname{add} \mathcal{A}$ is contravariantly finite.*

Proof. This follows immediately from Lemma 1.1 and the finiteness of $\mathcal{B}$. □

Let $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ be a full subcategory of $\operatorname{ind} \Lambda$ closed under successors such that $\operatorname{add} \mathcal{C}$ is covariantly finite. Denote by F the direct sum of all indecomposable Ext-projectives in $\operatorname{add} \mathcal{C}$ and by N the direct sum of all indecomposable injective Λ-modules lying in $\mathcal{C}$.

Lemma 2.2. *Let $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ be a full subcategory of* $\operatorname{ind}\Lambda$ *closed under successors such that* $\operatorname{add}\mathcal{C}$ *is covariantly finite. For each $M \in \mathcal{C}$, there exists a short exact sequence $0 \to F_2 \to F_1 \oplus I_1 \to M \to 0$ with $I_1 \in \operatorname{add} N$ and $F_1, F_2 \in \operatorname{add} F$ that is an* $\operatorname{add}(F \oplus N)$*-approximation resolution of length* 2 *of M.*

Proof. Since $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ by Lemma 1.3 we have $\mathcal{C} \subseteq \mathcal{R}_{(\Lambda_\mathcal{C})}$. And since $\operatorname{add}\mathcal{C}$ contains all the injective $\Lambda_\mathcal{C}$-modules it follows from Lemma 1.2 that $\Lambda_\mathcal{C}$ is a tilted algebra and F is a convex tilting $\Lambda_\mathcal{C}$-module.

Since $\operatorname{add}\mathcal{C} = \operatorname{Gen} F \subseteq \operatorname{Gen}(F \oplus N)$, then by [7] (1.4), for any $M \in \mathcal{C}$ there exists an exact sequence $0 \to K \to F_1 \oplus I_1 \to M \to 0$ with $F_1 \in \operatorname{add} F$, $I_1 \in \operatorname{add} N$ such that the short sequence

$$0 \to {}_\Lambda(-, K) \to {}_\Lambda(-, F_1 \oplus I_1) \to {}_\Lambda(-, M) \to 0$$

is exact in $\operatorname{add}(F \oplus N)$. Now by [7] (2.2) (f), we have $K \in \operatorname{add} F$ and therefore we have an $\operatorname{add}(F \oplus N)$-approximation resolution of length 2 of M. □

Lemma 2.3. *Let Λ be an artin algebra and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ a trisection of* $\operatorname{ind}\Lambda$ *with $\mathcal{B}$ finite, $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ and assume* $\operatorname{add}\mathcal{C}$ *is covariantly finite. Then,*

$$\operatorname{rep.dim}\Lambda \leq \max\{3, \operatorname{rep.dim}{}_\mathcal{A}\Lambda\}.$$

Proof. Denote ${}_\mathcal{A}\Lambda = A$ and suppose that $\operatorname{rep.dim} A = r + 1$. Let $\bar{X}$ be a generator–cogenerator of $\operatorname{mod} A$ which has the r-approximation property in $\operatorname{ind} A$. Consider the following modules:

- $\bar{X}'$ the direct sum of all indecomposable summands of $\bar{X}$ that lie in $\mathcal{A}$;
- Z the direct sum of all indecomposable Λ-modules lying in $\mathcal{B}$;
- F the direct sum of all indecomposable Ext-projectives in $\operatorname{add}\mathcal{C}$; and
- N the direct sum of all indecomposable injective Λ-modules lying in $\mathcal{C}$.

We will prove that $\bar{M} = \bar{X}' \oplus Z \oplus F \oplus N$ is a generator–cogenerator of $\operatorname{mod}\Lambda$ and that it has the $\max\{2, r\}$-approximation property in $\operatorname{ind}\Lambda$.

Let $P \in \operatorname{ind}\Lambda$ be a projective Λ-module. If P lies in $\mathcal{A}$ then P is a projective A-module and so it is a summand of $\bar{X}'$. If P lies in $\mathcal{B}$ then it is a summand of Z. And, if P lies in $\mathcal{C}$ then P is an Ext-projective in $\operatorname{add}\mathcal{C}$ and so a summand of F. Thus $\bar{M}$ is a generator of $\operatorname{mod}\Lambda$.

Let $I \in \operatorname{ind}\Lambda$ be an injective Λ-module. If I lies in $\mathcal{A}$ then I is an injective A-module and so it is a summand of $\bar{X}'$. If I lies in $\mathcal{B}$, then it is a summand of Z. And if I lies in $\mathcal{C}$, then it is a summand of N. Thus $\bar{M}$ is a cogenerator of $\operatorname{mod}\Lambda$.

In order to prove that $\bar{M}$ has the $\max\{2, r\}$-approximation property in $\operatorname{ind}\Lambda$, consider $M \in \operatorname{ind}\Lambda$. If $M \in \operatorname{add}\bar{M}$, there is nothing to do, then we can assume that $M \notin \operatorname{add}\bar{M}$ and, in this case, $M \in \mathcal{A} \cup \mathcal{C}$.

If $M \in \mathcal{A}$, then M is an A-module. Let

$$(1) \quad 0 \to X_r \xrightarrow{\varphi_r} X_{r-1} \to \cdots \to X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} M \to 0$$

be an add $\bar{X}$-approximation resolution of length r in mod A. Then, since $\mathrm{Hom}_\Lambda(L, N) = \mathrm{Hom}_A(L, N)$ for any $L, N \in \mathcal{A}$, the sequence of functors

$$(2) \quad 0 \to {}_\Lambda(-, X_r) \to {}_\Lambda(-, X_{r-1}) \to \cdots \to {}_\Lambda(-, X_1) \to {}_\Lambda(-, M) \to 0$$

is exact in add $\bar{X}$. Since add $\bar{X}' \subseteq$ add $\bar{X}$, it follows that (2) is exact in add $\bar{X}'$. The sequence (2) is zero in add$(Z \oplus F \oplus N)$ because all the indecomposable summands of $Z \oplus F \oplus N$ are in $\mathcal{B} \cup \mathcal{C}$ and $\mathcal{A}$ is closed under predecessors. This proves that (2) is exact in add $\bar{M}$ and therefore (1) is an add $\bar{M}$-approximation resolution of length r of M.

If $M \in \mathcal{C}$, then, by Lemma 2.2, there exists an add$(F \oplus N)$-approximation resolution of length 2 of M:

$$(3) \quad 0 \to F_2 \to F_1 \oplus I_1 \to M \to 0$$

with $F_1, F_2 \in$ add $F \subseteq$ add $\bar{M}$ and $I_1 \in$ add $N \subseteq$ add $\bar{M}$. Let $L \in$ ind Λ be a summand of $\bar{X}' \oplus Z$. If L is a projective Λ-module, then $0 \to {}_\Lambda(L, F_2) \to {}_\Lambda(L, F_1 \oplus I_1) \to {}_\Lambda(L, M) \to 0$ is exact. If L is not projective, then $\tau_\Lambda L \notin \mathcal{C}$, because $\mathcal{C}$ is closed under successors and $L \notin \mathcal{C}$, while $F_2 \in$ add $\mathcal{C}$ so we have

$$\mathrm{Ext}^1_\Lambda(L, F_2) \cong D\,\overline{\mathrm{Hom}}_\Lambda(F_2, \tau_\Lambda L) = 0.$$

Therefore, we have that the short sequence

$$0 \to {}_\Lambda(-, F_2) \to {}_\Lambda(-, F_1 \oplus I_1) \to {}_\Lambda(-, M) \to 0$$

is exact in add$(\bar{X}' \oplus Z)$ and so (3) is an add $\bar{M}$-approximation resolution of length 2 of M. This proves that rep.dim $\Lambda \leq \max\{3, r+1\}$ and completes the proof. $\square$

Lemma 2.4. *Let $\mathcal{A}$ be a convex full subcategory of* ind Λ *and* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* add $\mathcal{A}$. *If K is an Ext-injective in* add $\mathcal{A}$ *which is a summand of X, then it is isomorphic to a summand of Y.*

Proof. Let $p: X \to K$ and $i: K \to X$ be the natural morphisms such that $p \circ i = 1_K$. There exists a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \\ 0 \arrow[r] & K \arrow[r] & Q \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

where Q is the pushout of f and p. By convexity, Q is in $\operatorname{add}\mathcal{A}$. Since K is Ext-injective in $\operatorname{add}\mathcal{A}$ then the exact sequence $0 \to K \to Q \to Z \to 0$ splits. So we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "g"] \arrow[d, "\binom{h_1}{h_2}"] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \\
0 \arrow[r] & K \arrow[r, "\binom{1}{0}"'] & K \oplus Z \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

By the commutativity we have $\binom{h_1}{h_2} f = \binom{1}{0} p$, that is $h_1 \circ f = p$. So $h_1 \circ f \circ i = p \circ i = 1_K$. This proves that $h_1 : Y \to K$ is a retraction and therefore K is isomorphic to a summand of Y. $\square$

Lemma 2.5. *Let $\mathcal{A}$ be a full subcategory of* $\operatorname{ind}\Lambda$ *closed under predecessors and* $\bar{X} \in \operatorname{mod}\Lambda$. *If $f : X \to M$ is a minimal right* $\operatorname{add}\bar{X}$*-approximation of M, then* $\operatorname{Ker} f$ *has no Ext-injective direct summand in* $\operatorname{add}\mathcal{A}$.

Proof. Let K be a direct summand of $\operatorname{Ker} f$ which is Ext-injective in $\operatorname{add}\mathcal{A}$. By the last lemma, K is also a direct summand of X. But $f(K) = 0$ and this is a contradiction with the minimality of f by [11] (I.2.3). $\square$

Theorem 2.6. *Let Λ be a representation-infinite artin algebra and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a trisection of* $\operatorname{ind}\Lambda$ *with $\mathcal{B}$ finite.*

(a) *If $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ and* $\operatorname{add}\mathcal{C}$ *is covariantly finite, then*

$$\operatorname{rep.dim}\Lambda = \max\{3, \operatorname{rep.dim}{}_{\mathcal{A}}\Lambda\}.$$

(b) *If $\mathcal{A} \subseteq \mathcal{L}_\Lambda$ and* $\operatorname{add}\mathcal{A}$ *is contravariantly finite, then*

$$\operatorname{rep.dim}\Lambda = \max\{3, \operatorname{rep.dim}\Lambda_{\mathcal{C}}\}.$$

Proof. We will only prove part (a) because (b) is dual.

By Lemma 2.3, we have $\operatorname{rep.dim}\Lambda \leq \max\{3, \operatorname{rep.dim}{}_{\mathcal{A}}\Lambda\}$.

On the other hand, suppose $\operatorname{rep.dim}\Lambda = s + 1$. Note that $s \geq 2$ because Λ is a representation-infinite algebra. Let $\bar{M}$ be a generator–cogenerator of $\operatorname{mod}\Lambda$ which has the s-approximation property in $\operatorname{ind}\Lambda$. Denote $A = {}_{\mathcal{A}}\Lambda$, $\mathcal{B}' = \mathcal{B} \cap \operatorname{ind} A$ and $\mathcal{C}' = \mathcal{C} \cap \operatorname{ind} A$. Then $(\mathcal{A}, \mathcal{B}', \mathcal{C}')$ is clearly a trisection of $\operatorname{ind} A$ with $\mathcal{B}'$ finite. Consider the following modules:

– $\bar{M}'$ the direct sum of all indecomposable summands of $\bar{M}$ that lie in $\mathcal{A}$;
– E the direct sum of all indecomposable Ext-injectives in add $\mathcal{A}$ which are not injective in mod Λ;
– Z the direct sum of all indecomposable A-modules lying in $\mathcal{B}'$;
– F the direct sum of all indecomposable Ext-projectives in add $\mathcal{C}'$; and
– N the direct sum of all indecomposable injective A-modules lying in $\mathcal{C}'$.

We will prove that the A-module $\bar{X} = \bar{M}' \oplus E \oplus Z \oplus F \oplus N$ is a generator–cogenerator of mod A and that it has the s-approximation property in ind A.

If $P \in \operatorname{ind} \Lambda$ is a projective A-module, then P is a projective Λ-module lying in $\mathcal{A}$ and so P is a summand of $\bar{M}'$. Let $I \in \operatorname{ind} A$ be an injective A-module. If I lies in $\mathcal{A}$ then I is an Ext-injective in add $\mathcal{A}$ and so I is a summand of $\bar{M}'$ if it is injective in mod Λ, or I is a summand of E if it is not injective. If I lies in $\mathcal{B}'$ then it is a summand of Z and if I lies in $\mathcal{C}'$ then it is a summand of N. Therefore $\bar{X}$ is a generator–cogenerator of mod A.

To prove that $\bar{X}$ has the s-approximation property consider $M \in \operatorname{ind} A$ such that $M \notin \operatorname{add} \bar{X}$. Then $M \in \mathcal{A} \cup \mathcal{C}'$.

By Corollary 2.1, since add $\mathcal{C}$ is covariantly finite in mod Λ, then add $\mathcal{A}$ is contravariantly finite in mod Λ and hence it is contravariantly finite in mod A. Now, since $(\mathcal{A}, \mathcal{B}', \mathcal{C}')$ is a trisection of ind A with $\mathcal{B}'$ finite, then add $\mathcal{C}'$ is covariantly finite in mod A. Note that $\mathcal{C}'$ is closed under successors in ind A and, by Lemma 1.3, we have $\mathcal{C}' \subseteq \mathcal{R}_A$. Therefore, if $M \in \mathcal{C}'$, by Lemma 2.2, there is an exact sequence in mod A

$$0 \to F_2 \to F_1 \oplus I_1 \to M \to 0$$

with $F_1, F_2 \in \operatorname{add} F \subseteq \operatorname{add} \bar{X}$ and $I_1 \in \operatorname{add} N \subseteq \operatorname{add} \bar{X}$ such that the short sequence

$$0 \to {}_A(-, F_2) \to {}_A(-, F_1 \oplus I_1) \to {}_A(-, M) \to 0$$

is exact in add$(F \oplus N)$.

Let $L \in \operatorname{ind} A$ be a summand of $\bar{M}' \oplus Z \oplus E$ then $L \notin \mathcal{C}'$. If L is a projective A-module, then

$$0 \to {}_A(L, F_2) \to {}_A(L, F_1 \oplus I_1) \to {}_A(L, M) \to 0$$

is exact. If L is not A-projective, then $\tau_A L \notin \mathcal{C}'$ because $\mathcal{C}'$ is closed under successors while $F_2 \in \operatorname{add} \mathcal{C}'$ so we have

$$\operatorname{Ext}^1_A(L, F_2) \cong D\, \overline{\operatorname{Hom}}_A(F_2, \tau_A L) = 0.$$

Therefore, the short sequence

$$0 \to {}_A(-, F_2) \to {}_A(-, F_1 \oplus I_1) \to {}_A(-, M) \to 0$$

is exact in $\operatorname{add}(\bar{M}' \oplus Z \oplus E)$ and so $0 \to F_2 \to F_1 \oplus I_1 \to M \to 0$ is an $\operatorname{add} \bar{X}$-approximation resolution of length 2 of M.

If $M \in \mathcal{A}$ consider an $\operatorname{add} \bar{M}$-approximation resolution of M:

$$(1) \quad 0 \to M_s \overset{\varphi_s}{\to} M_{s-1} \to \cdots \to M_2 \overset{\varphi_2}{\to} M_1 \overset{\varphi_1}{\to} M \to 0.$$

Since $M \in \mathcal{A}$ and $\mathcal{A}$ is closed under predecessors each $M_i \in \operatorname{add} \mathcal{A}$ and so each $M_i \in \operatorname{add} \bar{M}' \subseteq \operatorname{add} \bar{X}$. Since $\operatorname{add} \bar{M}' \subseteq \operatorname{add} \bar{M}$ and $\operatorname{ind} A$ is a full subcategory of $\operatorname{ind} \Lambda$ then the induced sequence

$$(2) \quad 0 \to {}_{\Lambda}(-, M_s) \to {}_{\Lambda}(-, M_{s-1}) \to \cdots \to {}_{\Lambda}(-, M_1) \to {}_{\Lambda}(-, M) \to 0$$

is exact in $\operatorname{add} \bar{M}'$.

The sequence (2) is zero in $\operatorname{add}(Z \oplus F \oplus N)$ because $\mathcal{A}$ is closed under predecessors and $Z \oplus F \oplus N \in \operatorname{add}(\mathcal{B} \cup \mathcal{C})$.

Finally let $L \in \operatorname{add} E$ be an indecomposable module and denote $K_i = \operatorname{Ker} \varphi_i$ for $i \in \{1, ..., s-1\}$. Since L is Ext-injective in $\operatorname{add} \mathcal{A}$ and not injective then $\tau_{\Lambda}^{-1} L \notin \mathcal{A}$ and since M is not Ext-injective (because $M \notin \operatorname{add} \bar{X}$) then $\tau_{\Lambda}^{-1} M \in \mathcal{A}$. Therefore $\operatorname{Hom}_{\Lambda}(\tau_{\Lambda}^{-1} L, \tau_{\Lambda}^{-1} M) = 0$ and so $\overline{\operatorname{Hom}}_{\Lambda}(L, M) = 0$. If $f: L \to M$ is a morphism, then there exist an injective Λ-module I and morphisms $f_1: L \to I$, $f_2: I \to M$ such that $f = f_2 \circ f_1$. Since I is a summand of $\bar{M}$ then ${}_{\Lambda}(I, M_1) \to {}_{\Lambda}(I, M) \to 0$ is exact, so there is a morphism $g: I \to M_1$ such that $\varphi_1 \circ g = f_2$, that is $f = \varphi_1 \circ (g \circ f_1) = \operatorname{Hom}_{\Lambda}(L, \varphi_1)(g \circ f_1)$ and therefore $0 \to {}_{\Lambda}(L, K_1) \to {}_{\Lambda}(L, M_1) \to {}_{\Lambda}(L, M) \to 0$ is exact. Because of Remark 1.8 and Lemma 2.5, we can assume that each $K_i \in \operatorname{add} \mathcal{A}$ (for $i \in \{1, ..., s-1\}$) has no Ext-injective summand, so the same argument is valid replacing M by K_i. Therefore for each $i \in \{1, ..., s-1\}$ the sequence $0 \to {}_{\Lambda}(L, K_{i+1}) \to {}_{\Lambda}(L, M_{i+1}) \to {}_{\Lambda}(L, K_i) \to 0$ is exact. This proves that the sequence (2) is exact in $\operatorname{add} E$ and so (1) is an $\operatorname{add} \bar{X}$-approximation resolution of length s of M. This proves that $\operatorname{rep.dim} A \leq s + 1 = \operatorname{rep.dim} \Lambda$ and completes the proof of the theorem. □

Corollary 2.7. *Let Λ be a representation-infinite algebra.*

(a) *If $\mathcal{A}$ is a cofinite full subcategory of* $\operatorname{ind} \Lambda$ *closed under predecessors, then* $\operatorname{rep.dim} \Lambda = \operatorname{rep.dim} {}_{\mathcal{A}}\Lambda$.
(b) *If $\mathcal{C}$ is a cofinite full subcategory of* $\operatorname{ind} \Lambda$ *closed under successors, then* $\operatorname{rep.dim} \Lambda = \operatorname{rep.dim} \Lambda_{\mathcal{C}}$.

Proof. For (a) just take $\mathcal{B} = \mathcal{A}^c$ and $\mathcal{C} = \emptyset$. Since $\mathcal{A}$ is cofinite and $\operatorname{ind} \Lambda$ is infinite then $\mathcal{A} \subseteq \operatorname{ind}_{\mathcal{A}} \Lambda$ is infinite. Therefore $\operatorname{rep.dim} {}_{\mathcal{A}}\Lambda \geq 3$.

The item (b) is dual. □

Example 2.8. Let k be a field and Λ be the k-algebra given by the quiver

$$\begin{tikzcd}
& & & 4 \arrow[dl, "\gamma"'] & 5 \arrow[l, "\epsilon"'] & & & 8 \arrow[dl] \\
1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] & & & 6 \arrow[ul] \arrow[lll, "\delta"] & 7 \arrow[l, "\lambda"] & 9 \arrow[l] \\
& & & & & & & 10 \arrow[ul]
\end{tikzcd}$$

bound by the relations $\beta_i\alpha = \gamma\beta_i = \delta\beta_i = \epsilon\gamma = \lambda\delta = 0$, for $i = 1, 2$.

The algebra Λ is representation-infinite and so $\operatorname{rep.dim}\Lambda \geq 3$. The right part $\mathcal{R}_\Lambda$ consists of all the successors of $\tau^{-1}P_4$ and $\operatorname{add}\mathcal{R}_\Lambda$ is covariantly finite. The left part is $\mathcal{L}_\Lambda = \{P_1, P_2, S_2, P_3\}$. Its support algebra ${}_{(\mathcal{L}_\Lambda)}\Lambda$ is given by the objects 1, 2 and 3, that is, ${}_{(\mathcal{L}_\Lambda)}\Lambda$ is a tilted algebra that has the quiver $1 \xleftarrow{\alpha} 2 \overset{\beta_1}{\underset{\beta_2}{\leftleftarrows}} 3$ bound by $\beta_i\alpha = 0$ with $i = 1, 2$. Denote $\mathcal{A} = \operatorname{ind}{}_{(\mathcal{L}_\Lambda)}\Lambda$ which consists of all predecessors of S_3 (and so it is infinite). In this case, it is easy to see that $(\mathcal{A}, (\mathcal{A} \cup \mathcal{R}_\Lambda)^c, \mathcal{R}_\Lambda)$ is a trisection of $\operatorname{ind}\Lambda$ and $(\mathcal{A} \cup \mathcal{R}_\Lambda)^c$ is finite. By Theorem 2.6, $\operatorname{rep.dim}\Lambda = \operatorname{rep.dim}{}_{\mathcal{A}}\Lambda$. But ${}_{\mathcal{A}}\Lambda = {}_{(\mathcal{L}_\Lambda)}\Lambda$ and so $\operatorname{rep.dim}\Lambda = 3$.

There are other ways to see that the algebra in this example has representation dimension 3, for example using [27] (4.7). But the latter result cannot be used in the following Example 2.9 because it involves an algebra with representation dimension 4.

Example 2.9. Let k be a field and Λ be the k-algebra given by the quiver

$$\begin{tikzcd}
& & 3 \arrow[dl, "\beta"'] & & & \\
1 & 2 \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] & & 4 \arrow[ul, "\gamma"'] \arrow[ll, "\delta"] & 5 \arrow[l, shift right=2, "x_1"'] \arrow[l] \arrow[l, shift left=2, "x_2"] \arrow[l, shift left=2, phantom, "x_3"'] & 6 \arrow[l, shift right=2, "x_1"'] \arrow[l] \arrow[l, shift left=2, "x_2"] \arrow[l, shift left=2, phantom, "x_3"']
\end{tikzcd}$$

bound by the relations $\beta\alpha_l = \delta\alpha_l = \gamma\beta = x_i\gamma = x_i\delta = 0$ and $x_ix_j = x_jx_i$ for $l = 1, 2$ and for $1 \leq i, j \leq 3$.

Denote by $\mathcal{A}$ the subcategory of $\operatorname{ind}\Lambda$ which consists of all predecessors of S_2 and by $\mathcal{C}$ the subcategory of $\operatorname{ind}\Lambda$ which consists of all successors of S_4. Then $\mathcal{A} \subseteq \mathcal{L}_\Lambda$ and $\operatorname{add}\mathcal{A}$ is contravariantly finite, $(\mathcal{A}, (\mathcal{A} \cup \mathcal{C})^c, \mathcal{C})$ is a trisection of $\operatorname{ind}\Lambda$ and $(\mathcal{A} \cup \mathcal{C})^c$ is finite. By Theorem 2.6, $\operatorname{rep.dim}\Lambda = \operatorname{rep.dim}\Lambda_{\mathcal{C}}$. But $\Lambda_{\mathcal{C}}$ is the algebra given by the objects 4, 5 and 6, that is, $\Lambda_{\mathcal{C}}$ is the algebra given by

$$\begin{tikzcd}
4 & 5 \arrow[l, shift right=2, "x_1"'] \arrow[l] \arrow[l, shift left=2, "x_2"] \arrow[l, shift left=2, phantom, "x_3"'] & 6 \arrow[l, shift right=2, "x_1"'] \arrow[l] \arrow[l, shift left=2, "x_2"] \arrow[l, shift left=2, phantom, "x_3"']
\end{tikzcd}$$

bound by the relations $x_i x_j = x_j x_i$ for $1 \leq i, j \leq 3$ and by [21], Examples 7.3 and A.8, this algebra has representation dimension 4. Therefore $\operatorname{rep.dim} \Lambda = 4$.

3. The left and right parts and representation dimension

As a direct consequence of Theorem 2.6, we have the next corollary.

Corollary 3.1. *Let Λ be a representation-infinite artin algebra.*

(a) *If $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ is a full subcategory of* $\operatorname{ind} \Lambda$ *closed under successors such that* $\operatorname{add} \mathcal{C}$ *is covariantly finite, then*

$$\operatorname{rep.dim} \Lambda = \max\bigl\{3, \operatorname{rep.dim} {}_{(\mathcal{C}^c)}\Lambda\bigr\}.$$

(b) *If $\mathcal{A} \subseteq \mathcal{L}_\Lambda$ is a full subcategory of* $\operatorname{ind} \Lambda$ *closed under predecessors such that* $\operatorname{add} \mathcal{A}$ *is contravariantly finite, then*

$$\operatorname{rep.dim} \Lambda = \max\{3, \operatorname{rep.dim} \Lambda_{(\mathcal{A}^c)}\}.$$

Proof. For (a) just take $\mathcal{A} = \mathcal{C}^c$ and $\mathcal{B} = \emptyset$. The item (b) is dual. □

From this, we can prove a stronger result that does not require that the subcategory is homologically finite.

Proposition 3.2. *Let Λ be a representation-infinite artin algebra.*

(a) *If $\mathcal{C} \subseteq \mathcal{R}_\Lambda$ is a subcategory closed under successors, then*

$$\operatorname{rep.dim} \Lambda = \max\bigl\{3, \operatorname{rep.dim} {}_{(\mathcal{C}^c)}\Lambda\bigr\}.$$

(b) *If $\mathcal{A} \subseteq \mathcal{L}_\Lambda$ is a subcategory closed under predecessors, then*

$$\operatorname{rep.dim} \Lambda = \max\{3, \operatorname{rep.dim} \Lambda_{(\mathcal{A}^c)}\}.$$

Proof. If each projective indecomposable Λ-module lies in $\mathcal{C}^c$ then $\Lambda = {}_{(\mathcal{C}^c)}\Lambda$. Otherwise, let $\mathcal{D} = \operatorname{Succ} Y$ where Y is the sum of all projective indecomposable Λ-modules lying in $\mathcal{C}$. Then $\mathcal{D}$ is a full subcategory of $\mathcal{R}_\Lambda$ closed under successors. Denote by F the sum of all the Ext-projective indecomposable modules in $\operatorname{add} \mathcal{D}$. Since Y is Ext-projective in $\operatorname{add} \mathcal{D}$ we have that $\mathcal{D} = \operatorname{Succ} Y \subseteq \operatorname{Succ} F$. But $F \in \operatorname{add} \mathcal{D}$ and so $\operatorname{Succ} F \subseteq \mathcal{D}$, because $\mathcal{D}$ is closed under successors. Therefore $\mathcal{D} = \operatorname{Succ} F$. By [2] (8.2), we have that $\operatorname{add} \mathcal{D}$ is covariantly finite. By Corollary 3.1, it follows that $\operatorname{rep.dim} \Lambda = \max\{3, \operatorname{rep.dim} {}_{(\mathcal{D}^c)}\Lambda\}$. Finally, for a projective indecomposable Λ-module P, we have $P \in \mathcal{C}$ if and only if $P \in \mathcal{D}$, so $P \notin \mathcal{C}$ if and only if $P \notin \mathcal{D}$ and then ${}_{(\mathcal{C}^c)}\Lambda = {}_{(\mathcal{D}^c)}\Lambda$ and this completes the proof of (a). The item (b) is dual. $\square$

Corollary 3.3. *If Λ is a representation-infinite artin algebra, then*

$$\operatorname{rep.dim}\Lambda = \max\bigl\{3, \operatorname{rep.dim}{}_{(\mathcal{R}_\Lambda)^c}\Lambda\bigr\} = \max\{3, \operatorname{rep.dim}\Lambda_{(\mathcal{L}_\Lambda)^c}\}. \qquad \square$$

Applying the last corollary to the algebra $B = {}_{(\mathcal{R}_\Lambda)^c}\Lambda$ we conclude that

$$\operatorname{rep.dim}\Lambda = \max\{3, \operatorname{rep.dim} B_{(\mathcal{L}_B)^c}\}$$

and so the representation dimension of Λ depends just on an algebra that is a subcategory of ${}_{(\mathcal{R}_\Lambda)^c}\Lambda$ and of $\Lambda_{(\mathcal{L}_\Lambda)^c}$.

4. Proof of the second theorem

Now, even when $\mathcal{C}$ is not necessarily in $\mathcal{R}_\Lambda$ and $\mathcal{A}$ is not necessarily in $\mathcal{L}_\Lambda$ we can still find a relation between the representation dimension of Λ and the representation dimensions of ${}_{\mathcal{A}}\Lambda$ and of $\Lambda_{\mathcal{C}}$. For this, however, we need to suppose that $\operatorname{ind}\Lambda_{\mathcal{C}}$ is closed under successors or $\operatorname{ind}_{\mathcal{A}}\Lambda$ is closed under predecessors. To illustrate this hypothesis, we show an example.

Example 4.1. In Example 2.8 we have that $\operatorname{ind}{}_{(\mathcal{L}_\Lambda)}\Lambda$ consists of all predecessors of S_3, that is $\operatorname{ind}{}_{(\mathcal{L}_\Lambda)}\Lambda = \operatorname{Pred} S_3$ and so it is closed under predecessors. In the same example we have that $\Lambda_{(\mathcal{R}_\Lambda)}$ is the hereditary algebra

$$\begin{tikzcd}
 & & & & 8 \arrow[dl] \\
4 & 5 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] & 9 \arrow[l] \\
 & & & & 10 \arrow[ul]
\end{tikzcd}$$

and the module $\begin{smallmatrix} & 6 & \\ 5 & & 3 \end{smallmatrix} \notin \operatorname{ind}\Lambda_{(\mathcal{R}_\Lambda)}$ is a successor of $S_5 \in \operatorname{ind}\Lambda_{(\mathcal{R}_\Lambda)}$ so it is not closed under successors.

Theorem 4.2. *Let Λ be an artin algebra with a trisection $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\operatorname{ind}\Lambda$. If*

(a) $(\mathcal{A} \cup \operatorname{ind}\Lambda_{\mathcal{C}})^c$ *is finite and* $\operatorname{ind}\Lambda_{\mathcal{C}}$ *is closed under successors*

or

(b) $(\operatorname{ind}_{\mathcal{A}} \Lambda \cup \mathcal{C})^c$ *is finite and* $\operatorname{ind}_{\mathcal{A}} \Lambda$ *is closed under predecessors, then,*

$$\operatorname{rep.dim} \Lambda \leq \max\{\operatorname{rep.dim} {}_{\mathcal{A}}\Lambda, \operatorname{rep.dim} \Lambda_{\mathcal{C}}\}.$$

Proof. Suppose that $\operatorname{rep.dim} {}_{\mathcal{A}}\Lambda = r + 1$ and $\operatorname{rep.dim} \Lambda_{\mathcal{C}} = s + 1$. Let $\bar{Y}$ be a generator–cogenerator of $\operatorname{mod}_{\mathcal{A}} \Lambda$ which has the r-approximation property in $\operatorname{ind}_{\mathcal{A}} \Lambda$ and let $\bar{X}$ be a generator–cogenerator of $\operatorname{mod} \Lambda_{\mathcal{C}}$ which has the s-approximation property in $\operatorname{ind} \Lambda_{\mathcal{C}}$. Suppose (a) and consider the following modules:

- $\bar{Y}'$ the direct sum of all indecomposable summands of $\bar{Y}$ that lie in $\mathcal{A}$ but not in $\operatorname{ind} \Lambda_{\mathcal{C}}$, and
- Z the direct sum of the all indecomposable modules lying in $(\mathcal{A} \cup \operatorname{ind} \Lambda_{\mathcal{C}})^c$.

We will prove that $\bar{M} = \bar{Y}' \oplus Z \oplus \bar{X}$ is a generator–cogenerator of $\operatorname{mod} \Lambda$ and it has the $\max\{r, s\}$-approximation property in $\operatorname{ind} \Lambda$.

Let $P \in \operatorname{ind} \Lambda$ be a projective Λ-module. If P lies in $\mathcal{A} \setminus \operatorname{ind} \Lambda_{\mathcal{C}}$, then P is a projective ${}_{\mathcal{A}}\Lambda$-module and it is a summand of $\bar{Y}'$. If P lies in $(\mathcal{A} \cup \operatorname{ind} \Lambda_{\mathcal{C}})^c$, then it is a summand of Z. And if P lies in $\operatorname{ind} \Lambda_{\mathcal{C}}$, then P is a projective $\Lambda_{\mathcal{C}}$-module and so it is a summand of $\bar{X}$. Let $I \in \operatorname{ind} \Lambda$ be an injective Λ-module. If I lies in $\mathcal{A} \setminus \operatorname{ind} \Lambda_{\mathcal{C}}$, then I is an injective ${}_{\mathcal{A}}\Lambda$-module and it is a summand of $\bar{Y}'$. If I lies in $(\mathcal{A} \cup \operatorname{ind} \Lambda_{\mathcal{C}})^c$, then it is a summand of Z. And if I lies in $\operatorname{ind} \Lambda_{\mathcal{C}}$, then I is an injective $\Lambda_{\mathcal{C}}$-module and so it is a summand of $\bar{X}$. Therefore $\bar{M}$ is a generator–cogenerator of $\operatorname{mod} \Lambda$.

Consider $M \in \operatorname{ind} \Lambda$ such that $M \notin \operatorname{add} \bar{M}$. Then $M \in \mathcal{A} \cup \operatorname{ind} \Lambda_{\mathcal{C}}$.

Suppose $M \in \mathcal{A} \subseteq \operatorname{ind}_{\mathcal{A}} \Lambda$ such that $M \notin \operatorname{ind} \Lambda_{\mathcal{C}}$. There is an $\operatorname{add} \bar{Y}$-approximation resolution of length r in $\operatorname{mod}_{\mathcal{A}} \Lambda$:

$$(1) \quad 0 \to Y_r \xrightarrow{\varphi_r} Y_{r-1} \to \cdots \to Y_2 \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} M \to 0.$$

Since $\mathcal{A}$ is closed under predecessors and $\operatorname{ind} \Lambda_{\mathcal{C}}$ is closed under successors then any Y_i belongs to $\operatorname{add} \bar{Y}' \subseteq \operatorname{add} \bar{M}$. Since $\operatorname{mod} {}_{\mathcal{A}}\Lambda$ is a full subcategory of $\operatorname{mod} \Lambda$, the induced sequence

$$(2) \quad 0 \to {}_{\Lambda}(-, Y_r) \to {}_{\Lambda}(-, Y_{r-1}) \to \cdots \to {}_{\Lambda}(-, Y_1) \to {}_{\Lambda}(-, M) \to 0$$

is exact in $\operatorname{add} \bar{Y}'$.

Since $\mathcal{A} \setminus \operatorname{ind} \Lambda_{\mathcal{C}}$ is closed under predecessors and each indecomposable summand of $Z \oplus \bar{X}$ is not in $\mathcal{A} \setminus \operatorname{ind} \Lambda_{\mathcal{C}}$, then the sequence (2) is zero in $\operatorname{add}(Z \oplus \bar{X})$. This proves that (2) is exact in $\operatorname{add} \bar{M}$. Then (1) is an $\operatorname{add} \bar{M}$-approximation resolution of M.

If $M \in \operatorname{ind} \Lambda_{\mathcal{C}}$, there exists an $\operatorname{add} \bar{X}$-approximation resolution of length s in $\operatorname{mod} \Lambda_{\mathcal{C}}$:

$$(3) \quad 0 \to X_s \xrightarrow{\psi_s} X_{s-1} \to \cdots \to X_2 \xrightarrow{\psi_2} X_1 \xrightarrow{\psi_1} M \to 0.$$

Since $\operatorname{ind} \Lambda_{\mathcal{C}}$ is a full subcategory of $\operatorname{ind} \Lambda$, the induced sequence

$$(4) \quad 0 \to {}_{\Lambda}(-, X_s) \to {}_{\Lambda}(-, X_{s-1}) \to \cdots \to {}_{\Lambda}(-, X_1) \to {}_{\Lambda}(-, M) \to 0$$

is exact in $\operatorname{add} \bar{X}$. We have $N_i = \operatorname{Ker} \psi_i \in \operatorname{mod} \Lambda_{\mathcal{C}}$, for $i \in \{1, ..., s-1\}$. Let $N \in \operatorname{ind} \Lambda_{\mathcal{C}}$ be a non-injective summand of N_1, then since $\operatorname{ind} \Lambda_{\mathcal{C}}$ is closed under successors we have $\tau_{\Lambda}^{-1} N \in \operatorname{ind} \Lambda_{\mathcal{C}}$ and so

$$\operatorname{Ext}^1_{\Lambda}\big(\bar{Y}' \oplus Z, N\big) \cong D \, \underline{\operatorname{Hom}}_{\Lambda}\big(\tau_{\Lambda}^{-1} N, \bar{Y}' \oplus Z\big) = 0.$$

Then the short sequence

$$0 \to {}_{\Lambda}(-, N_1) \to {}_{\Lambda}(-, X_1) \to {}_{\Lambda}(-, M) \to 0$$

is exact in $\operatorname{add}(\bar{Y}' \oplus Z)$. The same argument holds true replacing M by N_i for $i \in \{1, ..., s-1\}$ and this proves that the sequence (4) is exact in $\operatorname{add}(\bar{Y}' \oplus Z)$ and so (3) is an $\operatorname{add} \bar{M}$-approximation resolution of M. Therefore

$$\operatorname{rep.dim} \Lambda \leq \max\{r+1, s+1\}.$$

The proof with the hypothesis (b) is dual. □

Example 4.3. In Example 2.8 we exhibit a trisection $(\mathcal{L}_{\Lambda}, (\mathcal{L}_{\Lambda} \cup \mathcal{R}_{\Lambda})^c, \mathcal{R}_{\Lambda})$ of $\operatorname{ind} \Lambda$ with $(\mathcal{L}_{\Lambda} \cup \mathcal{R}_{\Lambda})^c$ infinite. There $\operatorname{ind}_{(\mathcal{L}_{\Lambda})} \Lambda = \operatorname{Pred} S_3$ is closed under predecessors and $(\operatorname{ind}_{(\mathcal{L}_{\Lambda})} \Lambda \cup \mathcal{R}_{\Lambda})^c$ is finite. So, by Theorem 4.2 (b), we have $\operatorname{rep.dim} \Lambda \leq \max\{\operatorname{rep.dim} {}_{(\mathcal{L}_{\Lambda})}\Lambda, \operatorname{rep.dim} \Lambda_{(\mathcal{R}_{\Lambda})}\}$. Now, because $\Lambda_{(\mathcal{R}_{\Lambda})}$ is hereditary, ${}_{(\mathcal{L}_{\Lambda})}\Lambda$ is tilted and Λ is representation-infinite, we have $\operatorname{rep.dim} \Lambda = 3$.

5. Applications

5.1. *Laura algebras*

Following [4], we say that an artin algebra Λ is a **laura algebra** if $\mathcal{L}_{\Lambda} \cup \mathcal{R}_{\Lambda}$ is cofinite in $\operatorname{ind} \Lambda$ and it is a **strict laura algebra** if it is a laura but is not quasi-tilted. If Λ is a strict laura then Λ is left and right supported (see [5] (4.4)), that is, $\operatorname{add} \mathcal{L}_{\Lambda}$ is contravariantly finite and $\operatorname{add} \mathcal{R}_{\Lambda}$ is covariantly finite, respectively. As the first application of Theorem 2.6, we give another proof of the result of [7] (4.1) saying that if Λ is a strict laura algebra then $\operatorname{rep.dim} \Lambda \leq 3$.

Corollary 5.1. *If Λ is a laura algebra, then* $\operatorname{rep.dim} \Lambda \leq 3$.

Proof. If Λ is quasi-tilted, this follows from [22], hence we can assume that Λ is strict. Since Λ is left supported then $\operatorname{add} \mathcal{L}_{\Lambda}$ is contravariantly finite and by [5] (5.1) we have that ${}_{(\mathcal{L}_\Lambda)}\Lambda$ is a product of tilted algebras and so $\operatorname{rep.dim}{}_{(\mathcal{L}_\Lambda)}\Lambda \leq 3$. By Corollary 2.1, as $(\mathcal{L}_\Lambda, \mathcal{B}, \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ is a trisection of $\operatorname{ind}\Lambda$ where $\mathcal{B} = (\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c$ is finite then $\operatorname{add}(\mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda)$ is covariantly finite. By Lemma 2.3, we have $\operatorname{rep.dim}\Lambda \leq \max\{3, \operatorname{rep.dim}{}_{(\mathcal{L}_\Lambda)}\Lambda\} = 3$. $\square$

Let Λ be a strict laura algebra such that $\mathcal{L}_\Lambda \cap \mathcal{R}_\Lambda = \emptyset$, then $(\mathcal{L}_\Lambda, \mathcal{B}, \mathcal{R}_\Lambda)$, where $\mathcal{B} = (\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)^c$ is finite, is a trisection of $\operatorname{ind}\Lambda$. On the other hand, if E denotes the sum of all indecomposable Ext-injective modules of $\operatorname{add}\mathcal{L}_\Lambda$, then $\bar{X} = A \oplus DA \oplus E$ is a generator–cogenerator of $A = {}_{(\mathcal{L}_\Lambda)}\Lambda$ having the 2-approximation property, by [7] (2.3). Then, in this case, the generator–cogenerator constructed in Lemma 2.3 coincides with the one constructed in [7] (4.1).

5.2. Ada algebras

As the second application, we consider the class of ada algebras introduced in [3]. An artin algebra Λ is called an **ada algebra** if $\Lambda \oplus D\Lambda \in \operatorname{add}(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$. We have that for an ada algebra the representation dimension is less or equal to 3. This follows from the next consequence of Proposition 3.2.

Theorem 5.2. *Let Λ be a representation-infinite artin algebra. If $\Lambda \in \operatorname{add}(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$, then* $\operatorname{rep.dim}\Lambda = 3$.

Proof. For $\mathcal{C} = \mathcal{R}_\Lambda \setminus \mathcal{L}_\Lambda$ by Proposition 3.2 we have $\operatorname{rep.dim}\Lambda = \max\{3, \operatorname{rep.dim}{}_{(\mathcal{C}^c)}\Lambda\}$. But, in this case, a projective P lies in $\mathcal{C}^c$ if and only if $P \in \mathcal{L}_\Lambda$. Then, ${}_{(\mathcal{C}^c)}\Lambda = {}_{(\mathcal{L}_\Lambda)}\Lambda$. Moreover by [5] (2.3) the algebra ${}_{(\mathcal{L}_\Lambda)}\Lambda$ is a product of quasi-tilted algebras and then, by [22], we have $\operatorname{rep.dim}{}_{(\mathcal{L}_\Lambda)}\Lambda \leq 3$. Therefore $\operatorname{rep.dim}\Lambda = 3$. $\square$

Corollary 5.3. *If Λ is an ada algebra then* $\operatorname{rep.dim}\Lambda \leq 3$. $\square$

5.3. Nakayama oriented pullbacks

In this section, all algebras are basic, associative, finite dimensional algebras with identities over an algebraically closed field k.

Let A, B and C be algebras and let $f : A \to B$ and $g : C \to B$ be morphisms. The pullback of f and g is the algebra $R = \{(a, c) \in A \times C : f(a) = g(c)\}$. Consider the case where $A = kQ_A/I_A$, $C = kQ_C/I_C$ and Q_B is a full and convex subquiver of Q_A and of Q_C such that $I_A \cap kQ_B = I_C \cap kQ_B =: I_B$. In this case, the algebra $B = kQ_B/I_B \cong e_BAe_B \cong e_BCe_B$ is a common quotient of A and of C. Let R be the pullback of the canonical projections $A \to B$ and $C \to B$. The following lemma describes the bound quiver of R in terms of the bound quivers of A, B and C.

Lemma 5.4. *(See [17,20].) Let Q_R be the pushout of the inclusion maps $Q_B \to Q_A$ and $Q_B \to Q_C$, and consider the ideal $I_R = I_A + I_C + I$ where I is the ideal generated by all paths linking $(Q_A)_0 \setminus (Q_B)_0$ and $(Q_C)_0 \setminus (Q_B)_0$. Then $R \cong kQ_R/I_R$.* □

It is easily seen that every indecomposable A-module has an R-module structure. We can assume that $\operatorname{ind} A$ is contained in $\operatorname{ind} R$. Similarly we can assume that $\operatorname{ind} B \subseteq \operatorname{ind} C \subseteq \operatorname{ind} R$.

Definition 5.5. (See [20].) Let $R \cong kQ_R/I_R$ be the pullback of $A \to B$ and $C \to B$. Then R is a **Nakayama oriented pullback** if its bound quiver (Q_R, I_R) satisfies the following conditions:

(i) There is no path from $(Q_B)_0$ to $(Q_C)_0 \setminus (Q_B)_0$ and from $(Q_A)_0 \setminus (Q_B)_0$ to $(Q_B)_0$.
(ii) B is a hereditary Nakayama algebra and the connected components $Q_{B1}, Q_{B2}, \ldots, Q_{Br}$ of Q_B are of the form $Q_{Bi} = a_{i,t_i} \to a_{i,t_{i-1}} \to \cdots \to a_{i,1}$ with $1 \leq i \leq r$ and $t_i \geq 1$.
(iii) In Q_B only sources are target of arrows of $(Q_C)_1 \setminus (Q_B)_1$ and only sinks are sources of arrows of $(Q_A)_1 \setminus (Q_B)_1$.
(iv) No minimal relation of R has its origin in $(Q_B)_0$.

By the shape of (Q_R, I_R), we have that, for any $i \in (Q_C)_0$, the injective R-module associated to i coincides with the injective C-module associated to i. And, for any $i \in (Q_B)_0$, the injective A-module associated to i coincides with the injective B-module associated to i.

So we have the following remark.

Remark 5.6. If M is a C-module then $\mathrm{id}_R M = \mathrm{id}_C M$, that is, the injective dimension of M over R coincides with the injective dimension of M over C. And, if M is a B-module then $\mathrm{id}_A M = \mathrm{id}_B M$.

It follows from [17,20] that $\operatorname{ind} R = \operatorname{ind} A \cup \operatorname{ind} C$ and $\operatorname{ind} B = \operatorname{ind} A \cap \operatorname{ind} C$ and, moreover, we have that $\operatorname{ind} C$ is closed under successors and $\operatorname{ind} A$ is closed under predecessors.

Now, we have an application of Proposition 3.2.

Corollary 5.7. *Let R be a representation-infinite Nakayama oriented pullback of $A \to B$ and $C \to B$.*

(a) *If C is hereditary then* $\operatorname{rep.dim} R = \max\{3, \operatorname{rep.dim} A\}$.
(b) *If A is hereditary then* $\operatorname{rep.dim} R = \max\{3, \operatorname{rep.dim} C\}$.

Proof. Denote $\mathcal{C} = \operatorname{ind} C \setminus \operatorname{ind} B$ which is closed under successors, so by Remark 5.6, as C is hereditary, it follows that $\mathcal{C} \subseteq \mathcal{R}_R$. By Proposition 3.2, we have that $\operatorname{rep.dim} R =$

$\max\{3, \operatorname{rep.dim}_{(\mathcal{C}^c)}R\}$. But $\mathcal{C}^c = \operatorname{ind} A$ and so ${}_{(\mathcal{C}^c)}R = A$. This shows that $\operatorname{rep.dim} R = \max\{3, \operatorname{rep.dim} A\}$.

The proof of (b) is dual. $\square$

Finally, as an application of Theorem 4.2, we have a more general result for Nakayama oriented pullbacks.

Corollary 5.8. *Let R be the Nakayama oriented pullback of $A \to B$ and $C \to B$. Then* $\operatorname{rep.dim} R \leq \max\{\operatorname{rep.dim} A, \operatorname{rep.dim} C\}$.

Proof. If A and C are representation-finite algebras, then so is R, because $\operatorname{ind} R = \operatorname{ind} A \cup \operatorname{ind} C$. Suppose that A is representation-infinite. In Theorem 4.2, take $\mathcal{A} = \operatorname{ind} A \setminus \operatorname{ind} B$, $\mathcal{B} = \emptyset$ and $\mathcal{C} = \operatorname{ind} C$. Then $R_{\mathcal{C}} = C$ and $\operatorname{rep.dim} R \leq \max\{\operatorname{rep.dim}_{\mathcal{A}} R, \operatorname{rep.dim} C\}$.

Note that ${}_{\mathcal{A}}R = {}_{\mathcal{A}}A$ and that, for $M \in \operatorname{ind} B$, by Remark 5.6, we have $\operatorname{id}_A M = \operatorname{id}_B M = 1$ because B is hereditary. So $\operatorname{ind} B \subseteq \mathcal{R}_A$ and since A is a representation-infinite algebra, then, by Proposition 3.2, we have $\operatorname{rep.dim} A = \max\{3, \operatorname{rep.dim}_{\mathcal{A}} A\}$ and so $\operatorname{rep.dim}_{\mathcal{A}} A \leq \operatorname{rep.dim} A$.

Therefore, $\operatorname{rep.dim} R \leq \max\{\operatorname{rep.dim} A, \operatorname{rep.dim} C\}$.

A similar proof holds if we suppose that C is representation-infinite. $\square$

Example 5.9. The algebra Λ in Example 2.9 can be seen as a Nakayama oriented pullback where A is the algebra with radical square zero given by

$$\begin{tikzcd}
 & & 3 \arrow[dl, "\beta"'] & \\
1 & 2 \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] & & 4 \arrow[ll, "\delta"] \arrow[ul, "\gamma"']
\end{tikzcd}$$

and C is given by

$$\begin{tikzcd}
4 & 5 \arrow[l, shift right=2, "x_1"'] \arrow[l, "x_2"] \arrow[l, shift left=2, "x_3"] & 6 \arrow[l, shift right=2, "x_1"'] \arrow[l, "x_2"] \arrow[l, shift left=2, "x_3"]
\end{tikzcd}$$

bound by the relations $x_i x_j = x_j x_i$ for $1 \leq i, j \leq 3$. This two algebras have representation dimension 3 and 4, respectively. Then, the last corollary gives us the inequality $\operatorname{rep.dim} \Lambda \leq 4$.

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