On symmetric quotients of symmetric algebras ☆

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We investigate symmetric quotient algebras of symmetric algebras, with an emphasis on finite group algebras over a complete discrete valuation ring $\mathcal{O}$. Using elementary methods, we show that if an ordinary irreducible character χ of a finite group G gives rise to a symmetric quotient over $\mathcal{O}$ which is not a matrix algebra, then the decomposition numbers of the row labelled by χ are all divisible by the characteristic p of the residue field of $\mathcal{O}$.

1. Introduction

Let p be a prime and $\mathcal{O}$ a complete discrete valuation ring having a residue field k of characteristic p and a quotient field K of characteristic zero. Unless stated otherwise, we assume that K and k are splitting fields for all finite groups under consideration. Let G be a finite group. Any subset M of the set $\mathrm{Irr}_K(G)$ of irreducible K-valued characters of G gives rise to an $\mathcal{O}$-free quotient algebra, namely the image of a structural homomorphism $\mathcal{O}G \to \mathrm{End}_{\mathcal{O}}(V)$, where V is an $\mathcal{O}$-free $\mathcal{O}G$-module having character

$\sum_{\chi \in M} \chi$. This image is isomorphic to $\mathcal{O}G(\sum_{\chi \in M} e(\chi))$, where $e(\chi)$ denotes the primitive idempotent in $Z(KG)$ corresponding to χ. Any $\mathcal{O}$-free quotient algebra of $\mathcal{O}G$ arises in this way; in particular, $\mathcal{O}G$ has only finitely many $\mathcal{O}$-free algebra quotients. Any quotient of $\mathcal{O}G$ admits a decomposition induced by the block decomposition, and hence finding symmetric quotients of $\mathcal{O}G$ is equivalent to finding symmetric quotients of the block algebras of $\mathcal{O}G$. We denote by $\mathrm{IBr}_k(G)$ the set of irreducible Brauer characters of G, and by $d_G : \mathbb{Z}\mathrm{Irr}_K(G) \to \mathbb{Z}\mathrm{IBr}_k(G)$ the decomposition map, sending a generalised character of G to its restriction to the set $G_{p'}$ of p'-elements in G. For B a block algebra of $\mathcal{O}G$, we denote by $\mathrm{Irr}_K(B)$ and $\mathrm{IBr}_k(B)$ the sets of irreducible K-characters and Brauer characters, respectively, associated with B. We denote by $d_B : \mathbb{Z}\mathrm{Irr}_K(B) \to \mathbb{Z}\mathrm{IBr}_k(B)$ the decomposition map obtained from restricting d_G. We denote by $D_G = (d_\varphi^\chi)$ the decomposition matrix of $\mathcal{O}G$, with rows indexed by $\chi \in \mathrm{Irr}_K(G)$ and columns indexed by $\varphi \in \mathrm{IBr}_k(G)$; that is, the d_φ^χ are the nonnegative integers satisfying $d_G(\chi) = \sum_{\varphi \in \mathrm{IBr}_k(G)} d_\varphi^\chi \cdot \varphi$, for any $\chi \in \mathrm{Irr}_K(G)$. Equivalently, $d_\varphi^\chi = \chi(i)$, where i is a primitive idempotent in $\mathcal{O}G$ such that $\mathcal{O}Gi$ is a projective cover of a simple kG-module with Brauer character φ. If χ, φ belong to different blocks, then $d_\varphi^\chi = 0$. For B a block algebra of $\mathcal{O}G$, we denote by D_B the submatrix of D_G labelled by $\chi \in \mathrm{Irr}_K(B)$ and $\varphi \in \mathrm{IBr}_k(B)$. We say that $\chi \in \mathrm{Irr}_K(G)$ lifts the irreducible Brauer character $\varphi \in \mathrm{IBr}_k(G)$ if $d_G(\chi) = \varphi$, or equivalently, if $d_\varphi^\chi = 1$ and $d_{\varphi'}^\chi = 0$ for all $\varphi' \in \mathrm{IBr}_k(G)$ different from φ. In that case, $\mathcal{O}Ge(\chi)$ is a matrix algebra over $\mathcal{O}$, hence trivially symmetric (see Proposition 4.1 below).

Theorem 1.1. *Let G be a finite group and $\chi \in \mathrm{Irr}_K(G)$. Suppose that $\mathcal{O}Ge(\chi)$ is symmetric. Then either χ lifts an irreducible Brauer character, or d_φ^χ is divisible by p for all $\varphi \in \mathrm{IBr}_k(G)$.*

This is a special case of the slightly more general result Theorem 4.6 below. We note some immediate consequences.

Corollary 1.2. *Let G be a finite group and $\chi \in \mathrm{Irr}_K(G)$ such that d_φ^χ is prime to p for some $\varphi \in \mathrm{IBr}_k(G)$. Then the $\mathcal{O}$-algebra $\mathcal{O}Ge(\chi)$ is symmetric if and only if χ lifts φ.*

By a result of Dade [1], all decomposition numbers of blocks with cyclic defect are either 0 or 1.

Corollary 1.3. *Let G be a finite group and B a block with cyclic defect groups. Let $\chi \in \mathrm{Irr}_K(B)$. Then $\mathcal{O}Ge(\chi)$ is symmetric if and only if χ corresponds to a nonexceptional vertex at the end of a branch of the Brauer tree of B.*

By Erdmann's results in [2], every row of the decomposition matrix of a nonnilpotent tame block contains an entry equal to 1.

Corollary 1.4. *Suppose that $p = 2$. Let G be a finite group and B a nonnilpotent block of $\mathcal{O}G$ having a defect group P which is either dihedral, generalised quaternion, or semidihedral. Then for any $\chi \in \mathrm{Irr}_K(B)$, the algebra $\mathcal{O}Ge(\chi)$ is symmetric if and only if χ lifts an irreducible Brauer character in $\mathrm{IBr}_k(B)$.*

Many blocks of quasi-simple groups also seem to have the property that every row has an entry prime to p. Any row of the decomposition matrix of a finite group corresponding to a height zero character has at least one entry which is prime to p. Hence Corollary 1.2 yields the following.

Corollary 1.5. *Let G be a finite group, B a block of $\mathcal{O}G$, and $\chi \in \mathrm{Irr}_K(B)$. Suppose that χ has height zero. Then the $\mathcal{O}$-algebra $\mathcal{O}Ge(\chi)$ is symmetric if and only if χ lifts an irreducible Brauer character.*

By the main result of the first author and Malle [4], every irreducible character of a finite group lying in a p-block with abelian defect groups has height zero (the proof in [4] requires the classification of finite simple groups). Hence any irreducible character of a finite group lying in a p-block with abelian defect groups gives rise to a symmetric quotient if and only if the character lifts an irreducible Brauer character.

The symmetric algebras arising in the corollaries above are matrix algebras (see Proposition 4.1 below). By contrast, the symmetric algebras obtained from nonlinear characters in the next result are not isomorphic to matrix algebras.

Proposition 1.6. *Let P be a finite p-group having a normal cyclic subgroup of index p. Then $\mathcal{O}Pe(\chi)$ is symmetric for any $\chi \in \mathrm{Irr}_K(P)$.*

This will be proved in Section 3. Since a nilpotent block is isomorphic to a matrix algebra over one of its defect group algebras, this proposition, specialised to $p = 2$, has the following consequence (which includes nilpotent tame blocks).

Corollary 1.7. *Suppose that $p = 2$. Let G be a finite group and B a nilpotent block of $\mathcal{O}G$ having a defect group P which is either dihedral, generalised quaternion or semidihedral. Then for any $\chi \in \mathrm{Irr}_K(B)$, the algebra $\mathcal{O}Ge(\chi)$ is symmetric.*

Further examples of characters χ with symmetric quotient $\mathcal{O}Ge(\chi)$ which are not isomorphic to matrix algebras can be obtained from characters of central type. An irreducible character χ of a finite group G is *of central type* if $\chi(1)^2 = |G : Z(G)|$.

Proposition 1.8. *Let G be a finite group and $\chi \in \mathrm{Irr}_K(G)$ a character of central type. Then $\mathcal{O}Ge(\chi)$ is symmetric.*

This is shown as a special case of a slightly more general situation in Proposition 2.4. As a consequence of Propositions 1.6 and 1.8, if P is a finite p-group of order at most p^3,

then $\mathcal{O}Pe(\chi)$ is symmetric for all $\chi \in \mathrm{Irr}_K(P)$. In Section 6 we give an example showing that this is not the case in general for irreducible characters of finite p-groups of order p^4.

As is explained in the remarks following Proposition 2.5, for G a finite group, the set of ideals I of $\mathcal{O}G$ such that $\mathcal{O}G/I$ is $\mathcal{O}$-free corresponds bijectively to the set of subsets of $\mathrm{Irr}_K(G)$. If χ is an ordinary irreducible character of G then $\mathcal{O}Ge(\chi)$ is the quotient of $\mathcal{O}G$ by the ideal which corresponds under the above bijection to the subset $\mathrm{Irr}_K(G) - \{\chi\}$ of $\mathrm{Irr}_K(G)$. The next two propositions consider the complementary case of quotients $\mathcal{O}G/I$, where I corresponds to a one element subset of $\mathrm{Irr}_K(G)$. We first consider the case that I corresponds to the trivial character of G, in which case I can be explicitly described as $\mathcal{O}(\sum_{x \in G} x)$. The hypothesis of K, k being large enough is not necessary for the following result.

Proposition 1.9. *Let G be a finite group. The following are equivalent.*

(i) *The $\mathcal{O}$-algebra $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$ is symmetric.*
(ii) *The group G is p-nilpotent and has cyclic Sylow-p-subgroups.*

Corollary 1.10. *Let G be a finite group. The $\mathbb{Z}$-algebra $\mathbb{Z}G/\mathbb{Z}(\sum_{x \in G} x)$ is symmetric if and only if G is cyclic.*

The arguments used in the proof of Proposition 1.9 can be adapted to yield the following block theoretic version; we need k to be large enough for the block B in the next result.

Proposition 1.11. *Let G be a finite group, B a block algebra of $\mathcal{O}G$, and let $\chi \in \mathrm{Irr}_K(B)$. Suppose that χ lifts an irreducible Brauer character. Set $I = B \cap (K \otimes_{\mathcal{O}} B)e(\chi)$, where we identify B with its image $1 \otimes B$ in $K \otimes_{\mathcal{O}} B$. The following are equivalent.*

(i) *The algebra B/I is symmetric.*
(ii) *The block B is nilpotent with cyclic defect groups.*

2. Notation and basic facts

If A is an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module, we denote by $\mathrm{Irr}_K(A)$ the set of characters of the simple $K \otimes_{\mathcal{O}} A$-modules. Taking characters of $K \otimes_{\mathcal{O}} A$-modules yields an isomorphism between the Grohendieck group $R_K(A)$ of finitely generated $K \otimes_{\mathcal{O}} A$-modules and the free abelian group with basis $\mathrm{Irr}_K(A)$, inducing a bijection between the isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules and $\mathrm{Irr}_K(A)$. If in addition the K-algebra $K \otimes_{\mathcal{O}} A$ is semisimple, hence a direct product of simple K-algebras corresponding to the isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules, we denote by $e(\chi)$ the primitive idempotent of $Z(K \otimes_{\mathcal{O}} A)$ which acts as identity on the simple $K \otimes_{\mathcal{O}} A$-modules with character χ and which annihilates all other simple $K \otimes_{\mathcal{O}} A$-modules. In this case

we have $K \otimes_{\mathcal{O}} A \cong \prod_{\chi \in \mathrm{Irr}_K(A)} (K \otimes_{\mathcal{O}} A)e(\chi)$, and each factor $(K \otimes_{\mathcal{O}} A)e(\chi)$ is a simple K-algebra. An $\mathcal{O}$-algebra A is *symmetric* if A is finitely generated projective as an $\mathcal{O}$-module and if A is isomorphic to its $\mathcal{O}$-dual $A^{\vee} = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as an A–A-bimodule (this definition makes sense with $\mathcal{O}$ replaced by an arbitrary commutative ring). The image s in $A^{\vee}$ of 1_A under some bimodule isomorphism $A \cong A^{\vee}$ is called a *symmetrising form of A*. If s, s' are two symmetrising forms of A, then $s' = z \cdot s$ for a unique $z \in Z(A)^{\times}$, because any bimodule automorphism of A is given by left or right multiplication with an invertible element in $Z(A)$. The ring $\mathcal{O}$ has the property that every finitely generated projective $\mathcal{O}$-module is free. In particular, if A is a symmetric $\mathcal{O}$-algebra, then A is free as an $\mathcal{O}$-module, and hence any symmetrising form $s \in A^{\vee}$ of A satisfies $s(A) = \mathcal{O}$. Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. Since $\mathcal{O}$ is Noetherian, a submodule V of a finitely generated A-module U is $\mathcal{O}$-*pure* if and only if V is a direct summand of U as an $\mathcal{O}$-module (see e.g. [5, §4J] for more details on the notion of pure submodules over arbitrary commutative rings). We use without further comments the following well-known facts. See e.g. [3, 17.2], and also [9, Theorem 1] for an application in the context of blocks with cyclic defect groups.

Lemma 2.1. *Let A be an $\mathcal{O}$-algebra and let U be an A-module which is finitely generated free as an $\mathcal{O}$-module. Let V be a submodule of U. Then $U \cap (K \otimes_{\mathcal{O}} V)$ is the unique minimal $\mathcal{O}$-pure submodule of U containing V, where we identify U with its image $1 \otimes U$ in $K \otimes_{\mathcal{O}} U$. Moreover, the following are equivalent.*

- *(i) The A-module V is $\mathcal{O}$-pure in U.*
- *(ii) The A-module U/V is $\mathcal{O}$-free.*
- *(iii) We have $J(\mathcal{O})V = J(\mathcal{O})U \cap V$.*
- *(iv) The image of V in $k \otimes_{\mathcal{O}} U$ is isomorphic to $k \otimes_{\mathcal{O}} V$.*
- *(v) We have $V = U \cap (K \otimes_{\mathcal{O}} V)$, where we identify U to its image $1 \otimes U$ in $K \otimes_{\mathcal{O}} U$.*

Thus if I is an ideal in an $\mathcal{O}$-algebra A which is finitely generated free as an $\mathcal{O}$-module, then the quotient algebra A/I is $\mathcal{O}$-free if and only if I is $\mathcal{O}$-pure in A. Any $\mathcal{O}$-pure ideal I of A is equal to $A \cap M_I$ for a unique ideal M_I in $K \otimes_{\mathcal{O}} A$, where we have identified A to its canonical image $1_K \otimes A$ in $K \otimes_{\mathcal{O}} A$. If $K \otimes_{\mathcal{O}} A$ is in addition semisimple, hence a direct product of simple algebras, then every ideal of $K \otimes_{\mathcal{O}} A$ is a product of a subset of those simple algebras. In particular, in that case the set of $\mathcal{O}$-pure ideals in A is finite and corresponds bijectively to the set of subsets of a set of representatives of the isomorphism classes of simple $K \otimes_{\mathcal{O}} A$-modules.

The next result holds with $\mathcal{O}$ replaced by an arbitrary commutative Noetherian ring. For symmetric algebras over a field, this result is due to Nakayama [7]. The generalisation to algebras over commutative Noetherian rings is straightforward (we include a proof for the convenience of the reader). Note that the left and right annihilators of an ideal I in a symmetric algebra A are always equal, denoted by $\mathrm{ann}(I)$.

Proposition 2.2. (Cf. [7, Theorem 13].) *Let A be a symmetric $\mathcal{O}$-algebra, and let I be an $\mathcal{O}$-pure ideal in A. The quotient algebra A/I is a symmetric $\mathcal{O}$-algebra if and only if there is an element $z \in Z(A)$ such that $\mathrm{ann}(I) = Az$.*

Proof. Set $\bar{A} = A/I$, and denote by $\pi : A \to \bar{A}$ the canonical surjection. Since I is $\mathcal{O}$-pure, it follows that A/I is finitely generated projective as an $\mathcal{O}$-module. Let $s : A \to \mathcal{O}$ be a symmetrising form for A; that is, s is symmetric and the map sending $a \in A$ to the form $a \cdot s$ defined by $(a \cdot s)(b) = s(ab)$ for all $b \in A$ is an isomorphism $A \cong A^{\vee} = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$. (Since s is symmetric, this map is automatically a homomorphism of A–A-bimodules.) Suppose that $\mathrm{ann}(I) = Az$ for some $z \in Z(A)$. Then $t = z \cdot s$ annihilates I, hence induces a form $\bar{t} \in \bar{A}$ defined by $\bar{t}(\bar{a}) = s(za)$, where $a \in A$ and $\bar{a} = a + I \in \bar{A}$. Since s is symmetric and $z \in Z(A)$, the forms t and $\bar{t}$ are again symmetric. It suffices to show that the map sending $\bar{a} \in \bar{A}$ to $\bar{a} \cdot \bar{t} \in \bar{A}^{\vee}$ is surjective. Let $\bar{u} \in \bar{A}^{\vee}$. Then $u = \bar{u} \circ \pi \in A^{\vee}$, hence $u = a \cdot s$ for a uniquely determined element $a \in A$. Since $I \subseteq \ker(u)$, we have $s(aI) = \{0\}$, hence $a \in \mathrm{ann}(I)$. Thus $a = cz$ for some $c \in A$. It follows that $u = c \cdot (z \cdot s) = c \cdot t$, and hence $\bar{u} = \bar{c} \cdot \bar{t}$, which shows that $\bar{A}$ is symmetric. Suppose conversely that $\bar{A}$ is symmetric. Let $\bar{t} : \bar{A} \to \mathcal{O}$ be a symmetrising form, and set $t = \bar{t} \circ \pi$. Then t is symmetric (because $\bar{t}$ is) and hence $t = z \cdot s$ for some $z \in Z(A)$. Since $I \subseteq \ker(t)$ we have $s(zI) = \{0\}$, hence $z \in \mathrm{ann}(I)$. Let $b \in \mathrm{ann}(I)$. Then $u = b \cdot s$ has I in its kernel, hence induces a form $\bar{u}$ on $\bar{A}$ satisfying $\bar{u}(\bar{a}) = s(ab)$. Since $\bar{A}$ is symmetric, there is $\bar{c} \in \bar{A}$ such that $\bar{u} = \bar{c} \cdot \bar{t}$, hence such that $u = c \cdot t = (cz) \cdot s$. Thus $(cz) \cdot s = b \cdot s$, and hence $b = cz \in Az$, whence the equality $\mathrm{ann}(I) = Az$. $\square$

Let A be a symmetric $\mathcal{O}$-algebra. If $z \in Z(A)$ such that Az is the annihilator of an $\mathcal{O}$-pure ideal I in A, then Az is $\mathcal{O}$-pure. Thus finding symmetric quotients of A is equivalent to finding elements $z \in Z(A)$ with the property that Az is $\mathcal{O}$-pure in A.

Corollary 2.3. *Let A be a symmetric $\mathcal{O}$-algebra and $z \in Z(A)$. If Az is $\mathcal{O}$-pure, then the annihilator $I = \mathrm{ann}(z) = \mathrm{ann}(Az)$ is an $\mathcal{O}$-pure ideal satisfying $\mathrm{ann}(I) = Az$, and the $\mathcal{O}$-algebra A/I is symmetric. Moreover, any symmetric $\mathcal{O}$-algebra quotient of A arises in this way.*

Proof. This follows from Proposition 2.2 and the preceding remarks. $\square$

Proposition 2.4. *Let G be a finite group and N a normal subgroup of G. Suppose that K is a splitting field for N. Let $\eta \in \mathrm{Irr}_K(N)$, and suppose that η is G-stable. If $\mathcal{O}Ne(\eta)$ is symmetric, then $\mathcal{O}Ge(\eta)$ is symmetric.*

Proof. Suppose that $\mathcal{O}Ne(\eta)$ is symmetric. By Proposition 2.2 there exists an element $z \in Z(\mathcal{O}N)$ such that $\mathcal{O}Nz$ is the annihilator of the kernel of the map $\mathcal{O}N \to \mathcal{O}Ne(\eta)$. Thus $z \in Z(\mathcal{O}N)e(\eta) = \mathcal{O}e(\eta)$, where we use that K is large enough for N. In particular, there is $\lambda \in \mathcal{O}$ such that $z = \lambda e(\eta)$, and hence $z \in Z(\mathcal{O}G)$. Since $\mathcal{O}G$ is free as a right $\mathcal{O}N$-module and $\mathcal{O}Nz$ is $\mathcal{O}$-pure in $\mathcal{O}N$, it follows that $\mathcal{O}Gz$ is $\mathcal{O}$-pure in $\mathcal{O}G$. The

annihilator of the kernel of the map $\mathcal{O}G \to \mathcal{O}Ge(\eta)$ is therefore equal to $\mathcal{O}Gz$. The result follows from Proposition 2.2. $\square$

Proof of Proposition 1.8. Since χ is of central type, there is a unique (hence G-stable) linear character $\zeta : Z(G) \to \mathcal{O}^\times$ such that $e(\chi) = e(\zeta)$. We have $\mathcal{O}Z(G)e(\zeta) \cong \mathcal{O}$, which is trivially symmetric, and hence $\mathcal{O}Ge(\zeta) = \mathcal{O}Ge(\chi)$ is symmetric by Proposition 2.4. $\square$

The argument in the proof of Proposition 2.4 to describe a central element z which generates a pure ideal admits the following generalisation.

Proposition 2.5. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_\mathcal{O} A$ is semisimple. Let $\chi \in \operatorname{Irr}_K(A)$ be the character of an absolutely simple $K \otimes_\mathcal{O} A$-module, and denote by $e(\chi)$ the corresponding primitive idempotent in $Z(K \otimes_\mathcal{O} A)$. Let $\lambda \in \mathcal{O}$ be an element having minimal valuation such that $\lambda e(\chi) \in A$, where we identify A with its image $1 \otimes A$ in $K \otimes_\mathcal{O} A$. The following are equivalent.*

(i) The $\mathcal{O}$-algebra $Ae(\chi)$ is symmetric.
(ii) The A-module $A\lambda e(\chi)$ is $\mathcal{O}$-pure in A.
(iii) We have $A\lambda e(\chi) = A \cap (K \otimes_\mathcal{O} A)e(\chi)$.

Proof. The equivalence of (ii) and (iii) is a general fact (cf. Lemma 2.1). Let I be the kernel of the algebra homomorphism $A \to Ae(\chi)$ sending $a \in A$ to $ae(\chi)$. Multiplication by $e(\chi)$ in $K \otimes_\mathcal{O} A$ yields the projection of $K \otimes_\mathcal{O} A = \prod_{\psi \in \operatorname{Irr}_K(A)} (K \otimes_\mathcal{O} A)e(\psi)$ onto the factor $(K \otimes_\mathcal{O} A)e(\chi)$; this is a matrix algebra as χ is the character of an absolutely simple $K \otimes_\mathcal{O} A$-module. Thus I is the $\mathcal{O}$-pure ideal corresponding to the complement of $\{\chi\}$ in $\operatorname{Irr}_K(A)$; that is, $I = A \cap \prod_\psi (K \otimes_\mathcal{O} A)e(\psi)$, where ψ runs over the set $\operatorname{Irr}_K(A) - \{\chi\}$. It follows that the annihilator of I is equal to $A \cap (K \otimes_\mathcal{O} A)e(\chi)$, and hence $Z(A) \cap (K \otimes_\mathcal{O} A)e(\chi) \subseteq \mathcal{O}e(\chi)$. Thus A/I is symmetric if and only if Az is $\mathcal{O}$-pure for some $z \in A \cap \mathcal{O}e(\chi)$, hence if and only if $A\lambda e(\chi)$ is $\mathcal{O}$-pure for some $\lambda \in \mathcal{O}$. In that case, λ has necessarily the smallest possible valuation such that $\lambda e(\chi) \in A$. The result follows. $\square$

Remark 2.6. Let A be an $\mathcal{O}$-algebra which is finitely generated free as an $\mathcal{O}$-module, and let I be an $\mathcal{O}$-pure ideal in A. Then the image of the canonical map $A \to \operatorname{End}_\mathcal{O}(A/I)$ sending $a \in A$ to left multiplication by $a + I$ in A/I has kernel I, hence image isomorphic to A/I. Thus any $\mathcal{O}$-free algebra quotient of A is isomorphic to the image of the structural homomorphism $A \to \operatorname{End}_\mathcal{O}(V)$ sending $a \in A$ to left multiplication by a on V, for some A-module V which is free of finite rank as an $\mathcal{O}$-module. Since V is $\mathcal{O}$-free, this image is isomorphic to the canonical image of A in $\operatorname{End}_K(K \otimes_\mathcal{O} V)$. Thus, if $K \otimes_\mathcal{O} V$ is a semisimple $K \otimes_\mathcal{O} A$-module, then this image depends only on the isomorphism classes of simple $K \otimes_\mathcal{O} A$-modules occurring in a decomposition of $K \otimes_\mathcal{O} V$, but not on the multiplicity of the simple factors of $K \otimes_\mathcal{O} V$.

Remark 2.7. It follows from the formal properties of Morita equivalences that a Morita equivalence between two algebras induces a bijection between quotients of the two algebras, in such a way that quotients corresponding to each other are again Morita equivalent. In particular, symmetric quotients are preserved under Morita equivalences. We describe this briefly for the convenience of the reader. Let A, B be Morita equivalent $\mathcal{O}$-algebras; that is, there is an A–B-bimodule M and a B–A-bimodule N such that M, N are finitely generated projective as left and right modules, and such that we have isomorphisms of bimodules $M \otimes_B N \cong A$ and $N \otimes_A M \cong B$. The functor $M \otimes_B -$ induces an equivalence between the categories of B–B-bimodules and of A–B-bimodules, sending B to M. Thus this functor induces a bijection between ideals in B and subbimodules of M, sending an ideal J in B to the subbimodule MJ. Note that since M is finitely generated projective as a right B-module, we have $MJ \cong M \otimes_B J$. Similarly, we have a bijection between ideals in A and subbimodules in M sending an ideal in I in A to the subbimodule $IM \cong I \otimes_A M$. Combining these bijections yields a bijection between ideals in A and ideals in B, with the property that the ideal I in A corresponds to the ideal J in B if and only if $IM = MJ$, which in turn holds if and only if $JN = NI$. If I, J correspond to each other through this bijection, then M/IM and N/JN induce a Morita equivalence between A/I and B/J. Indeed, M/IM is finitely generated projective as a left A/I-module, since M is finitely generated as a left A-module. Using $IM = MJ$, this argument shows that M/IM is finitely generated projective as a right B/J-module. Similarly, N/JN is finitely generated projective as a left and right module. Moreover, we have $M/IM \otimes_{B/J} N/JN \cong M/MJ \otimes_B N/JN \cong M \otimes_B N/JN \cong M \otimes_B N/M \otimes_B JN \cong M \otimes_B N/M \otimes_B NI \cong A/I$. The same argument with reversed roles yields $N/NI \otimes_{A/I} M/IM \cong B/J$. Since Morita equivalences preserve the property of being symmetric, this shows that A/I is symmetric if and only if B/J is symmetric.

3. Proof of Proposition 1.6

Let G be a finite p-group having a cyclic normal subgroup H of index p, and let $\chi \in \mathrm{Irr}_K(G)$. In order to prove Proposition 1.6 we may assume that G is nonabelian, hence H has order at least p^2.

Suppose first that p is odd. Then the automorphism of H induced by an element $t \in G - H$ acts trivially on the subgroup H^p of index p in H, hence $Z(G) = H^p$ has index p^2 in G. Any nonlinear character of G has degree p, hence is a character of central type. It follows from Proposition 1.8 that $\mathcal{O}Ge(\chi)$ is symmetric.

Suppose now that $p = 2$. The previous arguments remain valid so long as the action of G on the cyclic normal 2-subgroup H of index 2 is trivial on the subgroup H^2 of index 2 in H. This includes the case of semidihedral 2-groups (where t is an involution which acts on the cyclic subgroup H of order 2^n by sending a generator s of H to $s^{1+2^{n-1}}$). If $n = 2$, then $|G| = 8$, hence χ is a character of central type, and so the symmetry of $\mathcal{O}Ge(\chi)$ follows from Proposition 1.8. Suppose that $n \geq 3$ and that G does not act trivially on the subgroup of index 2 in H. Then G is dihedral or generalised quaternion

(corresponding in both cases to the action of an element $t \in G - H$ of order either 2 or 4 on H sending s to s^{-1}) or quasidihedral (corresponding to the action of t sending s to $s^{-1+2^{n-1}}$). We will need the following elementary facts (a proof is included for the convenience of the reader).

Lemma 3.1. *Let $n \geq 3$ and let ζ be a primitive 2^n-th root of unity in $\mathcal{O}$. Let a, $b \in \mathbb{Z}$ such that $b - a$ is even. The following hold.*

(i) *The numbers $\zeta^a \pm \zeta^b$ are divisible by $(1 - \zeta)^2$ in $\mathcal{O}$.*
(ii) *The numbers $\frac{\zeta \pm \zeta^{-1}}{(1-\zeta)^2}$ are invertible in $\mathcal{O}$.*

Proof. The integer 2 is divisible by $(1 - \zeta)^3$ in $\mathcal{O}$, as $n \geq 3$. In particular, $\frac{2}{(1-\zeta)^2} \in J(\mathcal{O})$, and hence, in order to prove (i), it suffices to show that $\zeta^a + \zeta^b$ is divisible by $(1-\zeta)^2$. Write $b - a = 2c$ for some integer c. Then $\zeta^a + \zeta^b = \zeta^a(1 + \zeta^{b-a}) = \zeta^a(1 + \zeta^{2c})$. Thus we may assume that $a = 0$ and $b = 2c$. We have $1 + \zeta^{2c} = 1 - \zeta^{2c} + 2\zeta^{2c} = (1 - \zeta^c)(1 + \zeta^c) + 2\zeta^{2c} = (1 - \zeta^c)(1 - \zeta^c + 2\zeta^c) + 2\zeta^{2c}$, which is divisible by $(1 - \zeta)^2$ because $1 - \zeta$ divides $1 - \zeta^c$ and $(1 - \zeta)^2$ divides 2. This shows (i). Again, since $\frac{2}{(1-\zeta)^2} \in J(\mathcal{O})$, in order to prove (ii), it suffices to prove that $\frac{\zeta - \zeta^{-1}}{(1-\zeta)^2}$ is invertible in $\mathcal{O}$. Observe that $\frac{1-\zeta^2}{(1-\zeta)^2} = \frac{1+\zeta}{1-\zeta} = 1 + \frac{2\zeta}{1-\zeta}$ is invertible in $\mathcal{O}$. Multiplying this by ζ^{-1} shows (ii). $\quad\square$

We complete the proof of Proposition 1.6. Let G be a dihedral or generalised quaternion group of order 2^{n+1}. In order to show that $\mathcal{O}Ge(\chi)$ is symmetric, we may assume that $\chi(1) > 1$, hence $\chi(1) = 2$. Let H be the cyclic subgroup of order 2^n of G. Since $\chi(1) = 2$ we have $\chi = \mathrm{Ind}_H^G(\eta)$ for some nontrivial linear character $\eta \in \mathrm{Irr}_K(H)$. Arguing by induction, we may assume that η is faithful. Let ζ be a root of unity of order 2^n and let s be a generator of H such that $\eta(s) = \zeta$. Let $\bar{\eta}$ be the character of H sending s to ζ^{-1}. Then $\mathrm{Res}_H^G(\chi) = \eta + \bar{\eta}$, and χ vanishes outside H. In particular,

$$e(\chi) = e(\eta) + e(\bar{\eta}) = \frac{1}{2^n} \sum_{a=0}^{2^n-1} (\zeta^a + \zeta^{-a})s^a.$$

By Lemma 3.1 the coefficients $\zeta^a + \zeta^{-a}$ in this sum are all divisible by $(1 - \zeta)^2$. Set $\lambda = \frac{2^n}{(1-\zeta)^2}$, and set $z = \lambda e(\chi)$. By Lemma 3.1(i) we have $z \in \mathcal{O}G$ and by Lemma 3.1(ii) we have that λ has minimal valuation such that $\lambda e(\chi) \in \mathcal{O}G$. Thus, by Proposition 2.5, it suffices to show that $\mathcal{O}Gz$ is $\mathcal{O}$-pure in $\mathcal{O}G$. Since $e(\chi) \in \mathcal{O}H$, and hence $z \in \mathcal{O}H$, it suffices to show that $\mathcal{O}Hz$ is $\mathcal{O}$-pure in $\mathcal{O}H$. For any a such that $0 \leq a \leq 2^n - 1$ we have

$$s^a z = \lambda\big(s^a e(\eta) + s^a e(\bar{\eta})\big) = \lambda\big(\zeta^a e(\eta) + \zeta^{-a} e(\bar{\eta})\big).$$

We claim that the set $\{z, 2^n e(\eta)\}$ is an $\mathcal{O}$-basis of $\mathcal{O}Hz$. Note first that by Proposition 2.5

$$sz - \zeta^{-1}z = \zeta^{-1}\big(\zeta^2 - 1\big)\lambda e(\eta) = \mu 2^n e(\eta)$$

for some $\mu \in \mathcal{O}^\times$. Thus both z and $2^n e(\eta)$ belong to $\mathcal{O}Hz$. We need to show that any of the elements $s^a z$ with a as before is an $\mathcal{O}$-linear combination of z and $2^n e(\eta)$. We have

$$s^a z - \zeta^{-a} z = \lambda \zeta^{-a} (\zeta^{2a}) e(\eta) = \nu 2^n e(\eta)$$

for some $\nu \in \mathcal{O}$, whence the claim. It remains to show that $\mathcal{O}Hz$ is $\mathcal{O}$-pure in $\mathcal{O}H$. Let $u = \sum_{a=0}^{2^n-1} \mu_a s^a$ be an element in $\mathcal{O}Hz$ such that all coefficients $\mu_a \in \mathcal{O}$ are divisible by $1 - \zeta$. Write $u = \beta z + \gamma 2^n e(\eta)$ with $\beta, \gamma \in \mathcal{O}$. We need to show that β, γ are divisible by $1 - \zeta$. Comparing coefficients for the two expressions of u above yields

$$\mu_a = \beta \frac{\zeta^a + \zeta^{-a}}{(1-\zeta)^2} + \gamma \zeta^a$$

for $0 \leq a \leq 2^n - 1$. If $a = 2^{n-2}$, then $\zeta^a + \zeta^{-a} = 0$, hence $\mu_a = \gamma \zeta^a$, which implies that γ is divisible by $1 - \zeta$, as μ_a is divisible by $1 - \zeta$. We consider the above equation for $a = 1$, which reads

$$\mu_1 = \beta \frac{\zeta + \zeta^{-1}}{(1-\zeta)^2} + \gamma \zeta.$$

By Lemma 3.1 we have $\frac{\zeta + \zeta^{-1}}{(1-\zeta)^2} \in \mathcal{O}^\times$. Since μ_1 and γ are divisible by $1 - \zeta$, this implies that β is divisible by $1 - \zeta$. Thus $\mathcal{O}Hz$ is $\mathcal{O}$-pure in $\mathcal{O}H$, and hence $\mathcal{O}Ge(\chi)$ is symmetric.

Let finally G be quasidihedral; that is, conjugation by the involution t sends s to $s^{-1+2^{n-1}}$. The calculations are similar to the previous case; we sketch the modifications. Let $\bar{\eta}$ be the character of H sending s to $\zeta^{-1+2^{n-1}} = -\zeta^{-1}$. Then $e(\chi) = e(\eta) + e(\bar{\eta}) = \sum_{a=0}^{2^n-1} (\zeta^a + (-1)^a \zeta^{-a})$. As before, setting $z = \frac{2^{n-1}}{(1-\zeta)^2} e(\chi)$, it suffices to show that $\mathcal{O}Hz$ is $\mathcal{O}$-pure in $\mathcal{O}H$. One verifies as before, that $\{z, 2^n e(\eta)\}$ is an $\mathcal{O}$-basis of $\mathcal{O}Hz$. Let $u = \sum_{a=0}^{2^n-1} \mu_a s^a$ be an element in $\mathcal{O}Hz$ such that all coefficients $\mu_a \in \mathcal{O}$ are divisible by $1 - \zeta$. Write $u = \beta z + \gamma 2^n e(\eta)$ with $\beta, \gamma \in \mathcal{O}$. We need to show that β, γ are divisible by $1 - \zeta$. Comparing coefficients yields $\mu_a = \beta \frac{\zeta^a + (-1)^a \zeta^{-a}}{(1-\zeta)^2} + \gamma \zeta^a$. If $a = 2^{n-2}$, then a is even and $\zeta^a + \zeta^{-a} = 0$, implying $\mu_a = \gamma \zeta^a$, which in turn implies that γ is divisible by $1 - \zeta$. Comparing coefficients for $a = 1$ yields that β is divisible by $1 - \zeta$. Thus $\mathcal{O}Hz$ is $\mathcal{O}$-pure in $\mathcal{O}H$, and hence $\mathcal{O}Ge(\chi)$ is symmetric. This completes the proof of Proposition 1.6.

4. On symmetric subalgebras of matrix algebras

Let G be a finite group. By the remarks following Lemma 2.1, the $\mathcal{O}$-free $\mathcal{O}$-algebra quotients of $\mathcal{O}G$ correspond to $\mathcal{O}$-pure ideals, hence to subsets of $\mathrm{Irr}_K(G)$, and by Remark 2.6 any $\mathcal{O}$-free quotient of $\mathcal{O}G$ is the image of a structural map $\alpha : \mathcal{O}G \to \mathrm{End}_\mathcal{O}(V)$ for some finitely generated $\mathcal{O}$-free $\mathcal{O}G$-module V. Moreover, the kernel of this map depends only on the set of irreducible characters of G arising as constituents of the character of V. If the character χ of V is irreducible, then the image of α is isomorphic to the algebra $\mathcal{O}Ge(\chi)$, where $e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1}) x$. In that case, $\mathrm{Im}(\alpha) \cong \mathcal{O}Ge(\chi)$ has

the same $\mathcal{O}$-rank $\chi(1)^2$ as $\mathrm{End}_{\mathcal{O}}(V)$. If $\chi \in \mathrm{Irr}_K(G)$ has degree one, then $\mathcal{O}Ge(\chi) \cong \mathcal{O}$ is trivially symmetric. If χ has defect zero, then $\mathcal{O}Ge(\chi)$ is a matrix algebra over $\mathcal{O}$, hence also symmetric. Both examples are special cases of the following well-known situation:

Proposition 4.1. *Let G be a finite group, K a splitting field for G, and $\chi \in \mathrm{Irr}_K(G)$. The algebra $\mathcal{O}Ge(\chi)$ is isomorphic to a matrix algebra over $\mathcal{O}$ if and only if χ lifts an irreducible Brauer character.*

Proof. We include a proof for the convenience of the reader. Let X be an $\mathcal{O}$-free $\mathcal{O}G$-module with character χ. The character χ lifts an irreducible Brauer character if and only if $k \otimes_{\mathcal{O}} X$ is a simple kG-module. The kG-module $k \otimes_{\mathcal{O}} X$ is simple if and only if the structural map $kG \to \mathrm{End}_k(k \otimes_{\mathcal{O}} X)$ is surjective. By Nakayama's Lemma, this is the case if and only if the structural map $\mathcal{O}G \to \mathrm{End}_{\mathcal{O}}(X)$ is surjective. The result follows. $\square$

If $\mathcal{O}Ge(\chi)$ is symmetric but not a matrix algebra, then the following observation narrows down the possible symmetrising forms.

Proposition 4.2. *Let V be a free $\mathcal{O}$-module of finite rank n, and let A be a symmetric subalgebra of $\mathrm{End}_{\mathcal{O}}(V)$ of rank n^2. Then $Z(A) \cong \mathcal{O}$, and there is an integer $r \geq 0$ such that the restriction to A of the map $\pi^{-r} \mathrm{tr}_V : \mathrm{End}_{\mathcal{O}}(V) \to K$ sends A to $\mathcal{O}$ and induces a symmetrising form on A. Moreover, r is the smallest nonnegative integer satisfying $\pi^r \mathrm{End}_{\mathcal{O}}(V) \subseteq A$.*

Proof. Since the $\mathcal{O}$-rank of A is n^2, we have $K \otimes_{\mathcal{O}} A \cong \mathrm{End}_K(K \otimes_{\mathcal{O}} V)$, whence $Z(A) \cong \mathcal{O}$. Any symmetrising form on $\mathrm{End}_K(K \otimes_{\mathcal{O}} V)$ is a nonzero linear multiple of the trace map $\mathrm{tr}_{K \otimes_{\mathcal{O}} V}$, and hence a symmetrising form on A is of the form $\pi^{-r} \mathrm{tr}_V$ for some integer r such that $\pi^{-r} \mathrm{tr}_V(A) = \mathcal{O}$. Since $\mathrm{tr}_V(A) \subseteq \mathcal{O}$, this forces $r \geq 0$. Let s be the smallest nonnegative integer satisfying $\pi^s \mathrm{End}_{\mathcal{O}}(V) \subseteq A$. Let e be a primitive idempotent in $\mathrm{End}_{\mathcal{O}}(V)$. Then $\mathrm{tr}_V(e) = 1$. We have $\pi^s e \in A$, hence $\pi^{-r} \mathrm{tr}_V(\pi^s e) = \pi^{s-r} \in \mathcal{O}$. This implies that $s \geq r$. Since $\pi^{s-1} \mathrm{End}_{\mathcal{O}}(V)$ is not contained in A, there is an element $c \in A$ such that $\pi^{-1}c \notin A$ and $c \in \pi^s \mathrm{End}_{\mathcal{O}}(V)$. Thus $\mathcal{O}c$ is a pure $\mathcal{O}$-submodule of A. Thus there is an $\mathcal{O}$-basis of A containing c. By considering the dual $\mathcal{O}$-basis with respect to the symmetrising form $\pi^{-r} \mathrm{tr}_V$, it follows that $\pi^{-r} \mathrm{tr}_V(cA) = \mathcal{O}$. Since $c \in \pi^s \mathrm{End}_{\mathcal{O}}(V)$, we have $\mathrm{tr}_V(cA) \subseteq \pi^s \mathcal{O}$, hence $\pi^{-r} \mathrm{tr}_V(cA) \subseteq \pi^{s-r} \mathcal{O}$. This yields the inequality $s \leq r$, whence the equality $s = r$. $\square$

Corollary 4.3. *Let V be a free $\mathcal{O}$-module of finite rank n, and let A be a proper symmetric subalgebra of $\mathrm{End}_{\mathcal{O}}(V)$ of rank n^2. Then for any idempotent $i \in A$, the integer $\mathrm{tr}_V(i)$ is divisible by p. In particular, p divides n.*

Proof. If A is a proper symmetric subalgebra of $\mathrm{End}_{\mathcal{O}}(V)$, then the smallest nonnegative integer r satisfying $\pi^r \mathrm{End}_{\mathcal{O}}(V) \subseteq A$ is positive. It follows from Proposition 4.2 that A has

a symmetrising form which is the restriction to A of $\pi^{-r} \operatorname{tr}_V$ for some positive integer r. In particular, if i is an idempotent in A, then $\pi^{-r} \operatorname{tr}_V(i)$ is an element in $\mathcal{O}$. Since $\operatorname{tr}_V(i)$ is an integer and $r > 0$, this implies that p divides $\operatorname{tr}_V(i)$. Applied to $i = 1_A = \operatorname{Id}_V$ yields that p divides $n = \operatorname{tr}_V(\operatorname{Id}_V)$. $\square$

Corollary 4.4. *Let V be a free $\mathcal{O}$-module of finite rank n. Suppose that $(n, p) = 1$. Then $\operatorname{End}_{\mathcal{O}}(V)$ has no proper symmetric subalgebra of $\mathcal{O}$-rank n^2.*

Proof. This is clear by Corollary 4.3. $\square$

Corollary 4.5. *Let V be a free $\mathcal{O}$-module of finite rank p. Every proper symmetric subalgebra of rank p^2 of $\operatorname{End}_{\mathcal{O}}(V)$ is local.*

Proof. If i is an idempotent in a proper symmetric subalgebra A of $\operatorname{End}_{\mathcal{O}}(V)$, then p divides $\operatorname{tr}_V(i) \leq \operatorname{tr}_V(\operatorname{Id}_V) = p$. Thus $i = \operatorname{Id}_V$. The result follows. $\square$

In order to prove a slightly more general version of Theorem 1.1, we use the following notation. Let A be an $\mathcal{O}$-algebra which is finitely generated free as an $\mathcal{O}$-module, such that $K \otimes_{\mathcal{O}} A$ is split semisimple and $k \otimes_{\mathcal{O}} A$ is split. Let X be a simple $K \otimes_{\mathcal{O}} A$-module. Then there is an $\mathcal{O}$-free A-module V such that $K \otimes_{\mathcal{O}} V \cong X$. It is well-known that for any simple $k \otimes_{\mathcal{O}} A$-module S the number d_S^X of composition factors isomorphic to S in a composition series of $k \otimes_{\mathcal{O}} V$ does not depend on the choice of V. This follows, for instance, from the fact that $d_S^X = \dim_K(iX)$, where i is a primitive idempotent in A such that $Ai/J(A)i \cong S$. We denote by $e(X)$ the central primitive idempotent in $K \otimes_{\mathcal{O}} A$ which acts as identity on X. We say that X lifts a simple $k \otimes_{\mathcal{O}} A$-module S if $k \otimes_{\mathcal{O}} V \cong S$, or equivalently, if $d_S^X = 1$ and $d_{S'}^X = 0$ for any simple $k \otimes_{\mathcal{O}} A$-module S' not isomorphic to S. Theorem 1.1 is the special case $A = \mathcal{O}G$ of the following result.

Theorem 4.6. *Let A be an $\mathcal{O}$-algebra which is finitely generated free as an $\mathcal{O}$-module, such that $K \otimes_{\mathcal{O}} A$ is split semisimple and $k \otimes_{\mathcal{O}} A$ is split. Let X be a simple $K \otimes_{\mathcal{O}} A$-module. Suppose that the $\mathcal{O}$-algebra $Ae(X)$ is symmetric. Then either X lifts a simple $k \otimes_{\mathcal{O}} A$-module, or d_S^X is divisible by p for any simple $k \otimes_{\mathcal{O}} A$-module S.*

Proof. Let V be an $\mathcal{O}$-free A-module such that $K \otimes_{\mathcal{O}} V \cong X$. Suppose that X does not lift a simple $k \otimes_{\mathcal{O}} A$-module; that is, $k \otimes_{\mathcal{O}} V$ is not simple. Thus the image, denoted by E, of the structural map $A \to \operatorname{End}_{\mathcal{O}}(V)$ is a proper subalgebra of $\operatorname{End}_{\mathcal{O}}(V)$. Note that $E \cong Ae(X)$. Moreover, for any idempotent $i \in A$ we have $\operatorname{tr}_V(i) = \operatorname{rank}_{\mathcal{O}}(iV) = \dim_K(iX)$. As $Ae(X)$ is assumed to be symmetric, it follows from Corollary 4.3 that for any idempotent $i \in A$, the integer $\dim_K(iX)$ is divisible by p. Applied to primitive idempotents, this shows that d_S^X is divisible by p for any simple $k \otimes_{\mathcal{O}} A$-module S. $\square$

5. Proofs of Propositions 1.9 and 1.11

If $\chi : G \to \mathcal{O}^\times$ is a linear character, then the corresponding $\mathcal{O}$-pure ideal $I_\chi = KGe(\chi) \cap \mathcal{O}G$ has $\mathcal{O}$-rank 1 and is equal to $\mathcal{O}(\sum_{x \in G} \chi(x^{-1})x)$. There is a unique $\mathcal{O}$-algebra automorphism of $\mathcal{O}G$ sending $x \in G$ to $\chi(x)x$. This automorphism sends $\sum_{x \in G} \chi(x^{-1})x$ to $\sum_{x \in G} x$. Thus the linear characters of $\mathcal{O}G$ are permuted transitively by the group of $\mathcal{O}$-algebra automorphisms of $\mathcal{O}G$, and therefore, in order to address the question whether $\mathcal{O}G/I_\chi$ is symmetric, it suffices to consider the case where $\chi = 1$ is the trivial character, in which case the corresponding pure ideal is $I_1 = \mathcal{O}G(\sum_{x \in G} x) = \mathcal{O}(\sum_{x \in G} x)$. The annihilator of I_1 in $\mathcal{O}G$ is the augmentation ideal $I(\mathcal{O}G)$. Combining these observations with Proposition 2.2 and some block theory yields a proof of Proposition 1.9, which we restate in a slightly more precise way.

Proposition 5.1. *Let G be a finite group. The following are equivalent.*

(i) *The $\mathcal{O}$-algebra $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$ is symmetric.*
(ii) *There exists an element $z \in Z(\mathcal{O}G)$ such that $I(\mathcal{O}G) = \mathcal{O}Gz$.*
(iii) *The group G is p-nilpotent and has a cyclic Sylow-p-subgroup.*

Proof. The equivalence of (i) and (ii) is clear by Proposition 2.2. Suppose that (ii) holds. Let b be the principal block idempotent of $\mathcal{O}G$. Then $I(\mathcal{O}G)b = \mathcal{O}Gzb$ is a proper ideal in $\mathcal{O}Gb$; in particular, zb is not invertible in $Z(\mathcal{O}Gb)$. Since $Z(\mathcal{O}Gb)$ is local, it follows that zb is in the radical of $Z(\mathcal{O}Gb)$, and hence the ideal $\mathcal{O}Gzb$ is contained in $J(\mathcal{O}Gb)$. Since $\mathcal{O}Gb/I(\mathcal{O}G)b \cong \mathcal{O}G/I(\mathcal{O}G) \cong \mathcal{O}$, this implies that $\mathcal{O}Gb$ is a local algebra, and that $kG\bar{z}\bar{b} = J(kG\bar{b})$, where $\bar{z}$, $\bar{b}$ are the canonical images of z, b in kG, respectively. It follows that the finite group G is p-nilpotent (see [6, Chapter 5, Theorems 8.1 and 8.2]). Since $J(kG\bar{b}) = kG\bar{z}\bar{b}$ is a principal ideal, it follows from a result of Nakayama [8] that $kG\bar{b}$ is uniserial, and hence P is cyclic. Thus (ii) implies (iii). Conversely, if (iii) holds, then the principal block algebra $\mathcal{O}Gb$ of $\mathcal{O}G$ is isomorphic to $\mathcal{O}P$, where P is a Sylow-p-subgroup of G, and if y is a generator of P, then $I(\mathcal{O}P) = \mathcal{O}P(y-1)$. Note that $I(\mathcal{O}G)$ contains all nonprincipal block algebras of $\mathcal{O}G$. Set $z = (y-1)b + \sum_{b'} b'$, where in the sum b' runs over all nonprincipal block idempotents. By the above, this is an element in $Z(\mathcal{O}G)$ satisfying $I(\mathcal{O}G) = \mathcal{O}Gz$, completing the proof. $\square$

Proof of Corollary 1.10. Suppose that $\mathbb{Z}G/\mathbb{Z}(\sum_{x \in G} x)$ is symmetric. Tensoring by $\mathcal{O}$ over $\mathbb{Z}$ implies that the $\mathcal{O}$-algebra $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$ is symmetric. Thus G is p-nilpotent with a cyclic p-Sylow subgroup. This holds for any prime p. It follows that G is a direct product of its Sylow subgroups all of which are cyclic, and hence G is cyclic. Suppose conversely that G is cyclic. Let y be a generator of G. Then the augmentation ideal of $\mathbb{Z}G$ is equal to $(y-1)\mathbb{Z}G$, and this is also equal to the annihilator of $\mathbb{Z}(\sum_{x \in G} x)$.

It follows from Proposition 2.2, applied with the Noetherian ring $\mathbb{Z}$ instead of $\mathcal{O}$, that $\mathbb{Z}G/\mathbb{Z}(\sum_{x \in G} x)$ is symmetric. □

Proof of Proposition 1.11. Suppose that B/I is symmetric. By Proposition 2.2 the annihilator J of I is of the form $J = Bz$ for some $z \in Z(B)$. Let X be an $\mathcal{O}$-free B-module with character χ. By the assumptions, $\bar{X} = k \otimes_{\mathcal{O}} X$ is a simple module over $\bar{B} = k \otimes_{\mathcal{O}} B$. By Proposition 4.1, and using that k is large enough, the structural map $B \to \mathrm{End}_{\mathcal{O}}(X)$ is surjective, and the kernel of this map is J. Since $J = Bz$ is a proper ideal in B and $Z(B)$ is local, it follows that $z \in J(B)$. Thus the image $\bar{J}$ of J in $\bar{B}$ is equal to $J(\bar{B})$, and $\bar{B}$ has a single isomorphism class of simple modules. Since $J(\bar{B}) = \bar{B}\bar{z}$, where $\bar{z}$ is the image of z in $Z(\bar{B})$, it follows as before from [8] that $\bar{B}$ is uniserial. Thus B has cyclic defect groups. A block with cyclic defect and a single isomorphism class of simple modules is nilpotent, which shows that (i) implies (ii). Conversely, if (ii) holds, then B is Morita equivalent to $\mathcal{O}P$, where P is a defect group of B (and P is cyclic by the assumptions). Since χ lifts an irreducible Brauer character it follows that under some Morita equivalence, χ corresponds to the trivial character of $\mathcal{O}P$, and hence (i) holds by Proposition 1.9 applied to P. □

6. An Example

As was pointed out after Proposition 1.8, all irreducible characters of finite p-groups of order at most p^3 have the symmetric quotient property. The next example shows that for any odd prime p there is a finite p-group of order p^{p+1} having at least one irreducible character which does not have the symmetric quotient property.

Example 6.1. Let p be an odd prime. $G = Q \wr R = H \rtimes R$, where Q, R are cyclic of order p, and where H is a direct product of p copies of Q which are transitively permuted by R. Let s be a generator of Q and ζ be a primitive p-th root of unity. For $1 \le i \le p$ let $\psi_i : H \to \mathcal{O}^\times$ be the linear character sending $(s^{a_1}, s^{a_2}, .., s^{a_p}) \in H$ to ζ^{a_i}; that is, the kernel of ψ_i contains all but the i-th copy of Q in H, the ψ_i are pairwise different, and they are permuted transitively by the action of G. Set $\chi = \mathrm{Ind}_H^G(\psi_1)$. Then $\chi \in \mathrm{Irr}_K(G)$, and the $\mathcal{O}$-algebra $\mathcal{O}Ge(\chi)$ is not symmetric.

The irreducibility of χ is a standard result. In order to show the symmetry of $\mathcal{O}Ge(\chi)$, observe first that $\mathrm{Res}_H^G(\chi) = \sum_{i=1}^p \psi_i$ because the ψ_i form a G-orbit in $\mathrm{Irr}_K(H)$. We have

$$e(\chi) = \sum_{i=1}^p e(\psi_i) = \frac{1}{|H|} \sum_{h \in H} \left(\sum_{i=1}^p \psi_i(h^{-1}) \right) h.$$

The coefficients $\sum_{i=1}^p \psi_i(h^{-1})$ are divisible by $1 - \zeta$ because they are sums of p (arbitrary) powers of ζ. Moreover, for $h = (s, 1, .., 1) \in H$ one sees that $1 - \zeta$ is the highest power of $1 - \zeta$ dividing this coefficient. Thus if $\mathcal{O}Ge(\chi)$ were symmetric, then $\mathcal{O}Gz$ would have

to pure in $\mathcal{O}G$, where $z = \frac{|H|}{1-\zeta} e(\chi)$. Since $z \in \mathcal{O}H$, this is the case if and only if $\mathcal{O}Hz$ is pure in $\mathcal{O}H$. We will show that $\mathcal{O}Hz$ is not pure in $\mathcal{O}H$. If $u = (s^{a_1}, s^{a_2}, .., s^{a_p}) \in H$, then

$$uz = \frac{|H|}{1-\zeta} \sum_{i=1}^{p} ue(\psi_i) = \frac{|H|}{1-\zeta} \sum_{i=1}^{p} \zeta^{a_i} e(\psi_i).$$

Applied to the identity element and $v = (1, 1, ..1, s, 1, ..1)$, with s in the i-th component, and taking the difference yields $z - vz = |H|e(\psi_i) \in \mathcal{O}Gz$. For any $u \in H$, the above formula yields $uz - z \in \oplus_{i=1}^{p} \mathcal{O}|H|e(\psi)$. Thus the set

$$\left\{ z, |H|e(\psi_i) \ (2 \leq i \leq p) \right\}$$

is an $\mathcal{O}$-basis of $\mathcal{O}Hz$. Since p is odd, this basis has at least three elements. Suppose that $w = \sum_{h \in H} \mu_h h$ is an element in $\mathcal{O}Hz$. Write $w = \alpha z + \sum_{i=2}^{p} \beta_i |H|e(\psi_i)$ with $\alpha, \beta_i \in \mathcal{O}$. Thus

$$\mu_h = \alpha \frac{\sum_{i=1}^{p} \psi(h^{-1})}{1-\zeta} + \sum_{i=2}^{p} \beta_i \psi_i (h^{-1}).$$

Note that if $\sum_{i=2}^{p} \beta_i$ is divisible by $1 - \zeta$ then any sum of the form $\sum_{i=2}^{p} \beta_i \psi_i(h^{-1})$ is divisible by $1 - \zeta$ because any character value $\psi_i(h^{-1})$ is a power of ζ. This shows that if $1 - \zeta$ divides both α and the sum $\sum_{i=2}^{p} \beta_i$, then $1 - \zeta$ divides μ_h for all $h \in H$. But since $p > 2$ we may choose invertible elements β_i satisfying $\sum_{i=2}^{p} \beta_i = 0$. This shows that even if all μ_h are divisible by $1 - \zeta$, this does not imply that α and all β_i are divisible by $1 - \zeta$, hence $\mathcal{O}Hz$ is not $\mathcal{O}$-pure in $\mathcal{O}H$.

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