



ELSEVIER

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



McKay correspondence for semisimple Hopf actions on regular graded algebras, I

K. Chan^a, E. Kirkman^b, C. Walton^c, J.J. Zhang^a

^a Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA

^b Department of Mathematics, P.O. Box 7388, Wake Forest University, Winston-Salem, NC 27109, USA

^c Department of Mathematics, The University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

Article history:

Received 25 November 2017

Available online 16 May 2018

Communicated by Nicolás Andruskiewitsch

MSC:

16T05

16E10

14B05

Keywords:

Artin–Schelter regular algebras

Auslander’s theorem

Hopf algebra action

McKay correspondence

McKay quiver

Trivial homological determinant

ABSTRACT

In establishing a more general version of the McKay correspondence, we prove Auslander’s theorem for actions of semisimple Hopf algebras H on noncommutative Artin–Schelter regular algebras A of global dimension two, where A is a graded H -module algebra, and the Hopf action on A is inner faithful with trivial homological determinant. We also show that each fixed ring A^H under such an action arises as an analogue of a coordinate ring of a Kleinian singularity.

© 2018 Elsevier Inc. All rights reserved.

E-mail addresses: kenhchan@math.washington.edu (K. Chan), kirkman@wfu.edu (E. Kirkman), notlaw@illinois.edu (C. Walton), zhang@math.washington.edu (J.J. Zhang).

0. Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. All algebraic structures in this article are over $\mathbb{k}$, and we take the unadorned $\otimes$ to mean $\otimes_{\mathbb{k}}$.

The classical McKay correspondence provides deep and elegant ties between:

- finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}_2(\mathbb{C})$ (also called *binary polyhedral groups*);
- simple Lie algebras of type ADE;
- Platonic solids;
- Kleinian or DuVal singularities (that is, rational double points or quotient singularities $\mathbb{C}^2/\tilde{\Gamma}$ where $\tilde{\Gamma}$ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$);
- $\tilde{\Gamma}$ -Hilbert schemes;
- preprojective algebras of the reduced McKay quivers;

as well as other entities in algebra, geometry and string theory. At first glance these items are unrelated, yet they are all governed by the ADE Dynkin diagrams; see Buchweitz [5] for an excellent survey on the classical McKay correspondence, as well as McKay's original work [23]. Moreover, a very nice review of the McKay correspondence and noncommutative crepant resolutions is given by Leuschke [19]. Further, in this direction, different versions of a quantum McKay correspondence have been studied by many researchers, e.g. [4,7,15,26]. The goal of this paper (and the sequel [8]) is to produce an analogue of the McKay correspondence in the context of finite dimensional Hopf actions on noncommutative regular algebras; this is applicable in both noncommutative algebraic geometry and noncommutative invariant theory.

Consider the setting below.

Hypothesis 0.1. Unless indicated otherwise, H is a semisimple Hopf algebra (hence finite dimensional), A is an *Artin–Schelter (AS) regular algebra* [Definition 1.1] generated in degree one, which by definition is connected graded. Further, A is a *graded H -module algebra* [Definition 1.3] under an H -action that is *inner faithful* [Definition 1.5] with *trivial homological determinant* [Definition 1.6]. We also assume that $A^H \neq A$ throughout.

When A is commutative, it is a commutative polynomial ring, and when H is cocommutative, it is the group algebra of a finite subgroup of $SL_n(\mathbb{k})$, with G acting as graded automorphisms of A . Hence our non(co)commutative setting generalizes the setting of the classical McKay correspondence.

This paper can be viewed as a continuation of the project started in [9], where the finite dimensional Hopf actions on AS regular algebras of dimension 2 with trivial homological determinant were classified. The AS regular algebras of dimension 2 are isomorphic to one of the two families of algebras given in Example 1.2. The classification of such semisimple Hopf actions on AS regular algebras of dimension 2 is repeated in Theorem 0.4 below.

Table 1

Quantum binary polyhedral groups and their module algebras.

Case	A	H
(a)	$\mathbb{k}[u, v]$	$\mathbb{k}\tilde{\Gamma}$ for $\tilde{\Gamma}$ a finite subgroup of $\mathrm{SL}_2(\mathbb{k})$
(b)	$\mathbb{k}_{-1}[u, v]$	$\mathbb{k}C_n$ for $n \geq 2$ (diagonal action)
(c)	$\mathbb{k}_{-1}[u, v]$	$\mathbb{k}C_2$ (nondiagonal action)
(d)	$\mathbb{k}_{-1}[u, v]$	$\mathbb{k}D_{2n}$ for $n \geq 3$
(e)	$\mathbb{k}_{-1}[u, v]$	$(\mathbb{k}D_{2n})^\circ$ for $n \geq 3$
(f)	$\mathbb{k}_{-1}[u, v]$	$\mathcal{D}(\tilde{\Gamma})^\circ$ for $\tilde{\Gamma}$ a finite nonabel. subgroup of $\mathrm{SL}_2(\mathbb{k})$
(g)	$\mathbb{k}_q[u, v]$ with $q^2 \neq 1$	$\mathbb{k}C_n$ for $n \geq 2$ (diagonal action)
(h)	$\mathbb{k}_J[u, v]$	$\mathbb{k}C_2$ (diagonal action)

A key ingredient in the classical McKay correspondence is Auslander's Theorem, which states that if G is a small finite group acting on a commutative polynomial ring S , then $S\#\mathbb{k}G$ is isomorphic to $\mathrm{End}_{S^G}(S)$ as algebras [1, Proposition 3.4] (see also [20, Theorem 5.12]). We conjecture that a version of this result holds for semisimple Hopf actions on noetherian AS regular algebras.

Conjecture 0.2. *Let H be a semisimple Hopf algebra and suppose that A is a noetherian AS regular H -module algebra satisfying Hypothesis 0.1. Then there is a natural graded algebra isomorphism*

$$A\#H \cong \mathrm{End}(A_{A^H}). \quad (0.2.1)$$

Our main result is to show that Conjecture 0.2 is true in dimension 2, which answers [9, Question 8.5].

Theorem 0.3 (Theorem 4.1). *If A is a noetherian AS regular algebra of dimension 2, then Conjecture 0.2 is true.*

In [9] we gave the classification of pairs (A, H) , where H is a finite dimensional Hopf algebra, and A is an AS regular algebra of global dimension 2, generated in degree one, that admits a graded, inner faithful H -action with trivial homological determinant. We call such a Hopf algebra H a *quantum binary polyhedral group*. The pairs (A, H) , where H is semisimple, are described in the following theorem, on which the proof of Theorem 0.3 is based.

Theorem 0.4. ([9, Theorem 0.4]). *Suppose that (A, H) is a pair satisfying Hypothesis 0.1 with $\mathrm{gldim} A = 2$. Then (A, H) is one from Table 1.*

Here C_n is a cyclic group of order n , D_{2n} is a dihedral group of order $2n$, and $\mathcal{D}(\tilde{\Gamma})$ is a cocycle deformation of $(\mathbb{k}\tilde{\Gamma})^\circ$ [22]. The H -action on A is defined by the H -module (or equivalently, the H° -comodule) structure of $A_1 = \mathbb{k}u \oplus \mathbb{k}v$, as given below.

(a, d) A_1 is a faithful simple 2-dimensional G -module with $G = \tilde{\Gamma}$ or D_{2n} , resp.

- (b, g, h) $\mathbb{k}u$ is a faithful C_n -module and $\mathbb{k}v$ is its dual.
- (c) $\mathbb{k}(u + v)$ is the trivial C_2 -module, and $\mathbb{k}(u - v)$ is the sign representation.
- (e) Let $v_1 = u + \sqrt{-1}v$ and $v_2 = \sqrt{-1}u + v$. Then $\mathbb{k}v_1$ (resp. $\mathbb{k}v_2$) is 1-dimensional $\mathbb{k}D_{2n}$ -comodule with structure map $\rho(v_1) = v_1 \otimes g$ (resp. $\rho(v_2) = v_2 \otimes h$), where $D_{2n} = \langle g, h \mid g^2 = h^2 = (gh)^n = 1 \rangle$.
- (f) $\mathcal{D}(\tilde{\Gamma})$ is a finite dimensional Hopf quotient of the quantum special linear group $\mathcal{O}_{-1}(SL_2(\mathbb{k}))$ generated by $(e_{i,j})_{1 \leq i,j \leq 2}$, which coacts on A_1 from the right via $\rho(u) = u \otimes e_{1,1} + v \otimes e_{2,1}$ and $\rho(v) = u \otimes e_{1,2} + v \otimes e_{2,2}$. $\square$

We also have the following result for the fixed subrings A^H . These rings are referred to as *quantum Kleinian singularities* [Definition 5.1].

Theorem 0.5. ([9, Theorem 0.6] and Theorem 5.2). *Suppose that (A, H) is a pair satisfying Hypothesis 0.1 with $\text{gldim } A = 2$. Then A^H is not AS regular, and is isomorphic to $C/C\Omega$, for C an AS regular algebra of dimension 3 and Ω a normal element in C .*

We also define in a natural way, the *McKay quiver* of a semisimple Hopf action on a connected algebra A in Definition 2.3. In the sequel to this work [8], we obtain a correspondence between this quiver and the *Gabriel quiver* defined in terms of projective $A\#H$ -modules. Various correspondences between module categories over A^H , over $A\#H$, and over $\text{End}(A_{A^H})$ are presented, as well, in [8].

The paper is organized as follows. Section 1 contains definitions and preliminary results. In Section 2, we define the McKay quiver as discussed above, and we recall a result from [9] concerning the McKay quivers that can occur under our standing hypotheses with $\text{gldim } A = 2$. Section 3 contains results on the fixed subrings arising from actions with McKay quiver of types $\mathbb{D}$ and $\mathbb{E}$; this is one of key steps in the proof of Theorem 0.3. Section 4 contains the complete proof of our version of Auslander’s Theorem for semisimple Hopf actions on AS regular algebras of dimension 2 [Theorem 0.3] by considering the possible pairs (A, H) that satisfy our standing hypotheses (from Theorem 0.4). In Section 5, we prove Theorem 0.5.

1. Preliminaries

In this section, we provide background material on Artin–Schelter regular algebras, Hopf algebra actions on graded algebras, and the homological determinant of such actions. Note that we usually work with left modules, and a right A -module is identified with a (left) A^{op} -module, where A^{op} is the opposite ring of A . Let $A\text{-Mod}$ denote the category of left modules over an algebra A . Moreover, for every $\mathbb{Z}$ -graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$, and for each positive integer d , the d -th Veronese subring of A is defined to be $A^{(d)} := \bigoplus_{i \in \mathbb{Z}} A_{id}$.

1.1. Artin–Schelter regular algebras

An algebra A is said to be *connected graded* if $A = \bigoplus_{i \in \mathbb{N}} A_i$ with $A_0 = \mathbb{k}$ and $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$, and a graded algebra is called *locally finite* if $\dim_{\mathbb{k}} A_i < \infty$ for all i . In this case, the *Hilbert series* of A is defined to be $H_A(t) := \sum_{i \in \mathbb{N}} (\dim_{\mathbb{k}} A_i) t^i$.

Here we consider a class of noncommutative graded algebras that serve as noncommutative analogues of commutative polynomial rings. These algebras are defined as follows.

Definition 1.1. A connected graded algebra A is called *Artin–Schelter (AS) Gorenstein* if the following conditions hold:

- (i) A has finite injective dimension $d < \infty$ on both sides,
- (ii) $\text{Ext}_A^i(\mathbb{k}, A) = \text{Ext}_{A^{\text{op}}}^i(\mathbb{k}, A) = 0$ for all $i \neq d$ where $\mathbb{k} = A/A_{\geq 1}$, and
- (iii) $\text{Ext}_A^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$ and $\text{Ext}_{A^{\text{op}}}^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$ for some integer ℓ .

The integer ℓ is called the *AS index* of A . If moreover,

- (iv) A has finite global dimension d ,
- (v) A has finite Gelfand–Kirillov dimension,

then A is called *Artin–Schelter (AS) regular* of dimension d .

Example 1.2. The AS regular algebras of global dimension 2, that are generated in degree one, are listed below (up to isomorphism):

- (i) the *Jordan plane*: $\mathbb{k}_J[u, v] := \mathbb{k}\langle u, v \rangle / (vu - uv - u^2)$, and
- (ii) the *skew polynomial ring*: $\mathbb{k}_q[u, v] := \mathbb{k}\langle u, v \rangle / (vu - quv)$ for $q \in \mathbb{k}^\times$.

It is well-known that these algebras are noetherian domains.

1.2. Hopf algebra actions

Throughout H stands for a Hopf algebra over $\mathbb{k}$ with structural notation $(H, m, u, \Delta, \epsilon, S)$, and we write $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler notation). We recommend [24] as a basic reference for the theory of Hopf algebras.

Definition 1.3. Let A be an algebra and H a Hopf algebra.

- (1) We say that A is a (*left*) H -*module algebra*, or that H *acts on* A , if A is an algebra in the category of left H -modules. Equivalently, A is a left H -module such that

$$h(ab) = \sum h_1(a)h_2(b) \text{ and } h(1_A) = \epsilon(h)1_A$$

for all $h \in H$, and all $a, b \in A$.

- (2) We say that A is a *graded H -module algebra* if A is an algebra in the category of $\mathbb{N}$ -graded (left) H -modules (with elements in H having degree zero), or equivalently, each homogeneous component of A is a left H -submodule.
- (3) Given a H -module algebra A , the *smash product algebra* $A\#H$ is $A\otimes H$ as a $\mathbb{k}$ -vector space, with multiplication defined by

$$(a\#h)(a'\#h') = \sum ah_1(a')\#h_2h'$$

for all $h, h' \in H$ and $a, a' \in A$.

We identify H (resp. A) with a subalgebra of $A\#H$ via the map $i_H : h \mapsto 1\#h$ for all $h \in H$ (resp. the map $i_A : a \mapsto a\#1$ for all $a \in A$).

If A is a graded H -module algebra, then $A\#H$ is graded with $\deg h = 0$ for all $0 \neq h \in H$ and A is a graded subalgebra of $A\#H$. If, further, A is locally finite and H is finite dimensional, then $A\#H$ is locally finite and $(A\#H)_i = A_i\#H$ for all i .

Remark 1.4. One can form a right-handed version of the smash product algebra, denoted by $H\#A$, for a right H -module algebra A . When the antipode S is bijective as assumed, these two versions can be exchanged [28, Lemma 2.1]. We work with the left-handed version of smash product and would like to remark that all statements can be transformed from the left-handed version to the right-handed one.

We want to restrict ourselves to Hopf (H -)actions that do not factor through the action of a proper Hopf quotient of H .

Definition 1.5. Let M be a left H -module. We say that M is an *inner faithful H -module*, or H *acts inner faithfully* on M , if $IM \neq 0$ for every nonzero Hopf ideal I of H . The same terminology applies to H -module algebras A .

We now recall the homological determinant of a Hopf algebra action on an Artin–Schelter regular algebra below. Recall that by [21, Corollary D], a connected graded algebra A is AS regular if and only if the Ext-algebra $E = \bigoplus_{i \geq 0} \text{Ext}_A^i(A\mathbb{k}, A\mathbb{k})$ of A is Frobenius.

Definition 1.6. [16] Retain the notation above. Let A be an AS regular algebra with Frobenius Ext-algebra E . Suppose ϵ is a nonzero element in $\text{Ext}_A^d(A\mathbb{k}, A\mathbb{k})$ where $d = \text{gldim } A$. Let H be a Hopf algebra acting on A from the left. By [16, Lemma 5.9], H acts on E from the left.

- (1) The *homological determinant* of the H -action on A , denoted by $\text{hdet}_H A$, is defined to be $\eta \circ S$ where $\eta : H \rightarrow k$ is determined by $h \cdot \epsilon = \eta(h)\epsilon$.
- (2) The homological determinant is *trivial* if $\text{hdet}_H A = \epsilon$.

In the case that A is a commutative polynomial ring $\mathbb{k}[x_1, \dots, x_n]$, and $H = kG$, for G a finite subgroup of $GL_n(k)$, $\text{hdet}_H A$ becomes the ordinary determinant $\det_G : \mathbb{k}G \rightarrow \mathbb{k}$ [16, Remark 3.4(c)]. In this case, the homological determinant of the $\mathbb{k}G$ -action on A is trivial if and only if $G \leq SL_n(\mathbb{k})$.

2. McKay quivers

One of the most important objects in the McKay correspondence is the McKay quiver. In this section we present a version of such a diagram in the semisimple Hopf action setting. First, we begin with a discussion of fusion rings.

Given a Hopf algebra H over $\mathbb{k}$, we define a ring associated to a set of nonisomorphic finite dimensional H -modules.

Definition 2.1. Let H be a Hopf algebra. Let $\{[V_i]\}_{i \in J}$ be the set of all isomorphism classes of finite dimensional left H -modules, indexed by some set J . We define the *Grothendieck ring* $G_0(H) := G_0({}_H H)$ of H to be an associative $\mathbb{Z}$ -algebra generated by $\{[V_i]\}_{i \in J}$, subject to relations $[V] = [U] + [W]$ for each short exact sequence of H -modules

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0.$$

Then $G_0(H)$ has a $\mathbb{Z}$ -basis $\{[V_i]\}_{i \in I}$ consisting of isomorphism classes of all finite dimensional simple left H -modules where I is a subset of J . Moreover, for $i, j \in I$, the multiplication in $G_0(H)$ is determined by

$$[V_i] \cdot [V_j] = \sum_{k \in I} n_{i,j}^k [V_k], \quad (2.1.1)$$

where $n_{i,j}^k \in \mathbb{N}$ is the multiplicity of the finite dimensional simple left H -module V_k in the composition series of the H -module $V_i \otimes V_j$. The unit of $G_0(H)$ is $[\mathbb{k}]$.

We arrive at our definition of a fusion ring.

Definition 2.2. Let F be an associative ring over $\mathbb{Z}$, with multiplicative identity, equipped with an augmentation map $\pi : F \rightarrow \mathbb{Z}$. We call F a *fusion ring* if there is a $\mathbb{Z}$ -module basis $\{X_i\}_{i \in I}$ of F , for some index set I , such that

- (a) $1_F \in \{X_i\}_{i \in I}$;
- (b) $\pi(X_i) \geq 1$, for all $i \in I$; and
- (c) for all $i, j \in I$, we have $X_i X_j = \sum_{k \in I} n_{i,j}^k X_k$ for some $n_{i,j}^k \in \mathbb{N}$.

The coefficients $n_{i,j}^k$ are called the *fusion rule coefficients*.

The Grothendieck ring $G_0(H)$ is an example of a fusion ring, where $X_i = [V_i]$ and $\pi([V_i]) = \dim_{\mathbb{k}} V_i$ for $i \in I$.

Now we present the definition of the McKay quiver for a semisimple Hopf algebra action on a connected graded algebra. This is achieved by defining the McKay quiver for an *effective* element Y in a fusion ring F , where Y is a nonzero element in F so that $Y = \sum_{i \in I} \alpha_i X_i$ for some $\alpha_i \geq 0$.

Definition 2.3. Let F be a fusion ring with a $\mathbb{Z}$ -basis $\{X_i\}_{i \in I}$, and let $Y \in F$ be an effective element.

- (1) The *right McKay quiver* $\mathcal{M}(Y)$ is defined as follows:
 - (a) the vertices of $\mathcal{M}(Y)$ are X_i , where $i \in I$,
 - (b) there are exactly m_{ij} arrows from X_i to X_j in $\mathcal{M}(Y)$, where m_{ij} is determined by $X_j Y = \sum_{i \in I} m_{ij} X_i$.
 The *left McKay quiver* $(Y)\mathcal{M}$ is defined similarly, where m_{ij} is determined by $Y X_j = \sum_{i \in I} m_{ij} X_i$.
- (2) If $m_{ij} = m_{ji}$ for all $i, j \in I$, then the *right Euclidean diagram*, still denoted by $\mathcal{M}(Y)$, is an undirected graph defined as follows:
 - (a) the vertices of $\mathcal{M}(Y)$ are X_i where $i \in I$,
 - (b) the multiplicity of the edge connecting X_i to X_j is m_{ij} .
 The *left Euclidean diagram* $(Y)\mathcal{M}$ is defined similarly.
- (3) The connected component containing 1_F is called the *principal component* of $\mathcal{M}(Y)$ (resp. $(Y)\mathcal{M}$) and is denoted by $\mathcal{M}_0(Y)$ (resp. $(Y)_0\mathcal{M}$).
- (4) The positive number $\pi(Y)$ [Definition 2.2] is called the *rank* of the McKay quiver $\mathcal{M}(Y)$, and of the principal component $\mathcal{M}_0(Y)$.

If F is noncommutative, then the left McKay quiver could be different from the right one, although these two versions are isomorphic in many natural examples. On the other hand, if F is commutative, then there is no difference between left and right McKay quivers.

The McKay quivers for the actions of the quantum binary polyhedral groups on AS regular algebras of dimension 2 are discussed briefly in [9, Section 7]. We repeat that result below.

Proposition 2.4. [9, Proposition 7.1] *Let H be a semisimple quantum binary polyhedral group that acts on an AS regular algebra A of dimension 2 generated in degree one as in Theorem 0.4. Take the effective element Y to be the degree one graded piece A_1 of A in the fusion ring $G_0(H)$. Then the left Euclidean diagrams (and hence the left McKay quivers, which have an arrow in both directions for each edge in the diagram) $\mathcal{M}(Y)$ for these actions are given in Table 2. $\square$*

We refer the reader to [13] for illustrations of the Euclidean diagrams listed above.

Table 2

Left McKay quivers for actions of quantum binary polyhedral groups.

Case	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Left Euclidean diagram	$\tilde{A}-\tilde{D}-\tilde{E}$	$\tilde{A}_{n-1}$	$\tilde{L}_1$	$\begin{cases} \tilde{D}_{\frac{n+4}{2}}, & n \text{ even} \\ \tilde{D}_{\frac{n+1}{2}}, & n \text{ odd} \end{cases}$	$\tilde{A}_{2n-1}$	$\tilde{D}-\tilde{E}$	$\tilde{A}_{n-1}$	$\tilde{A}_1$

3. Fixed subrings

The proof of Theorem 0.3 is delicate, and, among other things, requires a deep understanding of the fixed subrings in types $\mathbb{D}$ and $\mathbb{E}$. A direct calculation of the fixed subrings in types $\mathbb{D}$ and $\mathbb{E}$ is too complicated (or even impossible) in the noncommutative case, so our approach is to use the McKay quivers to convert the noncommutative case to the commutative case.

3.1. Fixed subrings under Hopf actions of type $\mathbb{E}$

In this subsection we study type $\mathbb{E}$. In type $\mathbb{E}$ we have $A = \mathbb{k}[u, v]$ or $\mathbb{k}_{-1}[u, v]$ by Theorem 0.4 and Proposition 2.4. We will show that for Hopf algebras H of type $\mathbb{E}$ in Theorem 0.4 we have that $\mathbb{k}_{-1}[u, v]^H$ is isomorphic to $\mathbb{k}[u, v]^{\tilde{\Gamma}}$ for a corresponding subgroup $\tilde{\Gamma}$ of $\mathrm{SL}_2(\mathbb{k})$. Recall that $\tilde{\Gamma}$ arises as a nonsplit central extension of $\mathbb{Z}_2$ by the corresponding polyhedral group Γ : $0 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 0$.

By Theorem 0.4, for each given diagram $\tilde{\mathbb{E}}_i$, there are at most three Hopf algebras H associated to it (see cases (a,f) in Theorem 0.4), and each Hopf algebra H satisfies

$$\mathbb{k} \rightarrow \mathbb{k}\mathbb{Z}_2 \rightarrow H \rightarrow \mathbb{k}\Gamma \rightarrow \mathbb{k} \quad (3.0.1)$$

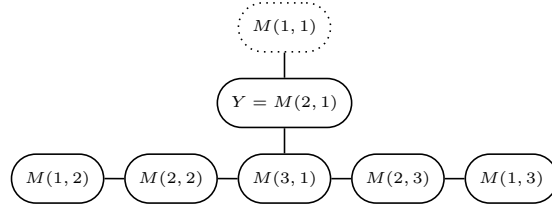
where Γ is the (unique) corresponding polyhedral group of type $\tilde{\mathbb{E}}_i$.

Notation. Let $\{M(d, j)\}_{j=1}^{s_d}$ be the complete set of isomorphism classes of simple left H -modules of dimension d , for some $s_d \geq 0$. Let $\mathbb{k}\langle M(d, j) \rangle$ (resp. $\mathbb{k}[M(d, j)]$) be the free algebra (resp. the commutative polynomial ring) generated by $M(d, j)$.

For example, considering the diagram $\tilde{\mathbb{E}}_6$ as in [13, p. 5], there are seven simple left H -modules

$$M(1, 1), M(1, 2), M(1, 3), M(2, 1), M(2, 2), M(2, 3), M(3, 1).$$

We choose $M(1, 1)$ to be the trivial module, and $M(2, 1)$ to be the 2-dimensional simple module that is connected with $M(1, 1)$. Using the notation in Section 2, $Y = M(2, 1) = \mathbb{k}u \oplus \mathbb{k}v$, which is the generating space of the AS regular algebra A of dimension two. Let $M(3, 1)$ be the unique 3-dimensional simple module that is connected with $M(2, 1)$ in the diagram $\tilde{\mathbb{E}}_6$.

$\widetilde{\mathbb{E}}_6$:

We have the following results.

Lemma 3.1. Suppose the McKay quiver corresponding to (A, H) is $\widetilde{\mathbb{E}}_6$. Then the following statements hold.

- (1) $\mathbb{k}\langle M(3, 1) \rangle_2 = M(3, 1) \otimes M(3, 1) = M(3, 1)^{\oplus 2} \oplus M(1, 1) \oplus M(1, 2) \oplus M(1, 3)$ as H -modules.
- (2) $\mathbb{k}[M(3, 1)]_2 = M(3, 1) \oplus M(1, 1) \oplus M(1, 2) \oplus M(1, 3)$ as Γ -modules.
- (3) There is a unique 1-dimensional subspace $\mathbb{k}r$ of $\mathbb{k}[M(3, 1)]_2$, such that $A^{(2)}$ is isomorphic to $\mathbb{k}[M(3, 1)]/(r)$ as graded Γ -module algebras.
- (4) For each H of type $\widetilde{\mathbb{E}}_6$, $A^{(2)}$ is isomorphic to $\mathbb{k}[u, v]^{(2)}$ as graded Γ -module algebras. As a consequence, $A^H \cong \mathbb{k}[u, v]^{\widetilde{\Gamma}}$ as graded algebras.

Proof. (1) Note that $Y \otimes Y = M(3, 1) \oplus M(1, 1)$ according to the diagram $\widetilde{\mathbb{E}}_6$. Further, by the diagram $\widetilde{\mathbb{E}}_6$, we have

$$\begin{aligned}
 [M(3, 1) \otimes M(3, 1)] \oplus M(3, 1) &= M(3, 1) \otimes [M(3, 1) \oplus M(1, 1)] \\
 &= M(3, 1) \otimes [Y \otimes Y] = [M(3, 1) \otimes Y] \otimes Y \\
 &= [M(2, 1) \oplus M(2, 2) \oplus M(2, 3)] \otimes Y \\
 &= M(3, 1)^{\oplus 3} \oplus M(1, 1) \oplus M(1, 2) \oplus M(1, 3).
 \end{aligned}$$

The assertion follows from the above equation by cancelling a copy of $M(3, 1)$.

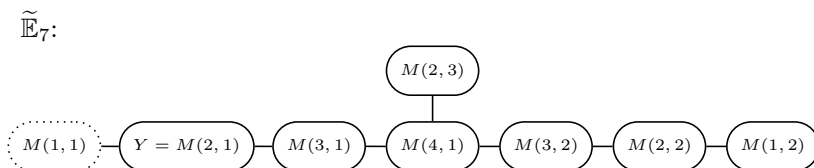
(2) Note that $\mathbb{k}[M(3, 1)]_2$ is a quotient of $\mathbb{k}\langle M(3, 1) \rangle_2$. Moreover, $Y \otimes Y = M(3, 1) \oplus M(1, 1)$ implies that $M(3, 1) = A_2$ is the generating space of the Veronese subring $A^{(2)}$. Now $A = \mathbb{k}[u, v]$ or $\mathbb{k}_{-1}[u, v]$ by Theorem 0.4 and Proposition 2.4, and $A^{(2)}$ is respectively $\mathbb{k}[a, b, c]/(ab - c^2)$ or $\mathbb{k}[a, b, c]/(ab + c^2)$ for $a = u^2$, $b = v^2$, $c = uv$. So $A^{(2)}$ is isomorphic as Γ -module algebras to $\mathbb{k}[M(3, 1)]/(r)$ for some relation r in degree two. This implies that $\mathbb{k}[M(3, 1)]_2$ has a quotient Γ -module of dimension 5. The assertion follows by a $\mathbb{k}$ -vector space dimension argument.

(3) As a commutative algebra, $A^{(2)}$ is isomorphic to $\mathbb{k}[M(3, 1)]/(r)$ for some r in degree two by the arguments in (2). Since $A^{(2)}$ is a Γ -module algebra, $\mathbb{k}r$ is a left Γ -module. Hence, $\mathbb{k}r$ is isomorphic to either $M(1, 1)$, or $M(1, 2)$ or $M(1, 3)$. Note that 1-dimensional

Γ -modules $M(1, 1)$, $M(1, 2)$ and $M(1, 3)$ are all non-isomorphic. The choice of r is uniquely determined by using the result [14, Proposition 2.4] and the fact that the Γ -action on $A^{(2)}$ has trivial homological determinant [9, Lemma 2.6(c)].

(4) This follows from (3) and (3.0.1); see also [9, Lemma 6.27(b)]. $\square$

The next two lemmas are similar to the previous one. We prove the first one and provide the corresponding diagrams for both results.



Lemma 3.2. *Suppose the McKay quiver corresponding to (A, H) is $\tilde{\mathbb{E}}_7$. Then the following statements hold.*

- (1) $\mathbb{k}\langle M(3, 1) \rangle_2 = M(3, 1) \otimes M(3, 1) = M(3, 1) \oplus M(3, 2) \oplus M(2, 3) \oplus M(1, 1)$ as H -modules.
- (2) $\mathbb{k}[M(3, 1)]_2 = M(3, i) \oplus M(2, 3) \oplus M(1, 1)$ where $i = 1$ or 2 as Γ -modules.
- (3) *There is a unique 1-dimensional subspace $\mathbb{k}r$ of $\mathbb{k}[M(3, 1)]_2$, such that $A^{(2)}$ is isomorphic to $\mathbb{k}[M(3, 1)]/(r)$ as graded Γ -module algebras. In fact, $\mathbb{k}r$ is the submodule $M(1, 1)$ in part (2).*
- (4) *For each H of type $\tilde{\mathbb{E}}_7$, $A^{(2)}$ is isomorphic to $\mathbb{k}[u, v]^{(2)}$ as graded Γ -module algebras. As a consequence, $A^H \cong \mathbb{k}[u, v]^{\tilde{\Gamma}}$ as graded algebras.*

Proof. (1) By the diagram $\tilde{\mathbb{E}}_7$, we have

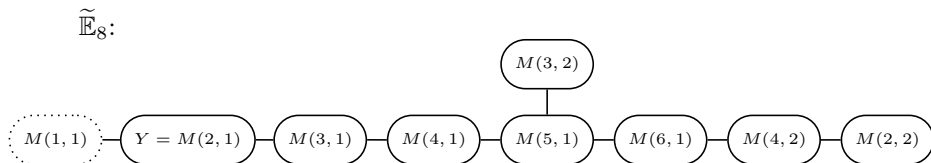
$$\begin{aligned}
 [M(3, 1) \otimes M(3, 1)] \oplus M(3, 1) &= M(3, 1) \otimes [M(3, 1) \oplus M(1, 1)] \\
 &= [M(3, 1) \otimes Y] \otimes Y \\
 &= [M(2, 1) \oplus M(4, 1)] \otimes Y \\
 &= M(1, 1) \oplus M(3, 1) \oplus M(3, 1) \oplus M(3, 2) \oplus M(2, 3).
 \end{aligned}$$

The assertion follows from the above equation by cancelling a copy of $M(3, 1)$.

(2) Note that $(\mathbb{k}[M(3, 1)])_2$ is a quotient of $(\mathbb{k}\langle M(3, 1) \rangle)_2$ and that $(\mathbb{k}[M(3, 1)])_2$ has a quotient Γ -module of dimension 5. Hence the assertion follows.

(3) As a commutative algebra $A^{(2)}$ is isomorphic to $\mathbb{k}[M(3, 1)]/(r)$ for some r in degree two. By part (2), the only possible 1-dimensional Γ -submodule in $(\mathbb{k}[M(3, 1)])_2$ is $M(1, 1)$. The assertion follows.

(4) This follows from (3). $\square$



Lemma 3.3. *Suppose the McKay quiver corresponding to (A, H) is $\widetilde{\mathbb{E}}_8$. Then the following statements hold.*

- (1) $\mathbb{k}\langle M(3,1) \rangle_2 = M(3,1) \otimes M(3,1) = M(5,1) \oplus M(3,1) \oplus M(1,1)$ as H -modules.
- (2) $\mathbb{k}[M(3,1)]_2 = M(5,1) \oplus M(1,1)$ as Γ -modules.
- (3) *There is a unique 1-dimensional subspace $\mathbb{k}r$ of $\mathbb{k}[M(3,1)]_2$, such that $A^{(2)}$ is isomorphic to $\mathbb{k}[M(3,1)]/(r)$ as graded Γ -module algebras. In fact, $\mathbb{k}r$ is the submodule $M(1,1)$ in part (2).*
- (4) *For each H of type $\widetilde{\mathbb{E}}_8$, $A^{(2)}$ is isomorphic to $\mathbb{k}[u, v]^{(2)}$ as graded Γ -module algebras. As a consequence, $A^H \cong \mathbb{k}[u, v]^{\bar{\Gamma}}$ as graded algebras. $\square$*

3.2. Fixed subrings under Hopf actions of type $\mathbb{D}$

In this subsection we study type $\mathbb{D}$. In type $\mathbb{D}$ we have $A = \mathbb{k}[u, v]$ or $\mathbb{k}_{-1}[u, v]$ by Theorem 0.4 and Proposition 2.4. Let D_{2n} be the dihedral group of order $2n$ for some integer $n \geq 2$. One presentation of D_{2n} is

$$\langle s_+, s_- \mid s_+^2 = s_-^2 = (s_+ s_-)^n = 1 \rangle.$$

The binary dihedral group of order $4n$, denoted by $\widetilde{D}_{2n}$, has a presentation

$$\langle \sigma, \tau \mid \sigma^n = \tau^2, \tau^4 = 1, \tau^{-1} \sigma \tau = \sigma^{-1} \rangle.$$

There is a short exact sequence of groups

$$1 \rightarrow \{\tau^2, 1\} \rightarrow \widetilde{D}_{2n} \rightarrow D_{2n} \rightarrow 1$$

where $\{\tau^2, 1\} (\cong \mathbb{Z}/(2))$ is a central subgroup of $\widetilde{D}_{2n}$. The map $\widetilde{D}_{2n} \rightarrow D_{2n}$ sends σ to $s_+ s_-$ and τ to s_+ . We can embed $\widetilde{D}_{2n}$ to $\mathrm{SL}_2(\mathbb{k})$ by setting

$$\sigma \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad \text{and} \quad \tau \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (3.3.1)$$

where ω is a primitive $2n$ -th root of unity and $i^2 = -1$. Using (3.3.1), $\widetilde{D}_{2n}$ acts on the commutative polynomial ring $T := \mathbb{k}[u, v]$ naturally. The lemma below follows by an easy computation.

Lemma 3.4. *With the basis $\{u, v\}$ of T_1 and the action defined by (3.3.1), the following hold.*

(1) *The σ and τ actions on the basis $\{u^2, uv, v^2\}$ of T_2 are given by*

$$\sigma = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^{-2} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3.4.1)$$

So, T_2 is a direct sum of two $\tilde{D}_{2n}$ -submodules $\mathbb{k}u^2 + \mathbb{k}v^2$ and $\mathbb{k}uv$.

(2) *Let $a = u^2, b = v^2$ and $c = uv$. Then $T^{(2)} = \mathbb{k}[a, b, c]/(ab - c^2)$, where the D_{2n} -action on $T^{(2)}$ is determined by*

$$s_+ = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad s_- = \begin{pmatrix} 0 & -\omega^{-2} & 0 \\ -\omega^2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad \square \quad (3.4.2)$$

By [22], there are two different noncocommutative cocycle deformations of $\mathbb{k}D_{4n}$ (or of $\mathbb{k}\tilde{D}_{2n}$), denoted by $\mathcal{A}(\tilde{D}_{2n})^\circ$ and $\mathcal{B}(\tilde{D}_{2n})^\circ$. By [6, Proposition 2.5 and Theorem 4.4] and [22, Theorem 4.1(1)], $\mathcal{B}(\tilde{D}_{2n})$ is self-dual and has no non-trivial cocycle twist (cotwist). Here we will consider $\mathcal{B}(\tilde{D}_{2n})$, or equivalently, $\mathcal{B}(\tilde{D}_{2n})^\circ$.

By [22, Definition 3.3(2)], for $n \geq 2$, $\mathcal{B}(\tilde{D}_{2n})$ is the Hopf algebra generated by a central grouplike element a of order 2, and s_+ and s_- , subject to the relations

$$a^2 = 1, \quad as_\pm = s_\pm a, \quad s_\pm^2 = 1, \quad (s_+s_-)^n = a, \quad (3.4.3)$$

with the structure

$$\Delta(s_\pm) = s_\pm \otimes e_0 s_\pm + s_\mp \otimes e_1 s_\pm, \quad \epsilon(s_\pm) = 1, \quad S(s_\pm) = e_0 s_\pm + e_1 s_\mp \quad (3.4.4)$$

where $e_0 = \frac{1}{2}(1 + a)$ and $e_1 = \frac{1}{2}(1 - a)$.

By [3, Lemma 5.15, Remarks 5.22 and 5.23], the Hopf algebra $\mathcal{B}(\tilde{D}_{2n})$ has four non-isomorphic 1-dimensional simple modules and $(n - 1)$ non-isomorphic 2-dimensional simple modules. We list these 2-dimensional simple modules next. Let ξ be a primitive $2n$ -th root of unity, and let $\omega = \xi^l$ for some l . Let $M(2, l)$, for $l = 1, 2, \dots, n - 1$, be the 2-dimensional simple left $\mathcal{B}(\tilde{D}_{2n})$ -module determined by its action on a 2-dimensional $\mathbb{k}$ -space $\mathbb{k}s \oplus \mathbb{k}t$ with matrix presentation

$$s_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_- = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad a = \begin{pmatrix} (-1)^l & 0 \\ 0 & (-1)^l \end{pmatrix} \quad (3.4.5)$$

where $\omega = \xi^l$. By using the relations of the algebra (3.4.3) and the group characters, one can easily show that these $\{M(2, l)\}_{l=1}^{n-1}$ are non-isomorphic 2-dimensional simple modules. When l is even, $M(2, l)$ is killed by the Hopf ideal generated by $(a - 1)$. Hence

$M(2, l)$ is not inner faithful. When l is odd and $g := \gcd(l, 2n) > 1$, then $M(2, l)$ is killed by the Hopf ideal generated by $(s_+s_-)^d - 1, (s_-s_+)^d - 1$, where $d = 2n/g$. Hence $M(2, l)$ is not inner faithful. Therefore the inner faithful modules are precisely those $M(2, l)$, where $\gcd(l, 2n) = 1$ (or equivalently, ω is primitive $2n$ -th root of unity).

For every inner faithful 2-dimensional representation of $\mathcal{B}(\tilde{D}_{2n})$ (where $\omega = \xi^l$ with $\gcd(l, 2n) = 1$), we consider the action on the free algebra $\mathbb{k}\langle M(2, l) \rangle = \mathbb{k}\langle s, t \rangle$ extended by (3.4.5). Using the coproduct, these elements have the following matrix presentations when using the basis $\{s^2, st, ts, t^2\}$ of $\mathbb{k}\langle s, t \rangle_2$:

$$s_+ = \begin{pmatrix} 0 & 0 & 0 & \omega^{-1} \\ 0 & 0 & \omega^{-1} & 0 \\ 0 & \omega & 0 & 0 \\ \omega & 0 & 0 & 0 \end{pmatrix}, \quad s_- = \begin{pmatrix} 0 & 0 & 0 & \omega^{-1} \\ 0 & 0 & \omega & 0 \\ 0 & \omega^{-1} & 0 & 0 \\ \omega & 0 & 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.5. Suppose $n \geq 2$. Given the $\mathcal{B}(\tilde{D}_{2n})$ -action on $\mathbb{k}\langle s, t \rangle$, we obtain a unique 1-dimensional left trivial $\mathcal{B}(\tilde{D}_{2n})$ -module inside $\mathbb{k}\langle s, t \rangle_2$, which is spanned by $r := \omega s^2 + t^2$.

Proof. The subspace spanned by st and ts is a simple $\mathcal{B}(\tilde{D}_{2n})$ -module. The subspace spanned by s^2 and t^2 decomposes into $s_{\pm}$ -eigenspaces with eigenvalues 1 and -1 . By a direct computation, $r := \omega s^2 + t^2$ is preserved by $s_{\pm}$ and a . Hence $\mathbb{k}r$ is a trivial $\mathcal{B}(\tilde{D}_{2n})$ -module. $\square$

Let Y be the 2-dimensional $\mathcal{B}(\tilde{D}_{2n})$ -module determined by the action given in (3.4.5). By Lemma 3.5 and [9, Theorem 2.1], there is a unique graded $\mathcal{B}(\tilde{D}_{2n})$ -action on $\mathbb{k}\langle s, t \rangle / (\omega s^2 + t^2)$ so that the degree one piece with the $\mathcal{B}(\tilde{D}_{2n})$ -action is isomorphic to Y . Since $\mathbb{k}\langle s, t \rangle / (\omega s^2 + t^2)$ is isomorphic to $\mathbb{k}_{-1}[u, v]$, there is a unique graded $\mathcal{B}(\tilde{D}_{2n})$ -action on $\mathbb{k}_{-1}[u, v]$ so that the degree one piece with the $\mathcal{B}(\tilde{D}_{2n})$ -action is isomorphic to Y .

Let $\alpha = st$, $\beta = -\omega^{-1}ts$ and $\gamma = is^2$ (where $i^2 = -1$) in $B := \mathbb{k}\langle s, t \rangle / (\omega s^2 + t^2)$. Then $B^{(2)}$ is a commutative algebra generated by α, β and γ , and subject to one relation $\alpha\beta + \gamma^2 = 0$. With respect to the basis $\{\alpha, \beta, \gamma\}$, $s_{\pm}$ have the following matrix presentation (with a being the identity)

$$s_+ = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad s_- = \begin{pmatrix} 0 & -\omega^2 & 0 \\ -\omega^{-2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.5.1)$$

Now we are ready to prove the following.

Lemma 3.6. Let $n \geq 2$ and let $H := \mathcal{B}(\tilde{D}_{2n}) \cong \mathcal{B}(\tilde{D}_{2n})^{\circ}$ act on $A = \mathbb{k}_{-1}[u, v]$ inner faithfully with trivial homological determinant. Then $A^{(2)}$ is isomorphic to $\mathbb{k}[u, v]^{(2)}$ as graded D_{2n} -module algebras. So, $A^H \cong \mathbb{k}[u, v]^{\tilde{D}_{2n}}$ as graded algebras.

Proof. By Lemma 3.5, such an H -action on A is uniquely determined by $\omega := \xi^l$. The induced D_{2n} -action on $A^{(2)} \cong \mathbb{k}[\alpha, \beta, \gamma]/(\alpha\beta + \gamma^2)$ is determined by (3.5.1). The main assertion follows by comparing (3.5.1) with (3.4.2).

As a consequence

$$(A^{(2)})^{D_{2n}} \cong (\mathbb{k}[u, v]^{(2)})^{D_{2n}}. \quad (3.6.1)$$

The final statement follows by [9, Lemma 6.27(b)] and (3.6.1). $\square$

4. Auslander's theorem for quantum binary polyhedral groups

In this section we prove Conjecture 0.2 for every pair (A, H) in Theorem 0.4.

Theorem 4.1. *Conjecture 0.2 is true for AS regular algebras of dimension 2.*

Some partial results on Auslander's Theorem for group actions on AS regular algebras were obtained by Mori–Ueyama in [27]. For A of dimension 2 we complete the remaining cases where the action is a group action, as well as handle the cases where the Hopf algebra is not a group algebra. The proof consists of the following case-by-case analysis. We start with a well-known fact, which is a special case of the original Auslander's Theorem.

4.1. Auslander's Theorem: Case (a):

Proposition 4.2. [1, Proposition 3.4] *Let (A, H) be in case (a) of Theorem 0.4. Then Conjecture 0.2 holds. $\square$*

The following cases were proved by Mori–Ueyama [27].

4.2. Auslander's Theorem: Cases (b), (g), and (h):

Proposition 4.3. [27, Theorem 4.5(1)] *Let (A, H) be in cases (b), (g), (h) of Theorem 0.4. Then Conjecture 0.2 holds. $\square$*

4.3. Auslander's Theorem: Cases (c) and (d):

To complete the next cases we need a result from [2], which is a generalization of [27, Theorem 3.10]. Throughout let t be an integral of a semisimple Hopf algebra H .

Lemma 4.4. [2, Theorem 0.3] *Let A be a noetherian AS regular algebra of dimension 2, and H act on A inner faithfully and homogeneously. Let $e = 1 \# t \in A \# H$. The following conditions are equivalent*

- (1) $(A\#H)/(e)$ is finite dimensional.
- (2) The natural algebra homomorphism

$$A\#H \rightarrow \text{End}_{(A\#H)^{\text{op}}}(A)$$

is an isomorphism. Namely, (0.2.1) holds. $\square$

Proposition 4.5. *Let (A, H) be as in Theorem 0.4(c). Then Conjecture 0.2 holds.*

Proof. In this case $A = \mathbb{k}_{-1}[u, v]$ and $G = C_2$. Let I be the ideal (e) . By Lemma 4.4, it suffices to show that $(A\#\mathbb{k}C_2)/I$ is finite dimensional.

We claim that $u^2\#1, v^2\#1 \in I$. Supposing that the claim holds, since $A_{\geq 3}$ is in the ideal generated by u^2, v^2 , then I contains $(A\#\mathbb{k}C_2)_{\geq 3}$. Therefore $(A\#\mathbb{k}C_2)/I$ is finite dimensional, as desired.

To prove the claim, write $C_2 = \{1, g\}$, where 1 is the identity of C_2 and $g^2 = 1$, and, replacing e by $2e$, it suffices to let $e := 1\#(1 + g)$. Using the notation in Theorem 0.4, $g(u) = v$ and $g(v) = u$, $e(u\#1) = u\#1 + v\#g$ and $(v\#1)e = v\#1 + v\#g$, and hence, both are in I . Hence $(u - v)\#1 \in I$. Now $(u\#1)((u - v)\#1) = (u^2 - uv)\#1$ and $((u - v)\#1)(u\#1) = (u^2 + uv)\#1$ are in I . Thus $u^2\#1 \in I$, and by symmetry, $v^2\#1 \in I$. Therefore we have verified the claim, and, consequently, the assertion of the proposition. $\square$

The next lemma provides some of the computation needed in the proof of case (d), which is similar to case (c), but more complicated. In case (d) $A = \mathbb{k}_{-1}[u, v]$ and $H = \mathbb{k}D_{2n}$. Let

$$\sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g := \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$$

where ξ is a primitive n -th root of unity. The dihedral group D_{2n} of order $2n$ is generated by σ and g . Denote

$$G_d = 1 + (\xi)^d g + (\xi)^{2d} g^2 + \cdots + (\xi)^{(n-1)d} g^{n-1},$$

$$F_d = G_d \sigma.$$

Recall that $e = 1\#(G_0 + F_0)$ and I is the ideal (e) .

Lemma 4.6. *Retain the notation above and assume the hypotheses as in case (d) of Theorem 0.4.*

- (1) $G_d = G_{n+d}$ and $F_d = F_{n+d}$ for all d .
- (2) $(1\#G_d)(u^i\#1) = u^i\#G_{d+i}$ and $(1\#G_d)(v^i\#1) = v^i\#G_{d-i}$ for all i .
- (3) $(1\#F_d)(u^i\#1) = v^i\#F_{d-i}$ and $(1\#F_d)(v^i\#1) = u^i\#F_{d+i}$ for all i .

- (4) $e(u^i \# 1) = u^i \# G_i + v^i \# F_{-i}$ for all i .
- (5) $(u^{n-i} \# 1)e(u^i \# 1) = u^n \# G_i + u^{n-i} v^i \# F_{-i}$ for all i .
- (6) $(v^i \# 1)e(u^{n-i} \# 1) = v^i u^{n-i} \# G_{-i} + v^n \# F_i$ for all i .
- (7) $(u^n - (-1)^{i(n-i)} v^n) \# G_i \in I$ for all i .
- (8) $e(uv \# 1) = uv \# G_0 - uv \# F_0$.
- (9) $u^{n+1} v \# 1 \in I$, and by symmetry, $uv^{n+1} \# 1 \in I$.
- (10) If n is even, $(u^{2n} - v^{2n}) \# 1 \in I$. If n is odd, $(u^{2n} + v^{2n}) \# 1 \in I$.
- (11) $u^{2n+1} \# 1 \in I$, and by symmetry, $v^{2n+1} \# 1 \in I$.

Proof. (1) This holds because ξ is an n -th root of unity.

(2) This follows from the equations

$$(1 \# (\xi^d)^n g^n)(u^i \# 1) = u^i \# (\xi^{d+i})^n g^n \quad \text{and} \quad (1 \# (\xi^d)^n g^n)(v^i \# 1) = v^i \# (\xi^{d-i})^n g^n$$

for all i, d, n .

(3) The proof is similar to (2).

(4) By parts (2,3), for all i ,

$$e(u^i \# 1) = (1 \# G_0)(u^i \# 1) + (1 \# F_0)(u^i \# 1) = u^i \# G_i + v^i \# F_{-i}.$$

(5) This follows from part (4).

(6) This follows from part (4), replacing i by $n - i$.

(7) By parts (5,6), and the fact that $uv = -vu$,

$$\begin{aligned} & (u^n - (-1)^{i(n-i)} v^n) \# G_i \\ &= u^n \# G_i + u^{n-i} v^i \# F_{-i} - (-1)^{i(n-i)} (v^i u^{n-i} \# G_{-i} + v^n \# F_i) (1 \# \sigma) \\ &= (u^{n-i} \# 1) e(u^i \# 1) - (-1)^{i(n-i)} (v^i \# 1) e(u^{n-i} \# 1) (1 \# \sigma) \in I. \end{aligned}$$

(8) This is easy to check.

(9) Note that $(uv \# 1)e = uv \# G_0 + uv \# F_0$. Combining this with part (8), we have $uv \# G_0, uv \# F_0 \in I$. Now, for every i ,

$$(-1)^i u^{n+1} v \# G_i = u^{n-i+1} v u^i \# G_i = (u^{n-i} \# 1)(uv \# G_0)(u^i \# 1) \in I.$$

Since $\sum_{i=0}^{n-1} G_i = n$, $u^{n+1} v \# 1 \in I$. By symmetry, $uv^{n+1} \# 1 \in I$.

(10) First we assume n is even, and then u^n commutes with v^n . By part (7),

$$(u^{2n} - v^{2n}) \# G_i = ((u^n + (-1)^{i(n-i)} v^n) \# 1)(u^n - (-1)^{i(n-i)} v^n) \# G_i \in I,$$

for all i , which implies that $(u^{2n} - v^{2n}) \# 1 \in I$.

Second we assume that n is odd. By parts (1,2), $1 \# G_i$ commutes with $u^n \# 1$ and $v^n \# 1$. Since $u^n v^n = -v^n u^n$,

$$((u^n - (-1)^{i(n-i)}v^n)\#G_i)^2 = (u^n - (-1)^{i(n-i)}v^n)^2\#(G_i)^2 = (u^{2n} + v^{2n})\#nG_i.$$

By part (7), $(u^{2n} + v^{2n})\#G_i \in I$ for all i . Hence $(u^{2n} + v^{2n})\#1 \in I$.

(11) This follows from parts (9,10). $\square$

Proposition 4.7. *Let (A, H) be as in Theorem 0.4(d). Then Conjecture 0.2 holds.*

Proof. By Lemma 4.6(11), $u^{2n+1}\#1, v^{2n+1}\#1 \in I$. Since $A_{\geq 4n+2}$ is in the ideal generated by u^{2n+1} and v^{2n+1} , I contains $(A\#\mathbb{k}G)_{\geq 4n+2}$. Therefore $(A\#\mathbb{k}G)/I$ is finite dimensional, as desired. $\square$

4.4. Auslander's Theorem: Case (e):

To handle case (e) in Theorem 0.4, we introduce some different techniques. Let H be the dual of the group algebra $\mathbb{k}G$ (where $G = D_{2n}$). Let $\{g_1, \dots, g_m\}$ be the set G , where g_1 is the unit of G . The dual basis $\{p_{g_1}, \dots, p_{g_m}\}$ is the complete set of orthogonal idempotents of H that forms a $\mathbb{k}$ -linear basis of H . It is well-known that p_{g_1} is an integral of H , and $\sum_{s=1}^m p_{g_s}$ is the algebra identity of H . The coproduct of H is determined by

$$\Delta(p_g) = \sum_{h \in G} p_h \otimes p_{h^{-1}g}$$

for all $g \in G$. The (left) H -action on A is equivalent to the (right) $\mathbb{k}G$ -coaction on A , that is equivalent to a G -grading on A . Therefore A has a natural G -grading induced by the H -action. For a G -homogeneous element f in A , its G -grading is denoted by $\deg_G(f)$. For the next two results, set $e = 1\#t = 1\#p_{g_1}$ and $I = (e)$.

Lemma 4.8. *Retain the notation above.*

(1) *Let f be a G -homogeneous element of A of degree $g \in G$. Then*

$$e(f \otimes 1) = f \otimes p_{g^{-1}}.$$

(2) *Suppose $f_1, \dots, f_m$ are homogeneous elements in A such that the set*

$$\{\deg_G(f_d) \cdots \deg_G(f_m)\}_{d=1}^m$$

equals G . Then the product $(f_1 \cdots f_m)\#1 \in I$.

Proof. (1) By definition,

$$e(f\#1) = (1\#p_{g_1})(f\#1) = \sum_{h \in G} p_h(f)\#p_{h^{-1}} = p_g(f)\#p_{g^{-1}} = f\#p_{g^{-1}}.$$

(2) Let $h_d = \deg_G(f_d) \cdots \deg_G(f_m)$. By part (1),

$$e((f_d \cdots f_m) \# 1) = (f_d \cdots f_m) \# p_{h_d^{-1}} \in I$$

for all d . Hence $(f_1 \cdots f_m) \# p_{h_d^{-1}} \in I$ for all d . By hypothesis, $\{h_d\}_{d=1}^m$ equals G . Then $1 = \sum_{d=1}^m p_{h_d^{-1}}$, which implies that $(f_1 \cdots f_m) \# 1 \in I$. $\square$

Now we are ready to prove Theorem 4.1 for case (e).

Proposition 4.9. *Let (A, H) be in Theorem 0.4(e). Then Conjecture 0.2 holds.*

Proof. As in the statement of Theorem 0.4(e), we recycle the variables u and v to denote the generators for A satisfying the relation $u^2 + v^2 = 0$. (These are v_1 and v_2 in Theorem 0.4.) Write $G := D_{2n}$ as $\langle g, h \mid g^2 = h^2 = (gh)^n = 1 \rangle$ and $\deg_G u = g$ and $\deg_G v = h$.

For $1 \leq i \leq m := 2n$, let f_i be either u if i is odd, or v if i is even. Then the hypotheses of Lemma 4.8(2) hold. As a consequence, $f_1 \cdots f_m \# 1 \in I$, or $(uv)^n \# 1 \in I$. By symmetry, $(vu)^n \# 1 \in I$. Since $(uv)^n (vu)^n = u^{2n} v^{2n} = (-1)^n u^{4n}$, we have $u^{4n} \# 1 \in I$. Since every element in A is a linear combination of $u^a (vu)^b v^c$ where c is either 0 or 1, I contains the ideal $(A \# H)_{\geq (6n+1)}$. This implies that $(A \# H)/I$ is finite dimensional. The assertion follows by Lemma 4.4. $\square$

4.5. Auslander's Theorem: Case (f):

It remains to consider case (f). We need to understand the structure of the fixed subrings, and we will use the results in Section 3. Let $A = \mathbb{k}_{-1}[u, v]$ as in case (f), H be one of the Hopf algebras in case (f), and let $B = A^H$. Then A is a left and a right B -module. We will be using the right B -module structure on A , which is equivalent to a left B^{op} -module structure of A . The following lemma is well-known.

Lemma 4.10. [2, Lemma 3.10] *Retain the notation above. If $A \# H$ is prime, then the natural algebra map $\theta : A \# H \rightarrow \text{End}_{B^{\text{op}}}(A)$ is injective.* $\square$

Moreover, we will use the following well-known lemma in the results below.

Lemma 4.11. *Let R be a connected graded algebra admitting an action by a semisimple Hopf algebra H . Let M be a graded, bounded below, left $R \# H$ -module. A minimal projective resolution of the $R \# H$ -module M is also a minimal free resolution of left R -module M .* $\square$

Let H be the Hopf algebra $(\mathcal{A}(\widetilde{D}_{2m}))^\circ$, which is one of the Hopf algebras of type $\mathbb{D}$ in case (f). Note that there is other notation for the dicyclic group (or binary dihedral group) $\widetilde{D}_{2m}$; in [22] it is denoted by T_{4m} ; in [3], it is denoted by $\widetilde{D}_m$; in other papers

it is denoted by BD_{4m} . By [22, Theorem 4.1(2)], H is a *dual cocycle twist* of the group algebra $\mathbb{k}D_{4m}$. A dual cocycle twist is also called a *cotwist* in [25, Definition 2.5]. The next lemma lists some general facts about cotwists, some of which is from Montgomery [25], Davies [12,11], or Chirvasitu–Smith [10].

Lemma 4.12. *Let R be a connected graded algebra and H be a semisimple Hopf algebra such that R is a graded H -module algebra. Let $\Omega \in H \otimes H$ be a dual cocycle [25, Section 2], and H^Ω be a cotwist of H with respect to Ω [25, Definition 2.5]. Then the following statements hold.*

- (1) H is a cotwist $(H^\Omega)^{\Omega^{-1}}$ of H^Ω .
- (2) The twist of R , denoted by R_Ω , as defined in [25, Definition 2.6], is a graded H^Ω -module algebra.
- (3) There is a graded algebra isomorphism

$$\phi : R_\Omega \# H^\Omega \longrightarrow R \# H$$

defined by $a \# h \mapsto \sum \Omega^1 \cdot a \# \Omega^2 h$ for all $a \in R$ and $h \in H$.

- (4) There is an equivalence of graded module categories

$$F_\phi : R_\Omega \# H^\Omega\text{-Mod} \cong R \# H\text{-Mod}$$

induced by the isomorphism ϕ in part (3).

- (5) If R is noetherian, then so is R_Ω .
- (6) $\text{gldim } R = \text{gldim } R_\Omega$ and $\text{GKdim } R = \text{GKdim } R_\Omega$.
- (7) R is AS regular if and only if R_Ω is AS regular.
- (8) Suppose R is AS regular. If the H -action on R has trivial homological determinant, then so does the H^Ω -action on R_Ω .
- (9) Let R and R_Ω be locally finite graded domains. If $R \# H \rightarrow \text{End}_{(R^H)^{\text{op}}}(R)$ is an isomorphism, then $R_\Omega \# H^\Omega \rightarrow \text{End}_{(R_\Omega^{H^\Omega})^{\text{op}}}(R_\Omega)$ is an isomorphism.

Proof. (1) This is well-known.

(2,3) See [25, Section 2] and references therein.

(4) For every $M \in R \# H\text{-Mod}$, define the left $R_\Omega \# H^\Omega$ -module structure on M , denoted by M_Ω , by

$$(a \# h) \bullet_\Omega m = \sum ((\Omega^1 \cdot a) \# \Omega^2 h) m$$

for all $a \in R$ and $h \in H$ and $m \in M$. The result then follows from part (3).

(5) See [25, Proposition 3.1].

(6) Consider a minimal projective resolution of the left trivial module $\mathbb{k}$ over $R_\Omega \# H^\Omega$:

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{k} \rightarrow 0 \quad (4.12.1)$$

where $P_0 = R_\Omega$ as a left $R_\Omega \# H^\Omega$ -module. By Lemma 4.11, (4.12.1) is a minimal projective (free) resolution of the trivial R_Ω -module $\mathbb{k}$. Applying the equivalence F_ϕ of categories in part (4), we obtain

$$\cdots \rightarrow F_\phi(P_2) \rightarrow F_\phi(P_1) \rightarrow F_\phi(P_0) \rightarrow \mathbb{k} \rightarrow 0 \quad (4.12.2)$$

which is a minimal projective resolution of the trivial $R \# H$ -module $\mathbb{k}$. By Lemma 4.11, (4.12.2) is a minimal projective (free) resolution of the trivial R -module $\mathbb{k}$. Combining these facts, we have that the shape of the minimal free resolution of the trivial R -module $\mathbb{k}$ is exactly the shape of the minimal free resolution of the trivial R_Ω -module $\mathbb{k}$. From this, we have the following consequences:

- (i) $\text{gldim } R = \text{gldim } R_\Omega$.
- (ii) R is Koszul (respectively, N -Koszul) if and only if R_Ω is.
- (iii) R and R_Ω have the same Hilbert series.
- (iv) R and R_Ω have the same Gelfand–Kirillov dimension.

(7) Following part (4), F_ϕ maps R_Ω to R , and by [16, Lemma 5.2] and part (4), we have

$$\text{Ext}_R^i(\mathbb{k}, R) \cong \text{Ext}_{R \# H}^i(\mathbb{k} \# H, R) \cong \text{Ext}_{R_\Omega \# H^\Omega}^i(\mathbb{k} \# H^\Omega, R_\Omega) \cong \text{Ext}_{R_\Omega}^i(\mathbb{k}, R_\Omega).$$

With part (6), we obtain that R is AS regular if and only if R_Ω is.

(8) Let $n = \text{gldim } R$. Suppose R (or R_Ω) is AS regular. Since F_ϕ maps R_Ω to R (and F_ϕ commutes with the shift), $P_n \cong R_\Omega(\ell)$ as a graded left $R_\Omega \# H^\Omega$ -module. It follows from Definition 1.6 that the H -action on R has trivial homological determinant if and only if $F_\phi(P_n) \cong R(\ell)$ as a graded left $R \# H$ -module, or the H -action on the lowest degree of $F_\phi(P_n)$ is trivial. This implies that the H^Ω -action on R_Ω has trivial homological determinant.

(9) Since $R \# H \rightarrow \text{End}_{(R^H)^{\text{op}}}(R)$ is an isomorphism, $R \# H$ is prime by [2, Lemma 3.10]. By part (3), $R_\Omega \# H^\Omega$ is prime. Hence, by Lemma 4.10, the map $R_\Omega \# H^\Omega \rightarrow \text{End}_{(R_\Omega^{H^\Omega})^{\text{op}}}(R_\Omega)$ is injective. It remains to show that these algebras have the same Hilbert series. It is clear that $R \# H$ and $R_\Omega \# H^\Omega$ have the same Hilbert series. By the hypothesis, it suffices to show that $\text{End}_{(R^H)^{\text{op}}}(R)$ and $\text{End}_{(R_\Omega^{H^\Omega})^{\text{op}}}(R_\Omega)$ have the same Hilbert series. By definition, $R = R_\Omega$ as vector spaces and the H -action on R agrees with the H^Ω -action on R_Ω . So $R^H = R_\Omega^{H^\Omega}$ as vector spaces and as algebras. Further, it is easy to check that the left R^H -multiplication on R is exactly the left $R_\Omega^{H^\Omega}$ -multiplication on R_Ω . As a consequence, R_Ω , as a left $R_\Omega^{H^\Omega}$ -module, is equal to R , as a left R^H -modules. Hence $\text{End}_{(R^H)^{\text{op}}}(R) = \text{End}_{(R_\Omega^{H^\Omega})^{\text{op}}}(R_\Omega)$, as required. Combining these facts, we obtain that $R_\Omega \# H^\Omega \cong \text{End}_{(R_\Omega^{H^\Omega})^{\text{op}}}(R_\Omega)$, as desired. $\square$

Now we prove part of Theorem 0.4 for case (f).

Proposition 4.13. *Let $H = (\mathcal{A}(\widetilde{D}_{2m}))^\circ$ and $A = \mathbb{k}_{-1}[u, v]$ be as in case (f) of Theorem 0.4. Then Conjecture 0.2 holds.*

Proof. By Lemma 4.12, it suffices to show that Conjecture 0.2 holds for some cotwist of H . By [22, Theorem 4.1], H is a cotwist $(\mathbb{k}D_{2m})^{\Omega^{-1}}$ of $\mathbb{k}D_{2m}$. Further, A_Ω is noetherian AS regular of dimension 2 and admits an H^Ω -action with trivial homological determinant by Lemma 4.11(2,5,6,7,8). Since $H^\Omega = \mathbb{k}D_{2m}$, we have by Theorem 0.4(d) that $A_\Omega \cong A$ as graded H^Ω -module algebras. Hence, Conjecture 0.2 holds by case (d). $\square$

We consider the situation where $H = \mathcal{D}(\widetilde{\Gamma})^\circ$ for the other algebras in case (f) of Theorem 0.4. There is a short exact sequence of Hopf algebras

$$\mathbb{k} \rightarrow \mathbb{k}\mathbb{Z}_2 \rightarrow H \rightarrow \mathbb{k}\Gamma \rightarrow \mathbb{k}, \quad (4.13.1)$$

and by [9, Lemma 6.27(b)], $A^H = (A^{(2)})^\Gamma$, where $A = \mathbb{k}_{-1}[u, v]$. There is a canonical Hopf algebra map $\gamma : H \rightarrow \mathbb{k}\Gamma$ given in (4.13.1). Let t be the integral of H such that $\epsilon(t) = 1$. Then $t' := \gamma(t)$ is the integral of $\mathbb{k}\Gamma$ such that $\epsilon(t') = 1$.

Lemma 4.14. *Retain the notation above. Let $(1\#t)$ and $(1\#t')$ be the 2-sided ideal generated by $1\#t$ and $1\#t'$, respectively. Then $(A\#H)/(1\#t)$ is finite dimensional if and only if $(A^{(2)}\#\mathbb{k}\Gamma)/(1\#t')$ is finite dimensional.*

Proof. By (4.13.1), $H/(g-1) \cong \mathbb{k}\Gamma$, where $g \in H$ is a central grouplike element of order 2. Let $\{f_1 := \frac{1}{2}(1+g), f_2, \dots, f_m, h_1, \dots, h_m\}$ be a $\mathbb{k}$ -basis of H such that:

- (i) $h_i \in (g-1)H$ for all i ,
- (ii) $gf_i = f_i$ for all i , and
- (iii) $\{\gamma(f_i)\}_{i=1}^m$ is the set of grouplike elements in Γ .

Suppose $(A\#H)/(1\#t)$ is finite dimensional. Then u^{2d}, v^{2d} are in the ideal I of $A\#H$ generated by $1\#t$. Write

$$u^{2d}\#1 = \sum x_i(1\#t)y_i = \sum_{\deg x_i = \text{odd}} x_i(1\#t)y_i + \sum_{\deg x_i = \text{even}} x_i(1\#t)y_i$$

where x_i and y_i are homogeneous in $A\#H$ and $\deg(x_i) + \deg(y_i) = 2d$ for all i . Multiplying the above equation by $f_1 = \frac{1}{2}(1+g)$ we obtain

$$\begin{aligned} u^{2d}\#f_1 &= (1\#f_1)(u^{2d}\#1) = (1\#\tfrac{1}{2}(1+g))(u^{2d}\#1) \\ &= \sum_{\deg x_i = \text{odd}} (1\#\tfrac{1}{2}(1+g))x_i(1\#t)y_i + \sum_{\deg x_i = \text{even}} (1\#f_1)x_i(1\#t)y_i \\ &= \sum_{\deg x_i = \text{odd}} x_i(1\#\tfrac{1}{2}(1-g))(1\#t)y_i + \sum_{\deg x_i = \text{even}} x_i(1\#f_1)(1\#t)y_i \end{aligned}$$

$$= \sum_{\deg x_i = \text{even}} x_i(1\#t)y_i.$$

Let γ denote also the induced surjective algebra homomorphism $A^{(2)}\#H \rightarrow A^{(2)}\#\mathbb{k}\Gamma$. Applying γ to the above, we obtain

$$u^{2d}\#1 = \sum_{\deg x_i = \text{even}} \gamma(x_i)(1\#t')\gamma(y_i),$$

which holds in $A^{(2)}\#\mathbb{k}\Gamma$. Thus $u^{2d}\#1$ is in the ideal I' of $A^{(2)}\#\mathbb{k}\Gamma$ generated by $1\#t'$. Similarly, $v^{2d}\#1$ is in the ideal I' of $A^{(2)}\#\mathbb{k}\Gamma$ generated by $1\#t'$. This implies that $(A^{(2)}\#\mathbb{k}\Gamma)/I'$ is finite dimensional.

Conversely, suppose that $(A^{(2)}\#\mathbb{k}\Gamma)/I'$ is finite dimensional. Then there is an integer $d \geq 1$ such that $u^{2d}\#1$ and $v^{2d}\#1$ are in the ideal I' generated by $1\#t'$. Write

$$u^{2d}\#1 = \sum_i \gamma(x_i)(1\#t')\gamma(y_i)$$

where x_i, y_i are elements in $A^{(2)}\#H$. Let F be the element $(\sum_i x_i(1\#t)y_i)(1\#f_1)$ in $A\#H$. Then

$$\gamma(F) = \sum_i \gamma(x_i)(1\#t')\gamma(y_i) = u^{2d}\#1.$$

Write $F = \sum z_i\#f_i + \sum t_i\#g_i$. Since $g_i \in (g-1)$ and $F = F(1\#g)$, we have $t_i = 0$. Hence $u^{2d}\#1 = \gamma(F) = \sum z_i\#\gamma(f_i)$. Since $\{\gamma(f_i)\}_{i=1}^m$ are linearly independent, $z_1 = u^{2d}$ and $z_i = 0$ for all $i \neq 1$. Thus $F = u^{2d}\#f_1$. Therefore, we have that $(u\#1)F + F(u\#1) = u^{2d+1}\#1$, which must be in I . Similarly, $v^{2d+1}\#1 \in I$. This implies that $(A\#H)/I$ is finite dimensional. $\square$

If (A, H) is in the case of Theorem 0.4(f) then $A = \mathbb{k}_{-1}[u, v]$ and $H = (\mathcal{A}(\tilde{D}_{2m}))^\circ$ or equation 4.13.1 holds.

Proposition 4.15. *Let (A, H) be in the case of Theorem 0.4(f). Then Conjecture 0.2 holds.*

Proof. When $H = (\mathcal{A}(\tilde{D}_{2m}))^\circ$, the assertion is Proposition 4.13. For the other cases, by Lemmas 4.4 and 4.14, it suffices to show that $(A^{(2)}\#\mathbb{k}\Gamma)/(1\#t')$ is finite dimensional. By Lemmas 3.1(4), 3.2(4), 3.3(4) and 3.6, it is equivalent to show that $(\mathbb{k}[u, v]^{(2)}\#\mathbb{k}\Gamma)/(1\#t')$ is finite dimensional. This last statement follows by Proposition 4.2, Lemma 4.4, and the commutative version of Lemma 4.14. $\square$

Proof of Theorem 4.1. It follows by combining Propositions 4.2, 4.3, 4.5, 4.7, 4.9, and 4.15. $\square$

5. Quantum Kleinian singularities as noncommutative hypersurfaces

Recall that if a semisimple Hopf algebra H acts on an Artin–Schelter (AS) regular algebra A inner faithfully, preserving the grading of A , with trivial homological determinant, and if further $A^H \neq A$, then A^H is not AS regular [9, Theorem 0.6]. This motivates the following definition.

Definition 5.1. We say that a ring R has *quantum Kleinian singularities* if it is isomorphic to a fixed ring A^H where:

- (i) H is a finite dimensional Hopf algebra, and
- (ii) A is an AS regular algebra of dimension 2 which is also a graded inner-faithful H -module algebra, and
- (iii) the H -action on A has trivial homological determinant.

The classification of the quantum Kleinian singularities follows from Theorem 0.4. Here we describe these fixed subrings as noncommutative hypersurface rings in the sense of [18].

Theorem 5.2. *Let H be a semisimple Hopf algebra and A be an H -module algebra satisfying Hypothesis 0.1 with $\text{gldim } A = 2$. Then A^H is isomorphic to $C/C\Omega$, where C is an AS-regular algebra of dimension 3 and Ω is a normal element in C .*

Proof. The proof of this theorem consists of a case-by-case analysis, the results of which are summarized in Table 3. We defer the verification that Ω is normal in C to the end of this proof.

In case (a), the fixed subring $\mathbb{k}[u, v]^G$, where $G \leq SL_2(\mathbb{k})$, is a (commutative) Kleinian singularity, and is well known to be a hypersurface singularity in affine 3-space. The cases of group actions (cases (b), (c), (d), (g), (h)) on noncommutative AS regular algebras holds by [16, Theorem 0.1] and [17, Theorem 0.1]. It remains to verify the result in cases (e) and (f). (Again, we show that Ω is normal in C at the end.)

Case (e): We show that $A^H = A^{\text{co } \mathbb{k}D_{2n}}$ as follows. Let $v_1 = u + \sqrt{-1}v$ and $v_2 = \sqrt{-1}u + v$. Then

$$A := \mathbb{k}_{-1}[u, v] \cong \mathbb{k}\langle v_1, v_2 \rangle / (v_1^2 + v_2^2).$$

We use the following presentation for

$$D_{2n} = \langle g, h \mid g^2 = h^2 = (gh)^n = 1 \rangle.$$

The $\mathbb{k}D_{2n}$ -coaction on A is given by $\rho(v_1) = v_1 \otimes g$ and $\rho(v_2) = v_2 \otimes h$.

Let $s_{r,k} := v_1^{r-2k}(v_1v_2)^k$ and $s'_{r,k} := v_1^{r-2k}(v_2v_1)^k$. We choose the following basis $\{s_{2m,k}, s'_{2m,k} \mid k = 1, \dots, m\} \cup \{v_1^{2m}\}$ for A_{2m} and the following basis $\{s_{2m+1,k}, s'_{2m+1,k} \mid k = 1, \dots, m\} \cup \{v_1(v_1v_2)^{2m}, v_2(v_1v_2)^{2m}\}$ for A_{2m+1} . It is immediate that $A^H = A^{\text{co } \mathbb{k}D_{2n}}$.

Next we compute the $\mathbb{k}D_{2n}$ -coinvariants in degree $2m$. Clearly v_1^{2m} is $\mathbb{k}D_{2n}$ -coinvariant, and $\rho(s_{2m,k}) = s_{2m,k} \otimes (gh)^k$ implies that $s_{2m,k}$ is $\mathbb{k}D_{2n}$ -coinvariant if and only if $n \mid k$. Similarly $s'_{2m,k}$ is coinvariant if and only if $n \mid k$. We obtain a $\mathbb{k}$ -basis of $\mathbb{k}D_{2n}$ -coinvariant elements $\{s_{2m,pn}\}_{p \geq 0} \cup \{s'_{2m,pn}\}_{p \geq 0}$. This argument shows that

$$a_1 = (v_1v_2)^n, \quad a_2 = v_1^2 = -v_2^2, \quad a_3 = (v_2v_1)^n$$

are $\mathbb{k}D_{2n}$ -coinvariant. These elements commute and satisfy $\Omega := a_2^n - (-1)^n a_1 a_3$ equal to 0. So letting $C = \mathbb{k}[a_1, a_2, a_3]$, we have a surjective ring homomorphism

$$\theta : C/(\Omega) \twoheadrightarrow A^{\text{co } \mathbb{k}D_{2n}}.$$

The $\mathbb{k}$ -basis for $\mathbb{k}D_{2n}$ -coinvariants given above shows that

$$\begin{aligned} H_{A^{\text{co } \mathbb{k}D_{2n}}}(t) &= \sum_{j=0}^{\infty} (1 + 2\lfloor \frac{j}{n} \rfloor) t^{2j} = (1 + t^2 + \dots + t^{2n-2}) \sum_{j=0}^{\infty} (1 + 2j) t^{2jn} \\ &= \frac{(1 + t^{2n})(1 + t^2 + \dots + t^{2n-2})}{(1 - t^{2n})^2} = \frac{1 + t^2 + \dots + t^{4n-2}}{(1 - t^{2n})^2}. \end{aligned}$$

The Hilbert series of $C/(\Omega)$ follows from noting that the generators a_1, a_2, a_3 are in degrees $2n, 2, 2n$, respectively, with a relation Ω in degree $4n$. Hence

$$H_{C/(\Omega)}(t) = \frac{1 - t^{4n}}{(1 - t^{2n})^2(1 - t^2)}.$$

Since $H_{A^H}(t) = H_{A^{\text{co } D_{2n}}}(t) = H_{C/(\Omega)}(t)$, we obtain that θ is an isomorphism, as desired.

Case (f): This case follows from Lemmas 3.1(4), 3.2(4), 3.3(4), and 3.6(4), and completes the proof.

We summarize the cases in Table 3.

Finally, in the cases when C is noncommutative, the normality of Ω in C holds due to the computations below. The reader may wish to refer to Table 1.

(b)	$a_1\Omega = (-1)^{-n^2}\Omega a_1$	$a_2\Omega = \Omega a_2$	$a_3\Omega = (-1)^{n^2}\Omega a_3$
(c)	$a_1\Omega = \Omega a_1$	$a_2\Omega = \Omega a_2$	
(d, n odd)	$a_1\Omega = \Omega a_1$	$a_2\Omega = \Omega a_2$	$a_3\Omega = \Omega a_3$
(g)	$a_1\Omega = q^{-n^2}\Omega a_1$	$a_2\Omega = \Omega a_2$	$a_3\Omega = q^{n^2}\Omega a_3$
(h)	$a_1\Omega = \Omega a_1$	$a_2\Omega = \Omega(2a_1 + a_2)$	$a_3\Omega = \Omega(6a_1 + 4a_2 + a_3). \quad \square$

Table 3

Quantum Kleinian singularities, as noncommutative hypersurface singularities.

	$R = C/(\Omega)$	(a_1, a_2, a_3)
(a)	Commutative Kleinian singularity	
(b)	$C = \mathbb{k}_{\mathbf{q}}[a_1, a_2, a_3]$ where $q_{12} = q_{13} = (-1)^{n^2}$, $q_{23} = (-1)^n$ $\Omega = a_2^n - (-1)^{\frac{n(n-1)}{2}} a_1 a_3$	(u^n, uv, v^n)
(c)	$C = \mathbb{k}\langle a_1, a_2 \rangle / ([a_1^2, a_2], [a_2^2, a_1])$ $\Omega = a_1^6 - a_2^2$	$(u + v, (u - v)(u + v)(u - v), 0)$ (two generators)
(d)	n even $C = \mathbb{k}[a_1, a_2, a_3]$ $\Omega = a_3 a_1 + a_1 a_3 - 4(-1)^{(n+2)/2} a_2^{(n+2)/2}$ n odd $C = \frac{\mathbb{k}\langle a_1, a_2, a_3 \rangle}{\left(\begin{array}{c} a_3 a_1 + a_1 a_3 - 4(-1)^{(n+1)/2} a_2^{(n+1)/2} \\ a_2 \text{ central} \end{array} \right)}$ $\Omega = a_3^2 + a_2 a_1^2$	$(u^n + v^n, (uv)^2, u^{n+1}v - uv^{n+1})$
(e)	$C = \mathbb{k}[a_1, a_2, a_3]$ $\Omega = a_2^{2n} - (-1)^n a_1 a_3$	$((v_1 v_2)^n, v_1^2, (v_2 v_1)^n)$
(f)	Commutative Kleinian singularity	
(g)	$C = \mathbb{k}_{\mathbf{q}}[a_1, a_2, a_3]$ where $q_{12} = q_{13} = q^{n^2}$, $q_{23} = q^n$ $\Omega = a_2^n - q^{\frac{n(n-1)}{2}} a_1 a_3$	(u^n, uv, v^n)
(h)	$C = \frac{\mathbb{k}\langle a_1, a_2, a_3 \rangle}{\left(\begin{array}{c} a_2 a_1 - a_1 a_2 - 2a_1^2 \\ a_3 a_2 - a_2 a_3 - 2a_2^2 \\ a_3 a_1 - a_1 a_3 - 4a_1 a_2 - 6a_1^2 \end{array} \right)}$ $\Omega = a_2^2 - a_1 a_2 - a_1 a_3$	(u^2, uv, v^2)

Acknowledgments

The authors would like to thank W. Frank Moore for supplying corrections to some computations presented in Table 3, and we thank the referee for providing several helpful suggestions. C. Walton and J.J. Zhang were supported by the US National Science Foundation: NSF grants DMS-1550306, 1663775 and DMS-1402863 respectively. C. Walton is also supported by a research fellowship from the Sloan Foundation. E. Kirkman was supported by Simons Foundation Grant #208314.

References

- [1] M. Auslander, On the purity of the branch locus, *Amer. J. Math.* 84 (1962) 116–125.
- [2] Y.-H. Bao, J.-W. He, J. Zhang, Pertinency of Hopf actions and quotient categories of Cohen–Macaulay algebras, preprint available at <http://arxiv.org/abs/1603.02346>, 2016.
- [3] J. Bichon, S. Natale, Hopf algebra deformations of binary polyhedral groups, *Transform. Groups* 16 (2) (2011) 339–374.

- [4] J. Bryan, A. Gholampour, The quantum McKay correspondence for polyhedral singularities, *Invent. Math.* 178 (3) (2009) 655–681.
- [5] R.-O. Buchweitz, From Platonic Solids to Preprojective Algebras via the McKay Correspondence, *Oberwolfach Jahresbericht Annual Report*, 2012, pp. 18–28.
- [6] C. Călinescu, S. Dăscălescu, A. Masuoka, C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras, *J. Algebra* 273 (2) (2004) 753–779.
- [7] D. Chan, McKay correspondence for canonical orders, *Trans. Amer. Math. Soc.* 362 (4) (2010) 1765–1795.
- [8] K. Chan, E. Kirkman, C. Walton, J.J. Zhang, McKay correspondence for semisimple Hopf actions on regular graded algebras, part II, *J. Noncommut. Geom.* (2018), in press, preprint available at <http://arxiv.org/abs/1607.06977>, 2016.
- [9] K. Chan, E. Kirkman, C. Walton, J.J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, *J. Reine Angew. Math.* 719 (2016) 211–252.
- [10] A. Chirvasitu, S.P. Smith, Exotic elliptic algebras of dimension 4, *Adv. Math.* 309 (2017) 558–623. With an appendix by Derek Tomlin.
- [11] A. Davies, Cocycle Twists of Algebras, Ph.D. Thesis, University of Manchester, 2014, Thesis available at <http://arxiv.org/abs/1502.06101>.
- [12] A. Davies, Cocycle twists of algebras, *Comm. Algebra* 45 (3) (2017) 1347–1363.
- [13] D. Happel, U. Preiser, C.M. Ringel, Binary polyhedral groups and Euclidean diagrams, *Manuscripta Math.* 31 (1–3) (1980) 317–329.
- [14] N. Jing, J.J. Zhang, Gorensteinness of invariant subrings of quantum algebras, *J. Algebra* 221 (2) (1999) 669–691.
- [15] A. Kirillov Jr., V. Ostrik, On a q -analogue of the McKay correspondence and the ADE classification of sl_2 conformal field theories, *Adv. Math.* 171 (2) (2002) 183–227.
- [16] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Gorenstein subrings of invariants under Hopf algebra actions, *J. Algebra* 322 (10) (2009) 3640–3669.
- [17] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Invariant theory of finite group actions on down-up algebras, *Transform. Groups* 20 (1) (2015) 113–165.
- [18] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Noncommutative complete intersections, *J. Algebra* 429 (2015) 253–286.
- [19] G.J. Leuschke, Non-commutative crepant resolutions: scenes from categorical geometry, in: *Progress in Commutative Algebra 1*, de Gruyter, Berlin, 2012, pp. 293–361.
- [20] G.J. Leuschke, R. Wiegand, *Cohen–Macaulay Representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012.
- [21] D.M. Lu, J.H. Palmieri, Q. Wu, J.J. Zhang, Koszul equivalences in A_∞ -algebras, *New York J. Math.* 14 (2008) 325–378.
- [22] A. Masuoka, Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension, in: *New Trends in Hopf Algebra Theory*, La Falda, 1999, in: *Contemp. Math.*, vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 195–214.
- [23] J. McKay, Graphs, singularities, and finite groups, in: *The Santa Cruz Conference on Finite Groups*, Univ. California, Santa Cruz, Calif., 1979, in: *Proc. Sympos. Pure Math.*, vol. 37, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186.
- [24] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [25] S. Montgomery, Algebra properties invariant under twisting, in: *Hopf Algebras in Noncommutative Geometry and Physics*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 239, Dekker, New York, 2005, pp. 229–243.
- [26] I. Mori, McKay-type correspondence for AS-regular algebras, *J. Lond. Math. Soc.* (2) 88 (1) (2013) 97–117.
- [27] I. Mori, K. Ueyama, Ample group action on AS-regular algebras and noncommutative graded isolated singularities, *Trans. Amer. Math. Soc.* 368 (10) (2016) 7359–7383.
- [28] M. Reyes, D. Rogalski, J.J. Zhang, Skew Calabi–Yau algebras and homological identities, *Adv. Math.* 264 (2014) 308–354.