Selfinjective algebras having a generalized standard family of quasi-tubes maximally saturated by simple and projective modules

Alicja Jaworska-Pastuszak [a],[*], Marta Kwiecień [b], Andrzej Skowroński [a]

[a] *Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland*
[b] *Faculty of Mathematics and Computer Science, University of Warmia and Mazury, ul. Słoneczna 54, 10-710 Olsztyn, Poland*

A R T I C L E   I N F O

A B S T R A C T

We give a complete description of finite dimensional self-injective algebras over an algebraically closed field whose Auslander–Reiten quiver admits a generalized standard family of quasi-tubes maximally saturated by simple and projective modules. In particular, we show that these algebras are self-injective algebras of strictly canonical type.

© 2018 Elsevier Inc. All rights reserved.

* Corresponding author.
    E-mail addresses: jaworska@mat.uni.torun.pl (A. Jaworska-Pastuszak), marta.kwiecien@uwm.edu.pl (M. Kwiecień), skowron@mat.uni.torun.pl (A. Skowroński).

1. Introduction and the main results

Throughout the article K will denote a fixed algebraically closed field. By an algebra is meant an associative finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional (over K) right A-modules, by $\operatorname{ind} A$ its full subcategory formed by the indecomposable modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$. Given a module M in $\operatorname{mod} A$, we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if the modules M and N have the same simple composition factors including the multiplicities. An algebra A is called *selfinjective* if A_A is an injective module, or equivalently, the projective and injective modules in $\operatorname{mod} A$ coincide. An important class of selfinjective algebras is formed by the orbit algebras R/G, where R is a selfinjective locally bounded K-category and G is an admissible group of automorphisms of R. Then we have a Galois covering $R \to R/G$ which frequently allows us to reduce the representation theory of R/G to the representation theory of R. In the theory, the selfinjective orbit algebras $\widehat{B}/G$ given by the repetitive categories $\widehat{B}$ of triangular algebras B and infinite cyclic admissible groups G of automorphisms of $\widehat{B}$ are of particular interest. We also note that for the algebras B of finite global dimension, the stable module category $\underline{\operatorname{mod}} \, \widehat{B}$ of $\widehat{B}$ is equivalent (as a triangulated category) to the derived category $D^b(\operatorname{mod} B)$ of bounded complexes over $\operatorname{mod} B$ [13].

An important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A is its Auslander–Reiten quiver Γ_A. The vertices of Γ_A are the isoclasses $[X]$ of modules X in $\operatorname{ind} A$, and the number of arrows starting at $[X]$ and ending in $[Y]$ in Γ_A is the number of linearly independent irreducible morphisms from X to Y in $\operatorname{mod} A$. Moreover, we have the Auslander–Reiten translations $\tau_A = \operatorname{DTr}$ and $\tau_A^- = \operatorname{TrD}$. We shall identify a vertex $[X]$ of Γ_A with the module X. By a *component* of Γ_A we mean the connected component of Γ_A. For a component Γ of Γ_A we denote by Γ^s its *stable part*, that is the subquiver of Γ obtained by deleting the τ_A-orbits of both the projective and the injective modules. The Auslander–Reiten quiver Γ_A describes the structure of the quotient category $\operatorname{mod} A/\operatorname{rad}_A^\infty$, where $\operatorname{rad}_A^\infty$ is the infinite Jacobson radical of $\operatorname{mod} A$ (the intersection of all powers rad_A^i, $i \geqslant 1$, of the Jacobson radical rad_A of $\operatorname{mod} A$). By a result due to Auslander [5], A is of finite representation type if and only if $\operatorname{rad}_A^\infty = 0$ (see also [16] for an alternative proof of this result). On the other hand, if A is of infinite representation type, then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a result proved in [8]. In general, it is important to study the shape and behavior of the components of Γ_A in the category $\operatorname{mod} A$. Recall that a translation quiver $\Gamma = (\Gamma, \tau)$ is said to be a *tube* if it contains a cyclic path and its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$, where S^1 is the unit circle and $\mathbb{R}^+$ is the non-negative real line [11], [26]. Then Γ is a stable tube if and only if it is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some integer $r \geqslant 1$. By a *quasi-tube* we mean a translation quiver Γ such that its full translation subquiver formed by all vertices which are not projective–injective is a tube [29].

Following [30], a family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of components of Γ_A is said to be *generalized standard* if $\mathrm{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$. It was proved in [30, (2.3)] that every generalized standard family $\mathcal{C}$ of components in Γ_A is almost periodic, that is, all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic. In particular, for a selfinjective algebra A, every infinite generalized standard component $\mathcal{C}$ of Γ_A is either acyclic with finitely many τ_A-orbits or a quasi-tube.

In the paper we are concerned with the structure of selfinjective algebras A for which the Auslander–Reiten quiver Γ_A admits a generalized standard component. A distinguished class of such algebras is formed by the selfinjective algebras of finite representation type. By general theory (see [32, Section 3]) these algebras are socle deformations of the orbit algebras $\widehat{B}/G$, for tilted algebras B of Dynkin type and infinite cyclic groups G of automorphisms of $\widehat{B}$. Further, it was proved in [34–36] that every selfinjective algebra A having an acyclic generalized standard component in Γ_A is of the form $\widehat{B}/G$, for a tilted algebra B of Euclidean or wild type and an infinite cyclic group G of automorphisms of $\widehat{B}$. On the other hand, the description of selfinjective algebras A whose Auslander–Reiten quiver admits a generalized standard quasi-tube is an exciting but difficult problem. Namely, every algebra is a quotient algebra of a selfinjective algebra A with Γ_A having a generalized standard stable tube (see [30], [31]). We refer to [6], [14], [15], [17] for some work on the structure of selfinjective algebras having generalized standard families of quasi-tubes.

In order to formulate the main result of the paper we need to present some concepts.

Let A be an algebra. Recall that a *smooth* quasi-tube is a quasi-tube whose all nonstable vertices are projective–injective. For a smooth quasi-tube $\mathcal{T}$ of Γ_A we denote by $s(\mathcal{T})$ the number of simple modules in $\mathcal{T}$, by $p(\mathcal{T})$ the number of projective modules in $\mathcal{T}$, and by $r(\mathcal{T})$ the rank of the stable tube $\mathcal{T}^s$. Obviously, if A is selfinjective then each quasi-tube is smooth. Moreover, in this case we know that $s(\mathcal{T}) + p(\mathcal{T}) \leqslant r(\mathcal{T}) - 1$ for any quasi-tube $\mathcal{T}$ in Γ_A [24]. A family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of smooth quasi-tubes in Γ_A is said to be *maximally saturated by simple and projective modules* if there exist two simple right A-modules S and T which are not in $\mathcal{C}$ and the following conditions are satisfied:

(MS1) $s(\mathcal{C}_\lambda) + p(\mathcal{C}_\lambda) = r(\mathcal{C}_\lambda) - 1$ for any $\lambda \in \Lambda$;

(MS2) the simple composition factors of indecomposable modules in $\mathcal{C}$ are: S, T, the simple modules in $\mathcal{C}$, and the socles and tops of indecomposable projective modules in $\mathcal{C}$;

(MS3) $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ consists of all quasi-tubes such that $\mathcal{C}_\lambda$ admits an indecomposable module E_λ with $\mathrm{soc}\, E_\lambda = S$ and $\mathrm{top}\, E_\lambda = T$.

In particular, if $p(\mathcal{C}_\lambda) = 0$ for any $\lambda \in \Lambda$, we say that a family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of stable tubes is *maximally saturated by simple modules*.

The following main result of the paper describes the structure of all selfinjective algebras whose Auslander–Reiten quiver admits a generalized standard family of quasi-tubes maximally saturated by simple and projective modules.

Theorem 1.1. *Let A be a basic, connected, finite dimensional selfinjective algebra over an algebraically closed field K. The following statements are equivalent.*

(i) *Γ_A admits a generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of quasi-tubes maximally saturated by simple and projective modules.*

(ii) *A is isomorphic to an orbit algebra $\widehat{B}/G$ of the repetitive category $\widehat{B}$ of a branch extension B of a canonical algebra C with respect to the canonical $\mathbb{P}_1(K)$-family of stable tubes of Γ_C and G is an infinite cyclic group of automorphisms of $\widehat{B}$ of one of the forms:*
   (a) *$G = (\varphi \nu_{\widehat{B}})$, for φ a strictly positive automorphism of $\widehat{B}$,*
   (b) *$G = (\varphi \nu_{\widehat{B}})$, for B a canonical algebra and φ a rigid automorphism of $\widehat{B}$,*
   where $\nu_{\widehat{B}}$ is the Nakayama automorphism of $\widehat{B}$.

We note that the selfinjective algebras occurring in the second statement of the above theorem are selfinjective algebras of strictly canonical type investigated in [18], [19], [20]. In particular, the structure and homological properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type were described in [18].

The following direct consequence of Theorem 1.1 and [25, Theorem 2] provides a characterization of the trivial extensions of the canonical algebras by their minimal injective cogenerators.

Corollary 1.2. *Let A be a basic, connected, finite dimensional symmetric algebra over an algebraically closed field K. The following statements are equivalent.*

(i) *Γ_A admits a generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of quasi-tubes maximally saturated by simple and projective modules.*

(ii) *Γ_A admits a generalized standard family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \Lambda}$ of stable tubes maximally saturated by simple modules.*

(iii) *A is isomorphic to the trivial extension $B \ltimes D(B)$ of a canonical algebra B.*

The paper is organized in the following way. In Section 2 we recall the canonical algebras and describe their canonical family of stable tubes. Section 3 presents quasi-tube enlargements of algebras. In Section 4 we show the needed facts on repetitive algebras and their orbit algebras. Section 5 contains results on selfinjective algebras of strictly canonical type which allow us to state that the implication (ii) ⇒ (i) of Theorem 1.1 is true. In Section 6 we complete the proof of Theorem 1.1. The final Section 7 is devoted to some examples illustrating the main results of the paper.

For basic background on the representation theory of algebras applied in the paper we refer to the books [1], [26], [27], [28], [37], [38].

2. Canonical algebras

The aim of this section is to introduce the canonical algebras and show that they are exactly the algebras whose Auslander–Reiten quiver contains a faithful generalized standard family of stable tubes maximally saturated by simple modules.

Throughout the paper for a vertex x in the Gabriel quiver Q_A of an algebra A, by $S(x)$, $P(x)$ and $I(x)$ we denote a simple, indecomposable projective and indecomposable injective A-module at the vertex x, respectively. Moreover, by $M = (M_a, M_\alpha)_{a \in Q_0, \alpha \in Q_1}$, where Q_0 is a set of vertices and Q_1 is a set of arrows in Q_A, we denote a K-linear representation of Q_A. If I is an admissible ideal of KQ_A, then by representation $M = (M_a, M_\alpha)$ of Q_A we will mean a representation satisfying the relations in I.

Let $m \geq 2$ be an integer number, $\boldsymbol{p} = (p_1, \ldots, p_m)$ a sequence of positive integer numbers and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ a sequence of pairwise different elements of the projective line $\mathbb{P}_1(K) = K \cup \{\infty\}$ normalized in such a way that $\lambda_1 = \infty$ and $\lambda_2 = 0$. Consider the quiver $\Delta(\boldsymbol{p})$ of the form
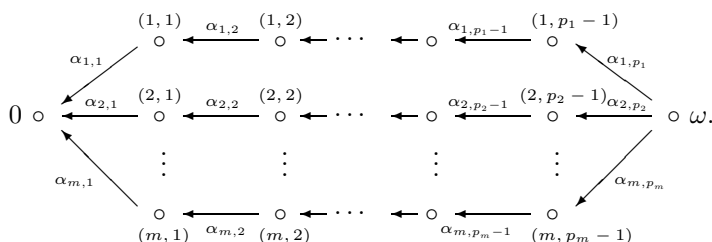
For $m = 2$, $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is defined to be the path algebra $K\Delta(\boldsymbol{p})$ of the quiver $\Delta(\boldsymbol{p})$ over K. For $m \geq 3$, $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is defined to be the quotient algebra $K\Delta(\boldsymbol{p})/I(\boldsymbol{p}, \boldsymbol{\lambda})$ of the path algebra $K\Delta(\boldsymbol{p})$ by the ideal $I(\boldsymbol{p}, \boldsymbol{\lambda})$ of $K\Delta(\boldsymbol{p})$ generated by the elements

$$\alpha_{j,p_j} \ldots \alpha_{j,1} + \alpha_{1,p_1} \ldots \alpha_{1,1} + \lambda_j \alpha_{2,p_2} \ldots \alpha_{2,1}, \quad \text{where } j \in \{3, \ldots, m\}.$$

Following [26], $C(\boldsymbol{p}, \boldsymbol{\lambda})$ is said to be a *canonical algebra* of type $(\boldsymbol{p}, \boldsymbol{\lambda})$, $\boldsymbol{p}$ the *weight sequence* of $C(\boldsymbol{p}, \boldsymbol{\lambda})$, and $\boldsymbol{\lambda}$ the *(normalized) parameter sequence* of $C(\boldsymbol{p}, \boldsymbol{\lambda})$. It follows from [26, (3.7)] that, for a canonical algebra $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$, the Auslander–Reiten quiver Γ_C of C is of the form

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$$

where $\mathcal{P}^C$ is a family of components containing all indecomposable projective C-modules (hence the unique simple projective C-module $S(0)$ associated with the vertex 0 of $\Delta(\boldsymbol{p})$),

$\mathcal{Q}^C$ is a family of components containing all indecomposable injective C-modules (hence the unique simple injective C-module $S(\omega)$ associated with the vertex ω of $\Delta(\boldsymbol{p})$), and $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \mathbb{P}_1(K)}$ is a *canonical* $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes separating $\mathcal{P}^C$ from $\mathcal{Q}^C$ and containing all simple C-modules except $S(0)$ and $S(\omega)$. Moreover, if r_λ^C denotes the rank of the stable tube $\mathcal{T}_\lambda^C$, then $r_{\lambda_i}^C = p_i$, for any $i \in \{1, \ldots, m\}$, and $r_\lambda^C = 1$, for $\lambda \in \mathbb{P}_1(K) \setminus \{\lambda_1, \ldots, \lambda_m\}$.

Let $C = C(\boldsymbol{p}, \boldsymbol{\lambda})$ be a canonical algebra. We recall the description of modules lying on the mouths of stable tubes of the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \mathbb{P}_1(K)}$ of Γ_C:

(a) For $\lambda = \lambda_1 = \infty$, the mouth of $\mathcal{T}_\lambda^C = \mathcal{T}_\infty^C$ consists of the simple C-modules $S(1,1), \ldots, S(1, p_1 - 1)$ at the vertices $(1,1), \ldots, (1, p_1 - 1)$ of $\Delta(\boldsymbol{p})$, if $p_1 \geq 2$, and the nonsimple C-module $E^{(\infty)}$ of the form
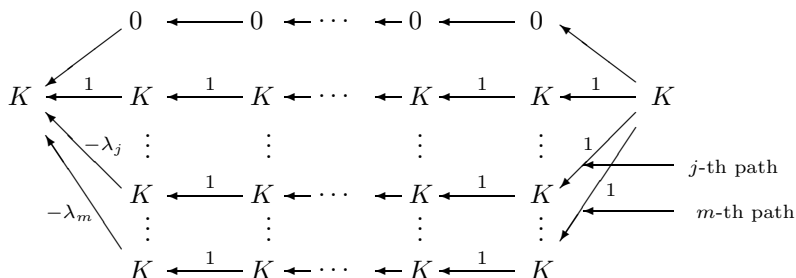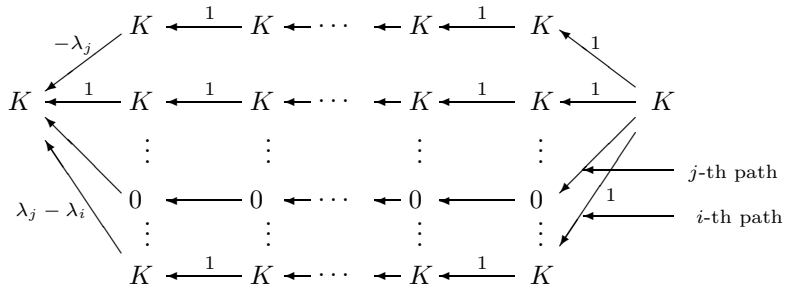

with $j \in \{3, \ldots, m\}$;

(b) For $\lambda = \lambda_2 = 0$, the mouth of $\mathcal{T}_\lambda^C = \mathcal{T}_0^C$ consists of the simple C-modules $S(2,1), \ldots, S(2, p_2 - 1)$ at the vertices $(2,1), \ldots, (2, p_2 - 1)$ of $\Delta(\boldsymbol{p})$, if $p_2 \geq 2$, and the nonsimple C-module $E^{(0)}$ of the form
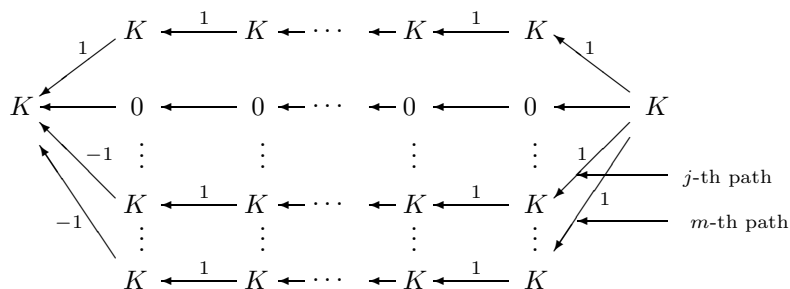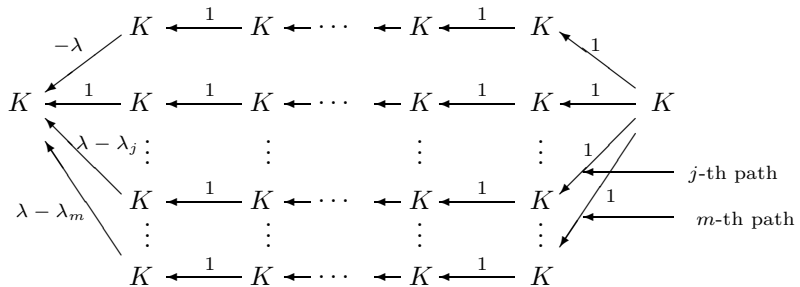

with $j \in \{3, \ldots, m\}$;

(c) For $\lambda = \lambda_j$ with $j \in \{3, \ldots, m\}$, the mouth of $\mathcal{T}_\lambda^C$ consists of the simple C-modules $S(j,1), \ldots, S(j, p_j - 1)$ at the vertices $(j,1), \ldots, (j, p_j - 1)$ of $\Delta(\boldsymbol{p})$, if $p_j \geq 2$, and the nonsimple C-module $E^{(\lambda_j)}$ of the form

$$
\begin{array}{c}
\quad K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K \\
K \xleftarrow{\;1\;} K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K \xleftarrow{\;1\;} K \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow 0 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K
\end{array}
$$

with arrows labelled $-\lambda_j$, $\lambda_j - \lambda_i$, 1, j-th path, i-th path

for $i \in \{3, \ldots, m\} \setminus \{j\}$;

(d) For $\lambda \in \mathbb{P}_1(K) \setminus \{\lambda_1, \ldots, \lambda_m\}$, the mouth of $\mathcal{T}_\lambda^C$ consists of one nonsimple C-module $E^{(\lambda)}$ of the form

$$
\begin{array}{c}
\quad K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K \\
K \xleftarrow{\;1\;} K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K \xleftarrow{\;1\;} K \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
K \xleftarrow{\;1\;} K \leftarrow \cdots \leftarrow K \xleftarrow{\;1\;} K
\end{array}
$$

with arrows labelled $-\lambda$, $\lambda - \lambda_j$, $\lambda - \lambda_m$, 1, j-th path, m-th path

with $j \in \{3, \ldots, m\}$.

The following results describing generalized standard stable tubes of an Auslander–Reiten quiver were established in [30, Corollary 5.3].

Proposition 2.1. *Let A be an algebra and Γ a stable tube of Γ_A. Then the following statements are equivalent.*

(i) *Γ is generalized standard.*
(ii) *Γ is standard.*
(iii) *The mouth of Γ consists of pairwise orthogonal bricks.*
(iv) *$\operatorname{rad}_A^\infty(X, X) = 0$ for any module X in Γ.*

Recall that an indecomposable A-module X is called a *brick* if its endomorphism algebra $\operatorname{End}_A(X)$ is isomorphic to K.

We now give the characterization of canonical algebras by means of a family of stable tubes which is maximally saturated by simple modules.

Theorem 2.2. *Let B be an algebra. Then the following statements are equivalent.*

(i) Γ_B *contains a faithful generalized standard family* $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \Lambda}$ *of stable tubes maximally saturated by simple modules.*

(ii) B *is a canonical algebra.*

Proof. We prove only that (i) implies (ii), as the other implication is obvious in view of the above description.

Let n be the rank of $K_0(B)$ and $S = S(0)$, $T = S(\omega)$ be simple B-modules at the vertices 0 and ω of the Gabriel quiver Q_B of B, respectively. Since $\mathcal{C}$ is a generalized standard family we have that the modules S and T are not isomorphic and hence $0 \neq \omega$. For a module $M \in \mathcal{C}$, by $l_{\mathcal{C}}(M)$ we shall denote the length of M in the additive category $\mathrm{add}\,\mathcal{C}$ of $\mathcal{C}$ in $\mathrm{mod}\,B$, that is, the length l of a chain $M = M_0 \supset M_1 \supset \ldots \supset M_l = 0$ of submodules of M which belong to $\mathcal{C}$ and such that M_{j-1}/M_j is a module from the mouth of $\mathcal{C}_\lambda$, for $1 \leq j \leq l$ and some $\lambda \in \Lambda$. If X is a module from the mouth of $\mathcal{C}_\lambda$, for some $\lambda \in \Lambda$, by $X[j]$ we shall denote a module M which belongs to infinite ray starting at X such that $l_{\mathcal{C}}(M) = j$.

We have two cases to consider.

(1) Assume that the family $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \Lambda}$ consists entirely of homogeneous tubes. Since $\mathcal{C}$ is faithful family maximally saturated by simple modules, we have that S and T are the only simple B-modules. Moreover, by [30, Lemma 5.9] $\mathrm{pd}_A X \leq 1$ and $\mathrm{id}_A X \leq 1$ for any indecomposable module $X \in \mathcal{C}$. Therefore, the Gabriel quiver Q_B of B does not admit any oriented cycle. Then, by (MS3), we conclude that Q_B is of the form:

$$0 \circ \xleftarrow[\alpha_m]{\overset{\alpha_1}{\underset{\vdots}{}}} \circ\, \omega$$

for some $m \geq 1$. Assume that $\alpha_1, \ldots, \alpha_m$ are linearly independent elements of the K-vector space $e_\omega B e_0$ and hence B is the path algebra KQ_B. Note that $m \geq 2$ since B is representation-infinite. If $m \geq 3$, then B is wild and Γ_B does not contain tubes (see [28, XVIII.1.6]). Hence B is the path algebra KQ_B of the Kronecker quiver:

$$0 \circ \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \circ\, \omega$$

and $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \Lambda}$ is a unique $\mathbb{P}_1(K)$-family of homogeneous tubes of Γ_B (see [27, (XI.4.6)]).

(2) Assume that $\{\mathcal{C}_1 \ldots, \mathcal{C}_m\}$, for some $m \geqslant 1$, is a complete set of stable tubes of rank at least two in the family $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \Lambda}$. Note that all simple modules from $\mathcal{C}$ lie on mouths of these tubes. Since $\mathcal{C}$ is faithful, simple modules from $\mathcal{C}$ together with S and T form the set of all simple B-modules. We mention also that the number m of nonhomogeneous tubes in Γ_B is no greater than n, by [30, (2.2)].

Let $\mathcal{C}_i$, for some $i \in \{1, \ldots, m\}$, be a tube of rank $r(\mathcal{C}_i) = p_i \geq 2$. Denote by $S_{(i,1)}, \ldots, S_{(i,p_i-1)}$ all simple modules which belong to $\mathcal{C}_i$ and by F_i the remaining module

from the mouth of $\mathcal{C}_i$. Note that F_i satisfies the condition (MS3), otherwise an epimorphism $f : E_\lambda \to T$ shall not factorize by a mouth module (see [27, Lemma X.2.9]).

We start with describing the Gabriel quiver Q_B of B. Since $\mathcal{C}$ is generalized standard family of stable tubes, by Proposition 2.1, we have that

$$\dim_K \operatorname{Ext}_B^1(S', S'') = \dim_K \operatorname{Hom}_B(S'', \tau_B S') = \begin{cases} 1, & \text{if} \quad \tau_B S' = S'' \\ 0, & \text{if} \quad \tau_B S' \neq S'' \end{cases},$$

for any simple modules S', S'' in $\mathcal{C}$. Consider now the tube $\mathcal{C}_i$. Without loss of generality we assume that $\tau_B S_{(i,l)} = S_{(i,j)}$ if $l = j + 1$, for $j \in \{1, ..., p_i - 2\}$ and $\tau_B S_{(i,1)} = F_i$, $\tau_B F_i = S_{(i,p_i-1)}$. Observe that:

- $\dim_K \operatorname{Ext}_B^1(T, S_{(i,p_i-1)}) = \dim_K \underline{\operatorname{Hom}}_B(\tau_B^- S_{(i,p_i-1)}, T) = \dim_K \underline{\operatorname{Hom}}_B(F_i, T) = \dim_K \operatorname{Hom}_B(F_i, T) = 1$, since $T = \operatorname{top} F_i$,
- $\dim_K \operatorname{Ext}_B^1(T, S_{(i,j)}) = \dim_K \underline{\operatorname{Hom}}_B(\tau_B^- S_{(i,j)}, T) = \dim_K \underline{\operatorname{Hom}}_B(S_{(i,j+1)}, T) = 0$, since $S_{(i,j+1)}, T$ are nonisomorphic simple B-modules, for $1 \leq j < p_i - 1$,
- $\dim_K \operatorname{Ext}_B^1(S_{(i,1)}, S) = \dim_K \overline{\operatorname{Hom}}_B(S, \tau_B S_{(i,1)}) = \dim_K \overline{\operatorname{Hom}}_B(S, F_i) = 1$, because $S = \operatorname{soc} F_i$,
- $\dim_K \operatorname{Ext}_B^1(S_{(i,j)}, S) = \dim_K \overline{\operatorname{Hom}}_B(S, \tau_B S_{(i,j)}) = \dim_K \overline{\operatorname{Hom}}_B(S, S_{(i,j-1)}) = 0$, since $S, S_{(i,j-1)}$ are nonisomorphic simple B-modules, for any $1 < j \leq p_i - 1$,
- $\dim_K \operatorname{Ext}_B^1(S_{(i,j)}, T) = 0$, for any $1 \leq j \leq p_i - 1$, because $\operatorname{Hom}_B(T, \tau_B S_{(i,j)}) = \operatorname{Hom}_B(T, S_{(i,j-1)}) = 0$, for $j > 1$, and $\operatorname{Hom}_B(T, \tau_B S_{(i,1)}) = \operatorname{Hom}_B(T, F_i) = \operatorname{Hom}_B(\operatorname{top} F_i, F_i) = 0$,
- $\dim_K \operatorname{Ext}_B^1(S, S_{(i,j)}) = 0$, for any $1 \leq j \leq p_i - 1$, because $\operatorname{Hom}_B(\tau_B^- S_{(i,j)}, S) = \operatorname{Hom}_B(S_{(i,j+1)}, S) = 0$, for $j < p_i - 1$, and $\operatorname{Hom}_B(\tau_B^- S_{(i,p_i-1)}, S) = \operatorname{Hom}_B(F_i, S) = \operatorname{Hom}_B(F_i, \operatorname{soc} F_i) = 0$.

Denote by (i, j), for $1 \leq i \leq m$ and $1 \leq j \leq p_i - 1$, the vertex of Q_B for which $S_{(i,j)}$ is the simple B-module at this vertex, that is $S_{(i,j)} = S(i, j)$. Then by [1, (III.2.12)], we obtain that Q_B has a subquiver Q of the form
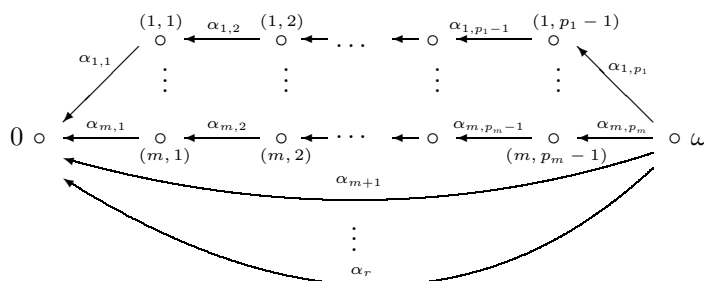
where $r \geq m$, there are no other vertices in Q_B, no other arrows starting at or ending in vertices (i, j) and no other arrows starting at ω and ending in 0. Since $\mathcal{C}$ form a hereditary family of modules in $\operatorname{mod} B$ (see [30, Lemma 5.9]), then any path $\alpha_{i,p_i} \ldots \alpha_{i,1}$,

for $1 \leq i \leq m$, is a nonzero element of B, and Q does not admit any oriented cycle. Therefore, we conclude that $Q_B = Q$.

Consider now the algebra $B' = eBe$, where $e = e_0 + e_\omega$ is the sum of the primitive orthogonal idempotents e_0, e_ω corresponding to the vertices 0 and ω, respectively. There is the canonical restriction functor

$$\mathrm{res}_e : \mathrm{mod}\, B \to \mathrm{mod}\, B'$$

which assigns to a module M in $\mathrm{mod}\, B$ the module $\mathrm{res}_e(M) = Me$ in $\mathrm{mod}\, B'$ and to a homomorphism $f : M \to N$ in $\mathrm{mod}\, B$ its restriction $\mathrm{res}_e(f) : \mathrm{res}_e(M) \to \mathrm{res}_e(N)$ to Me. Note that $Q_{B'}$ is an enlarged Kronecker quiver with r arrows

$$0 \circ \underset{\alpha_r}{\overset{\alpha_1}{\xrightleftharpoons{\vdots}}} \circ \omega \; .$$

Now we apply the functor res_e to the additive subcategory $\mathrm{add}(\mathcal{C}_i)$ of $\mathrm{mod}\, B$. Observe that $\mathrm{res}_e(S_{(i,1)}) = \mathrm{res}_e(S_{(i,2)}) = ... = \mathrm{res}_e(S_{(i,p_i-1)}) = 0$. Moreover, since $F_i = \tau_B S_{(i,1)}$, we have $\mathrm{res}_e(F_i) = Z = (Z_x, Z_\alpha)_{x,\alpha}$, where $Z_x = K$ for the vertices 0, ω, and $Z_\alpha = 0$ only for $\alpha = \alpha_i$. Clearly, $\mathrm{res}_e(F_i)$ is an indecomposable B'-module. We shall now fix $E_i = S_{(i,1)}[p_i]$. Let $M_i = F_i[p_i + 1]$. Since $\tau_B^- F_i = S_{(i,1)}$, there exists an exact sequence of B-modules

$$0 \longrightarrow F_i \overset{f}{\longrightarrow} M_i \overset{g}{\longrightarrow} E_i \longrightarrow 0 \; .$$

Then we have an exact sequence of B'-modules

$$0 \longrightarrow \mathrm{res}_e(F_i) \overset{\mathrm{res}_e(f)}{\longrightarrow} \mathrm{res}_e(M_i) \overset{\mathrm{res}_e(g)}{\longrightarrow} \mathrm{res}_e(E_i) \longrightarrow 0$$

because the restriction functor res_e is exact (see [1, Theorem I.6.8]). Observe that $\mathrm{res}_e(E_i) = \mathrm{res}_e(F_i)$. Moreover, $W = \mathrm{res}_e(M_i)$ is indecomposable, since M_i is indecomposable and, for $j \in \{1,...,m\}$, $W_{\alpha_j} : K^2 \to K^2$ is given by $W_{\alpha_j} = (M_i)_\varrho$, where $\varrho = \alpha_{j,p_j} \alpha_{j,p_{j-1}} ... \alpha_{j,2} \alpha_{j,1}$.

We are now in the position to show that $\mathrm{res}_e(f)$ is a left almost split homomorphism in $\mathrm{mod}\, B'$. Since $\mathrm{res}_e(M_i)$ is indecomposable and $\mathrm{res}_e(g) \neq 0$, $\mathrm{res}_e(f)$ is not a section. Let $u : \mathrm{res}_e(F_i) \to U$, for some indecomposable B'-module U, be a non-zero homomorphism which is not a section. Since $\mathrm{End}_{B'}(\mathrm{res}_e(F_i)) \cong K$, it follows that $U \not\cong \mathrm{res}_e(F_i)$. Invoking the extension functor $L_e : \mathrm{mod}\, B' \to \mathrm{mod}\, B$, $L_e(-) = \mathrm{Hom}_{B'}(Be, -)$, which is right adjoint to res_e, we obtain that there exists a homomorphism $v : F_i \to L_e(U)$ of B-modules such that $u = \mathrm{res}_e(v)$. The functor L_e preserves indecomposability of modules, thus $L_e(U)$ is indecomposable. Moreover, there is a functorial isomorphism $\mathrm{res}_e L_e \cong 1_{\mathrm{mod}\, B}$ (see [1, Theorem I.6.8]). Hence, v is not a section. We claim that $L_e(U)$ is not of the form $F_i[j]$ for $j \in \{1,...,p_i\}$. Observe that $\mathrm{res}_e(F_i[j]) = \mathrm{res}_e(F_i)$

for any $j \in \{1, ..., p_i\}$. Then the claim follows from the facts that $\mathrm{res}_e L_e(U) \cong U$ and $U \not\cong \mathrm{res}_e(F_i)$. Therefore, $v : F_i \to L_e(U)$ is a composition $w f_{p_i}...f_1$ for some homomorphism $w : M_i \to L_e(U)$ and irreducible homomorphisms $f_j : F_i[j] \to F_i[j+1]$ for $j \in \{1, ..., p_i\}$ (see [1, Lemma IV.5.1]). Thus v factorizes through $M_i = F_i[p_i + 1]$. Invoking now the restriction functor res_e, we conclude that $u = \mathrm{res}_e(v)$ factorizes through $\mathrm{res}_e(M_i)$. It shows that u is left almost split and, by [1, Theorem IV.1.13],

$$0 \longrightarrow \mathrm{res}_e(F_i) \xrightarrow{\mathrm{res}_e(f)} \mathrm{res}_e(M_i) \xrightarrow{\mathrm{res}_e(g)} \mathrm{res}_e(E_i) \longrightarrow 0$$

is an almost split sequence in $\mathrm{mod}\, B'$. Consequently, the image of $\mathcal{C}_i$ by the functor res_e is a homogeneous tube of $\Gamma_{B'}$. Again, by [28, (XVIII.1.6)], we conclude that $r = 2$ since B' is a tame algebra. Hence $\dim_K e_\omega B e_0 = 2$.

Let us denote by ϱ_i the path $\alpha_{i,p_i}...\alpha_{i,1}$ if $1 \le i \le m$, and the arrow α_i if $m + 1 \le i \le r$. Assume now that $\lambda \varrho_i = \varrho_j$ for some $j \ne i$, $1 \le j, i \le r$ and non-zero $\lambda \in K$. Fix a stable tube $\mathcal{C}_i \in \mathcal{C}$ which contains the simple modules $S(x)$ at the vertices $x \in \{(i, 1), (i, 2), ..., (i, p_i - 1)\}$. For simplicity, denote the unique non-simple module F_i which lies on the mouth of $\mathcal{C}_i$ by F. Observe that $F_{\alpha_{i,1}} = ... = F_{\alpha_{i,p_i}} = 0$ and hence $F_{\varrho_i} = F_{\alpha_{i,p_i}...\alpha_{i,1}} = 0$. Then $F_{\varrho_j} = F_{\lambda \varrho_i} = 0$ which is impossible since $F = \tau_B S_{(i,1)}$.

Without loss of generality we may now assume that ϱ_1, ϱ_2 form a basis of $e_\omega B e_0$. From the above consideration we conclude that the equations which describe $\varrho_3, ..., \varrho_m, \alpha_{m+1}, ..., \alpha_r$ define the set Ω of generic relations (in the sense of [26]) in KQ_B. Then, by [26, (3.7)], $B \cong KQ_B/\langle\Omega\rangle$ is a canonical algebra and $\mathcal{C}$ is a canonical separating $\mathbb{P}_1(K)$-family of stable tubes of Γ_B. Note that B is a canonical algebra of type $(\boldsymbol{p}, \boldsymbol{\lambda})$, where the weight sequence $\boldsymbol{p}$ contains a subsequence $(p_1, ..., p_m)$ and the parameter sequence $\boldsymbol{\lambda}$ is determined by relations in Ω. Following [18, Lemma 1.1] (see also [26, (3.7)]), if $m = r \ge 3$, we may assume that $p_i \ge 2$ for each $i \in \{1, ..., m\}$. $\quad\square$

3. Quasi-tube enlargements of algebras

In this section we introduce quasi-tube enlargements of algebras which are essential in the paper. For convenience of the reader, we shall start with evoking the definition of admissible operations on a translation quiver and then present the corresponding admissible operation on an algebra. We consider only the admissible operations of type $(\mathrm{ad}\,1)$, $(\mathrm{ad}\,2)$, $(\mathrm{ad}\,1^*)$ and $(\mathrm{ad}\,2^*)$ (see [3], [4], [22] for more).

For a field K and $t \ge 1$, we denote by $H_t(K)$ the $t \times t$-lower triangular matrix algebra

$$\begin{bmatrix} K & 0 & 0 & \cdots & 0 \\ K & K & 0 & \cdots & 0 \\ K & K & K & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K & K & K & \cdots & K \end{bmatrix}.$$

Then by Γ_t we denote a translation quiver isomorphic to the Auslander–Reiten quiver $\Gamma_{H_t(K)}$ of algebra $H_t(K)$:
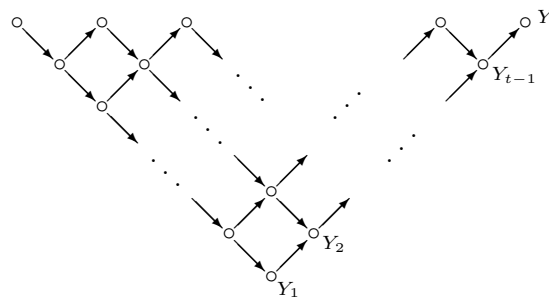

Let now $\Gamma = (\Gamma, \tau)$ be a translation quiver (with trivial valuations). For the vertex X in Γ, called the *pivot*, the following operations, called the *admissible operations*, modify Γ to a new translation quiver $\Gamma' = (\Gamma', \tau')$ whose shape depends on the shape of paths in Γ starting at X, and dually ending at X.

(ad 1) Suppose that Γ admits an infinite sectional path starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

(that is, $X_i \neq \tau X_{i+2}$, for any $i \geq 0$), and assume that every sectional path in Γ starting at X is a subpath of the above path. We then let Γ' be the translation quiver having as vertices those of Γ, those of Γ_t, additional vertices Z_{ij} and X_i', where $i \geq 0, 1 \leq j \leq t$, and having arrows as in the figure below:
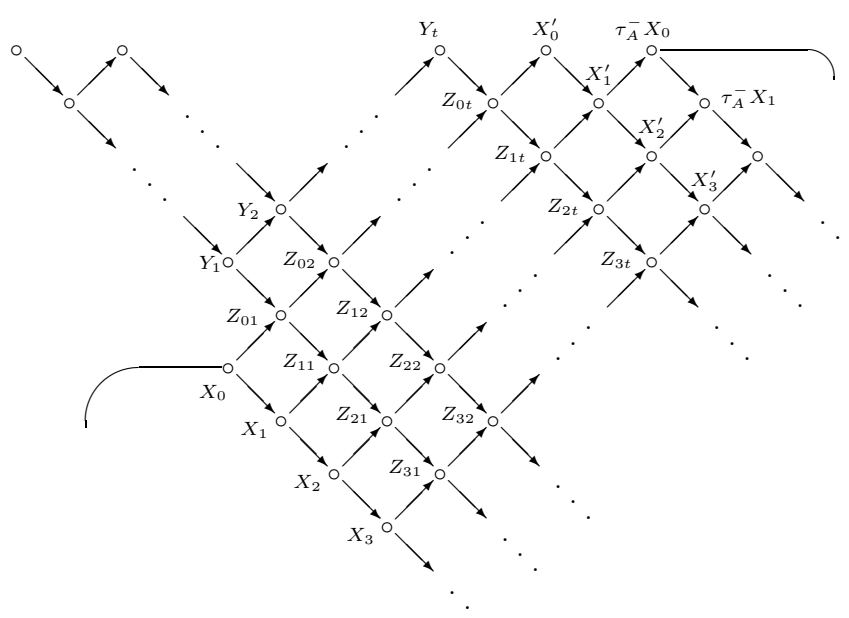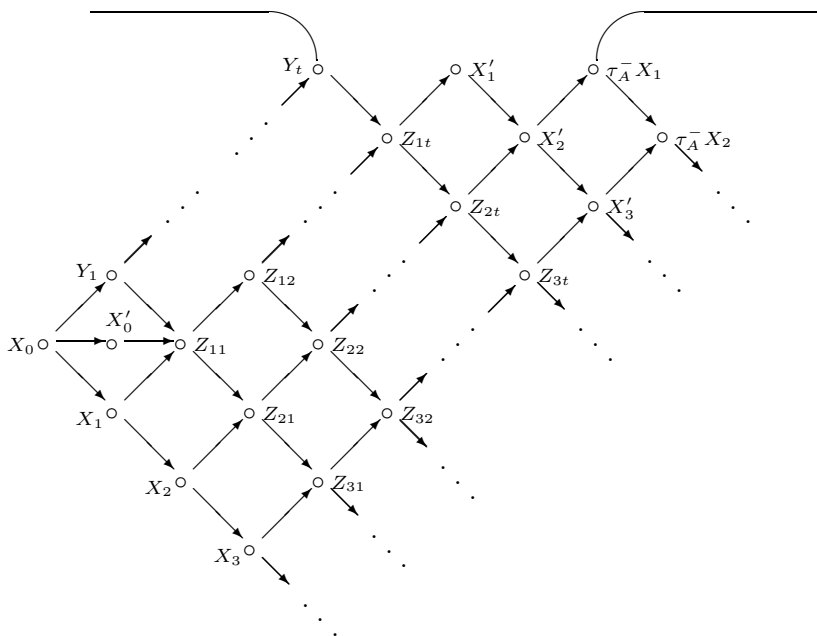
The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$ and Z_{01} is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 1$ and $\tau'(\tau^- X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ or Γ_t, respectively. For $t = 0$, the new translation quiver Γ' is obtained from Γ by inserting only the sectional path consisting of the vertices X_i', $i \geq 0$.

(ad 2) Suppose that Γ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow

$$Y_t \leftarrow Y_{t-1} \leftarrow \ldots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \ldots$$

such that any sectional path starting at X is a subpath of one of these paths and X_0 is injective. Then Γ' is the translation quiver having as vertices those of Γ, additional vertices X_0', Z_{ij}, X_j' (where $1 \leq i$, $1 \leq j \leq t$), and having arrows as in the figure below:

The translation τ' of Γ' is defined as follows: X_0' is projective–injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X_1' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 2$ and $\tau'(\tau^- X_i) = X_i'$ provided X_i is not injective in Γ, otherwise X_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

The admissible operations (ad 1*) and (ad 2*) are dual to (ad 1), (ad 2), respectively.

Observe that a quasi-tube in the sense of [29], can be defined in a constructing way. Namely, a connected translation quiver Γ is a quasi-tube if it can be obtained from a stable tube by an iterated application of admissible operations (ad 1), (ad 2), (ad 1*),

or $(\mathrm{ad}\,2^*)$. Moreover, note that any operation of type $(\mathrm{ad}\,1)$ (respectively, $(\mathrm{ad}\,1^*)$) is an iterated ray insertion (respectively, coray insertion) in the sense of [11], [26]. Thus, we have that a *ray tube* (respectively, a *coray tube*), in the sense of [11], [26], is a connected translation quiver obtained from a stable tube by an iterated (possibly zero) application of admissible operations $(\mathrm{ad}\,1)$ (respectively, $(\mathrm{ad}\,1^*)$).

The following proposition provides a characterization of quasi-tubes in Auslander–Reiten quivers of selfinjective algebras ([21], [22], [39]).

Proposition 3.1. *Let A be a selfinjective algebra and Γ a connected component of Γ_A. The following statements are equivalent.*

(i) Γ *is a quasi-tube.*
(ii) *The stable part Γ^s of Γ is a stable tube.*
(iii) Γ *contains an oriented cycle.*

Let A be an algebra and X a module in $\mathrm{mod}\,A$. The *one-point extension* of A by X is the 2×2-matrix algebra

$$A[X] = \begin{bmatrix} A & 0 \\ {}_K X_A & K \end{bmatrix} = \left\{ \begin{pmatrix} a & 0 \\ x & \lambda \end{pmatrix} ; \, a \in A, \, \lambda \in K, \, x \in X \right\}$$

with the usual addition of matrices and the multiplication induced from the canonical K-A-bimodule structure ${}_K X_A$ of X. The quiver $Q_{A[X]}$ of $A[X]$ contains the quiver Q_A of A as a full convex subquiver, and there is a single additional vertex in $Q_{A[X]}$, which is a source. Dually, the *one-point coextension* of A by X is the 2×2-matrix algebra

$$[X]A = \begin{bmatrix} K & 0 \\ {}_A \mathrm{D}(X)_K & A \end{bmatrix} = \left\{ \begin{pmatrix} \lambda & 0 \\ f & a \end{pmatrix} ; \, a \in A, \, \lambda \in K, \, f \in \mathrm{D}(X) \right\}$$

with the usual addition of matrices and the multiplication induced from the canonical A-K-bimodule structure ${}_A \mathrm{D}(X)_K$ of $\mathrm{D}(X)$. The quiver $Q_{[X]A}$ of $[X]A$ contains the quiver Q_A of A as a full convex subquiver, and there is a single additional vertex in $Q_{[X]A}$, which is a sink.

Suppose now that Γ is a generalized standard component of Γ_A. Recall that by a *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X, -)_{|\Gamma}$, for an indecomposable module $X \in \Gamma$, we mean the K-linear category defined as follows. Let $\mathcal{H}_X$ denote the full subcategory of $\mathrm{mod}\,A$ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X, M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}(X)$) such that $\mathrm{Hom}_A(X, f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}(X)/\mathcal{I}(X)$. We shall identify the K-linear category $\mathcal{S}(X)$ with its quiver.

For each admissible operation on Γ of type $(\mathrm{ad}\,1), (\mathrm{ad}\,2), (\mathrm{ad}\,1^*)$, or $(\mathrm{ad}\,2^*)$, we shall now define the corresponding admissible operation on A in such a way that the modified translation quiver Γ' is a component of the Auslander–Reiten quiver $\Gamma_{A'}$ of the modified algebra A' (see [3], [4]). Let X be an indecomposable brick in Γ.

Assume that $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

and let $t \geq 1$. Then X is the pivot of an admissible operation (ad 1) on quiver Γ and we get a new translation quiver Γ'. We define the *modified algebra* A' of A to be the one-point extension $A' = (A \times H_t)[X \oplus Y]$, where Y is the unique indecomposable projective–injective H_t-module. Moreover, the new translation quiver Γ' with vertices $Z_{ij} = (K, X_i \oplus Y_j, \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right])$ for $i \geq 0, t \geq j \geq 1$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$ is called the *modified translation quiver*. For $t = 0$ we define the modified algebra A' to be the one-point extension $A' = A[X]$ and the modified translation quiver Γ' to be the translation quiver obtained from Γ by inserting the sectional path consisting of the vertices X_i', $i \geq 0$.

If now $\mathcal{S}(X)$ consists of two sectional paths starting at X, one infinite and one finite with at least one arrow $(t \geq 1)$

$$Y_t \leftarrow Y_{t-1} \leftarrow \ldots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \ldots,$$

then X is a pivot of an admissible operation (ad 2) on quiver Γ and we get a new translation quiver Γ'. Note that X is injective in this case. We define the *modified algebra* A' of A to be the one-point extension $A' = A[X]$. Moreover, the new translation quiver Γ' with vertices $Z_{ij} = (K, X_I \oplus Y_j, \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right])$ for $i \geq 1, t \geq j \geq 1$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$ is called the *modified translation quiver*.

Dually, invoking the one-point coextensions, one defines the modified algebra A', if X is a pivot of an admissible operation of type (ad 1*) or (ad 2*). Then the following fact mentioned above holds (see [3, Section 2]).

**Lemma 3.2.** *The modified translation quiver Γ' of Γ is a component of $\Gamma_{A'}$.*

Let now C be an algebra and $\mathcal{T}$ a generalized standard family of stable tubes in Γ_C. Following [4], an algebra B is said to be a *quasi-tube enlargement* of C using modules from $\mathcal{T}$ if there is a finite sequence of algebras $A_0 = C, A_1, ..., A_m = B$ such that, for each $0 \leq j < m$, A_{j+1} is obtained from A_j by an admissible operation of type (ad 1), (ad 2), (ad 1*), or (ad 2*), with pivot either in a stable tube of $\mathcal{T}$ or in a quasi-tube of Γ_{A_j} obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations (of types (ad 1), (ad 2), (ad 1*), (ad 2*)) done so far. Then a $\mathcal{T}$-*tubular extension* (respectively, $\mathcal{T}$-*tubular coextension*) of C, in the sense of [28, (XV.2)], is just a quasi-tube enlargement of C invoking only admissible operations of type (ad 1) (respectively, of type (ad 1*)) [28, XV.2.9].

Proposition 3.3. *Let B be a quasi-tube enlargement of an algebra C using modules from a generalized standard family $\mathcal{T}$ of stable tubes of Γ_C, and $\mathcal{B}$ the family of components*

of Γ_B obtained from $\mathcal{T}$ by means of admissible operations leading from C to B. Then $\mathcal{B}$ is a generalized standard family of quasi-tubes of Γ_B.

For the proof of the above proposition we refer to [3, Lemma 2.2, 2.3] and [23, Theorem C].

We end this section with recalling some information about branch extension and coextension of an algebra. A *branch* is a finite connected full bounded subquiver $\mathcal{L} = (Q_{\mathcal{L}}, I_{\mathcal{L}})$ of the following infinite tree
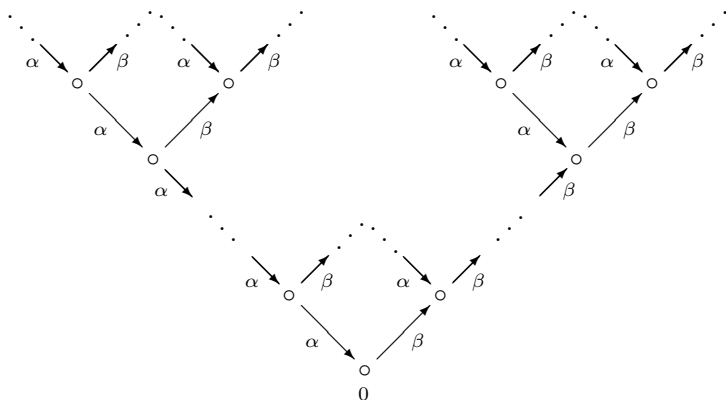
which contains the lowest vertex 0 and $I_{\mathcal{L}}$ is generated by all paths $\alpha\beta$ contained in $Q_{\mathcal{L}}$. The lowest vertex 0 of $\mathcal{L}$ is called the *germ* of $\mathcal{L}$.

Let C be an algebra and $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda\in\Lambda}$ is a family of pairwise orthogonal standard stable tubes in Γ_C. Let $E_1, \ldots, E_s$ be a set of pairwise different modules lying on the mouths of the tubes of $\mathcal{T}^C$. Consider the multiple one-point extension of C

$$C[E_1, \ldots, E_s] = \begin{bmatrix} C & 0 \\ E_1 \oplus \ldots \oplus E_s & K_1 \times \ldots \times K_s \end{bmatrix}$$

and the multiple one-point coextension of C

$$[E_1, \ldots, E_s]C = \begin{bmatrix} K_1 \times \ldots \times K_s & 0 \\ D(E_1 \oplus \ldots \oplus E_s) & C \end{bmatrix}$$

where $K_1 = \ldots = K_s = K$ and the left module structure of $E_1 \oplus \ldots \oplus E_s$ over $K_1 \times \ldots \times K_s$ is given by $(\lambda_1, \ldots, \lambda_s)(u_1, \ldots, u_s) = (\lambda_1 u_1, \ldots, \lambda_s u_s)$ for $\lambda_1, \ldots, \lambda_s \in K$, $u_1 \in E_1, \ldots, u_s \in E_s$. Observe that $C[E_1, \ldots, E_s]$ is an iterated one-point extension $C[E_1][E_2] \ldots [E_s]$ and $[E_1, \ldots, E_s]C$ is an iterated one-point coextension $[E_1][E_2] \ldots [E_s]C$. Moreover, let $C[E_1, \ldots, E_s] = KQ_{C[E_1, \ldots, E_s]}/I_{C[E_1, \ldots, E_s]}$ and $[E_1, \ldots, E_s]C = KQ_{[E_1, \ldots, E_s]C}/I_{[E_1, \ldots, E_s]C}$ be the bound quiver presentations of $C[E_1, \ldots, E_s]$ and $[E_1, \ldots, E_s]C$.

Denote by $0_1^+, \ldots, 0_s^+$ (respectively, $0_1^-, \ldots, 0_s^-$) the extension vertices of $Q_{C[E_1,\ldots,E_s]}$ (respectively, coextension vertices of $Q_{[E_1,\ldots,E_s]C}$) corresponding to the extensions (respectively, coextensions) by the modules $E_1, \ldots, E_s$. Choose now branches $\mathcal{L}_1 = (Q_{\mathcal{L}_1}, I_{\mathcal{L}_1}), \ldots, \mathcal{L}_s = (Q_{\mathcal{L}_s}, I_{\mathcal{L}_s})$ with the germs $0_1^*, \ldots, 0_s^*$, respectively. The *branch extension* of C (*branch $\mathcal{T}^C$-extension* of C in the sense of [28, (XV.3)]), with respect to the mouth modules $E_1, \ldots, E_s$ and the branches $\mathcal{L}_1, \ldots, \mathcal{L}_s$, is the bound quiver algebra

$$C[E_1, \mathcal{L}_1, \ldots, E_s, \mathcal{L}_s] = KQ_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}/I_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]},$$

where the bound quiver $(Q_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]}, I_{C[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]})$ is obtained from the bound quiver $(Q_{C[E_1,\ldots,E_s]}, I_{C[E_1,\ldots,E_s]})$ of $C[E_1, \ldots, E_s]$ by adding the bound quivers of the branches $\mathcal{L}_1, \ldots, \mathcal{L}_s$ and making the identification of the vertices 0_i^+ with 0_i^* for $i \in \{1, \ldots, s\}$. Dually, the *branch coextension* of C (*branch $\mathcal{T}^C$-coextension* of C in the sense of [28, (XV.3)]), with respect to the mouth modules $E_1, \ldots, E_s$ and the branches $\mathcal{L}_1, \ldots, \mathcal{L}_s$, is the bound quiver algebra

$$[E_1, \mathcal{L}_1, \ldots, E_s, \mathcal{L}_s]C = KQ_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}/I_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C},$$

where the bound quiver $(Q_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C}, I_{[E_1,\mathcal{L}_1,\ldots,E_s,\mathcal{L}_s]C})$ is obtained from the bound quiver $(Q_{[E_1,\ldots,E_s]C}, I_{[E_1,\ldots,E_s]C})$ of $[E_1, \ldots, E_s]C$ by adding the bound quivers of the branches $\mathcal{L}_1, \ldots, \mathcal{L}_s$ and making the identification of the vertices 0_i^- with 0_i^* for $i \in \{1, \ldots, s\}$.

The following proposition shows that the concepts of the tubular extension and tubular coextension coincide with the concept of the branch extension and branch coextension, respectively [28, Theorem XV.3.9].

Proposition 3.4. *Let C be an algebra and $\mathcal{T}^C$ be a family of pairwise orthogonal standard stable tubes of Γ_C. For an algebra A the following equivalences hold.*

(i) *A is a $\mathcal{T}^C$-tubular extension of C if and only if A is a branch $\mathcal{T}^C$-extension of C.*

(ii) *A is a $\mathcal{T}^C$-tubular coextension of C if and only if A is a branch $\mathcal{T}^C$-coextension of C.*

Let now C be a canonical algebra. Recall from [4, Section 2], [26, Section 4] that for a branch extension B of C the Auslander–Reiten quiver Γ_B has a disjoint union decomposition

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where $\mathcal{P}^B = \mathcal{P}^C$ is a family of components consisting of C-modules and containing all indecomposable projective C-modules, $\mathcal{Q}^B$ is a family of components containing all indecomposable injective B-modules but no projective B-module, and $\mathcal{T}^B$ is a $\mathbb{P}_1(K)$-family

$(\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard ray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$. Respectively, for branch coextension B, $\mathcal{P}^B$ is a family of components containing all indecomposable projective B-modules but no injective B-modules, $\mathcal{Q}^B = \mathcal{Q}^C$ is a family of components consisting of C-modules and containing all indecomposable injective C-modules, and $\mathcal{T}^B$ is a $\mathbb{P}_1(K)$-family $(\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard coray tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$.

4. Selfinjective orbit algebras

In this section we recall needed background on selfinjective orbit algebras.

Let B be an algebra and $\mathcal{E}_B = \{e_i \,|\, 1 \leq i \leq n\}$ be a fixed set of orthogonal primitive idempotents of B with $1_B = e_1 + \ldots + e_n$. Then the *repetitive category* $\widehat{B}$ of B is the category with $\widehat{\mathcal{E}}_B = \{e_{m,i} \,|\, m \in \mathbb{Z}, \, 1 \leq i \leq n\}$ as a set of objects of $\widehat{B}$ and the morphism spaces defined by

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i, & r = m \\ D(e_i B e_j), & r = m+1 \\ 0, & \text{otherwise} \end{cases}$$

and the composition of morphisms given by the multiplication in B and the canonical B-B-bimodule structure of $\mathrm{D}(B) = \mathrm{Hom}_K(B, K)$. For each $m \in \mathbb{Z}$, we denote by B_m the full subcategory of $\widehat{B}$ given by the objects $e_{m,i}$ for all $i \in \{1, ..., n\}$. Observe that $\widehat{B}$ is a selfinjective locally bounded K-category. An automorphism φ of $\widehat{B}$ is said to be

- *positive* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1, \ldots, n\}$;
- *rigid* if, for each pair $(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{m,j}$ for some $j \in \{1, \ldots, n\}$;
- *strictly positive* if it is positive but not rigid.

An important role is played by the *Nakayama automorphism* $\nu_{\widehat{B}}$ of $\widehat{B}$ which is defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}, \quad \text{for all} \quad (m,i) \in \mathbb{Z} \times \{1, ..., n\}.$$

Note that the Nakayama automorphism $\nu_{\widehat{B}}$ is a strictly positive automorphism of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if it acts freely on the set $\widehat{\mathcal{E}}_B$ and has finitely many orbits.

Let B be an algebra and G be an admissible group of automorphisms of $\widehat{B}$. Following Gabriel [12], we may consider the finite orbit K-category $\widehat{B}/G$ defined as follows. The objects of $\widehat{B}/G$ are the elements $a = Gx$ of the set $\widehat{\mathcal{E}}_B/G$ of G-orbits in $\widehat{\mathcal{E}}_B$ and the morphism spaces are given by

$$(\widehat{B}/G)(a,b) = \left\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} \widehat{B}(x,y) \mid \quad g\cdot f_{y,x} = f_{gy,gx} \text{ for all } g\in G,\ x\in a,\ y\in b \right\},$$

for all objects a, b of $\widehat{B}/G$. Then we have a canonical *Galois covering functor* $F : \widehat{B} \to \widehat{B}/G$ which assigns to each object x of $\widehat{B}$ its G-orbit Gx, and, for any objects x of $\widehat{B}$ and a of $\widehat{B}/G$, F induces natural K-linear isomorphisms

$$\bigoplus_{y\in\widehat{\mathcal{E}}_B, Fy=a} \widehat{B}(x,y) \xrightarrow{\sim} (\widehat{B}/G)(Fx,a),$$

$$\bigoplus_{y\in\widehat{\mathcal{E}}_B, Fy=a} \widehat{B}(y,x) \xrightarrow{\sim} (\widehat{B}/G)(a,Fx).$$

The finite dimensional algebra $\bigoplus_{a,b\in\widehat{\mathcal{E}}/G}(\widehat{B}/G)(a,b)$ associated to the orbit category $\widehat{B}/G$ is a selfinjective algebra, denoted by $\widehat{B}/G$ and called an *orbit algebra* of $\widehat{B}$, with respect to the admissible automorphism group G of $\widehat{B}$. The group G acts also on the category $\operatorname{mod}\widehat{B}$ of right $\widehat{B}$-modules (identified with contravariant functors from $\widehat{B}$ to $\operatorname{mod}K$ with finite support) by $gM = M\circ g^{-1}$ for any $M\in\operatorname{mod}\widehat{B}$ and $g\in G$. Further, we have the *push-down functor* $F_\lambda : \operatorname{mod}\widehat{B} \to \operatorname{mod}\widehat{B}/G$ such that $F_\lambda(M)(a) = \bigoplus_{x\in a} M(x)$ for a module M in $\operatorname{mod}\widehat{B}$ and an object a of $\widehat{B}/G$.

The following theorem is a consequence of [12, Lemma 3.5, Theorem 3.6].

Theorem 4.1. *Let B be an algebra and G a torsion-free admissible group of K-linear automorphisms of $\widehat{B}$. Then*

(i) *The push-down functor $F_\lambda : mod\,\widehat{B} \to mod\,\widehat{B}/G$ induces an injection from the set of G-orbits of isomorphism classes of indecomposable modules in $mod\,\widehat{B}$ into the set of isomorphism classes of indecomposable modules in $mod\,\widehat{B}/G$.*

(ii) *The push-down functor $F_\lambda : mod\,\widehat{B} \to mod\,\widehat{B}/G$ preserves the Auslander–Reiten sequences.*

In general, the push-down functor $F_\lambda : \operatorname{mod}\widehat{B} \to \operatorname{mod}\widehat{B}/G$, associated to a Galois covering $F : \widehat{B} \to \widehat{B}/G$ is not dense (see [2]). Following [9], a repetitive category $\widehat{B}$ is said to be *locally support-finite*, if for any object x of $\widehat{B}$, the full subcategory of $\widehat{B}$ given by the supports $\operatorname{supp}M$ of all indecomposable modules M in $\operatorname{mod}\widehat{B}$ with $M(x)\neq 0$, is finite. Here, by a *support* of a module M in $\operatorname{mod}\widehat{B}$ we mean the full subcategory of $\widehat{B}$ given by all objects z with $M(z)\neq 0$.

The following theorem is a consequence of [10, Proposition 2.5] (see also [9, Theorem]).

Theorem 4.2. *Let B be an algebra with locally support-finite repetitive category $\widehat{B}$, and G be a torsion-free admissible group of automorphisms of $\widehat{B}$. Then the push-down functor $F_\lambda : mod\,\widehat{B} \to mod\,\widehat{B}/G$ is dense. In particular, F_λ induces an isomorphism of the orbit*

translation quiver $\Gamma_{\widehat{B}}/G$ of the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$, with respect to the action of G, and the Auslander–Reiten quiver $\Gamma_{\widehat{B}/G}$ of $\widehat{B}/G$.

Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $e = e_1 + ... + e_n$, where $\{e_i; 1 \leq i \leq n\}$ is a complete set of orthogonal primitive idempotents of A which are not contained in I. Then such an idempotent e is uniquely determined by I up to an inner automorphism of A, and we call it a *residual identity* of B [33]. Note that $B \cong eAe/eIe$. For an ideal I of a selfinjective algebra A, we consider its *left annihilator* $\ell_A(I) = \{a \in A | ax = 0 \text{ for any } x \in I\}$ and its *right annihilator* $r_A(I) = \{a \in A | xa = 0 \text{ for any } x \in I\}$. Following [33, (2.1)], the ideal I is said to be *deforming* if $eIe = \ell_{eAe}(I) = r_{eAe}(I)$ and A/I is triangular. The lemma below was proved in [36, Lemma 5.1].

Lemma 4.3. *Let A be a selfinjective algebra, e an idempotent of A, and assume that $\ell_A(I) = Ie$ or $r_A(I) = eI$. Then e is a residual identity of the quotient algebra A/I.*

Moreover, the following result was obtained in [33, Proposition 2.3].

Proposition 4.4. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. Then the following conditions are equivalent.*

  (i) *Ie is an injective cogenerator in $\mathrm{mod}\, B$.*
  (ii) *eI is an injective cogenerator in $\mathrm{mod}\, B^{op}$.*
 (iii) *$\ell_A(I) = Ie$.*
 (iv) *$r_A(I) = eI$.*

Moreover, under these equivalent conditions, we have $\mathrm{soc}\, A \subseteq I$ and $eIe = \ell_{eAe}(I) = r_{eAe}(I)$.

We end this section with the criterion which is fundamental in the proof of the main Theorem 1.1 (see [34, Section 3 and 4] and [36, Theorem 5.3]).

Theorem 4.5. *Let A be a selfinjective algebra over an algebraically closed field K. The following conditions are equivalent.*

(1) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra over K with acyclic quiver Q_B and φ is a positive automorphism of $\widehat{B}$.*
(2) *There is an ideal I of A and an idempotent e of A such that*
    (i) *$r_A(I) = eI$,*
    (ii) *the quiver $Q_{A/I}$ of A/I is acyclic.*

Moreover, in this case, B is isomorphic to A/I.

5. Selfinjective algebras of strictly canonical type

The aim of this section is to introduce some results on selfinjective algebras of strictly canonical type. In particular, we give an answer to the question when the canonical family of quasi-tubes of such an algebra is generalized standard.

The following results were established in [18, Theorem 5.1].

Theorem 5.1. *Let B be a branch extension (respectively, branch coextension) of a canonical algebra C. Then there exist algebras C_q, B_q^-, B_q^+, B_q^* and $\overline{B}_q$, $q \in \mathbb{Z}$, and a decomposition*

$$\Gamma_{\widehat{B}} = \vee_{q \in \mathbb{Z}}(\mathcal{X}_q \vee \mathcal{C}_q)$$

of the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ such that the following statements hold:

(a) *for each $q \in \mathbb{Z}$, $\mathcal{X}_q$ is a family of components of $\Gamma_{\widehat{B}}$ containing exactly one simple $\widehat{B}$-module S_q;*

(b) *for each $q \in \mathbb{Z}$, $\mathcal{C}_q$ is a family $(\mathcal{C}_q(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes of $\Gamma_{\widehat{B}}$ with $s(\mathcal{C}_q(\lambda)) + p(\mathcal{C}_q(\lambda)) = r(\mathcal{C}_q(\lambda)) - 1$, for any $\lambda \in \mathbb{P}_1(K)$;*

(c) *for each pair $p, q \in \mathbb{Z}$ with $p < q$, we have $\mathrm{Hom}_{\widehat{B}}(\mathcal{X}_q, \mathcal{X}_p \vee \mathcal{C}_p) = 0$ and $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_q, \mathcal{X}_p \vee \mathcal{C}_p \vee \mathcal{X}_{p+1}) = 0$;*

(d) *for each $q \in \mathbb{Z}$, C_q is a canonical algebra, B_q^- is a maximal branch coextension of C_q in B_q^*, B_q^+ is a maximal branch extension of C_q in B_q^*, $\overline{B}_q$ is a one-point extension of B_{q-1}^+ and B_q^* is a quasi-tube enlargement of C_q; moreover, C_q, B_q^-, B_q^+, B_q^* and $\overline{B}_q$ are full convex subcategories of $\widehat{B}$ with $\widehat{B_q^-} = \widehat{B} = \widehat{B_q^+}$, $\nu_{\widehat{B}}(C_q) = C_{q+2}$, $\nu_{\widehat{B}}(B_q^-) = B_{q+2}^-$, $\nu_{\widehat{B}}(B_q^+) = B_{q+2}^+$, $\nu_{\widehat{B}}(B_q^*) = B_{q+2}^*$, $\nu_{\widehat{B}}(\overline{B}_q) = \overline{B}_{q+2}$.*

(e) *for each $q \in \mathbb{Z}$, $\mathcal{C}_q$ is the canonical $\mathbb{P}_1(K)$-family of quasi-tubes of $\Gamma_{B_q^*}$, obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}_q^-$ of coray tubes of $\Gamma_{B_q^-}$ by infinite rectangle insertions, and from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}_q^+$ of ray tubes of $\Gamma_{B_q^+}$ by infinite rectangle coinsertions;*

(f) *for each $q \in \mathbb{Z}$, $\mathcal{X}_q$ consists of indecomposable $\overline{B}_q$-modules;*

(g) *for each $q \in \mathbb{Z}$, we have $\nu_{\widehat{B}}(\mathcal{X}_q) = \mathcal{X}_{q+2}$ and $\nu_{\widehat{B}}(\mathcal{C}_q) = \mathcal{C}_{q+2}$;*

(h) *for each $q \in \mathbb{Z}$, $\mathrm{Hom}_{\widehat{B}}(S_q, \mathcal{C}_q(\lambda)) \neq 0$ for all $\lambda \in \mathbb{P}_1(K)$, and $\mathrm{Hom}_{\widehat{B}}(S_p, \mathcal{C}_q) = 0$ for $p \neq q$ in $\mathbb{Z}$;*

(i) *for each $q \in \mathbb{Z}$, $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_q(\lambda), S_{q+1}) \neq 0$ for all $\lambda \in \mathbb{P}_1(K)$, and $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_q, S_p) = 0$ for $p \neq q+1$ in $\mathbb{Z}$;*

(j) *for each $q \in \mathbb{Z}$, we have $\Omega_{\widehat{B}}(\mathcal{C}_{q+1}^s) = \mathcal{C}_q^s$ and $\Omega_{\widehat{B}}(\mathcal{X}_{q+1}^s) = \mathcal{X}_q^s$, where $\Omega_{\widehat{B}}$ is the syzygy operator.*

(k) *$\widehat{B}$ is locally support-finite.*

Moreover, there is the following description of torsion-free admissible groups of automorphisms of $\widehat{B}$ [18, Proposition 5.2].

Proposition 5.2. *Let B be a branch extension (respectively, branch coextension) of a canonical algebra C. Then there exists a strictly positive automorphism $\varrho_{\widehat{B}}$ of $\widehat{B}$ such that the following statements hold:*

(i) *$\varrho_{\widehat{B}} = \nu_{\widehat{B}}$ or $\varrho_{\widehat{B}}^2 = \nu_{\widehat{B}}$;*

(ii) *every torsion-free admissible group G of automorphisms of $\widehat{B}$ is an infinite cyclic group generated by a strictly positive automorphism $\sigma\varrho_{\widehat{B}}^s$, for some integer $s \geq 1$ and some rigid automorphism σ of $\widehat{B}$.*

Preserving the above notation, for a canonical algebra C, it follows from Proposition 3.3 that Γ_B contains a generalized standard $\mathbb{P}_1(K)$-family $\mathcal{C}$ of quasi-tubes. We consider $\mathcal{C} = \mathcal{C}_0$ as a family of components of $\Gamma_{\widehat{B}}$.

Recall that, following [18], a selfinjective algebra A of the form $\widehat{B}/G$, where B is a branch extension (equivalently, branch coextension) of a canonical algebra C and G is an infinite cyclic group generated by a strictly positive automorphism of $\widehat{B}$, is called a *selfinjective algebra of strictly canonical type*. The structure and homological properties of the Auslander–Reiten quivers of selfinjective algebras of strictly canonical type were described in [18, Theorem 5.3]. In particular, for a selfinjective algebra A of strictly canonical type its Auslander–Reiten quiver has a decomposition

$$\Gamma_A = \bigvee_{q \in \mathbb{Z}/n\mathbb{Z}} (\mathcal{X}_q^A \vee \mathcal{C}_q^A),$$

for some positive integer n, and, for each $q \in \mathbb{Z}/n\mathbb{Z}$, $\mathcal{C}_q^A = (\mathcal{C}_q^A(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of quasi-tubes with $s(\mathcal{C}_q^A(\lambda)) + p(\mathcal{C}_q^A(\lambda)) = r(\mathcal{C}_q^A(\lambda)) - 1$ for each $\lambda \in \mathbb{P}_1(K)$, and $\mathcal{X}_q^A$ is a family of components containing exactly one simple module S_q. Moreover, we have the following proposition which is an immediate consequence of properties of push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$.

Proposition 5.3. *Let $A = \widehat{B}/G$ be a selfinjective algebra of strictly canonical type. Then $\mathcal{C}_0^A = F_\lambda(\mathcal{C}_0)$ is a $\mathbb{P}_1(K)$-family of quasi-tubes in Γ_A maximally saturated by simple and projective modules.*

We are now in a position to prove the following equivalence.

Proposition 5.4. *Let $A = \widehat{B}/G$, where B is a branch extension (respectively, coextension) of a canonical algebra C with respect to the canonical $\mathbb{P}_1(K)$-family of stable tubes and G an admissible group of automorphisms of $\widehat{B}$ generated by a strictly positive automorphism of $\widehat{B}$. Then the following statements are equivalent.*

(i) *The canonical $\mathbb{P}_1(K)$-family $\mathcal{C}_0^A = F_\lambda(\mathcal{C}_0)$ of quasi-tubes of Γ_A is generalized standard.*

(ii) $G = (\varphi \nu_{\widehat{B}})$, where φ is strictly positive, or $G = (\varphi \nu_{\widehat{B}})$, where φ is rigid and B is a canonical algebra.

Proof. Let C be a canonical algebra and $\mathcal{T}^C$ the canonical $\mathbb{P}_1(K)$-family of pairwise orthogonal standard stable tubes of Γ_C. Since the classes of repetitive algebras of branch extensions and branch coextensions of C with respect to the canonical $\mathbb{P}_1(K)$-family of stable tubes coincide (see [18, Section 4]), invoking Proposition 3.4, we may assume that B is a branch coextension of C.

Let A be an orbit algebra $\widehat{B}/G$, where G is generated by a strictly positive automorphism g of $\widehat{B}$. Following Theorem 5.1(a) and (b), the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ has a decomposition:

$$\Gamma_{\widehat{B}} = \vee_{q \in \mathbb{Z}} (\mathcal{X}_q \vee \mathcal{C}_q),$$

such that, for each $q \in \mathbb{Z}$, $\mathcal{X}_q$ is a family of components containing exactly one simple $\widehat{B}$-module S_q, and $\mathcal{C}_q$ is a family $(\mathcal{C}_q(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard quasi-tubes. We set $C = C_0$ and $B = B_0^-$. Applying now Theorem 5.1(e), (h) and (i), we know that there exist, for any $q \in \mathbb{Z}$, an indecomposable C_q-module in $\mathcal{C}_q$ with S_{q+1} in its top and an indecomposable C_{q+1}-module in $\mathcal{C}_{q+1}$ which has S_{q+1} in the socle. Hence $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_0, \mathcal{C}_1) \neq 0$.

Assume $\mathcal{C}_0^A = F_\lambda(\mathcal{C}_0)$ is a generalized standard family of quasi-tubes. Since $\nu_{\widehat{B}}(\mathcal{C}_q) = \mathcal{C}_{q+2}$ and $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_0, \mathcal{C}_1) \neq 0$, applying Proposition 5.2, we conclude that $g = \varphi \nu_{\widehat{B}}$, where φ is a positive automorphism of $\widehat{B}$. Recall that $\mathcal{C}_0$ is the canonical $\mathbb{P}_1(K)$-family of quasi-tubes obtained from the canonical $\mathbb{P}_1(K)$-family $\mathcal{T}^B$ of coray tubes of Γ_B by iterated infinite rectangle insertions (see Theorem 5.1(e)). Observe that $\mathcal{C}_0$ contains no projective–injective modules if and only if $\mathcal{T}^B$ contains no injective modules, that is, $B = C$ (see [18, Section 2 and 3]). In this situation $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_0, \mathcal{C}_2) = 0$, since $\mathrm{supp}\, \mathcal{C}_0 \cap \mathrm{supp}\, \mathcal{C}_2 = \mathrm{supp}\, \mathcal{C}_0 \cap \mathrm{supp}\, \nu_{\widehat{B}}(\mathcal{C}_0) = \emptyset$. Suppose now that $\mathcal{C}_0$ contains a projective–injective module, equivalently $B \neq C$. Then by Theorem 5.1(g), $\mathcal{C}_p$, for any even $p \in \mathbb{Z}$, contains a projective–injective module. Let P be a projective–injective module which belongs to $\mathcal{C}_2$. Clearly, then $P/\mathrm{soc}\, P$ belongs to $\mathcal{C}_2$. From Theorem 5.1(j) we obtain that the simple socle $\mathrm{soc}\, P$ of P belongs to $\mathcal{C}_1$. Again, by Theorem 5.1(j), we get that the projective cover P' of $\mathrm{soc}\, P$ belongs to $\mathcal{C}_0$, because $\mathrm{rad}\, P' = \Omega_{\widehat{B}}(\mathrm{soc}\, P)$ belongs to $\mathcal{C}_0$. Hence, there is a non-zero homomorphism $f : P' \to P$ which implies that $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_0, \mathcal{C}_2) \neq 0$ for $B \neq C$. Therefore, if φ is rigid, then $B = C$. Summing up, we conclude that (i) implies (ii).

Assume now that (ii) holds. Suppose that M, N are indecomposable $\widehat{B}$-modules belonging to $\mathcal{C}_p$, for some $p \in \mathbb{Z}$. From the description of subcategories B_q^-, B_q^+ of $\widehat{B}$, we know that $\mathrm{supp}\, \mathcal{C}_q \cap \mathrm{supp}\, \mathcal{C}_{q+3} = \emptyset$ for any $q \in \mathbb{Z}$ (see the proof of [18, Theorem 5.1]). Then, by the assumption imposed on a strictly positive generator g of the group G, we obtain that $\mathrm{supp}\, {}^{g^i} M \cap \mathrm{supp}\, N = \emptyset$ for any integer $i \neq 0$. Since the push-down functor $F_\lambda : \mathrm{mod}\, \widehat{B} \to \mathrm{mod}\, A$ is dense, there are the following natural isomorphisms of K-vector spaces

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\widehat{B}}({}^{g^i}M, N) \cong \mathrm{Hom}_A(F_\lambda(M), F_\lambda(N)),$$

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\widehat{B}}(M, {}^{g^i}N) \cong \mathrm{Hom}_A(F_\lambda(M), F_\lambda(N)),$$

for any indecomposable modules M, N in mod $\widehat{B}$.

Let X, Y be modules in $\mathcal{C}_0^A = F_\lambda(\mathcal{C}_0)$. Then $X = F_\lambda(M)$, $Y = F_\lambda(N)$ for some $M \in \mathcal{C}_p$, $N \in \mathcal{C}_q$, and clearly $F_\lambda(\mathcal{C}_p) = F_\lambda(\mathcal{C}_q) = \mathcal{C}_0$. Without loss of generality we may assume that $p = q = 0$. Thus $\mathrm{Hom}_A(X, Y) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\widehat{B}}(M, {}^{g^i}N) \cong \mathrm{Hom}_{\widehat{B}}(M, N)$. Since $\mathcal{C}_0$ is a family of pairwise orthogonal standard quasi-tubes, we have $\mathrm{rad}_{\widehat{B}}^\infty(M, N) = 0$, and hence $\mathrm{rad}_A^\infty(X, Y) = 0$. This shows that (ii) implies (i). $\quad\square$

6. Proof of Theorem 1.1

The implication (ii) $\Rightarrow$ (i) of the main theorem is an immediate consequence of Propositions 5.3 and 5.4.

We prove now the implication (i) $\Rightarrow$ (ii). Suppose that $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ is a generalized standard family of quasi-tubes maximally saturated by simple and projective modules in the Auslander–Reiten quiver Γ_A of a selfinjective algebra A. We recall that the annihilator of the family of components $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ is the intersection $\mathrm{ann}_A(\mathcal{C}) = \bigcap_{X \in \mathcal{C}} \mathrm{ann}_A(X)$ of the annihilators of all indecomposable A-modules X belonging to $\mathcal{C}$. Consider the quotient algebra $D = A/\mathrm{ann}_A(\mathcal{C})$. Then the family $\mathcal{C}$ is a generalized standard faithful family of quasi-tubes in Γ_D maximally saturated by simple and projective modules. We claim that D is a quasi-tube enlargement of a canonical algebra C. Namely, a quasi-tube is a connected translation quiver obtained from a stable tube by an iterated application of admissible operations (ad 1), (ad 2) and their dual versions. Assume that, for each $\lambda \in \Lambda$, a quasi-tube $\mathcal{C}_\lambda$ is obtained, as a translation quiver, from a stable tube $\mathcal{T}_\lambda$ by means of the above operations. This allows us to consider the family $T(\mathcal{C}) = \bigcup_{\lambda \in \Lambda} T(\mathcal{C}_\lambda)$ of indecomposable modules in $\mathcal{C}$ such that, for each $\lambda \in \Lambda$, $T(\mathcal{C}_\lambda)$ corresponds to all vertices of the stable tube $\mathcal{T}_\lambda$. Let $C = D/\mathrm{ann}_D(T(\mathcal{C}))$ be a quotient algebra of D by the annihilator $\mathrm{ann}_D(T(\mathcal{C}))$ of the family $T(\mathcal{C})$ given as the intersection $\bigcap_{Y \in T(\mathcal{C})} \mathrm{ann}_D(Y)$ of the annihilators of all modules belonging to $T(\mathcal{C})$. Then D is a quasi-tube enlargement of C. Note that the modules from $T(\mathcal{C})$ form the family of stable tubes $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \Lambda}$ in Γ_C, where $\mathcal{T}_\lambda^C = \mathcal{T}_\lambda$ for every $\lambda \in \Lambda$. Clearly, $(\mathcal{T}_\lambda^C)_{\lambda \in \Lambda}$ is a faithful generalized standard family of stable tubes in Γ_C (maximally saturated by simple modules). Thus, invoking Theorem 2.2, we conclude that C is a canonical algebra and then D is a quasi-tube enlargement of the canonical algebra C. In particular, $\Lambda = \mathbb{P}_1(K)$ and $\mathcal{T}^C$ is the separating canonical $\mathbb{P}_1(K)$-family $(\mathcal{T}_\lambda^C)_{\lambda \in \mathbb{P}_1(K)} = (\mathcal{T}_\lambda^C)_{\lambda \in \Lambda}$ of stable tubes in Γ_C. Hence, applying [4, (3.5)], we infer that there exists a unique maximal branch extension B of C inside D, which is obtained from C by an iterated application of algebra admissible operations of type (ad 1) (see also [23, Theorem C]). Then the Auslander–Reiten quiver Γ_B of B contains a faithful $\mathbb{P}_1(K)$-family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda \in \mathbb{P}_1(K)}$ of

pairwise orthogonal generalized standard ray tubes (obtained from $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ by an iterated application of translation quiver admissible operations of type (ad 1)). Moreover, D is obtained from B by an iterated application of admissible algebra operations of types (ad 1*), (ad 2*), and $\mathcal{C}$ from $\mathcal{T}^B$ by an iterated application of translation quiver admissible operations of types (ad 1*), (ad 2*). Then $B = D/\mathrm{ann}_D(\mathcal{T}^B)$, where $\mathcal{T}^B$ is taken as a family of modules. Thus we conclude that $B = A/\mathrm{ann}_A(\mathcal{T}^B)$ since $\mathrm{ann}_D(\mathcal{T}^B) = \mathrm{ann}_A(\mathcal{T}^B) \cap D$.

Let $I = \mathrm{ann}_A(\mathcal{T}^B)$. Then $B = A/I$. We will show now that I satisfies the conditions (2) of Theorem 4.5. Observe that $Q_{A/I} = Q_B$ is acyclic, because B is a $\mathcal{T}^C$-branch extension of the canonical algebra C.

By J we shall denote the trace ideal of the family $\mathcal{T}^B$ in A, that is $J = \sum_h \mathrm{Im}\, h$, where $h \in \mathrm{Hom}_A(Y, A_A)$ for any $Y \in \mathcal{T}^B$. Since A_A is of finite dimension over K, we obtain that J is a finite sum $J = \sum_{i=1}^s \mathrm{Im}\, h_i$ for some homomorphisms $h_i \in \mathrm{Hom}_A(Y_i, A_A)$ with $Y_i \in \mathcal{T}^B$. Similarly, by J' we denote the trace ideal of the dual family $\mathrm{D}(\mathcal{T}^B)$ of left A-modules in A.

We may choose a complete set of pairwise orthogonal primitive idempotents $e_1, ..., e_r$ of A such that $1_A = e_1 + ... + e_r$ and $e = e_1 + ... + e_n$, for some $n \leqslant r$, is a residual identity of $B = A/I$. Observe that $B \cong eAe/eIe$. We will show that I is a deforming ideal of A with $\ell_A(I) = Ie$ and $r_A(I) = eI$. We will apply the strategy similar to the proof of [15, Theorem 7.14].

Proposition 6.1. *We have $J \cup J' \subseteq I$.*

Proof. We know that $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a generalized standard family of ray tubes in Γ_B and the generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of quasi-tubes in Γ_A is obtained from $\mathcal{T}^B$ by an iterated application of admissible translation quiver operations of types (ad 1*) and (ad 2*) corresponding to the admissible algebra operations of types (ad 1*) and (ad 2*) leading from B to D. Then, applying arguments as in the proof of [15, Proposition 7.1], we prove the required inclusion $J \cup J' \subseteq I$. $\square$

Applying arguments as in the proof of [15, Lemma 7.2] we obtain the following facts.

**Proposition 6.2.** *We have $\ell_A(I) = J$, $r_A(I) = J'$ and $I = r_A(J) = \ell_A(J')$.*

The following proposition is the key ingredient for proving that I is a deforming ideal of A such that $\ell_A(I) = Ie$ and $r_A(I) = eI$.

Proposition 6.3. *We have $eIe = eJe = eJ'e$. In particular, $(eIe)^2 = 0$.*

Proof. Observe that J is a right B-module since $1_A - e \in I$ implies that $J(1-e) \subseteq JI = 0$ and so $J = Je + J(1_A - e) = Je$. Hence eJ is an ideal of eAe with $eJ \subseteq eIe$, by Proposition 6.1. We denote by B' the algebra $B' = eAe/eJ$. Note that e is a residual identity of B'.

Consider the canonical restriction functor $res_e : \operatorname{mod} A \to \operatorname{mod} eAe$. Applying res_e to the (generalized) standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \Lambda}$ of quasi-tubes maximally saturated by simple and projective modules in Γ_A we obtain the family $\mathcal{T}^B$ of ray tubes in Γ_{eAe}. Moreover, $\mathcal{T}^B$ is sincere generalized standard in Γ_{eAe}. Further, $\mathcal{T}^B$ is also sincere generalized standard family for the quotient algebra $B' = eAe/eJ$ since $eJ \subseteq eIe$. We shall show that, in fact, the algebras B' and B are equal.

We shall compare the bound quivers (Q_B, I_B), $(Q_{B'}, I_{B'})$ of algebras B, B', respectively. Since Q_B is a subquiver of $Q_{B'}$ with the same set of vertices, suppose there exists an arrow $\eta : x \to y$ in $Q_{B'}$ which does not belong to Q_B. Recall that Q_B is of the form
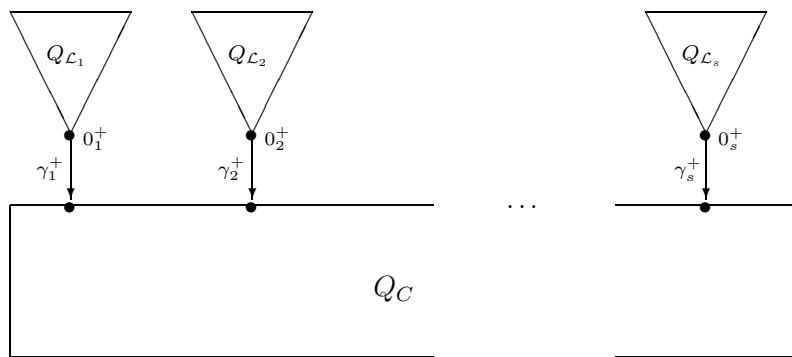
where $Q_{\mathcal{L}_1}, Q_{\mathcal{L}_2}, ..., Q_{\mathcal{L}_s}$ are the quivers of the branches $\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_s$, respectively, with the vertex 0_i^+ such that $\operatorname{rad} P(0_i^+)$, for $i \in \{1, ..., s\}$, are pairwise nonisomorphic modules lying on the mouths of stable tubes from canonical family $\mathcal{T}^C$ in Γ_C (see [28, Chapter XV.3]). For Q_C we shall use the notation from Section 2. By B'' we denote a quotient algebra of B' such that the set of arrows of $Q_{B''}$ consists of all arrows of Q_B and additionally the arrow η. Hence we have a sequence of algebra epimorphisms $B' \to B'' \to B$. This implies that $\mathcal{T}^B$ is a sincere generalized standard family of ray tubes in $\Gamma_{B''}$. We have the following cases to consider.

(1) Assume that $x \in Q_{\mathcal{L}_i}$ for some $i \in \{1, ..., s\}$. Since $\mathcal{T}^B$ contains all projective B-modules $P(a)$ for $a \in Q_{\mathcal{L}_i}$ and $\mathcal{T}^B$ is a family of ray tubes in $\Gamma_{B'}$, a projective B-module $P(x)$ is also a projective B'-module. Therefore, if $\eta : x \to y$ in $Q_{B'}$ then η belongs to Q_B, a contradiction with an assumption imposed on η. If now $y \in Q_{\mathcal{L}_i}$, for some $i \in \{1, ..., s\}$, then there exists a homomorphism $f : P(y) \to P(x)$ in $\operatorname{mod} B'$ given by the formula $f(-) = \eta \cdot -$. Hence $\operatorname{Im} f = \eta e_y B' = \eta eAe$, because $\mathcal{T}^B$ is a family of ray tubes in Γ_{eAe}. Therefore $\eta eAe \subseteq eJ$, contradiction with the assumption that $\eta \in Q_{B'}$.

(2) Let now $x \in Q_C$ and $y \in Q_C \backslash \{\omega\}$. Assume $y = (i, k)$ for some $i \in \{1, ..., m\}$ and $k \in \{1, ..., p_i - 1\}$. Then $S(i, k)$ has in $\operatorname{mod} B''$ a minimal injective presentation of the form

$$0 \to S(i, k) \to I(i, k) \to I(i, k + 1) \oplus I(x) \oplus I(0^+),$$

where $I(0^+) = 0$ if there is no branch extension of C at $S(i, k)$. Then using the quasi-inverse $\nu_{B''}^{-1}$ of the Nakayama functor $\nu_{B''}$, we obtain the following exact sequence

$$0 \to P(i, k) \to P(i, k+1) \oplus P(x) \oplus P(0^+) \to \tau_{B''}^- S(i, k) \to 0,$$

where we assume that $P(0^+) = 0$ if $I(0^+) = 0$. Then the socle of $\tau_{B''}^- S(i, k)$ contains an additional direct summand $S(x)$. Therefore, $\tau_{B''}^- S(i, k) \neq \tau_B^- S(i, k)$, and hence $\mathcal{T}_B$ is not a family of ray tubes in $\Gamma_{B''}$ and neither is in $\Gamma_{B'}$, a contradiction. In the case of $y = 0$, we repeat the above arguments for a nonsimple module F_i from the mouth of a stable tube from $\mathcal{T}^C$.

(3) Assume that $x = (i, k)$, for some $i \in \{1, ..., m\}$ and $k \in \{1, ..., p_i - 1\}$, and $y = \omega$. Then we show analogously that $\tau_{B''}^-(\tau_B S(x)) \neq S(x)$ since B''-module $\tau_{B''}^-(\tau_B S(x))$ contains ηB as a submodule. Again, we get a contradiction.

(4) Let $\eta : \omega \to \omega$. By (1), (2) and (3) we conclude that $\eta, \alpha_{1,p_1}, ..., \alpha_{m,p_m}$ are all arrows that start at ω in Q_{eAe}. Denote by ϱ_i the path $\alpha_{i,p_i}...\alpha_{i,2}\alpha_{i,1}$, for any $i \in \{1, ..., m\}$. Observe that $\eta \varrho_i$ belongs to the K-vector space $e_\omega A e_0$ generated by ϱ_1 and ϱ_2, otherwise using the canonical restriction functor $\mathrm{res}_{e'} : \mathrm{mod}\, eAe \to \mathrm{mod}\, e'Ae'$ for $e' = e_\omega + e_0$, we obtain that $e'Ae'$ is a wild algebra (see arguments from the proof of Theorem 2.2 and [28, (XVIII.1.6)]). Assume $\eta \varrho_i \neq 0$ in B'', for some $i \in \{1, \ldots, m\}$. Then $\eta \varrho_i = a_1 \varrho_1 + a_2 \varrho_2$ for some $a_1, a_2 \in K$ such that $a_1^2 + a_2^2 > 0$. Consider the nonsimple module $E^{(0)}$ from the mouth of the stable tube $\mathcal{T}_0^C$ from the family $\mathcal{T}^C$ as a B''-module (see Section 2). Note that $E_{\varrho_2}^{(0)} = 0$ and hence $E_{a_1\varrho_1}^{(0)} = E_{\eta \varrho_i}^{(0)} = 0$. But $E_{a_1\varrho_1}^{(0)}(e_\omega) = a_1\varrho_1 \neq 0$ and we conclude that $\eta \varrho_i = 0$ for any $i \in \{1, \ldots, m\}$. Let now $k_i \in \{0, 1, \ldots, p_i\}$ be the minimal integer such that $\eta \alpha_{i,p_i} \ldots \alpha_{i,k_i} \neq 0$ in B'' where we put $\alpha_{i,p_i} \ldots \alpha_{i,0} = e_\omega$. Then, for any $i \in \{1, \ldots, m\}$, there exists a nonzero homomorphism $f_i : S(i, k_i - 1)[p_i - k_i + 1] \to P(\omega)$ in $\mathrm{mod}\, eAe$, where $S(i, k_i - 1)[p_i - k_i + 1]$ is a module of $\mathcal{T}^C$-length $p_i - k_i + 1$ lying on a ray starting at the mouth module $S(i, k_i - 1)$ in $\mathcal{T}^C$. Note that $\mathrm{Im}\, f_i = \eta eAe$. Thus $\eta eAe \subset eJ$ and $\eta \notin Q_{B'}$.

(5) Let now $\eta_1, \eta_2, \ldots, \eta_r$, for some $r \geq 1$, be all arrows in $Q_{B'}$ which start at 0 and end in ω.

Consider the Galois covering $F : \widetilde{B'} \to B'$ with an infinite cyclic group $\mathbb{Z}$. Then $\widetilde{B'}$ is a locally bounded K-category and it follows from [7, Section 2] that $\widetilde{B'} \cong KQ_{\widetilde{B'}}/I_{\widetilde{B'}}$, where $Q_{\widetilde{B'}}$ is a connected, locally finite, acyclic quiver with $I_{\widetilde{B'}}$ an admissible ideal of the path category $KQ_{\widetilde{B'}}$ of $Q_{\widetilde{B'}}$. Thus quiver $\widetilde{Q_{B'}}$ is of the form
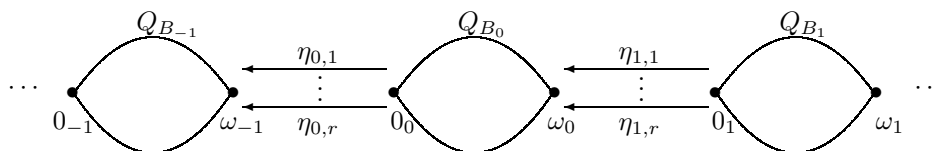
where, for all $k \in \mathbb{Z}$ and $1 \leq i \leq r$, we have $Q_{B_k} = Q_B$, $\eta_{k,i} : 0_k \to \omega_{k-1}$, and the generators of I_{B_k} belong to the set of generators of $I_{\widetilde{B'}}$. By B'_- we denote the full

subcategory of $\widetilde{B'}$ whose objects are the objects of B_k for all integer $k \leq 0$. Then the Auslander–Reiten quiver B'_- has a form

$$\Gamma_{B'_-} = \mathcal{P}^{B'_-} \cup \mathcal{T}^{B_0} \cup \mathcal{Q}^{B_0},$$

where $\mathcal{P}^{B'_-}$ is a family of components containing all indecomposable B'_--modules X such that $\mathrm{res}_e(X)$, for e being the residual identity of B_0, is zero or belongs to the family $\mathcal{P}^{B_0}$ of Γ_{B_0}, $\mathcal{T}^{B_0} = \mathcal{T}^B$ and $\mathcal{Q}^{B_0} = \mathcal{Q}^B$. By P we denote the projective $\widetilde{B'}$-module $P_{\widetilde{B'}}(0_1)$ at the vertex 0_1 (vertex 0 in Q_{B_1}). Let now $R = B'_-[\mathrm{rad}\, P]$ be a one-point extension of B'_- by the radical of P. Since $\mathcal{T}^{B_0}$ remains a family of ray tubes in Γ_R (and in $\Gamma_{\widetilde{B'}}$), we conclude that $\mathrm{rad}\, P \in \mathrm{add}\, \mathcal{Q}^{B_0}$ (see [28, Theorem XV.1.6]). Consider the projective cover $f : P' \to \mathrm{rad}\, P$ of $\mathrm{rad}\, P$ in $\mathrm{mod}\, B_0$. Then f factorizes through the additive subcategory $\mathrm{add}\, \mathcal{T}^{B_0}$ of $\mathrm{mod}\, B_0$, because $\mathcal{T}^{B_0}$ is a separating family of ray tubes in Γ_{B_0}. Therefore, there is a module $M \in \mathrm{add}\, \mathcal{T}^{B_0}$ and an epimorphism $h : M \to \mathrm{rad}\, P$ in $\mathrm{mod}\, \widetilde{B'}$. Further, there exists an epimorphism $h' : M \to \mathrm{rad}\, P$ in $\mathrm{mod}\, B'$ because for the push-down functor $F_\lambda : \mathrm{mod}\, \widetilde{B'} \to \mathrm{mod}\, B'$ we have $F_\lambda(M) = M$ and $F_\lambda(\mathrm{rad}\, P) = \mathrm{rad}\, P$. Observe that by (1)–(4), P is a projective eAe-module. Hence $\mathrm{Im}\, h' = \sum_{i=1}^r \eta_i eAe$ and $\eta_1, \ldots, \eta_r \in eJ$, a contradiction.

To sum up, we obtain that $eIe = eJ$. Hence $\mathcal{T}^B$ is faithful generalized standard family of ray tubes in $\mathrm{mod}\, B'$ because $eIe/eJ = \mathrm{ann}_{B'}(\mathcal{T}^B) = 0$. We show analogously that $eIe = J'e$. Applying now Proposition 6.2, we have $(eIe)^2 = eJeeIe = eJeIe = (eJe)Ie = eJIe = 0$. □

We note that although Proposition 6.3 is the analogue of Lemma 7.3 in [15] their proofs are different because the family $\mathcal{C}$ of quasi-tubes is assumed only to be generalized standard whereas in [15] the quasi-tubes in $\mathcal{C}$ consist of modules which do not lie on infinite short cycles. But having Proposition 6.3, we may proceed as in [15, Section 7], and, using Lemmas 7.4–7.12 of [15], prove the following analogue of [15, Proposition 7.13].

Proposition 6.4. *We have $Ie = J$, $eI = J'$, and $eIe = J \cap J'$.*

This allows us to prove the desired proposition.

Proposition 6.5. *I is a deforming ideal of A with $\ell_A(I) = Ie$ and $r_A(I) = eI$.*

Proof. From Proposition 6.2 and 6.4 we know that $\ell_A(I) = J = Ie$ and $r_A(I) = J' = eI$. In particular, we have $IeI = 0$. Therefore, applying Proposition 4.4, we get $eIe = \ell_{eAe}(I) = r_{eAe}(I)$. Since $Q_{A/I} = Q_B$ is acyclic, this shows that I is a deforming ideal of A. □

We complete now the proof of implication (i) $\Rightarrow$ (ii) of Theorem 1.1. Since the ideal I and the idempotent e satisfy condition (2) in Theorem 4.5, we conclude that A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where φ is a positive automorphism of $\widehat{B}$.

Finally, applying Proposition 5.4 we infer that either φ is strictly positive or φ is rigid and B is a canonical algebra, as required in (ii).

7. Examples

The following examples illustrate the statements of Theorem 1.1 and Corollary 1.2.

Example 7.1. Let $B = KQ_B/I_B$, where Q_B is the quiver
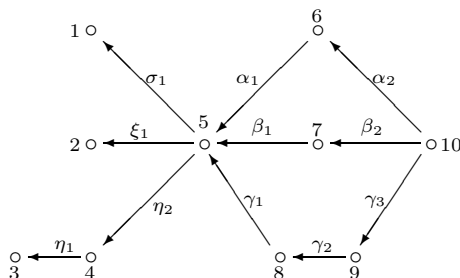
and I_B is the ideal of the path algebra KQ_B of Q_B generated by the elements $\alpha_1\sigma_1$, $\beta_1\xi_1$, $\gamma_1\eta_2$, $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1$. Denote by C the bound quiver algebra $C = KQ_C/I_C$, where Q_C is the full subquiver of Q given by the vertices 5, 6, 7, 8, 9, 10, and I_C is the ideal in the path algebra KQ_C of Q_C generated by $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1$. Then C is the canonical algebra $C(\boldsymbol{p}, \boldsymbol{\lambda})$ with the weight sequence $\boldsymbol{p} = (2, 2, 3)$ and the parameter sequence $\boldsymbol{\lambda} = (\infty, 0, 1)$. Further, B is the branch coextension of C in the sense of [28, XV.3]. Namely, $B = [E_1, \mathcal{L}_1, E_2, \mathcal{L}_2, E_3, \mathcal{L}_3]C$ with $E_1 = E^{(\infty)} \in \mathcal{T}_\infty^C$, $E_2 = E^{(0)} \in \mathcal{T}_0^C$, $E_3 = E^{(1)} \in \mathcal{T}_1^C$, $\mathcal{L}_1$ the branch given by the vertex 1, $\mathcal{L}_2$ the branch given by the vertex 2, $\mathcal{L}_3$ the branch given by the vertices 3, 4 and the arrow η_1. Consider the repetitive algebra $\widehat{B}$ of B. Then there exists a strictly positive automorphism $\varphi_{\widehat{B}}$ of $\widehat{B}$, with $\varphi_{\widehat{B}}^2 = \nu_{\widehat{B}}$ such that, for any $k \in \mathbb{Z}$, $\varphi_{\widehat{B}}(e_{k,l}) = e_{k,l+5}$, if $1 \leq l \leq 5$ and $\varphi_{\widehat{B}}(e_{k,l}) = e_{k+1,l-5} = \nu_{\widehat{B}}(e_{k,l-5})$, if $6 \leq l \leq 10$. Denote by A the orbit algebra $\widehat{B}/(\varphi_{\widehat{B}})$. Then A is the bound quiver algebra $A = KQ/I$, where Q is the quiver
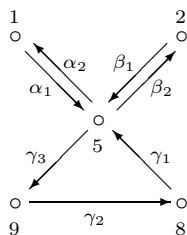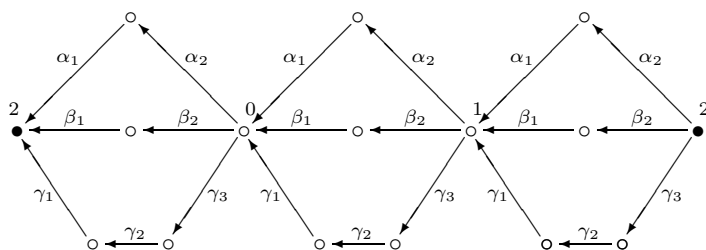

and I is the ideal of KQ generated by the elements $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1$, $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_3$, $\alpha_2\alpha_1\beta_2\beta_1 - \beta_2\beta_1\alpha_2\alpha_1$. Note that A is a symmetric algebra but not a trivial extension of C. Hence from Corollary 1.2, Γ_A does not admit a generalized standard family of quasi-tubes maximally saturated by simple and projective modules. Indeed, by [18, Theorem 5.3] Γ_A has a decomposition

$$\Gamma_A = \mathcal{X}^A \vee \mathcal{C}^A$$

where $\mathcal{C}^A = (\mathcal{C}^A(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is the unique $\mathbb{P}_1(K)$-family of quasi-tubes of Γ_A containing all simple modules and indecomposable projective modules, except the simple module $S(5)$ and the projective module $P(5)$ at the vertex 5, which belong to $\mathcal{X}^A$. For $\lambda = 1$ from the parameter sequence consider the module $E^{(1)}$ from the mouth of a stable tube $\mathcal{T}_1^C$ in Γ_C. Then $E^{(1)}$ belongs to $\mathcal{C}^A(1)$ and there is a nonzero homomorphism $f : E^{(1)} \to E^{(1)}$ which factors through $\operatorname{soc} E^{(1)} \cong S(5) \cong \operatorname{top} E^{(1)}$. Thus $f \in \operatorname{rad}^\infty(\mathcal{C}^A)$.
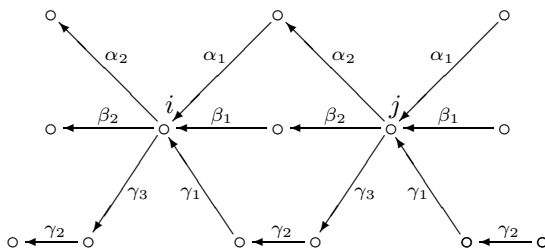
Example 7.2. Consider the selfinjective algebra $A = \widehat{B}/(\varphi_{\widehat{B}}^3)$, where B and $\varphi_{\widehat{B}}$ are as above. Then $A = KQ/I$ is the bound quiver algebra, where Q is the quiver

where we identify two vertices denoted by $\bullet$ and I is the ideal of KQ generated by the elements $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1, \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_3, \alpha_2\alpha_1\beta_2\beta_1 - \beta_2\beta_1\alpha_2\alpha_1$. Again, by [18, Theorem 5.3] Γ_A has a decomposition
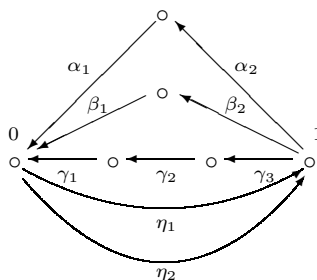
$$\Gamma_A = \mathcal{X}_0^A \vee \mathcal{C}_0^A \vee \mathcal{X}_1^A \vee \mathcal{C}_1^A \vee \mathcal{X}_2^A \vee \mathcal{C}_2^A$$

where, for each $0 \le i \le 2$, $\mathcal{C}_i^A = (\mathcal{C}_i^A(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is the $\mathbb{P}_1(K)$-family of quasi-tubes of Γ_A, $\mathcal{X}_i^A$ is a family of components containing exactly one simple module $S(i)$. Since $\varphi_{\widehat{B}}^3 = (\varphi_{\widehat{B}}^2)\varphi_{\widehat{B}} = \nu_{\widehat{B}}\varphi_{\widehat{B}}$ and $\varphi_{\widehat{B}}$ is strictly positive automorphism of $\widehat{B}$, by Theorem 1.1 we get that $\mathcal{C}_i^A$, for some $0 \le i \le 2$, is generalized standard family of quasi-tubes maximally saturated by simple and projective modules. Further, applying Theorem 5.1 (see also [18, Theorem 5.1]), we have that $\mathcal{C}_i^A$, for each $0 \le i \le 2$, is a canonical family of quasi-tubes of $\Gamma_{B_i^*}$, where B_i^* is a quasi-tube enlargement of canonical algebra $C_i = C$. Note that $B_i^* = KQ_{B_i^*}/I_{B_i^*}$, where $Q_{B_i^*}$ is of the form

and $I_{B_i^*}$ is the ideal of $KQ_{B_i^*}$ generated by $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1, \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_3$ and $j \equiv i + 1 (\mathrm{mod}\,3)$. In conclusion, Γ_A admits three generalized standard families $\mathcal{C}_i^A$, $0 \le i \le 2$, of quasi-tubes maximally saturated by simple and projective modules.

Example 7.3. Let $A = \widehat{B}/(\nu_{\widehat{B}})$ be a selfinjective algebra, where $B = C$ is the canonical algebra as above. Then $A = KQ/I$ is the bound quiver algebra, where Q is the quiver

and I is the ideal of KQ generated by $\gamma_3\gamma_2\gamma_1 + \alpha_2\alpha_1 + \beta_2\beta_1, \eta_1\alpha_2, \eta_2\beta_2, \alpha_1\eta_1, \beta_1\eta_2,$ $\eta_1\beta_2\beta_1 - \eta_2\alpha_2\alpha_1, \alpha_2\alpha_1\eta_2 - \beta_2\beta_1\eta_1$. Following Theorem 5.1 and [18, Theorem 5.3], Γ_A has a decomposition

$$\Gamma_A = \mathcal{X}_0^A \vee \mathcal{C}_0^A \vee \mathcal{X}_1^A \vee \mathcal{C}_1^A$$

where, for $i \in \{0, 1\}$, $\mathcal{C}_i^A = (\mathcal{C}_i^A(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ is the $\mathbb{P}_1(K)$-family of quasi-tubes of Γ_A, $\mathcal{X}_i^A$ is a family of components containing exactly one simple module $S(i)$. Further, $\mathcal{C}_0^A$ is the canonical $\mathbb{P}_1(K)$-family of stable tubes of Γ_C and $\mathcal{C}_1^A$ is the canonical $\mathbb{P}_1(K)$-family of quasi-tubes of $\Gamma_{B_1^*}$, where B_1^* is the quasi-tube enlargement of Kronecker algebra C_1 (given by vertices $0, 1$ and arrows η_1, η_2). Note that $\mathcal{C}_1^A$ contains projective A-module but no simple A-modules (see [18, Example 5.4]). Since $A = \widehat{B}/(\nu_{\widehat{B}})$ is symmetric algebra, there is a nonzero homomorphism $f : P \to P$ for projective module $P \in \mathcal{C}_1^A$ which factorizes through simple module $\mathrm{top}\,P$. Hence $f \in \mathrm{rad}_A^\infty$ and only $\mathcal{C}_0^A$ satisfies the condition of Theorem 1.1.

Acknowledgments

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, 2006.

[2] I. Assem, A. Skowroński, On tame repetitive algebras, Fund. Math. 142 (1993) 59–84.

[3] I. Assem, A. Skowroński, Coils and multicoil algebras, in: Representation Theory of Algebras and Related Topics, in: Conf. Proc., Can. Math. Soc., vol. 19, Amer. Math. Soc., 1996, pp. 1–24.

[4] I. Assem, A. Skowroński, B. Tomé, Coil enlargements of algebras, Tsukuba J. Math. 19 (1995) 453–479.
[5] M. Auslander, Representation theory of Artin algebras II, Comm. Algebra 1 (1974) 269–310.
[6] J. Białkowski, A. Skowroński, K. Yamagata, Cartan matrices of symmetric algebras having generalized standard stable tubes, Osaka J. Math. 45 (1) (2008) 15–171.
[7] K. Bongartz, P. Gabriel, Covering spaces in representation theory, Invent. Math. 65 (1982) 331–378.
[8] F.U. Coelho, E.N. Marcos, H.A. Merklen, A. Skowroński, Module categories with infinite radical square zero are of finite type, Comm. Algebra 22 (11) (1994) 4511–4517.
[9] P. Dowbor, A. Skowroński, On Galois coverings of tame algebras, Arch. Math. (Basel) 44 (1985) 52–529.
[10] P. Dowbor, A. Skowroński, Galois coverings of representation-infinite algebras, Comment. Math. Helv. 62 (1987) 311–337.
[11] G. d'Este, C.M. Ringel, Coherent tubes, J. Algebra 87 (1984) 150–171.
[12] P. Gabriel, The universal cover of a representation-finite algebra, in: Representation Theory I, in: Lecture Notes in Mathematics, vol. 903, Springer-Verlag, 1981, pp. 68–105.
[13] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge University Press, 1988.
[14] M. Karpicz, On selfinjective algebras without short cycles in the component quiver, Colloq. Math. 123 (2) (2011) 219–232.
[15] M. Karpicz, A. Skowroński, K. Yamagata, On selfinjective Artin algebras having generalized standard quasitubes, J. Pure Appl. Algebra 215 (11) (2011) 2738–2760.
[16] O. Kerner, A. Skowroński, On module categories with nilpotent infinite radical, Compos. Math. 77 (1991) 313–333.
[17] O. Kerner, A. Skowroński, K. Yamagata, Invariance of selfinjective algebras of quasitilted type under stable equivalences, Manuscripta Math. 119 (3) (2006) 359–381.
[18] M. Kwiecień, A. Skowroński, Selfinjective algebras of strictly canonical type, Colloq. Math. 116 (2009) 47–103.
[19] M. Kwiecień, A. Skowroński, A classification of symmetric algebras of strictly canonical type, Colloq. Math. 116 (2) (2009) 249–271.
[20] M. Kwiecień, A. Skowroński, A characterization of selfinjective algebras of strictly canonical type, J. Algebra 323 (2) (2010) 473–508.
[21] S. Liu, The degrees of irreducible maps and the shapes of the Auslander–Reiten quivers, J. Lond. Math. Soc. 45 (1992) 32–54.
[22] P. Malicki, A. Skowroński, Almost cyclic coherent components of an Auslander–Reiten quiver, J. Algebra 229 (2000) 695–749.
[23] P. Malicki, A. Skowroński, Algebras with separating almost cyclic coherent Auslander–Reiten components, J. Algebra 291 (2005) 208–237.
[24] P. Malicki, A. Skowroński, On the number of simple and projective modules in the quasi-tubes of self-injective algebras, Comm. Algebra 39 (1) (2011) 322–334.
[25] Y. Ohnuki, K. Takeda, K. Yamagata, Automorphisms of repetitive algebras, J. Algebra 232 (2000) 708–724.
[26] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., vol. 1099, Springer-Verlag, 1984.
[27] D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type, London Math. Soc. Stud. Texts, vol. 71, Cambridge Univ. Press, 2007.
[28] D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Math. Soc. Stud. Texts, vol. 72, Cambridge University Press, 2007.
[29] A. Skowroński, Algebras of polynomial growth, in: Topics in Algebra. Part 1: Rings and Representations of Algebras, in: Banach Center Publications, vol. 26, PWN, Warsaw, 1990, pp. 535–568.
[30] A. Skowroński, Generalized standard Auslander–Reiten components, J. Math. Soc. Japan 46 (1994) 517–543.
[31] A. Skowroński, Generalized canonical algebras and standard stable tubes, Colloq. Math. 90 (2001) 77–93.
[32] A. Skowroński, Selfinjective algebras: finite and tame type, in: Trends in Representation Theory of Algebras and Related Topics, in: Contemp. Math., vol. 406, Amer. Math. Soc., 2006, pp. 169–238.
[33] A. Skowroński, K. Yamagata, Socle deformations of self-injective algebras, Proc. Lond. Math. Soc. 72 (1996) 545–566.

[34] A. Skowroński, K. Yamagata, Galois coverings of selfinjective algebras by repetitive algebras, Trans. Amer. Math. Soc. 351 (1999) 715–734.
[35] A. Skowroński, K. Yamagata, On selfinjective Artin algebras having nonperiodic generalized standard Auslander–Reiten components, Colloq. Math. 96 (2003) 233–244.
[36] A. Skowroński, K. Yamagata, Positive Galois coverings of selfinjective algebras, Adv. Math. 194 (2005) 398–436.
[37] A. Skowroński, K. Yamagata, Frobenius Algebras I. Basic Representation Theory, EMS Textbooks in Mathematics, European Mathematical Society Publishing House, Zürich, 2011.
[38] A. Skowroński, K. Yamagata, Frobenius Algebras II. Tilted and Hochschild Extension Algebras, EMS Textbooks in Mathematics, European Mathematical Society Publishing House, Zürich, 2017.
[39] Y. Zhang, The structure of stable components, Canad. J. Math. 43 (1991) 652–672.