Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 321 (2009) 3335–3374
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Relative pro-ℓ completions of mapping class groups ☆

Richard Hain [a,*], Makoto Matsumoto [b]

[a] *Department of Mathematics, Duke University, Durham, NC 27708-0320, United States*
[b] *Department of Mathematics, Graduate School of Science, Hiroshima University, 739-8526, Japan*

Received 1 February 2008

Available online 5 March 2009

Communicated by Andrew Mathas and Jean Michel

Dedicated to Gus Lehrer on the occasion of his 60th birthday

Abstract

Fix a prime number ℓ. In this paper we develop the theory of relative pro-ℓ completion of discrete and profinite groups—a natural generalization of the classical notion of pro-ℓ completion—and show that the pro-ℓ completion of the Torelli group does not inject into the relative pro-ℓ completion of the corresponding mapping class group when the genus is at least 2. (See Theorem 1 below.) As an application, we prove that when $g \geqslant 2$, the action of the pro-ℓ completion of the Torelli group $T_{g,1}$ on the pro-ℓ fundamental group of a pointed genus g surface is not faithful.

The choice of a first-order deformation of a maximally degenerate stable curve of genus g determines an action of the absolute Galois group $G_{\mathbb{Q}}$ on the relative pro-ℓ completion of the corresponding mapping class group. We prove that for all g all such representations are unramified at all primes $\neq \ell$ when the first order deformation is suitably chosen. This proof was communicated to us by Mochizuki and Tamagawa.

Keywords: Mapping class group; Moduli space of curves; Pro-ℓ completion; Torelli group; Johnson homomorphism; Unramified Galois representation

☆ Supported in part by the Scientific Grants-in-Aid 16204002 and 19204002 and by the Core-to-Core grant 18005 from the Japan Society for the Promotion of Science; and grants DMS-0405440 and DMS-0706955 from the National Science Foundation.

* Corresponding author.
E-mail addresses: hain@math.duke.edu (R. Hain), m-mat@math.sci.hiroshima-u.ac.jp (M. Matsumoto).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.02.014

1. Introduction

Suppose that Γ is a discrete or profinite group and that P is a profinite group. Suppose that $\rho : \Gamma \to P$ is a continuous, dense homomorphism.[1] The *relative pro-ℓ completion $\Gamma^{(\ell),\rho}$ of Γ with respect to $\rho : \Gamma \to P$* is defined by

$$\Gamma^{(\ell),\rho} := \varprojlim G_\phi$$

where the limit is taken over the inverse system of commutative triangles

$$\begin{tikzcd} \Gamma \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & G_\phi \arrow[dl, "\rho_\phi"] \\ & P & \end{tikzcd}$$

where G_ϕ is a profinite group, ϕ is a continuous dense homomorphism, ρ_ϕ is continuous, and $\ker \rho_\phi$ is a pro-ℓ group. There are natural homomorphisms $\Gamma \to \Gamma^{(\ell),\rho} \to P$ whose composition is ρ.

Relative pro-ℓ completion with respect to the trivial representation ρ of a discrete or profinite group is simply the classical pro-ℓ completion of the group. Relative pro-ℓ completion with respect to non-trivial representations is natural in arithmetic geometry.

Since finite ℓ-groups are nilpotent, the classical pro-ℓ completion of a group Γ with vanishing $H_1(\Gamma, \mathbb{F}_\ell)$ is trivial. Since all mapping class groups in genus g are perfect when $g \geqslant 3$, their pro-ℓ completions are trivial.[2] In contrast, the natural relative pro-ℓ completions of mapping class groups are large and more closely reflect their structure. For example, when the number of marked points is at least 1, a mapping class group injects into its relative pro-ℓ completion.

Suppose that g, n, and r are non-negative integers satisfying $2g - 2 + n + r > 0$. Denote by $\Gamma_{g,n,\vec{r}}$ the mapping class group of a compact oriented surface of genus g with n distinct points and r distinct non-zero tangent vectors (or, alternatively, r boundary components) whose anchor points are distinct from the n marked points.[3,4] Denote by $T_{g,n,\vec{r}}$ its Torelli subgroup. Denote the group of symplectic $2g \times 2g$ matrices with entries in a ring R by $\mathrm{Sp}_g(R)$. Denote the relative pro-ℓ completion of $\Gamma_{g,n,\vec{r}}$ with respect to the natural homomorphism $\Gamma_{g,n,\vec{r}} \to \mathrm{Sp}_g(\mathbb{Z}_\ell)$ by $\Gamma^{(\ell)}_{g,n,\vec{r}}$. Denote the pro-$\ell$ completion of $T_{g,n,\vec{r}}$ by $T^{(\ell)}_{g,n,\vec{r}}$.

For $m \in \mathbb{N}$, the level m subgroup $\mathrm{Sp}_g(\mathbb{Z})[m]$ of $\mathrm{Sp}_g(\mathbb{Z})$ is defined to be the kernel of $\mathrm{Sp}_g(\mathbb{Z}) \to \mathrm{Sp}_g(\mathbb{Z}/m\mathbb{Z})$. The level m subgroup of $\Gamma_{g,n,\vec{r}}$ is defined to be the inverse image of $\mathrm{Sp}_g(\mathbb{Z})[m]$ in $\Gamma_{g,n,\vec{r}}$. Denote the relative pro-ℓ completion of $\Gamma_{g,n,\vec{r}}[m]$ with respect to the homomorphism to the closure of its image in $\mathrm{Sp}_g(\mathbb{Z}_\ell)$ by $\Gamma_{g,n,\vec{r}}[m]^{(\ell)}$.

Define a discrete group T to be *ℓ-regular* if every finite quotient of T that is an ℓ-group is the quotient of a torsion free nilpotent quotient of T. Free groups, surface groups and pure braid

[1] A homomorphism $\phi : \Gamma \to G$ from a group Γ to a topological group G is *dense* if its image is dense in G.

[2] The pro-ℓ completions of mapping class groups vanish in genus 1 when $\ell \neq 2, 3$ and in genus 2 when $\ell \neq 2, 5$ for similar reasons.

[3] A detailed definition is given in Section 3.1.

[4] We adopt the standard convention that n and $\vec{r}$ are omitted when they are zero. For example, $\Gamma_g = \Gamma_{g,0,\vec{0}}$, $\Gamma_{g,\vec{r}} = \Gamma_{g,0,\vec{r}}$ and $\Gamma_{g,n} = \Gamma_{g,n,\vec{0}}$.

groups are all ℓ-regular. We suspect, but were unable to prove, that when ℓ is odd and $n+r>0$, $T_{g,n,\vec{r}}$ is ℓ-regular.

Theorem 1. *Suppose that $g \geqslant 3$ and that r and n are non-negative integers. For each prime number ℓ and each positive integer m, the kernel of the natural homomorphism $T^{(\ell)}_{g,n,\vec{r}} \to \Gamma_{g,n,\vec{r}}[m]^{(\ell)}$ is central in $T^{(\ell)}_{g,n,\vec{r}}$ and contains a copy of $\mathbb{Z}_\ell$. If $T_{g,n,\vec{r}}$ is ℓ-regular, then there is an exact sequence*

$$0 \to \mathbb{Z}_\ell \to T^{(\ell)}_{g,n,\vec{r}} \to \Gamma_{g,n,\vec{r}}[m]^{(\ell)} \to \mathrm{Sp}_g(\mathbb{Z}_\ell)[m] \to 1.$$

The theorem is trivially false in genus 0 as $T_{0,n,\vec{r}} = \Gamma_{g,n,\vec{r}}$. The theorem is false in genus 1 as $\mathrm{SL}_2(\mathbb{Z})$ does not have the congruence subgroup property. In genus 2 the kernel of $T^{(\ell)}_{2,n,\vec{r}} \to \Gamma^{(\ell)}_{2,n,\vec{r}}$ is very large as we explain in Section 6.4.

Theorem 1 may be regarded as an integral analogue of results about the relative unipotent completion of mapping class groups that are proved in [11,12]. It is also an analogue for mapping class groups of a result of Deligne [6], which may be stated as follows. Suppose that S is either SL_n ($n \geqslant 3$) or Sp_g ($g \geqslant 2$). Suppose that Γ is a finite index subgroup of $S(\mathbb{Z})$ and that $\widetilde{\Gamma}$ is the restriction of the universal central extension of $S(\mathbb{Z})$ to Γ. Denote the relative pro-ℓ completion of Γ (resp. $\widetilde{\Gamma}$) with respect to the homomorphism to the closure of the image of $\Gamma \to S(\mathbb{Z}_\ell)$ (resp. $\widetilde{\Gamma} \to S(\mathbb{Z}_\ell)$) by $\Gamma^{(\ell)}$ (resp. $\widetilde{\Gamma}^{(\ell)}$). Deligne's result implies that there is an exact sequence

$$0 \to \mathbb{Z}_\ell \xrightarrow{\times e} \mathbb{Z}_\ell \to \widetilde{\Gamma}^{(\ell)} \to \Gamma^{(\ell)} \to 1$$

where $e \in \{1,2\}$. It and the basic structure result for the relative unipotent completion of mapping class groups [11] mentioned above are the two main tools in the proof of Theorem 1. The main technical ingredient in the proof is the existence (Corollary 6.10) of a central extension of a finite index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$ by $\mathbb{Z}$ in a subquotient of $\Gamma_{g,n,\vec{r}}$ whenever $g \geqslant 3$. This extension is a non-zero multiple of the restriction of the universal central extension of $\mathrm{Sp}_g(\mathbb{Z})$.

Suppose that $n \geqslant 0$ and that $(S, \{x_0, \dots, x_n\})$ is an $n+1$ marked surface of genus $g \geqslant 2$. Let $S' = S - \{x_1, \dots, x_n\}$ be the corresponding n-punctured surface. Since the action of the mapping class group $\Gamma_{g,n+1}$ on $\pi_1(S', x_0)$ induces an action of $\Gamma^{(\ell)}_{g,n+1}$ on $\pi_1(S', x_o)^{(\ell)}$, Theorem 1 implies that the action of the pro-ℓ Torelli group $T^{(\ell)}_{g,n+1}$ on $\pi_1(S', x_0)^{(\ell)}$ is not faithful. Similarly, since the outer action of $\Gamma_{g,n}$ on $\pi_1(S', x_0)$ induces an outer action of $\Gamma^{(\ell)}_{g,n}$ on $\pi_1(S', x_0)^{(\ell)}$, the outer action of $T^{(\ell)}_{g,n}$ on $\pi_1(S', x_0)^{(\ell)}$ is not faithful.[5]

Corollary 2. *If $g \geqslant 2$ and $n \geqslant 0$, then the natural homomorphisms*

$$T^{(\ell)}_{g,n+1} \to \mathrm{Aut}\,\pi_1(S', x_0)^{(\ell)} \quad \textit{and} \quad T^{(\ell)}_{g,n} \to \mathrm{Out}\,\pi_1(S', x_0)^{(\ell)}$$

are not injective. Their kernels each contain a copy of $\mathbb{Z}_\ell$.

The injectivity of the corresponding homomorphisms

$$\Gamma^{(\ell)}_{g,n+1} \to \mathrm{Aut}\,\pi_1(S', x_0)^{(\ell)} \quad \text{and} \quad \Gamma^{(\ell)}_{g,n} \to \mathrm{Out}\,\pi_1(S', x_0)^{(\ell)}$$

for mapping class groups is not known for any (g, n), with $g \geqslant 3$ and $n \geqslant 0$.

[5] When $g = 2$, the statement follows from Corollary 6.13.

The proof of the following result (in the case $r = 0$) was communicated to us by Mochizuki and Tamagawa.

Theorem 3. *Suppose that ℓ is a prime number. For all non-negative integers g, n, r satisfying $2g - 2 + n + r > 0$, the action of $G_{\mathbb{Q}}$ on $\Gamma^{(\ell)}_{g,n,\vec{r}}$, determined by the choice of a suitably chosen smoothing of a maximally degenerate stable curve of type (g, n, r) as a tangential base point, is unramified at all primes $\neq \ell$.*

We record the following consequence which follows directly from this result using the functoriality of relative unipotent completion and the fact (proved in [18] as a consequence of the "comparison theorem" proved there) that the $G_{\mathbb{Q}}$-action on $\Gamma^{(\ell)}_{g,n,\vec{r}}$ extends to the relative unipotent completion $\mathcal{G}_{g,n,\vec{r}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ of $\Gamma_{g,n,\vec{r}}$ over $\mathbb{Q}_\ell$.

Corollary 4. *For all prime numbers ℓ, the $G_{\mathbb{Q}}$ representation on the relative unipotent completion $\mathcal{G}_{g,n,\vec{r}} \otimes \mathbb{Q}_\ell$ is unramified outside ℓ. Consequently, the $G_{\mathbb{Q}}$ action on $\mathcal{G}_{g,n,\vec{r}} \otimes \mathbb{Q}_\ell$ induces an action of $\pi_1(\operatorname{Spec} \mathbb{Z}[1/\ell])$.*

This result should be a useful technical ingredient in the investigation of "Teichmüller motives"—relative motives over moduli spaces of curves—of which elliptic motives and elliptic polylogarithms [3] will be special cases.

2. Relative pro-ℓ completion

2.1. Basic properties

As in the introduction, we suppose that Γ is a discrete or profinite group, that P is a profinite group, and that $\rho : \Gamma \to P$ is a continuous, dense homomorphism.

The relative pro-ℓ completion of Γ with respect to ρ is characterized by a universal mapping property: if G is a profinite group, $\psi : G \to P$ a continuous homomorphism with pro-ℓ kernel, and if $\phi : \Gamma \to G$ is a continuous homomorphism whose composition with ψ is ρ, then there is a unique continuous homomorphism $\Gamma^{(\ell),\rho} \to G$ that extends ϕ:

$$\begin{tikzcd}
\Gamma \arrow[dr] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\phi"'] & & \\
& \Gamma^{(\ell),\rho} \arrow[r] \arrow[d, dotted] & P \\
& G \arrow[ur, "\psi"'] &
\end{tikzcd}$$

The following property is a direct consequence of the universal mapping property.

Proposition 2.1. *A dense homomorphism $\rho : \Gamma \to P$ from a discrete group to a profinite group induces a homomorphism $\overline{\rho} : \hat{\Gamma} \to P$ from the profinite completion of Γ to P. The natural homomorphism $\Gamma \to \hat{\Gamma}$ induces a natural isomorphism $\Gamma^{(\ell),\rho} \to \hat{\Gamma}^{(\ell),\overline{\rho}}$.*

Suppose that S is a linear group scheme over $\mathbb{Z}$. For a positive integer m and a commutative ring R, set

$$S(R)[m] = \ker\bigl\{S(R) \to S(R/mR)\bigr\}.$$

This is the *level m subgroup of $S(R)$*. Note that $S(\mathbb{Z}_\ell)[m] = S(\mathbb{Z}_\ell)[\ell^\nu]$, where $\nu = \mathrm{ord}_\ell(m)$.

We say that $S(\mathbb{Z})$ has the *congruence subgroup property* if every finite index subgroup of $S(\mathbb{Z})$ contains the level m subgroup $S(\mathbb{Z})[m]$ of $S(\mathbb{Z})$ for some $m > 0$, or equivalently, the natural homomorphism $S(\mathbb{Z})^\wedge \to S(\hat{\mathbb{Z}})$ is injective. In this case, the profinite completion of $S(\mathbb{Z})$ is the closure of its image in $S(\hat{\mathbb{Z}})$.

Proposition 2.2. *Suppose that S is a linear group scheme over $\mathbb{Z}$. If S satisfies*

(1) *$S(\mathbb{Z})$ has the congruence subgroup property, and*
(2) *$S(\mathbb{Z}) \to S(\mathbb{Z}/N)$ is surjective for every positive integer N,*

then the natural homomorphism $S(\mathbb{Z})^\wedge \to S(\hat{\mathbb{Z}})$ is an isomorphism. If, in addition, m is a positive integer and for every prime number p not dividing ℓm, $S(\mathbb{F}_p)$ has no non-trivial quotient ℓ-group, then, the relative pro-ℓ completion of $S(\mathbb{Z})[m]$ with respect to $S(\mathbb{Z})[m] \to S(\mathbb{Z}_\ell)[m]$ is the closure of $S(\mathbb{Z})[m]$ in $S(\mathbb{Z}_\ell)$, which equals $S(\mathbb{Z}_\ell)[\ell^\nu]$, where $\nu = \mathrm{ord}_\ell(m)$.

Proof. The morphism

$$S(\mathbb{Z}) \to S(\hat{\mathbb{Z}}) \cong \varprojlim_N S(\mathbb{Z}/N)$$

induces a homomorphism $S(\mathbb{Z})^\wedge \to S(\hat{\mathbb{Z}})$, which is injective by condition (1) and surjective by condition (2). Condition (2) also implies that the sequence

$$1 \to S(\mathbb{Z})[m] \to S(\mathbb{Z}) \to S(\mathbb{Z}/m\mathbb{Z}) \to 1$$

is exact. Since the right group is finite, the sequence

$$1 \to S(\mathbb{Z})[m]^\wedge \to S(\mathbb{Z})^\wedge \to S(\mathbb{Z}/m\mathbb{Z}) \to 1$$

of profinite completions is exact. On the other hand, the sequence

$$1 \to S(\hat{\mathbb{Z}})[m] \to S(\hat{\mathbb{Z}}) \to S(\mathbb{Z}/m\mathbb{Z}) \to 1 \tag{1}$$

is exact by definition. The isomorphism $S(\mathbb{Z})^\wedge \to S(\hat{\mathbb{Z}})$ therefore induces an isomorphism of $S(\mathbb{Z})[m]^\wedge$ with $S(\hat{\mathbb{Z}})[m]$.

Denote the order of m at a prime number p by ν_p. The short exact sequence (1) is the direct product over all prime numbers p of the exact sequences

$$1 \to S(\mathbb{Z}_p)\bigl[p^{\nu_p}\bigr] \to S(\mathbb{Z}_p) \to S\bigl(\mathbb{Z}/p^{\nu_p}\mathbb{Z}\bigr) \to 1.$$

Consequently, the profinite completion of $S(\mathbb{Z})[m]$ is

$$S(\hat{\mathbb{Z}})[m] = \prod_p S(\mathbb{Z}_p)\big[p^{\nu_p}\big]$$

where p ranges over all prime numbers. The homomorphism $\rho : S(\mathbb{Z})[m] \to S(\mathbb{Z}_\ell)[m]$ induces the projection

$$S(\hat{\mathbb{Z}})[m] \cong \prod_p S(\mathbb{Z}_p)\big[p^{\nu_p}\big] \to S(\mathbb{Z}_\ell)\big[\ell^{\nu_\ell}\big]$$

onto the factor corresponding to ℓ. Since the relative pro-ℓ completion $S(\mathbb{Z})[m]^{(\ell)}$ is the largest quotient G of the profinite completion $S(\hat{\mathbb{Z}})[m]$ such that the kernel of $G \to S(\mathbb{Z}_\ell)[m]$ is pro-ℓ, there is a surjection

$$\prod_{p\neq\ell} S(\mathbb{Z}_p)\big[p^{\nu_p}\big] \to \ker\big\{S(\mathbb{Z})[m]^{(\ell)} \to S(\mathbb{Z}_\ell)[m]\big\}$$

whose image is pro-ℓ. Thus, to prove the proposition, it suffices to show that $S(\mathbb{Z}_p)[p^{\nu_p}]$ has no non-trivial ℓ-quotient when $p \neq \ell$.

Now suppose that $p \neq \ell$. It is well known that $S(\mathbb{Z}_p)[p^N]$ is a pro-p group when $N > 0$. (It suffices to check this for GL_M.) So if $N > 0$, every homomorphism from $S(\mathbb{Z}_p)[p^N]$ to an ℓ-group is trivial. In particular, if $\nu_p > 0$ (or equivalently if p divides m), then $S(\mathbb{Z}_p)[p^{\nu_p}]$ has no non-trivial ℓ-quotients, as required. Suppose that p does not divide $m\ell$. In particular, $\nu_p = 0$. Since

$$1 \to S(\mathbb{Z}_p)[p] \to S(\mathbb{Z}_p) \to S(\mathbb{F}_p) \to 1$$

is exact by condition (2), every homomorphism $S(\mathbb{Z}_p) \to A$ to an ℓ-group ($\ell \neq p$) factors through a homomorphism $S(\mathbb{F}_p) \to A$. But we have assumed that there are no non-trivial such homomorphisms. Thus $S(\mathbb{Z}_p)[p^{\nu_p}]$ has no non-trivial ℓ-quotients, as required. □

The hypotheses of Proposition 2.2 are satisfied for all $m \geqslant 1$ when $S = \mathrm{Sp}_g$ when $g \geqslant 2$ and when $S = \mathrm{SL}_n$ when $n \geqslant 3$. (See [2] and [9, Theorem 1, p. 12].)

Define Group_f to be the category whose objects are topological groups that are either discrete or profinite; morphisms are continuous group homomorphisms. The category of discrete groups and the category of profinite groups are both full subcategories of Group_f. A sequence of group homomorphisms in Group_f is exact if its is exact in the category of groups.

Proposition 2.3 *(naturality). Suppose that Γ_1 and Γ_2 are objects of Group_f and that P_1 and P_2 are profinite groups. Suppose that $\rho_j : \Gamma_j \to P_j$ are continuous dense homomorphisms. If*

$$\begin{tikzcd} \Gamma_1 \arrow[r, "\rho_1"] \arrow[d, "\phi_\Gamma"'] & P_1 \arrow[d, "\phi_P"] \\ \Gamma_2 \arrow[r, "\rho_2"] & P_2 \end{tikzcd}$$

is a commutative diagram in Group_f*, then there is a unique continuous homomorphism* $\phi^{(\ell)}$: $\Gamma_1^{(\ell),\rho_1} \to \Gamma_2^{(\ell),\rho_2}$ *such that the diagram*

$$\begin{tikzcd}
\Gamma_1 \arrow[r] \arrow[d, "\phi_\Gamma"'] \arrow[rr, bend left, "\rho_1"] & \Gamma_1^{(\ell),\rho_1} \arrow[r] \arrow[d, "\phi^{(\ell)}"'] & P_1 \arrow[d, "\phi_P"'] \\
\Gamma_2 \arrow[r] \arrow[rr, bend right, "\rho_2"'] & \Gamma_2^{(\ell),\rho_2} \arrow[r] & P_2
\end{tikzcd}$$

commutes.

Proof. This follows easily from the universal mapping property. □

Proposition 2.4 *(right exactness). Suppose that* P_1*,* P_2 *and* P_3 *are profinite groups and that* ρ_j: $\Gamma_j \to P_j$ $(j \in \{1,2,3\})$ *are continuous dense homomorphisms in* Group_f*. Suppose that the* Γ_j *are all discrete groups or all profinite groups. If the diagram*

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_1 \arrow[r] \arrow[d, "\rho_1"'] & \Gamma_2 \arrow[r] \arrow[d, "\rho_2"'] & \Gamma_3 \arrow[r] \arrow[d, "\rho_3"'] & 1 \\
1 \arrow[r] & P_1 \arrow[r] & P_2 \arrow[r] & P_3 \arrow[r] & 1
\end{tikzcd}$$

in Group_f *commutes and has exact rows, then the sequence*

$$\Gamma_1^{(\ell),\rho_1} \to \Gamma_2^{(\ell),\rho_2} \to \Gamma_3^{(\ell),\rho_3} \to 1$$

is exact.

Proof. Denote $\Gamma_j^{(\ell),\rho_j}$ by $\Gamma_j^{(\ell)}$. Since $\Gamma_2 \to \Gamma_3$ is surjective, the homomorphism $\Gamma_2 \to \Gamma_3 \to \Gamma_3^{(\ell)}$ has dense image. This implies that $\Gamma_2^{(\ell)} \to \Gamma_3^{(\ell)}$ is surjective.

Denote the closure of the image of $\Gamma_1 \to \Gamma_2^{(\ell)}$ by N. Observe that this equals the image of $\Gamma_1^{(\ell)} \to \Gamma_2^{(\ell)}$. Since Γ_1 is normal in Γ_2, this is a closed normal subgroup of $\Gamma_2^{(\ell)}$. The image of N in P_2 is P_1. The homomorphism $\Gamma_2^{(\ell)} \to P_2$ induces a homomorphism $\overline{\rho} : \Gamma_2^{(\ell)}/N \to P_2/P_1 \cong P_3$. Since the kernel of the composite $\Gamma_2 \to \Gamma_2^{(\ell)} \to \Gamma_2^{(\ell)}/N$ contains Γ_1 it induces a continuous homomorphism $\rho : \Gamma_3 \to \Gamma_2^{(\ell)}/N$ whose composition with $\overline{\rho}$ is ρ_3. The universal mapping property of $\Gamma_3 \to \Gamma_3^{(\ell)}$ implies that ρ induces a homomorphism $\Gamma_3^{(\ell)} \to \Gamma_2^{(\ell)}/N$, which is necessarily an isomorphism. Since N is the image of $\Gamma_1^{(\ell)} \to \Gamma_2^{(\ell)}$, this implies the exactness at $\Gamma_2^{(\ell)}$. □

Relative pro-ℓ completion is not left exact. One dramatic illustration of this is Deligne's result [6], which we now describe.

Suppose that S is the $\mathbb{Z}$-group scheme SL_n or Sp_g where $n \geqslant 3$ or $g \geqslant 2$. Since $S(\mathbb{Z})$ is perfect, $S(\mathbb{Z})$ has a universal central extension. Denote the restriction of the universal central extension of $S(\mathbb{Z})$ to a finite index subgroup Γ by $\widetilde{\Gamma}$. This is an extension

$$0 \to \mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to 1.$$

Deligne [6] proves that every finite index subgroup of $\widetilde{\Gamma}$ contains $2\mathbb{Z}$, which implies that the profinite completion of $\widetilde{\Gamma}$ is an extension

$$0 \to \mathbb{Z}/e\mathbb{Z} \to \widetilde{\Gamma}^{\wedge} \to \Gamma^{\wedge} \to 1$$

where $e \in \{1, 2\}$. Since $S(\mathbb{Z})$ has the congruence subgroup property [2], the profinite completion of Γ is its closure in $S(\hat{\mathbb{Z}})$.

Deligne's result can be reformulated and extended to multiples of the universal central extension $\widetilde{\Gamma} \to \Gamma$.

Suppose as above that $S = \mathrm{SL}_n$ or Sp_g with $n \geqslant 3$ or $g \geqslant 2$. Suppose that Γ is a finite index subgroup of $S(\mathbb{Z})$. Denote the restriction to Γ of the dth power of the universal central extension of $S(\mathbb{Z})$ by $\widetilde{\Gamma}_d$. (When $d = 0$, $\widetilde{\Gamma}_d = \Gamma \times \mathbb{Z}$.) Note that $\widetilde{\Gamma} = \widetilde{\Gamma}_1$. For all non-zero $d \in \mathbb{Z}$ there is a short exact sequence

$$1 \to \widetilde{\Gamma} \to \widetilde{\Gamma}_d \to \mathbb{Z}/d\mathbb{Z} \to 0.$$

When the context is clear, we will often drop ρ from the notation and denote $\Gamma^{(\ell),\rho}$ by $\Gamma^{(\ell)}$. For example, we will denote the relative pro-ℓ completion of $\mathrm{Sp}_n(\mathbb{Z})$ with respect to reduction mod ℓ by $\mathrm{Sp}_n(\mathbb{Z})^{(\ell)}$.

Proposition 2.5. *For all prime numbers ℓ and all non-zero $d \in \mathbb{Z}$, there are short exact sequences*

$$1 \to \widetilde{\Gamma}^{(\ell)} \to \widetilde{\Gamma}_d^{(\ell)} \to \mathbb{Z}_\ell/d\mathbb{Z}_\ell \to 0$$

and

$$0 \to \mathbb{Z}_\ell/df\mathbb{Z}_\ell \to \widetilde{\Gamma}_d^{(\ell)} \to \Gamma^{(\ell)} \to 1,$$

where $f \in \{1, 2\}$ and $f \leqslant e$, the constant defined above.

Proof. Applying right exactness to the standard exact sequence

$$1 \to \widetilde{\Gamma} \to \widetilde{\Gamma}_d \to \mathbb{Z}/d\mathbb{Z} \to 0$$

implies that the sequence

$$\widetilde{\Gamma}^{(\ell)} \to \widetilde{\Gamma}_d^{(\ell)} \to \mathbb{Z}_\ell/d\mathbb{Z}_\ell \to 0$$

is exact. Denote the kernel of $\widetilde{\Gamma}_d \to \Gamma$ by K_d. Set $K = K_1$. Then each K_d is isomorphic to $\mathbb{Z}$ and K_1 is the subgroup of K_d of index d. Injectivity of $\widetilde{\Gamma}^{(\ell)} \to \widetilde{\Gamma}_d^{(\ell)}$ follows from the fact that if N is a normal subgroup of $\widetilde{\Gamma}$ whose intersection with K has ℓ-power index, then the subgroup

$\widetilde{N} := N \cdot f^{-1}(K \cap N)$ of $\widetilde{\Gamma}_d$, where $d = \ell^\nu f$ with $\nu = \mathrm{ord}_\ell(d)$, is normal in $\widetilde{\Gamma}_d$ and has the property that its intersection with K_d has ℓ-power index and $\widetilde{N} \cap \widetilde{\Gamma} = N$.

The diagram

$$\begin{tikzcd}
 & \mathbb{Z}/d\mathbb{Z} \arrow[r, equal] & \mathbb{Z}/d\mathbb{Z} & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, two heads] & \widetilde{\Gamma}_d \arrow[r] \arrow[u, two heads] & \Gamma \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, "\times d"'] & \widetilde{\Gamma} \arrow[r] \arrow[u] & \Gamma \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

has exact rows; the upper vertical maps are surjective and the lower vertical maps are injective. Deligne's result and the first part of this result imply that the relative pro-ℓ completion of this diagram is

$$\begin{tikzcd}
 & \mathbb{Z}_\ell/d\mathbb{Z}_\ell \arrow[r, equal] & \mathbb{Z}_\ell/d\mathbb{Z}_\ell & & \\
 & \mathbb{Z}_\ell/df\mathbb{Z}_\ell \arrow[r] \arrow[u, two heads] & \widetilde{\Gamma}_d^{(\ell)} \arrow[r] \arrow[u, two heads] & \Gamma^{(\ell)} \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z}_\ell/f\mathbb{Z}_\ell \arrow[r] \arrow[u, "\times d"'] & \widetilde{\Gamma}^{(\ell)} \arrow[r] \arrow[u] & \Gamma^{(\ell)} \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

where $f \leqslant e$ and $f \in \{1, 2\}$. The second assertion follows. $\square$

The following useful lemma is easily proved using the universal mapping property of relative pro-ℓ completion.

Lemma 2.6. *Suppose that*

$$1 \longrightarrow K \longrightarrow P \xrightarrow{\psi} \overline{P} \longrightarrow 1$$

is a short exact sequence of profinite groups. Suppose that Γ is a discrete or profinite group and that $\rho : \Gamma \to P$ is a continuous dense homomorphism. Denote $\psi \circ \rho$ by $\overline{\rho}$. If K is a pro-ℓ group, then the natural homomorphism $\Gamma^{(\ell),\rho} \to \Gamma^{(\ell),\overline{\rho}}$ is an isomorphism. $\square$

Example 2.7. Let S be the $\mathbb{Z}$-group scheme SL_{n+1} or Sp_n, where $n \geqslant 1$. Let $P = S(\mathbb{Z}_\ell)$ and $\overline{P} = S(\mathbb{F}_\ell)$. Suppose that Γ is a discrete or profinite group and that $\rho : \Gamma \to S(\mathbb{Z}_\ell)$ is a continuous, dense homomorphism. Suppose that $\psi : P \to \overline{P}$ is reduction mod ℓ. Since $\ker \psi$ is a pro-ℓ group, there is a natural isomorphism $\Gamma^{(\ell),\rho} \cong \Gamma^{(\ell),\overline{\rho}}$.

2.2. Remarks on pro-ℓ completion

To conclude this section, we establish some basic facts about standard (i.e., relative to the trivial representation) pro-ℓ completion. Denote the lower central series of a discrete group Γ by

$$\Gamma = L^1\Gamma \supseteq L^2\Gamma \supseteq L^3\Gamma \supseteq \cdots.$$

The following observation follows directly from the fact that every finite ℓ-group is nilpotent.

Lemma 2.8. *For all discrete groups Γ, the natural mapping*

$$\Gamma^{(\ell)} \to \varprojlim_n \big(\Gamma/L^n\big)^{(\ell)}$$

is an isomorphism.

This result can be refined as follows. For an integer m, let $\operatorname{supp}(m)$ denote the set of prime divisors of m. For a set S of prime numbers and a subgroup N of Γ, define

$$\sqrt[S]{N} = \big\{g \in \Gamma\colon \text{ there exists an integer } m > 0 \text{ with } \operatorname{supp}(m) \subseteq S \text{ and } g^m \in N\big\}.$$

The fact that the set of torsion elements of a nilpotent group is a characteristic subgroup, implies that if N is a normal (resp. characteristic) subgroup of Γ and Γ/N is nilpotent, then for all S, $\sqrt[S]{N}$ is a normal (resp. characteristic) subgroup of Γ.

Denote the complement of S in the set $\wp$ of prime numbers by S'. When S consists of a single prime number ℓ, we shall write $\sqrt[\ell]{N}$ for $\sqrt[S]{N}$ and $\sqrt[\ell']{N}$ for $\sqrt[S']{N}$. For simplicity, we denote $\sqrt[\wp]{N}$ by $\sqrt{N}$. This is the set of all elements of Γ that are torsion mod N.

Again, the fact that every finite ℓ group is nilpotent implies:

Lemma 2.9. *For all discrete groups Γ, the natural mapping*

$$\Gamma^{(\ell)} \to \varprojlim_n \big(\Gamma/\sqrt[\ell']{L^n}\big)^{(\ell)}$$

is an isomorphism.

For each $n \geqslant 1$, define $D^n\Gamma$ to be the inverse image in Γ of the torsion elements of $\Gamma/L^n\Gamma$. That is, $D^n\Gamma = \sqrt{L^n\Gamma}$, which is a characteristic subgroup of Γ. It is known as the nth *rational dimensional subgroup of* Γ.[6]

Recall from the introduction that a discrete group Γ is ℓ-regular if every finite quotient of Γ that is an ℓ-group is the quotient of a torsion free nilpotent quotient of Γ.

The condition that a discrete group Γ be ℓ-regular may be expressed in terms of the groups $D^m\Gamma$ and $\sqrt[\ell']{L^n\Gamma}$. Observe that for all positive integers n, $D^n\Gamma \supseteq \sqrt[\ell']{L^n\Gamma}$.

[6] The groups $D^n\Gamma$ can also be constructed as follows. Suppose that F is a field of characteristic zero. Denote the augmentation ideal of the group algebra $F\Gamma$ by J_F. Then $D^n\Gamma$ is the intersection of Γ with $1 + J_F^n$ in $F\Gamma$. For a proof see [34, Theorem 1.10, p. 474].

Proposition 2.10. *A discrete group Γ is ℓ-regular if for each positive integer n, there exists an integer $m > 0$ such that $D^m\Gamma \subseteq \sqrt[\ell']{L^n\Gamma}$. In this case, the mapping*

$$\Gamma^{(\ell)} \to \varprojlim_n \big(\Gamma/D^n\big)^{(\ell)}$$

is an isomorphism.

Proof. The natural mapping

$$\Gamma^{(\ell)} \cong \varprojlim_n \big(\Gamma/\sqrt[\ell']{L^n}\big)^{(\ell)} \to \varprojlim_n \big(\Gamma/D^n\big)^{(\ell)}$$

is surjective. The condition in the statement guarantees that it is also injective. □

Not all groups that inject into their pro-ℓ completions are ℓ-regular.

Example 2.11. Suppose that $S = \mathrm{SL}_{n+1}$ or Sp_n where $n \geqslant 2$. Suppose that $\Gamma = S(\mathbb{Z})[\ell^m]$ where $m \geqslant 1$. Even though this group injects into its pro-ℓ completion $S(\mathbb{Z}_\ell)[\ell^m]$, the homomorphism

$$\Gamma^{(\ell)} \to \varprojlim_k \big(\Gamma/D^k\big)^{(\ell)}$$

is trivial as Γ has no non-trivial torsion free abelian quotients [37], and therefore no non-trivial torsion free nilpotent quotients.

If the graded quotients of the lower central series of Γ have no ℓ-torsion, then $D^n\Gamma = \sqrt[\ell']{L^n\Gamma}$ for all $n > 0$. Thus we have:

Corollary 2.12. *Suppose that Γ is a discrete group. If the graded quotients of the lower central series of Γ have no ℓ-torsion, then Γ is ℓ-regular.*

We conclude this section with the following result, which follows easily from the fact that every finite ℓ-group is nilpotent.

Proposition 2.13. *If Γ is a discrete group such that $H_1(\Gamma) \otimes \mathbb{F}_\ell = 0$, then $\Gamma^{(\ell)}$ is trivial.*

3. Mapping class and Torelli groups

Suppose that g and n are non-negative integers satisfying $2g - 2 + n > 0$. Fix a closed, oriented surface S of genus g and a finite subset $\mathbf{x} = \{x_1, \ldots, x_n\}$ of n distinct points in S. The corresponding mapping class group will be denoted

$$\Gamma_{S,\mathbf{x}} = \pi_0 \operatorname{Diff}^+(S, \mathbf{x}),$$

where $\operatorname{Diff}^+(S, \mathbf{x})$ denotes the group of orientation preserving diffeomorphisms of S that fix $\mathbf{x}$ pointwise. By the classification of surfaces, the diffeomorphism class of $(S, \mathbf{x})$ depends only

on (g,n). Consequently, the group $\Gamma_{S,\mathbf{x}}$ depends only on the pair (g,n). It will be denoted by $\Gamma_{g,n}$ when we have no particular marked surface $(S,\mathbf{x})$ in mind.

For a commutative ring A, set $H_A = H_1(S;A)$. The intersection pairing $H_A^{\otimes 2} \to A$ is skew symmetric and unimodular. The choice of a symplectic basis of H_A gives an isomorphism $\mathrm{Sp}(H_A) \cong \mathrm{Sp}_g(A)$ of $2g \times 2g$ symplectic matrices with entries in A. The action of $\mathrm{Diff}^+(S,\mathbf{x})$ on S induces a homomorphism

$$\rho : \Gamma_{S,\mathbf{x}} \to \mathrm{Sp}(H_{\mathbb{Z}})$$

which is well-known to be surjective.

Let $\overline{\rho} : \Gamma_{S,\mathbf{x}} \to \mathrm{Sp}(H_{\mathbb{Z}/\ell\mathbb{Z}})$ be reduction of ρ mod ℓ. Define $\Gamma_{S,\mathbf{x}}^{(\ell)}$ to be the relative pro-ℓ completion of $\Gamma_{S,\mathbf{x}}$ with respect to $\overline{\rho}$. By Example 2.7, this is isomorphic to the relative pro-ℓ completion of $\Gamma_{S,\mathbf{x}}$ with respect to the natural homomorphism $\Gamma_{S,\mathbf{x}} \to \mathrm{Sp}(H_{\mathbb{Z}_\ell})$.

Proposition 3.1. *Suppose that $n \geqslant 0$, that S is a surface of genus $\geqslant 2$, and that $\mathbf{x} = \{x_0,\dots,x_n\}$ is a set of $n+1$ distinct points of S. Set $\mathbf{x}' = \mathbf{x} - \{x_0\}$ and $S' = S - \mathbf{x}'$. Then*

(1) *the natural homomorphism $\Gamma_{S,\mathbf{x}} \to \Gamma_{S,\mathbf{x}}^{(\ell)}$ is injective;*
(2) *the sequence $1 \to \pi_1(S',x_0)^{(\ell)} \to \Gamma_{S,\mathbf{x}}^{(\ell)} \to \Gamma_{S,\mathbf{x}'}^{(\ell)} \to 1$ of relative pro-ℓ completions is exact.*

Proof. Since $\pi_1(S',x_0)$ is residually nilpotent and since the graded quotients of its lower central series are torsion free (cf. [30] when $n=0$), the homomorphism $\pi_1(S',x_0) \to \pi_1(S',x_0)^{(\ell)}$ is injective. Thus $\mathrm{Aut}\,\pi_1(S',x_0) \to \mathrm{Aut}_{\mathrm{cts}}\,\pi_1(S',x_0)^{(\ell)}$ is also injective. Since $\Gamma_{S,\mathbf{x}}$ is a subgroup of $\mathrm{Aut}\,\pi_1(S',x_0)$, it is a subgroup of $\mathrm{Aut}_{\mathrm{cts}}\,\pi_1(S',x_0)^{(\ell)}$. The first assertion now follows as the inclusion of $\Gamma_{S,\mathbf{x}}$ into $\mathrm{Aut}_{\mathrm{cts}}\,\pi_1(S',x_0)^{(\ell)}$ factors through the completion homomorphism:

$$\Gamma_{S,\mathbf{x}} \to \Gamma_{S,\mathbf{x}}^{(\ell)} \to \mathrm{Aut}_{\mathrm{cts}}\,\pi_1(S',x_0)^{(\ell)}.$$

Since relative pro-ℓ completion is right exact, to prove the second assertion, we need only show that $\pi_1(S',x_0)^{(\ell)} \to \Gamma_{S,\mathbf{x}}^{(\ell)}$ is injective. But this follows as the composite

$$\pi_1(S',x_0)^{(\ell)} \to \Gamma_{S,\mathbf{x}}^{(\ell)} \to \mathrm{Aut}_{\mathrm{cts}}\,\pi_1(S',x_0)^{(\ell)}$$

is the conjugation action, which is injective as $\pi_1(S',x_0)^{(\ell)}$ has trivial center (cf. [1] when $n=0$). $\square$

The *Torelli group* $T_{S,\mathbf{x}}$ is defined to be the kernel of ρ. Its isomorphism class depends only on (g,n). It will be denoted by $T_{g,n}$ when we have no particular marked surface in mind.

Proposition 3.2. *Suppose that $n \geqslant 0$, that S is a surface of genus $\geqslant 2$, and that $\mathbf{x} = \{x_0,\dots,x_n\}$ is a set of $n+1$ distinct points of S. Set $\mathbf{x}' = \mathbf{x} - \{x_0\}$ and $S' = S - \mathbf{x}'$. Then the sequence*

$$1 \to \pi_1(S',x_0)^{(\ell)} \to T_{S,\mathbf{x}}^{(\ell)} \to T_{S,\mathbf{x}'}^{(\ell)} \to 1$$

of relative pro-ℓ completions is exact.

Proof. We need only prove the injectivity of the left-hand mapping. But this follows from the injectivity of the composite

$$\pi_1(S', x_0)^{(\ell)} \to T^{(\ell)}_{S,\mathbf{x}} \to \Gamma^{(\ell)}_{S,\mathbf{x}}$$

which was established in Proposition 3.1. □

3.1. Variant: Tangent vectors and boundary components

Suppose that $\mathbf{x}$ is a set of n distinct points in S and that v_j is a non-zero tangent vector at $y_j \in S$, and $\{y_1, \dots, y_r\}$ is a set of r distinct points, disjoint from $\mathbf{x}$. Set $\vec{\mathbf{v}} = \{v_1, \dots, v_r\}$. Define

$$\Gamma_{S,\mathbf{x},\vec{\mathbf{v}}} = \pi_0 \operatorname{Diff}^+(S, \mathbf{x}, \vec{\mathbf{v}}),$$

the group of connected components of the group of orientation preserving diffeomorphisms of S that fix $\mathbf{x}$ and $\vec{\mathbf{v}}$ pointwise. The corresponding Torelli group $T_{S,\mathbf{x},\vec{\mathbf{v}}}$ is the kernel of the homomorphism $\Gamma_{S,\mathbf{x},\vec{\mathbf{v}}} \to \operatorname{Sp}(H_1(S))$. For $m \in \mathbb{N}$, define its level m subgroup $\Gamma_{g,n,\vec{r}}[m]$ to be the kernel of the natural homomorphism $\Gamma_{g,n,\vec{r}} \to \operatorname{Sp}(H_1(S; \mathbb{Z}/m))$.

The group $\Gamma_{S,\mathbf{x},\vec{\mathbf{v}}}$ is isomorphic to the mapping class group of a genus g surface with n marked points and r boundary components. The isomorphism can be seen by replacing $S - \{x_1, \dots, x_n, y_1, \dots, y_r\}$ by the real oriented blow-up of $S - \{x_1, \dots, x_n\}$ at each of the y_j. (See [15, §4.1] for more details.)

The groups $\Gamma_{S,\mathbf{x},\vec{\mathbf{v}}}$ and $T_{S,\mathbf{x},\vec{\mathbf{v}}}$ depend only on g, n and r. We shall often denote them by $\Gamma_{g,n,\vec{r}}$ and $T_{g,n,\vec{r}}$. The indices n and $\vec{r}$ will be dropped when they are zero. For example, $\Gamma_{g,\vec{1}}$ denotes the mapping class group associated to a genus g surface with one tangent vector, while $\Gamma_{g,1}$ denotes the mapping class group of a genus g surface with one marked point.

Replacing each tangent vector by its anchor point defines a natural surjective homomorphism $\Gamma_{g,n,\vec{r}} \to \Gamma_{g,n+r}$ whose kernel consists of the free abelian group generated by the r Dehn twists about the anchor points of the tangent vectors. The extension

$$0 \to \mathbb{Z}^r \to \Gamma_{g,n,\vec{r}} \to \Gamma_{g,n+r} \to 1$$

is central and, by right exactness of relative pro-ℓ completion, induces an exact sequence

$$\mathbb{Z}^r_\ell \to \Gamma^{(\ell)}_{g,n,\vec{r}} \to \Gamma^{(\ell)}_{g,n+r} \to 1.$$

Proposition 3.3. *The sequence*

$$0 \to \mathbb{Z}^r_\ell \to \Gamma^{(\ell)}_{g,n,\vec{r}} \to \Gamma^{(\ell)}_{g,n+r} \to 1$$

is exact.

Proof. Suppose that $2g - 1 + m > 0$. Denote the fundamental group of a genus g surface S with m punctures and one boundary component (with base point lying on the boundary)

by $\Pi_{g,m,\vec{1}}$. It is a free group $\langle a_1, \dots, a_g, b_1, \dots, b_g, z_1, \dots, z_m \rangle$ of rank $2g + m$ generated by a standard set of generators. The boundary loop c is given by

$$c = z_1^{-1} \cdots z_m^{-1} \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}.$$

The action of the Dehn twist on the boundary is conjugation by c^{-1}, which is of infinite order as $\Pi_{g,m,\vec{1}}$ is free.

Since $\Pi_{g,m,\vec{1}}$ is free, it injects into its pro-ℓ completion $\Pi^{(\ell)}_{g,m,\vec{1}}$. Consequently, the action of a Dehn twist about the boundary of S on $\Pi^{(\ell)}_{g,m,\vec{1}}$ has infinite order. In other words, the restriction of the natural homomorphism

$$\psi : \Gamma_{g,m,\vec{1}} \to \operatorname{Aut} \Pi^{(\ell)}_{g,m,\vec{1}}$$

to the central $\mathbb{Z}$ generated by the Dehn twist about the boundary, is injective. This proves the result when $r = 1$.

When $r > 1$, consider the homomorphism $\phi_j : \Gamma_{g,n,\vec{r}} \to \Gamma_{g,n+r-1,\vec{1}}$ that replaces all but the jth ($1 \leqslant j \leqslant r$) tangent vector by its anchor point. The composite

$$\mathbb{Z}^r \longrightarrow \Gamma_{g,n,\vec{r}} \xrightarrow{\psi \circ \phi_j} \operatorname{Aut} \Pi^{(\ell)}_{g,n+r-1,\vec{1}}$$

is injective on the jth factor, and trivial on all others. This implies that the homomorphism $\mathbb{Z}^r_\ell \to \Gamma^{(\ell)}_{g,n,\vec{r}}$ is injective. □

3.2. Continuous actions

For later use, we show that when $g \geqslant 3$, even though $T^{(\ell)}_{g,n,\vec{r}}$ may not inject into $\Gamma^{(\ell)}_{g,n,\vec{r}}$, there is still a natural action of $\Gamma^{(\ell)}_{g,n,\vec{r}}$ on $T^{(\ell)}_{g,n,\vec{r}}$.

Proposition 3.4. *If $g \geqslant 3$, then for all $r, n \geqslant 0$, the conjugation action $\Gamma_{g,n,\vec{r}} \to \operatorname{Aut} T_{g,n,\vec{r}}$ induces a continuous action $\Gamma^{(\ell)}_{g,n,\vec{r}} \to \operatorname{Aut} T^{(\ell)}_{g,n,\vec{r}}$.*

Proof. Set $\Gamma = \Gamma_{g,n,\vec{r}}$, $T = T_{g,n,\vec{r}}$ and $S = \mathrm{Sp}_g$. Denote the genus g reference surface by X. Since pro-ℓ completion is functorial, the conjugation action induces an action $\Gamma \to \operatorname{Aut} T^{(\ell)}$. Since T acts trivially on the graded quotients of the lower central series of $T^{(\ell)}$, its image in $\operatorname{Aut} T^{(\ell)}$ lies in the kernel $\operatorname{Aut}_0 T^{(\ell)}$ of the natural homomorphism $\operatorname{Aut} T^{(\ell)} \to \operatorname{Aut} H_1(T^{(\ell)})$. Since $g \geqslant 3$, T is finitely generated by [25], which implies that $H_1(T^{(\ell)}) = H_1(T) \otimes \mathbb{Z}_\ell$. Since the sequence

$$1 \to \operatorname{Aut}_0 T^{(\ell)} \to \operatorname{Aut} T^{(\ell)} \to \operatorname{Aut} H_1(T) \otimes \mathbb{Z}_\ell$$

is exact and $\operatorname{Aut}_0 T^{(\ell)}$ is a pro-ℓ group, to prove the result, it suffices to show that the action of $S(\mathbb{Z})$ on $H_1(T) \otimes \mathbb{Z}_\ell$ factors through $S(\mathbb{Z}) \to S(\mathbb{Z}_\ell)$. Johnson [26] has shown that $H_1(T)$ has

only 2-torsion and that $H_1(T)/\text{torsion}$ is an algebraic representation of the group scheme S. This implies that when ℓ is odd, the $S(\mathbb{Z})$ action on $H_1(T) \otimes \mathbb{Z}_\ell$ factors through $S(\mathbb{Z}) \to S(\mathbb{Z}_\ell)$.

To establish the result when $\ell = 2$, we need another fundamental fact that was established by Johnson in [26]. Namely, $H_1(T;\mathbb{F}_2)$ is isomorphic to a space[7] of cubic boolean polynomials on the $\mathbb{F}_2$-valued quadratic forms on $H_1(X;\mathbb{F}_2)$ with respect to its intersection form. This implies that the action of $S(\mathbb{Z})$ on $H_1(T;\mathbb{F}_2)$ factors through $S(\mathbb{Z}) \to S(\mathbb{F}_2)$. Since the 2-torsion $H_1(T)_{(2)}$ of $H_1(T)$ is an $S(\mathbb{Z})$ submodule of $H_1(T;\mathbb{F}_2)$, it is also an $S(\mathbb{F}_2)$-module. To show that the action of $S(\mathbb{Z})$ on $H_1(T) \otimes \mathbb{Z}_2$ factors through $S(\mathbb{Z}) \to S(\mathbb{Z}_2)$, it suffices to show that the image of $S(\mathbb{Z})[2]$ in $\operatorname{Aut} H_1(T) \otimes \mathbb{Z}_2$ is contained in a pro-2 subgroup of $\operatorname{Aut} H_1(T) \otimes \mathbb{Z}_2$. This is because $S(\mathbb{Z}_2)$ is the relative pro-2 completion of $S(\mathbb{Z})$ with respect to $S(\mathbb{Z}) \to S(\mathbb{F}_2)$ by Proposition 2.2 and Example 2.7.

Write $H_1(T)$ as an extension

$$0 \to H_1(T)_{(2)} \to H_1(T) \to V \to 0.$$

Johnson's computation of $H_1(T)$ implies that V is a finitely generated, torsion free $\mathbb{Z}$-module that is an algebraic representation of S. The image of $S(\mathbb{Z})[2]$ in $\operatorname{Aut} H_1(T)$ preserves this sequence and is thus contained in a subgroup G of $\operatorname{Aut} H_1(T) \otimes \mathbb{Z}_2$ that is an extension

$$0 \to \operatorname{Hom}_{\mathbb{Z}}\bigl(V, H_1(T)_{(2)}\bigr) \to G \to S(\mathbb{Z}_2)[2] \to 1,$$

which is a pro-2 group and thus completes the proof. □

3.3. Notational convention

To avoid confusion, we make explicit our convention that the *relative* pro-ℓ completion of the mapping class group $\Gamma_{g,n,\vec{r}}[m]$ (resp. the arithmetic group $\operatorname{Sp}_g(\mathbb{Z})[m]$) with respect to the natural homomorphism to the closure of its image in $\operatorname{Sp}_g(\mathbb{Z}_\ell)$ will be denoted by $\Gamma_{g,n,\vec{r}}[m]^{(\ell)}$ (resp. $\operatorname{Sp}_g(\mathbb{Z})[m]^{(\ell)}$).

4. Relative unipotent completion of discrete groups

Relative unipotent completion is analogous to relative pro-ℓ completion, but often more computable as it is controlled by cohomology. In many situations, including the ones in this paper, the relative pro-ℓ completion of a group Γ maps to its relative unipotent completion. In such cases, relative unipotent completion can be used to give a lower bound for the size of the relative pro-ℓ completion of a discrete group; under very favorable conditions, it can be used to compute the relative pro-ℓ completion.

There are various kinds of relative unipotent completion, in this paper we need only the simplest kind, which is reviewed in this section. More details can be found in [11,15]. Throughout this section, F will denote a field of characteristic zero.

The basic data are:

(1) a discrete group Γ;

[7] The space depends on n and r.

(2) a reductive linear algebraic F-group R;
(3) a Zariski dense representation $\rho : \Gamma \to R(F)$.

The relative unipotent completion of Γ with respect to ρ is a proalgebraic group $\mathcal{G}$ over F which is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1,$$

where $\mathcal{U}$ is a prounipotent F-group, together with a homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}(F)$ that lifts ρ. It is characterized by the following universal mapping property: if G is a linear algebraic F-group that is an extension of R by a unipotent group and if $\tilde{\rho} : \Gamma \to G(F)$ is a homomorphism that lifts ρ, then there is a unique homomorphism $\phi : \mathcal{G} \to G$ of proalgebraic F-groups such that

$$\begin{tikzcd}
 & \mathcal{G}(F) \arrow[dd, "\phi"] \arrow[dr] & \\
\Gamma \arrow[ur, "\hat{\rho}"] \arrow[dr, "\tilde{\rho}"'] & & R(F) \\
 & G(F) \arrow[ur] &
\end{tikzcd}$$

commutes.

We shall denote the relative unipotent completion of Γ with respect to ρ by $\mathcal{G}(\Gamma, \rho)$, or simply $\mathcal{G}(\Gamma)$ when ρ is clear from context.

The relative unipotent completion of a discrete group can be constructed as an inverse limit of Zariski dense representations $\hat{\rho} : \Gamma \to G(F)$, where the F-group G is an extension of R by a unipotent group. Alternatively, $\mathcal{G}(\Gamma)$ can be constructed as the tannakian fundamental group of the category of finite dimensional $F\Gamma$-modules V that admit a filtration

$$0 = I_0 V \subseteq I_1 V \subseteq \cdots \subseteq I_{n-1} V \subseteq I_n V = V$$

by $F\Gamma$-submodules with the property that each graded quotient $I_k V / I_{k-1} V$ has a rational action of R for which the action of Γ on it factorizes through $\rho : \Gamma \to R(F)$. This is a neutral tannakian category $\mathcal{R}(\Gamma, \rho)$ over F. There is a natural isomorphism

$$\pi_1\big(\mathcal{R}(\Gamma, \rho), \omega\big) \cong \mathcal{G}(\Gamma, \rho)$$

where $\omega : \mathcal{R}(\Gamma, \rho) \to \mathsf{Vec}_F$ is the forgetful functor. See [7] for definitions and [17, §7] for discussion of tannakian description of weighted completion.

Example 4.1. If Γ is an arithmetic subgroup of an almost simple $\mathbb{Q}$-group G of real rank $\geqslant 2$, and F is any field of characteristic zero, then the relative unipotent completion of Γ with respect to the inclusion $\Gamma \hookrightarrow G(F)$ is $G_{/F}$, [11, p. 84]. In particular, if Γ is a finite index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$ and $g \geqslant 2$ (resp. of $\mathrm{SL}_n(\mathbb{Z})$ and $n \geqslant 3$), then the relative unipotent completion of Γ with respect to the inclusion $\rho : \Gamma \to \mathrm{Sp}_g(F)$ (resp. $\Gamma \to \mathrm{SL}_n(F)$) is $\mathrm{Sp}_{g/F}$ (resp. $\mathrm{SL}_{n/F}$). When $g = 1$, the completion of Γ has a very large prounipotent radical—it is free of countable rank, [12, Remark 3.9].

4.1. Unipotent completion

When R is the trivial group, relative unipotent completion reduces to *unipotent* (*also called Malcev*) *completion*. We shall denote the unipotent completion of Γ over F by $\Gamma^{\mathrm{un}}_{/F}$.

The unipotent completion of a finitely generated group is always defined over $\mathbb{Q}$. Suppose that Γ is finitely generated. Denote its unipotent completion $\Gamma^{\mathrm{un}}_{/\mathbb{Q}}$ over $\mathbb{Q}$ by Γ^{un}. Since $(\Gamma^{\mathrm{un}} \otimes_{\mathbb{Q}} F)(F) = \Gamma^{\mathrm{un}}(F)$, the homomorphism $\Gamma \to \Gamma^{\mathrm{un}}(F)$ induces a homomorphism $\Gamma^{\mathrm{un}}_{/F} \to \Gamma^{\mathrm{un}} \otimes_{\mathbb{Q}} F$. This homomorphism is an isomorphism. Proofs can be found in [11] and [16].

Denote the nth term of the lower central series of Γ^{un} by $L^n\Gamma^{\mathrm{un}}$. The following "well-known" result follows from [36] and [34], as explained in [21, (2.17)].

Proposition 4.2. *If Γ is a finitely generated group and F is a field of characteristic* 0*, then $D^n\Gamma$ equals the inverse image of $(L^n\Gamma^{\mathrm{un}})(F)$ under the natural homomorphism $\Gamma \to \Gamma^{\mathrm{un}}(F)$. Moreover, the image of $D^n\Gamma$ is Zariski dense in $L^n\Gamma^{\mathrm{un}}$ and there is a natural isomorphism*

$$\big(D^n\Gamma/D^{n+1}\big) \otimes_{\mathbb{Z}} F \cong \big(L^n\Gamma^{\mathrm{un}}\big)(F)/\big(L^{n+1}\Gamma^{\mathrm{un}}\big)(F).$$

4.2. Relative unipotent completion of mapping class groups

Suppose that S is a compact oriented surface of genus $g \geqslant 0$, that r and n are non-negative integers such that $2g-2+n+r>0$, that $\mathbf{x}$ is a finite subset of S of cardinality n, and that $\vec{\mathbf{v}}$ is a finite set of tangent vectors of S of cardinality r as in Section 3.1. The natural homomorphism

$$\rho : \Gamma_{S,\mathbf{x},\vec{\mathbf{v}}} \to \mathrm{Sp}\big(H_1(S;F)\big)$$

is Zariski dense. Denote the relative unipotent completion of $\Gamma_{S,\mathbf{x},\vec{\mathbf{v}}}$ with respect to ρ by $\mathcal{G}_{S,\mathbf{x},\vec{\mathbf{v}}}$. The relative completion of $\Gamma_{g,n,\vec{r}}$ will be denoted by $\mathcal{G}_{g,n,\vec{r}}$.

Note that, when $g=0$ and $n+r \geqslant 3$, $H_1(S)$ is trivial so that relative unipotent completion is unipotent completion: $\mathcal{G}_{0,n,\vec{r}} = \Gamma^{\mathrm{un}}_{0,n,\vec{r}/F}$.

We recall some basic facts about relative unipotent completion of mapping class groups. First, for all $g \geqslant 0$ and $n+r>0$, the natural homomorphism $\Gamma_{g,n,\vec{r}} \to \mathcal{G}_{g,n,\vec{r}}$ is injective.

For a non-negative integer m, the level m subgroup $\Gamma_{g,n,\vec{r}}[m]$ of $\Gamma_{g,n,\vec{r}}$ is defined to be the kernel of the mod m reduction $\rho_m : \Gamma_{g,n,\vec{r}} \to \mathrm{Sp}_g(\mathbb{Z}/m)$ of ρ. Note that $\Gamma_{g,n,\vec{r}} = \Gamma_{g,n,\vec{r}}[1]$ and $T_{g,n,\vec{r}} = \Gamma_{g,n,\vec{r}}[0]$. When $g \geqslant 3$, the relative unipotent completion is independent of the level.

Theorem 4.3. *(See Hain [12].) If $g \geqslant 3$, then for all non-negative integers r, n and all positive integers m, the inclusion $\Gamma_{g,n,\vec{r}}[m] \to \Gamma_{g,n,\vec{r}}$ induces an isomorphism on relative unipotent completions. When $g = 1, 2$, the relative completion of $\Gamma_{g,n,\vec{r}}[m]$ depends non-trivially on the level $m > 0$.*

Right exactness implies that the sequence

$$T^{\mathrm{un}}_{g,n,\vec{r}} \to \mathcal{G}_{g,n,\vec{r}} \to \mathrm{Sp}_g \to 1$$

is exact.

Theorem 4.4. *(See Hain [11,12].) If $g \geqslant 3$, then the kernel of $T^{\mathrm{un}}_{g,n,\vec{r}} \to \mathcal{G}_{g,n,\vec{r}}$ is isomorphic to the additive group $\mathbb{G}_a$. That is, there is an exact sequence*

$$0 \to \mathbb{G}_a \to T^{\mathrm{un}}_{g,n,\vec{r}} \to \mathcal{G}_{g,n,\vec{r}} \to \mathrm{Sp}_g \to 1.$$

5. Unipotent completion versus pro-ℓ completion

In this section we show that when Γ is a finitely generated ℓ-regular group, the natural homomorphism $\Gamma^{(\ell)} \to \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell)$ is injective, a fact we shall need in the proof of Theorem 1.

Lemma 5.1. *(See [16, Lemma A.7].) If Γ is a finitely generated discrete group and ℓ a prime number, then the natural homomorphism $\Gamma \to \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell)$ induces a homomorphism $\Gamma^{(\ell)} \to \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell)$.*

Next we give conditions under which this homomorphism is injective. In these cases, the unipotent completion of a group Γ over $\mathbb{Q}_\ell$ can be used to "compute" pro-ℓ completion of Γ.

As in previous sections, the lower central series of any group (discrete, profinite, proalgebraic) G will be denoted

$$G = L^1 G \supseteq L^2 G \supseteq L^3 G \supseteq \cdots.$$

Proposition 5.2. *If Γ is a finitely generated ℓ-regular group, then the natural homomorphism $\Gamma^{(\ell)} \to \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell)$ is injective.*

Proof. The unipotent completion of any group is the inverse limit of the unipotent completions of its torsion free nilpotent quotients

$$\Gamma^{\mathrm{un}} = \varprojlim_n \big(\Gamma/D^n\big)^{\mathrm{un}}.$$

By Proposition 2.10, the ℓ-regularity of Γ implies that

$$\Gamma^{(\ell)} \cong \varprojlim_n \big(\Gamma/D^n\big)^{(\ell)}.$$

It therefore suffices to prove the result when Γ is a finitely generated, torsion free nilpotent group. This is easily proved by induction on the length of the filtration $D^\bullet$ of Γ. If $D^2\Gamma$ is trivial, then Γ is a finitely generated abelian group, and thus isomorphic to $\mathbb{Z}^N$ for some N. In this case

$$\Gamma^{(\ell)} \cong \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \quad \text{and} \quad \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell) \cong \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

So the result follows. If D^{n+1} is trivial, then we can write Γ as an extension

$$1 \to D^n\Gamma \to \Gamma \to \Gamma' \to 1$$

where $\Gamma' = \Gamma/D^n$. Both Γ' and $D^n\Gamma$ are finitely generated, torsion free, nilpotent groups. By induction, we may assume that the result is true for Γ' and $D^n\Gamma$. To complete the proof, consider the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & D^n\Gamma \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \Gamma' \arrow[r] \arrow[d] & 1 \\
 & (D^n\Gamma)^{(\ell)} \arrow[r] \arrow[d, "j_n"'] & \Gamma^{(\ell)} \arrow[r] \arrow[d, "j"'] & (\Gamma')^{(\ell)} \arrow[r] \arrow[d, "j'"'] & 1 \\
1 \arrow[r] & (D^n\Gamma)^{\mathrm{un}}(\mathbb{Q}_\ell) \arrow[r, "i"] & \Gamma^{\mathrm{un}}(\mathbb{Q}_\ell) \arrow[r] & (\Gamma')^{\mathrm{un}}(\mathbb{Q}_\ell) \arrow[r] & 1
\end{tikzcd}$$

The third row is exact by Proposition 4.2; the second is exact by the right exactness of pro-ℓ completion. The maps j_n and j' are injective by induction. Since i is injective, j is as well. □

6. Proof of Theorem 1

Fix integers $g \geqslant 3$, $n \geqslant 0$, $r \geqslant 0$ and $m \geqslant 1$. Set $\Gamma = \Gamma_{g,n,\vec{r}}$, $\Gamma[m] = \Gamma_{g,n,\vec{r}}[m]$ and $T = T_{g,n,\vec{r}}$. Denote the $\mathbb{Z}$-group scheme Sp_g by S.

Fix a prime number ℓ. Denote the pro-ℓ completion of T by $T^{(\ell)}$ and the relative pro-ℓ completion of $\Gamma[m]$ with respect to the standard homomorphism to $S(\mathbb{Z}_\ell)[m]$ by $\Gamma[m]^{(\ell)}$. By Proposition 2.2, the relative pro-ℓ completion of $S(\mathbb{Z})[m]$ is $S(\mathbb{Z}_\ell)[m]$.

6.1. An upper bound

In this section, we prove that if T is ℓ-regular, then the kernel of $T^{(\ell)} \to \Gamma^{(\ell)}$ is either 0 or isomorphic to $\mathbb{Z}_\ell$.

The right exactness of relative pro-ℓ completion implies that the sequence

$$T^{(\ell)} \to \Gamma[m]^{(\ell)} \to S(\mathbb{Z}_\ell)[m] \to 1$$

is exact. Denote the kernel of $T^{(\ell)} \to \Gamma[m]^{(\ell)}$ by $K_{\ell,m}$. When T is ℓ-regular, we can give a tight upper bound on the size of $K_{\ell,m}$.

Lemma 6.1. *The kernel $K_{\ell,m}$ of $T^{(\ell)} \to \Gamma[m]^{(\ell)}$ is an abelian pro-ℓ group contained in the center of $T^{(\ell)}$. It has a natural $S(\mathbb{Z}_\ell)[m]$ action.*

Proof. By Proposition 3.4, the conjugation action $\Gamma \to \mathrm{Aut}\,T$ induces a continuous action $\Gamma[m]^{(\ell)} \to \mathrm{Aut}\,T^{(\ell)}$. The kernel $K_{\ell,m}$ of $T^{(\ell)} \to \Gamma[m]^{(\ell)}$ is thus a $\Gamma[m]^{(\ell)}$-module that is contained in the center of $T^{(\ell)}$. Since $T^{(\ell)}$ acts trivially on $K_{\ell,m}$, the $\Gamma[m]$-action on $K_{\ell,m}$ descends to a $S(\mathbb{Z}_\ell)[m]$-action. □

We suspect, but cannot prove, that the $S(\mathbb{Z})[m]$-action on $K_{\ell,m}$ is trivial. When T is ℓ-regular, we can prove this and give a strong upper bound on the size of $K_{\ell,m}$.

Proposition 6.2. *If T is ℓ-regular, then the kernel $K_{\ell,m}$ of $T^{(\ell)} \to \Gamma[m]^{(\ell)}$ is a compact subgroup of $\mathbb{Q}_\ell$ and is therefore either 0 or isomorphic to $\mathbb{Z}_\ell$. The $S(\mathbb{Z})[m]$-action on $K_{\ell,m}$ is trivial.*

Proof. Since T is ℓ-regular, $j : T^{(\ell)} \to T^{\mathrm{un}}(\mathbb{Q}_\ell)$ is injective. The diagram

$$\begin{tikzcd}
 & & T^{(\ell)} \arrow[r] \arrow[d, "j"'] & \Gamma[m]^{(\ell)} \arrow[r] \arrow[d] & S(\mathbb{Z}_\ell)[m] \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbb{Q}_\ell \arrow[r] & T^{\mathrm{un}}(\mathbb{Q}_\ell) \arrow[r] & \mathcal{G}(\mathbb{Q}_\ell) \arrow[r] & S(\mathbb{Q}_\ell) \arrow[r] & 1
\end{tikzcd}$$

commutes and has exact rows by Theorem 4.4 and the right exactness of relative pro-ℓ completion. Consequently, the kernel of $T^{(\ell)} \to \Gamma[m]^{(\ell)}$ is contained in the intersection of $\operatorname{im} j$ with $\mathbb{Q}_\ell$, which is compact as $T^{(\ell)}$ is profinite. The first assertion follows as the compact subgroups of $\mathbb{Q}_\ell$ are 0 and $\ell^n\mathbb{Z}_\ell$ where $n \in \mathbb{Z}$. The triviality of the $S(\mathbb{Z})[m]$-action on $K_{\ell,m}$ follows from the fact that the inclusion $\Gamma[m] \to \Gamma$ induces an isomorphism on relative unipotent completions [12, Proposition 3.3], and because the kernel of $T^{\mathrm{un}} \to \mathcal{G}$ is a trivial $\mathcal{G}$-module with respect to the action of $\mathcal{G}$ on T^{un} that is induced by the conjugation action of Γ on T. $\square$

All remaining sections of Section 6 are devoted to proving that, regardless of whether T is ℓ-regular or not, the kernel of $T^{(\ell)} \to \Gamma^{(\ell)}$ contains copy of $\mathbb{Z}_\ell$.

6.2. The groups G and $\hat{G}$.

The argument to prove the non-triviality of the kernel of $T^{(\ell)} \to \Gamma^{(\ell)}$ is somewhat intricate and uses the structure of some natural quotients of Γ.

Consider the filtration

$$\Gamma = D^0\Gamma \supseteq D^1\Gamma \supseteq D^2\Gamma \supseteq \cdots$$

of Γ defined, when $k \geqslant 1$, by $D^k\Gamma := D^kT$. Since T is finitely generated [25], the graded quotients D^1/D^{k+1} are finitely generated, torsion free nilpotent groups for all $k \geqslant 1$. Each graded quotient is an $S(\mathbb{Z})$-module.

Set

$$V = D^1\Gamma/D^2\Gamma = H_1(T;\mathbb{Z})/\text{torsion}.$$

Then for all positive integers m we have an extension

$$1 \to V \to \Gamma[m]/D^2 \to S(\mathbb{Z})[m] \to 1. \tag{2}$$

Denote the relative pro-ℓ completion of $\Gamma[m]/D^2$ with respect to the natural homomorphism to $S(\mathbb{Z}_\ell)[m]$ by $(\Gamma[m]/D^2)^{(\ell)}$. Set $V_\ell = V \otimes \mathbb{Z}_\ell$.

Lemma 6.3. *For all positive integers m, there is an injection*

$$\phi_m : \Gamma[m]/D^2 \to S(\mathbb{Z})[m] \ltimes V$$

that commutes with the projections to $S(\mathbb{Z})[m]$ and whose restriction to V is multiplication by 2. Its image has finite index.

Proof. It suffices to prove the case $m = 1$. The extension (2) is determined by a class $e \in H^2(S(\mathbb{Z}), V)$. Johnson's theorem implies (cf. [12]) that V is isomorphic to the sum of $\Lambda^3 H/H$ and $n + r$ copies of H, where H denotes the defining representation of Sp_g. Since $-I \in S(\mathbb{Z})$ acts as -1 on V and is central in $S(\mathbb{Z})$, it follows from the "center kills" argument that $H^2(S(\mathbb{Z}), V)$ is annihilated by 2. In particular, $2e = 0$. Consequently, the extension of $S(\mathbb{Z})$ by V obtained by pushing out the extension (2) along the multiplication by 2 map $\times 2 : V \to V$ splits. This implies that there is a map of extensions

$$\begin{tikzcd} 1 \arrow[r] & V \arrow[r] \arrow[d, "\times 2"] & \Gamma/D^2 \arrow[r] \arrow[d, "\phi"] & S(\mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & V \arrow[r] & S(\mathbb{Z}) \ltimes V \arrow[r] & S(\mathbb{Z}) \arrow[r] & 1. \end{tikzcd} \tag{3}$$

The result follows. □

Corollary 6.4. *If ℓ is odd, the relative pro-ℓ completion of $\Gamma[m]/D^2$ is isomorphic to $S(\mathbb{Z}_\ell)[m] \ltimes V_\ell$. When $\ell = 2$, $(\Gamma[m]/D^2)^{(2)}$ has finite index in $S(\mathbb{Z}_2)[m] \ltimes V_2$.*

Proof. We first show that the relative pro-ℓ-completion of $S(\mathbb{Z})[m] \ltimes V$ is $S(\mathbb{Z}_\ell)[m] \ltimes V_\ell$. The relative pro-$\ell$ completion of $S(\mathbb{Z})[m]$ is $S(\mathbb{Z}_\ell)[m]$ and the pro-ℓ completion of V is V_ℓ. The obvious homomorphism

$$\phi : S\big(\mathbb{Z}[m]\big) \ltimes V \to S(\mathbb{Z}_\ell)[m] \ltimes V_\ell$$

is continuous. It therefore induces a homomorphism ϕ_ℓ such that the diagram

$$\begin{tikzcd} & V_\ell \arrow[r] \arrow[d, equal] & \big(S(\mathbb{Z}) \ltimes V\big)^{(\ell)} \arrow[r] \arrow[d, "\phi_\ell"'] & S(\mathbb{Z}_\ell)[m] \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & V_\ell \arrow[r] & S(\mathbb{Z}_\ell)[m] \ltimes V_\ell \arrow[r] & S(\mathbb{Z}_\ell)[m] \arrow[r] & 1 \end{tikzcd}$$

commutes and has exact rows. It follows that ϕ_ℓ is an isomorphism.

When ℓ is odd, 2 is a unit in $\mathbb{Z}_\ell$. The result now follows from right exactness of relative pro-ℓ completion by taking the relative pro-ℓ completion of the diagram

$$\begin{tikzcd} 1 \arrow[r] & V \arrow[r] \arrow[d, "\times 2"] & \Gamma[m]/D^2 \arrow[r] \arrow[d, "\phi"] & S(\mathbb{Z})[m] \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & V \arrow[r] & S(\mathbb{Z})[m] \ltimes V \arrow[r] & S(\mathbb{Z})[m] \arrow[r] & 1 \end{tikzcd}$$

which is the restriction of the diagram (3) to the level m subgroups. □

The next step is to enlarge the quotient $\Gamma[m]/D^2$ of $\Gamma[m]$ by a central extension. Since $V \otimes \mathbb{Q}$ is a rational representation of S and since $(\mathrm{Gr}_D^\bullet T) \otimes \mathbb{Q}$ is a graded Lie algebra in the category of $S(\mathbb{Z})$-modules that is generated by $(\mathrm{Gr}_D^1 T) \otimes \mathbb{Q} \cong V \otimes \mathbb{Q}$, each $(D^k T/D^{k+1}) \otimes \mathbb{Q}$ is a rational representation of the algebraic group $S \otimes \mathbb{Q}$. By [12, Theorem 10.1], when $r = n = 0$,

$$\dim\bigl[\bigl(D^2 T_g/D^3 T_g\bigr) \otimes \mathbb{Q}\bigr]^{S(\mathbb{Z})} = 1.$$

Let $\alpha_0 : D^2 T_g/D^3 T_g \to \mathbb{Q}$ be an $S(\mathbb{Z})$-invariant projection. Since the action of $S(\mathbb{Z})$ is semi-simple, this projection is unique up to scalar multiplication. Define $\alpha_{\mathbb{Q}} : D^2 T/D^3 \to \mathbb{Q}$ be the composite

$$D^2 T/D^3 \to D^2 T_g/D^3 T_g \xrightarrow{\alpha_0} \mathbb{Q}.$$

Since $D^2 T/D^3$ is finitely generated, the image of $\alpha_{\mathbb{Q}}$ is isomorphic to $\mathbb{Z}$. Let $\alpha : D^2 T/D^3 \to \mathbb{Z}$ be the unique (up to sign) surjection whose composition with $\mathbb{Z} \to \mathbb{Q}$ is $\alpha_{\mathbb{Q}}$.

6.2.1. Construction of G

This is a key construction in the proof. Define G to be the group obtained by pushing out the extension

$$0 \to D^2 T/D^3 \to \Gamma/D^3 \to \Gamma/D^2 \to 1$$

along the surjection $\alpha : D^2 T/D^3 \to \mathbb{Z}$. It is an extension of Γ/D^2 by $\mathbb{Z}$. It can also be written as an extension

$$1 \to E \to G \to S(\mathbb{Z}) \to 1,$$

where the kernel E is a central extension of V by $\mathbb{Z}$, whose structure we compute below.

6.2.2. Heisenberg groups

Suppose that W and A are abelian groups, and that $b : W \otimes_{\mathbb{Z}} W \to A$ is a bilinear pairing. The Heisenberg group $\mathrm{Heis}_b(W, A)$ is the group whose underlying set is $W \times A$ and whose multiplication is

$$(w, a) \cdot (w', a') = \bigl(w + w', a + a' + b(w, w')\bigr).$$

This is a central extension

$$0 \to A \to \mathrm{Heis}_b(W, A) \to W \to 0 \tag{4}$$

of W by A. The commutator induces the skew symmetric bilinear pairing

$$c : W \otimes_{\mathbb{Z}} W \to A$$

defined by $c(w, w') = b(w, w') - b(w', w)$.[8] The abelianization of $\mathrm{Heis}_b(W, A)$ is thus an extension

$$0 \to A/\operatorname{im} c \to H_1\big(\mathrm{Heis}_b(W, A)\big) \to W \to 0. \tag{5}$$

When W is a free abelian group of finite rank, c can be regarded as an element of $H^2(W; A)$. It is the class of the extension (4) and determines it up to isomorphism of extensions. Thus, when W is torsion free of finite rank, every central extension of W by A is of the form $\mathrm{Heis}_b(W, A)$ for some bilinear pairing $b : W \otimes W \to A$.

Example 6.5. The classical integral Heisenberg group is the group $\mathrm{Heis}_b(\mathbb{Z}^2, \mathbb{Z})$, where $b((x, y), (x', y')) = xy'$. It is isomorphic to the group

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

The class of the extension is the class of the skew symmetrization $((x, y), (x', y')) = xy' - x'y$ of b, which is the generator of $H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$.

Lemma 6.6. *If $b : W \times W \to \mathbb{Z}$ is a bilinear pairing on a finitely generated, torsion free abelian group, then the natural homomorphism*

$$\mathrm{Heis}_b(W, \mathbb{Z}) \to \mathrm{Heis}_b(W \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell)$$

is the pro-ℓ completion of $\mathrm{Heis}_b(W, \mathbb{Z})$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{Z}_\ell \arrow[r] \arrow[d, equal] & \mathrm{Heis}_b(W, \mathbb{Z})^{(\ell)} \arrow[r] \arrow[d] & W \otimes \mathbb{Z}_\ell \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z}_\ell \arrow[r] & \mathrm{Heis}_b(W \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell) \arrow[r] & W \otimes \mathbb{Z}_\ell \arrow[r] & 1
\end{tikzcd}$$

Right exactness of pro-ℓ completion implies that the first row is exact. Since the first and third vertical maps are isomorphisms, the central map is also an isomorphism. □

Since V is a finitely generated, torsion free group, the kernel E of the homomorphism $G \to S(\mathbb{Z})$ is a Heisenberg group. The commutator of E induces a skew symmetric bilinear pairing $V \times V \to \mathbb{Z}$ which is $S(\mathbb{Z})$-invariant because it is invariant under the conjugation action of the mapping class group on T. The commutator pairing can be written as dq, where d is a positive integer and $q : V \times V \to \mathbb{Z}$ is a primitive pairing. (That is, it cannot be divided by any

[8] Note that if b is skew symmetric, then $c = 2b$.

integer > 1.) In other words, the class of the extension is $dq \in H^2(V, \mathbb{Z})$. It follows from the sequence (5) that the abelianization of E is an extension

$$0 \to \mathbb{Z}/d \to H_1(E) \to V \to 0.$$

Proposition 6.7. *The integer d divides 2 and E is a subgroup of* $\mathrm{Heis}_q(V, \mathbb{Z})$ *of index* $2/d$.

Proof. Since $T \to E$ is surjective, $H_1(T) \to H_1(E)$ is surjective. The definition of the rational dimension subgroups implies that D^2T surjects onto the torsion subgroup of $H_1(T)$. It follows that the torsion subgroup of $H_1(T)$ surjects onto $\mathbb{Z}/d$. Since the torsion subgroup of $H_1(T)$ has exponent 2 [26], it follows that $d = 1$ or 2. Standard facts about group cohomology imply that if $d = 2$, then $E \cong \mathrm{Heis}_q(V, \mathbb{Z})$. If $d = 1$, then E has index 2 in $\mathrm{Heis}_q(V, \mathbb{Z})$. □

Since 2 is a unit in $\mathbb{Z}_\ell$ when ℓ is odd, Lemma 6.6 and the previous result give the following computation of $E^{(\ell)}$.

Corollary 6.8. *If ℓ is an odd prime number, then the pro-ℓ completion of E is isomorphic to* $\mathrm{Heis}_q(V \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell)$. *If $\ell = 2$, then either $E^{(2)}$ is isomorphic to* $\mathrm{Heis}_q(V \otimes \mathbb{Z}_2, \mathbb{Z}_2)$ *or has index* 2 *in it. In either case, the completion $\mathbb{Z}_\ell \to E^{(\ell)}$ of the inclusion $\mathbb{Z} \hookrightarrow E$ of the central $\mathbb{Z}$ into E is injective.*

6.2.3. Construction of $\hat{G}$

It will be necessary to compute the relative pro-ℓ completion of G up to a finite group. In order to do this, we will construct another group $\hat{G}$ that contains G as a finite index subgroup.

Proposition 6.9. *There is a group $\hat{G}$ that is an extension*

$$0 \to \mathbb{Z} \to \hat{G} \to S(\mathbb{Z}) \ltimes V \to 1 \tag{6}$$

that contains G as a finite index subgroup. More precisely, there is a diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\times 8/d"] & G \arrow[r] \arrow[d] & \Gamma/D^2 \arrow[r] \arrow[d, "\phi_1"] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \hat{G} \arrow[r] & S(\mathbb{Z}) \ltimes V \arrow[r] & 1
\end{tikzcd}$$

where $d \in \{1, 2\}$ is the integer defined before Proposition 6.7 *and ϕ_1 is the homomorphism constructed in Lemma* 6.3. *The extension of $S(\mathbb{Z})$ by $\mathbb{Z}$ obtained by restricting the extension* (6) *to $S(\mathbb{Z})$ is $-(8g + 4)$ times the universal central extension $\widetilde{S(\mathbb{Z})}$ of $S(\mathbb{Z})$.*

It will follow from the proof of Corollary 6.10 that d is, in fact, 2.

Proof. The proof uses the geometry of moduli spaces of curves. We first review the construction of the biextension bundle; a detailed exposition can be found in [20]. In this proof, all moduli spaces are defined over $\mathbb{C}$ and are viewed as analytic orbifolds. The moduli space $\mathcal{A}_g$ (viewed

as an orbifold) is a model of the classifying space $BS(\mathbb{Z})$ of $S(\mathbb{Z})$. The bundle of intermediate jacobians $\mathcal{J}$ over $\mathcal{A}_g$ associated to the representation

$$V_0 := \Lambda^3 H/H \cong T_g/D^2 T_g$$

of $S(\mathbb{Z})$, where H denotes the defining representation of S, is a model of the classifying space $B(S(\mathbb{Z}) \ltimes V_0)$. A splitting of the extension is given by the zero section of $\mathcal{J}(V_0) \to \mathcal{A}_g$. There is a natural line bundle $\mathcal{B}_0 \to \mathcal{J}(V_0)$ whose associated $\mathbb{C}^*$-bundle $\mathcal{B}_0^*$ is a model of $B(S(\mathbb{Z}) \ltimes \mathrm{Heis}_q(V_0, \mathbb{Z}))$, where q is the S-invariant bilinear pairing $V \times V \to \mathbb{Z}$ defined before Proposition 6.7. The restriction of $\mathcal{B}_0$ to the zero section $\mathcal{A}_g$ is trivial. A detailed exposition of the construction of $\mathcal{B}_0$ is given in [20].

The period mapping $\mathcal{M}_g \to \mathcal{A}_g$ lifts to a holomorphic mapping $\nu_0 : \mathcal{M}_g \to \mathcal{J}(V_0)$, which is the normal function that takes the moduli point of the curve C to the point in the primitive intermediate jacobian of $\operatorname{Jac} C$ that corresponds to the algebraic cycle $C - C^-$ in $\operatorname{Jac} C$:

$$\begin{tikzcd} & \mathcal{J}(V_0) \arrow[d] \\ \mathcal{M}_g \arrow[ur, "\nu_0"] \arrow[r] & \mathcal{A}_g \end{tikzcd}$$

However, it does not lift to a section of $\mathcal{B}_0^* \to \mathcal{A}_g$. The obstruction is the first Chern class of $\nu_0^*\mathcal{B}_0$, which is $(8g+4)\lambda \in H^2(\mathcal{M}_g; \mathbb{Z}) \cong H^2(\mathcal{A}_g; \mathbb{Z})$, where λ is the first Chern class of the line bundle $\mathcal{L}$ over $\mathcal{A}_g$ corresponding to the universal central extension of $S(\mathbb{Z})$, pulled back to $\mathcal{M}_g$. This Chern class computation follows from results of Morita [32, (5.8)] and is proved directly in [19].

The Chern class computation implies that the pullback of $\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))}$ to $\mathcal{M}_g$ along ν_0 is trivial. This implies that $\nu_0 : \mathcal{M}_g \to \mathcal{J}(V_0)$ lifts to a section $\tilde{\nu}$ of the $\mathbb{C}^*$-bundle $(\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))})^*$:

$$\begin{tikzcd} & & \mathcal{M}_g \arrow[dll, "\tilde{\nu}"'] \arrow[dl, "\nu_0"] \arrow[d] \\ \left(\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))}\right)^* \arrow[r] & \mathcal{J}(V_0) \arrow[r] & \mathcal{A}_g \end{tikzcd}$$

At this stage it is useful to reinterpret this statement group theoretically. Set $\hat{G}_0 = \pi_1((\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))})^*, *)$. This is an extension

$$0 \to \mathbb{Z} \to \hat{G}_0 \to S(\mathbb{Z}) \ltimes V_0 \to 1$$

that is also an extension of $S(\mathbb{Z})$ by the Heisenberg group $\mathrm{Heis}_q(V_0, \mathbb{Z})$. Its restriction to $S(\mathbb{Z})$ (fundamental group of the zero section of $\mathcal{J}(V_0)$) is $\mathcal{L}^{\otimes(-(8g+4))}$, which has class $-(8g+4)\lambda \in H^2(S(\mathbb{Z}); \mathbb{Z})$. The lift $\tilde{\nu}$ of the normal function ν_0 induces a homomorphism $\Gamma_g \to \hat{G}_0$ whose image contains the center $\mathbb{Z}$; the induced homomorphism $T_g \to V_0$ is twice the Johnson homomorphism and has image $2V_0$.

This induces a homomorphism $\psi : E \to \mathrm{Heis}_q(V_0, \mathbb{Z})$. Noting that the commutator pairing of $\mathrm{Heis}_q(V_0, \mathbb{Z})$ is $2q$, we see that ψ must be multiplication by $8/d$ on the central $\mathbb{Z}$'s:

$$\begin{tikzcd} \Lambda^2 V_0 \arrow[r, "\times 4"] \arrow[d, "dq"'] & \Lambda^2 V_0 \arrow[d, "2q"'] \\ \mathbb{Z} \arrow[r, "\times 8/d"] & \mathbb{Z} \end{tikzcd}$$

Here the first column is the commutator pairing of E and the second is the commutator pairing of $\mathrm{Heis}_q(V_0, \mathbb{Z})$. This completes the proof of the result when $r = n = 0$ by taking $\hat{G} = \hat{G}_0$.

By replacing $\Gamma_{g,n,\vec{r}}$ by $\Gamma_{g,n+r}$, we may assume that $r = 0$. When $n = 1$, $V = \Lambda^3 H$. There is a family of intermediate jacobians $\mathcal{J}(\Lambda^3 H)$ over $\mathcal{A}_g$, which also fibers over $\mathcal{J}(V_0)$:

$$\mathcal{J}\big(\Lambda^3 H\big) \to \mathcal{J}(V_0) \to \mathcal{A}_g.$$

This has fundamental group $S(\mathbb{Z}) \ltimes \Lambda^3 H$ and the restriction of ν_1 to $T_{g,1}$ induces twice the Johnson homomorphism $T_{g,1} \to \Lambda^3 H$. The normal function lifts to a normal function $\nu_1 : \mathcal{M}_{g,1} \to \mathcal{J}(\Lambda^3 H)$ that induces the homomorphism $\Gamma_{g,1} \to S(\mathbb{Z}) \ltimes \Lambda^3 H$ of Lemma 6.3. It is the normal function that takes the moduli point of the pointed curve (C, x) to the point in the intermediate jacobian of $H_3(\mathrm{Jac}\, C)$ determined by the algebraic cycle $C_x - C_x^-$ in $\mathrm{Jac}\, C$.

This generalizes to the case $n > 1$ as follows: Denote the projection $\Lambda^3 H \to V_0$ by p. Then, by [14, Corollary 3],

$$V = \big\{(u_1, \ldots, u_n) \in \big(\Lambda^3 H\big)^n\colon\ p(u_1) = \cdots = p(u_n)\big\}.$$

Define $J(V)$ to be fibered product of n copies of $\mathcal{J}(\Lambda^3 H)$ over $\mathcal{J}(V_0)$. It is a model of the classifying space of $S(\mathbb{Z}) \ltimes V$. Each of the n points determines a normal function ν_j: $\mathcal{M}_{g,n} \to \mathcal{J}(\Lambda^3 H)$. These all project to the normal function

$$\mathcal{M}_{g,n} \to \mathcal{M}_g \xrightarrow{\nu_0} \mathcal{J}(V_0).$$

They therefore define a section $\nu_n : \mathcal{M}_{g,n} \to \mathcal{J}(V)$. This induces the homomorphism $\Gamma_{g,n} \to S(\mathbb{Z}) \ltimes V$ of Lemma 6.3.

Denote the projection $\mathcal{J}(V) \to \mathcal{J}(V_0)$ by π. Then one has a diagram

$$\begin{tikzcd} & \pi^*\big(\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))}\big)^* \arrow[r] \arrow[d] & \big(\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))}\big)^* \arrow[d] \\ & \mathcal{J}(V) \arrow[r, "\pi"] \arrow[d] & \mathcal{J}(V_0) \arrow[d] \\ \mathcal{M}_{g,n} \arrow[r] \arrow[uur, "\tilde{\nu}_n"] \arrow[ur, "\nu_n"'] & \mathcal{A}_g \arrow[r, equal] & \mathcal{A}_g \end{tikzcd}$$

The group $\hat{G}$ is defined to be

$$\hat{G} = \pi_1\big(\pi^*\big(\mathcal{B}_0 \otimes \mathcal{L}^{\otimes(-(8g+4))}\big)^*, *\big).$$

The homomorphism $\Gamma_{g,n} \to \hat{G}$ is induced by $\tilde{\nu}_n$. Its image is G. □

Denote the inverse image of $S(\mathbb{Z})[m]$ in G by $G[m]$ and by $\hat{G}[m]$ the inverse image of $S(\mathbb{Z})[m]$ under the natural projection $\hat{G} \to S(\mathbb{Z})$. Then $G[m]$ has finite index in $\hat{G}[m]$.

Corollary 6.10. *For all positive integers m, there is a subgroup $\widetilde{S}$ of $G[m]$ whose image S' under the quotient mapping $G[m] \to S(\mathbb{Z})[m]$ has finite index and which is, up to 4-torsion, $-(2g+1)$ times the restriction of the universal central extension of $S(\mathbb{Z})$ to S'. Moreover, $d = 2$ and E is isomorphic to* $\mathrm{Heis}_q(V, \mathbb{Z})$.

Proof. Denote by $\widehat{S(\mathbb{Z})}$ the restriction of the extension (6) to $S(\mathbb{Z})$. This is $-(8g+4)$ times the universal central extension of $S(\mathbb{Z})$. Let $\widetilde{S}$ be the intersection of $G[m]$ with $\widehat{S(\mathbb{Z})}$ in $\hat{G}$. The statement about the Chern class of the extension follows as $\widetilde{S}$ contains the center of $G[m]$, which has index $8/d$ in the center of $\widehat{S(\mathbb{Z})}$. Its Chern class is thus, up to $8/d$-torsion, $d/8$ times the restriction of the Chern class $-(8g+4)\lambda$ of the extension $\widehat{S(\mathbb{Z})}$ of $S(\mathbb{Z})$. Since λ generates $H^2(\mathrm{Sp}_g(\mathbb{Z}), \mathbb{Z})$, this forces $d = 2$. The remaining assertion follows from Proposition 6.7. □

Recall that $G[m]$ is an extension

$$1 \to E \to G[m] \to S(\mathbb{Z})[m] \to 1.$$

Corollary 6.11. *For all prime numbers ℓ, the kernel of the homomorphism*

$$E^{(\ell)} \to G[m]^{(\ell)}$$

is isomorphic to $\mathbb{Z}_\ell$.

Proof. Apply relative pro-ℓ completion to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \widetilde{S} \arrow[r] \arrow[d] & S' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & E \arrow[r, "\gamma"] & G[m] \arrow[r] & S(\mathbb{Z})[m] \arrow[r] & 1
\end{tikzcd}$$

to obtain the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_\ell \arrow[r, "\alpha^{(\ell)}"] \arrow[d, "\beta^{(\ell)}"'] & \widetilde{S}^{(\ell)} \arrow[r] \arrow[d] & S'^{(\ell)} \arrow[r] \arrow[d] & 1 \\
E^{(\ell)} \arrow[r, "\gamma^{(\ell)}"] & G[m]^{(\ell)} \arrow[r] & S(\mathbb{Z})[m]^{(\ell)} \arrow[r] & 1
\end{tikzcd}$$

whose rows are exact. Corollary 6.8 implies that $\beta^{(\ell)}$ is injective. Since $\widetilde{S}$ has index 4 in $\widehat{S(\mathbb{Z})}$, Proposition 2.5 implies that the image of $\alpha^{(\ell)}$ is finite, so that $\ker\alpha^{(\ell)} \cong \mathbb{Z}_\ell$. The commutativity of the diagram implies that $\beta^{(\ell)}(\ker\alpha^{(\ell)})$ is contained in $\ker\gamma^{(\ell)}$. Since $\ker\gamma^{(\ell)}$ is contained in $\ker\{E^{(\ell)} \to V^{(\ell)}\}$, which is isomorphic to $\mathbb{Z}_\ell$, this gives the result. □

6.3. *The kernel of $T^{(\ell)} \to \Gamma^{(\ell)}$ contains $\mathbb{Z}_\ell$*

We have the exact sequence

$$1 \to K \to T^{(\ell)} \to \Gamma[m]^{(\ell)} \to S(\mathbb{Z}_\ell)[m] \to 1$$

where $K := K_{\ell,m}$. Denote the kernel of the mapping $\Gamma \to G$ by C. This is a subgroup of T. Observe that T/C is isomorphic to $\mathrm{Heis}_q(V,\mathbb{Z})$ and that $\Gamma[m]/C$ is isomorphic to $G[m]$.

Denote the closure of the image of C in $T^{(\ell)}$ (resp., $\Gamma[m]^{(\ell)}$) by $\overline{C}_T$ (resp., $\overline{C}_\Gamma$). Since C is normal in $\Gamma[m]$, $\overline{C}_\Gamma$ is normal in $\Gamma[m]^{(\ell)}$. Right exactness of relative pro-ℓ completion implies that $G[m]^{(\ell)} \cong \Gamma[m]^{(\ell)}/\overline{C}_\Gamma$ and $(T/C)^{(\ell)} \cong T^{(\ell)}/\overline{C}_T$. It also implies that the sequence

$$1 \to K/(K \cap \overline{C}_T) \to (T/C)^{(\ell)} \to G[m]^{(\ell)} \to S(\mathbb{Z}_\ell)[m] \to 1$$

is exact. Corollary 6.11 implies that $K/(K \cap \overline{C}_T) \cong \mathbb{Z}_\ell$. It follows that K contains a copy of $\mathbb{Z}_\ell$.

6.4. *Genus* 2

When $g = 2$, the homomorphism $T^{(\ell)} \to \Gamma^{(\ell)}[m]$ has a large kernel. For simplicity, we restrict ourselves to the case $n = r = 0$. Non-injectivity in the case $r + n > 0$ follows by a similar argument. Details are left to the reader.

Set

$$\overline{T} := \mathrm{im}\{T_2^{(\ell)} \to \Gamma_2^{(\ell)}[m]\}.$$

Observe that its abelianization, $H_1(\overline{T})$ is a pro-ℓ group with a continuous $\mathrm{Sp}_2(\mathbb{Z}_\ell)[m]$ action.

Mess' computations [31] imply that, as an $\mathrm{Sp}_2(\mathbb{Z})$-module,

$$H_1(T_2) \cong \mathrm{Ind}_{C_2 \ltimes \mathrm{SL}_2(\mathbb{Z})^2}^{\mathrm{Sp}_2(\mathbb{Z})} \mathbb{Z}.$$

Here the group $C_2 \ltimes \mathrm{SL}_2(\mathbb{Z})^2$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Z})$ of the decomposition of the integral homology of an abelian surface which is the product of two elliptic curves. The cyclic group C_2 interchanges the two factors, and the two copies of $\mathrm{SL}_2(\mathbb{Z})$ act on the homology of the two elliptic factors.

It is easy to produce ℓ-adic quotients of $H_1(T_2)$ that cannot be quotients of $H_1(\overline{T})$. For example, choose a prime number p that does not divide $m\ell$. Let

$$M = \mathrm{Ind}_{C_2 \ltimes \mathrm{SL}_2(\mathbb{F}_p)^2}^{\mathrm{Sp}_2(\mathbb{F}_p)} \mathbb{Z}_\ell.$$

View this as an $\mathrm{Sp}_2(\mathbb{Z})[m]$-module via the projection $\mathrm{Sp}_2(\mathbb{Z})[m] \to \mathrm{Sp}_2(\mathbb{F}_p)$, which is surjective as p does not divide m.

Proposition 6.12. *The group M is not a continuous quotient of $H_1(\overline{T})$. Consequently, $H_1(T^{(\ell)}) \to H_1(\overline{T})$ is not injective.*

Proof. If there were a quotient mapping $\pi : H_1(\overline{T}) \to M$, it would have to be $\mathrm{Sp}_2(\mathbb{Z})[m]$-equivariant. Since $\mathrm{Sp}_2(\mathbb{Z})[p]$ acts trivially on M, $\mathrm{Sp}_2(\mathbb{Z}_\ell)[pm]$ would have to act trivially on M. But this is impossible as $\mathrm{Sp}_2(\mathbb{Z}_\ell)[pm] = \mathrm{Sp}_2(\mathbb{Z}_\ell)[m]$ and since $\mathrm{Sp}_2(\mathbb{Z})[m] \to \mathrm{Sp}_2(\mathbb{F}_p)$ is surjective. □

An improvement of the previous result and the following corollary were suggested by the referee.

Corollary 6.13. *The kernel of $T_2^{(\ell)} \to \Gamma_2[m]^{(\ell)}$ contains a copy of $\mathbb{Z}_\ell$.*

Proof. Set $K = \ker\{T^{(\ell)} \to \overline{T}\}$. The previous result implies that the composite $K \to T \to M$ is non-trivial. Since M is a free $\mathbb{Z}_\ell$-module, the image of $K \to M$ is a non-zero free $\mathbb{Z}_\ell$-module, which implies that K contains at least one copy of $\mathbb{Z}_\ell$. □

7. Proof of Theorem 3

First we prove the case $r = 0$, from which the case $r > 0$ follows by a group theoretic argument presented at the end of this section.

Suppose that $2g - 2 + n > 0$. Denote the moduli stack of smooth projective n-pointed genus g curves by $\mathcal{M}_{g,n/\mathbb{Z}}$ and its Deligne–Mumford compactification by $\overline{\mathcal{M}}_{g,n/\mathbb{Z}}$ [28,29].

Several times in this section we need to normalize a scheme in another scheme which is not irreducible. Specifically, suppose that $Y' \to Y$ is a finite flat representable 1-morphism of Deligne–Mumford stacks. Suppose that X is an integral scheme and that $X' \to Y$ is a morphism from a nonempty Zariski open subset of X to Y. Define the normalization of X with respect to

$$X' \to Y \leftarrow Y'$$

to be the disjoint union of the normalizations of X in the function fields of the irreducible components of $X' \times_Y Y'$.

7.1. Tangential base points

The notion of tangential base point was introduced by Deligne in [5]. A $\mathbb{Q}$-rational tangential base point of a connected scheme X yields an exact functor from the category of finite étale coverings of X to the category of finite étale coverings of $\operatorname{Spec}\mathbb{Q}$, and hence induces a homomorphism $\pi_1(\operatorname{Spec}\mathbb{Q}) \to \pi_1(X)$. Since we deal only with fundamental groups, we identify a tangential base point with the corresponding exact functor.

We consider the $\mathbb{Q}$-rational tangential base points constructed as follows. Let

$$C \to \operatorname{Spec}\mathbb{Z}[\![q_1, q_2, \ldots, q_{3g-3+n}]\!]$$

be the universal deformation of a maximally degenerate stable curve of type (g, n) over $\mathbb{Z}$, where the parameter q_i corresponds to a smoothing of the ith double point. Such deformations are constructed in [24, Remark 2.3.10].

Let

$$[C] : \operatorname{Spec} \mathbb{Z}[\![q_1, q_2, \ldots, q_{3g-3+n}]\!] \to \overline{\mathcal{M}}_{g,n/\mathbb{Z}}$$

be the classifying map. The choice of the set of parameters q_i determines a $\mathbb{Q}$-rational base point of $\mathcal{M}_{g,n/\mathbb{Z}}$ as follows.[9]

For each positive integer m, set

$$B = \mathbb{Q}[\![q_1, q_2, \ldots, q_{3g-3+n}]\!] \quad \text{and} \quad B^{1/m} = \mathbb{Q}[\![q_1^{1/m}, q_2^{1/m}, \ldots, q_{3g-3+n}^{1/m}]\!].$$

Then $[C]$ induces a map

$$[C]_m : \operatorname{Spec} B^{1/m}\big[(q_1 \cdots q_{3g-3+n})^{-1}\big] \to \mathcal{M}_{g,n/\mathbb{Q}}.$$

Suppose that $M \to \mathcal{M}_{g,n/\mathbb{Q}}$ is a finite étale covering and $\widetilde{M}_m$ be the normalization of $B^{1/m}$ with respect to

$$\operatorname{Spec} B^{1/m}\big[(q_1 \cdots q_{3g-3+n})^{-1}\big] \to \mathcal{M}_{g,n/\mathbb{Q}} \leftarrow M.$$

By Abhyankar's lemma [10, Exposé XIII], there is a positive integer m such that $\widetilde{M}_m$ is étale over $B^{1/m}$. By specializing each $q_i^{1/m}$ to 0, we obtain a finite etale cover of $\operatorname{Spec} \mathbb{Q}$. This covering is independent of the choice of m and defines an exact functor from the category of finite étale covers of $\mathcal{M}_{g,n/\mathbb{Q}}$ to the finite étale covers of $\operatorname{Spec} \mathbb{Q}$. We shall abuse notation and denote this tangential base point of $\mathcal{M}_{g,n}$ by $[C]$.

The tangential base point $[C]$ induces a section of the short exact sequence

$$1 \to \pi_1(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}) \to \pi_1(\mathcal{M}_{g,n} \otimes \mathbb{Q}) \to G_{\mathbb{Q}} \to 1 \tag{7}$$

and thus a Galois action

$$\rho_{[C]} : G_{\mathbb{Q}} \to \operatorname{Aut} \pi_1(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}).$$

The choice of an imbedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ determines an isomorphism

$$\pi_1(\mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}) \cong \Gamma_{g,n}^{\wedge}$$

of the geometric fundamental group of $\mathcal{M}_{g,n}$ with the profinite completion of the mapping class group $\Gamma_{g,n}$. We will make this identification. By Proposition 2.1, the relative pro-ℓ completion of $\Gamma_{g,n}^{\wedge}$ is isomorphic to $\Gamma_{g,n}^{(\ell)}$.

Our next task is to lift the base point $[C]$ of $\mathcal{M}_{g,n}$ to a base point $[C']$ of $\mathcal{M}_{g,n+1}$. The special fiber C_0 ($q_j = 0$, all j) of the curve C is a stable curve, each of whose components is isomorphic to $\mathbb{P}^1$; each $\mathbb{P}^1$ has exactly three distinguished points (i.e., singular points or marked points). Denote the n marked points of C by $x_1, \ldots, x_n$. Define C_0' to be the curve obtained by replacing the first node ($q_1 = 0$) by a copy of $\mathbb{P}^1$, where $0, \infty \in \mathbb{P}^1$ are nodes. Take x_{n+1} to be the point 1

[9] If q_i is replaced by $-q_i$ for example, the exact functor, and hence the base point, changes.

on the added $\mathbb{P}^1$. Then C_0' is an $n+1$ marked stable curve whose moduli point in $[C_0']$ in $\overline{\mathcal{M}}_{g,n+1}$ maps to the moduli point $[C_0]$ in $\overline{\mathcal{M}}_{g,n}$ by forgetting $(n+1)$st point.

Extend C_0' to an $(n+1)$-marked curve C' over $\mathbb{Z}[\![q_1,\dots,q_{3g-3+n+1}]\!]$ using the deformation theory construction in [24]. Label the parameters so that $q_{3g-3+n+1}$ is the parameter associated to the double point at the ∞ on the attached $\mathbb{P}^1$, q_1 the parameter associated to that at 0 on the attached $\mathbb{P}^1$, and q_j is the parameter corresponding to the jth node of C_0 when $1 < j \leqslant 3g-3+n$.

The curve C' corresponds to a morphism $[C']: \operatorname{Spec}\mathbb{Z}[\![q_1,\dots,q_{3g-3+n+1}]\!] \to \overline{\mathcal{M}}_{g,n+1}$. If we forget the $(n+1)$st marked point from C', we have a family of semi-stable curves. Its stabilization is isomorphic to $C \times \operatorname{Spec}\mathbb{Z}[\![q_1,\dots,q_{3g-3+n+1}]\!]$. Thus, the following diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{Z}[\![q_1,\dots,q_{3g-3+n+1}]\!] \arrow[r, "{[C']}"] \arrow[d] & \overline{\mathcal{M}}_{g,n+1} \arrow[d] \\
\operatorname{Spec}\mathbb{Z}[\![q_1,\dots,q_{3g-3+n}]\!] \arrow[r, "{[C]}"] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd}$$

commutes, where the left vertical map is obtained by $q_1 \mapsto q_1 q_{3g-3+n+1}$ and $q_i \mapsto q_i$ when $1 < i \leqslant 3g-3+n$.

This and the above construction of the exact functors give a homomorphism

$$\pi_1\big(\mathcal{M}_{g,n+1},[C']\big) \to \pi_1\big(\mathcal{M}_{g,n},[C]\big)$$

that preserves the section from $G_{\mathbb{Q}}$. On the other hand, the image of $[C']$ is in the fiber isomorphic to C, which gives a tangential base point of the smooth fiber locus $C_{\mathbb{Z}((q_1,\dots,q_{3g-3+n}))}$.[10] Let Ω be an algebraically closed field that contains $B^{1/m}$ for all positive integers m. Then we have a short exact sequence

$$1 \to \pi_1\big(C_\Omega,[C']\big) \to \pi_1\big(\mathcal{M}_{g,n+1/\mathbb{Q}},[C']\big) \to \pi_1\big(\mathcal{M}_{g,n/\mathbb{Q}},[C]\big) \to 1. \tag{8}$$

The left group is isomorphic to the profinite completion of $\Pi_{g,n}$, the fundamental group of an n-punctured, genus g reference surface. This short exact sequence gives the universal monodromy representation

$$\pi_1(\mathcal{M}_{g,n/\mathbb{Q}},[C]) \to \operatorname{Out}\Pi_{g,n}^{(\ell)}. \tag{9}$$

Restricting to the geometric fundamental group gives the geometric monodromy representation

$$\Gamma_{g,n}^{\wedge} \cong \pi_1\big(\mathcal{M}_{g,n/\overline{\mathbb{Q}}},[C]\big) \to \operatorname{Out}\Pi_{g,n}^{(\ell)}.$$

Further restricting to the action on the abelianization of $\pi_1^{(\ell)}(C_\Omega)$, we obtain a representation $\Gamma_{g,n}^{\wedge} \to \operatorname{Sp}_g(\mathbb{Z}_\ell)$. The $G_{\mathbb{Q}}$-actions (induced from (8) and the tangential section) on $\Gamma_{g,n}^{\wedge}$ and

[10] For a ring R we denote $R[\![t_1,\dots,t_r]\!][t_1^{-1},\dots,t_r^{-1}]$ by $R((t_1,\dots,t_r))$.

$\mathrm{Sp}_g(\mathbb{Z}_\ell)$ are $G_{\mathbb{Q}}$-equivariant. Thus, the functoriality of relative pro-ℓ completion implies that there is a Galois action

$$\rho_{[C]}^{(\ell)} : G_{\mathbb{Q}} \to \mathrm{Aut}\, \Gamma_{g,n}^{(\ell)}.$$

Theorem 7.1 *(Mochizuki–Tamagawa). For all prime numbers ℓ and all (g,n) satisfying $2g-2+n>0$, the representation $\rho_{[C]}^{(\ell)}$ is unramified outside ℓ. That is, it factors through $\pi_1(\mathrm{Spec}\,\mathbb{Z}[1/\ell])$:*

$$G_{\mathbb{Q}} \to \pi_1\big(\mathrm{Spec}\,\mathbb{Z}[1/\ell]\big) \to \mathrm{Aut}\, \Gamma_{g,n}^{(\ell)}.$$

The case $g=0$ is proved by Ihara [23].

7.2. *Étale coverings of moduli spaces*

Suppose that A is a ring contained in an algebraically closed field Ω of characteristic 0, and hence that A is an integral domain. We have $\Gamma_{g,n}^{\wedge} \cong \pi_1(M_{g,n/\Omega}) \to \pi_1(\mathcal{M}_{g,n/A})$. We say that a finite étale covering $M \to \mathcal{M}_{g,n/A}$ is *geometrically relative-ℓ*, if the corresponding $\Gamma_{g,n}^{\wedge}$-action on fibers factors through $\Gamma_{g,n}^{(\ell)}$. This is independent of the choice of $A \to \Omega$. The category of such coverings is a Galois category. By restricting to the geometrically relative-ℓ coverings, we obtain the quotient

$$1 \to \Gamma_{g,n}^{(\ell)} \to \pi_1(\mathcal{M}_{g,n} \otimes \mathbb{Q})' \to G_{\mathbb{Q}} \to 1 \tag{10}$$

of (7).

The existence of suitable finite coverings of $\mathcal{M}_{g,n}$ that have a compactification that is smooth over $\mathbb{Z}[1/\ell]$ was established by de Jong and Pikaart when $n=0$ for all ℓ in [27], when $n>0$ and ℓ is odd by Boggi and Pikaart in [4], and when $n>0$ and $\ell=2$ by Pikaart in [35]. Their results needed here are summarized in the following statement.

Proposition 7.2. *For all prime numbers ℓ and all (g,n) satisfying $2g-2+n>0$, there is a geometrically relative-ℓ Galois covering $M \to \mathcal{M}_{g,n}$ defined over $\mathbb{Z}[1/\ell]$ that satisfies*:

(0) *M is an integral scheme*;
(1) *the normalization $\overline{M}$ of $\overline{\mathcal{M}}_{g,n}$ in the function field of M is proper and smooth over $\mathbb{Z}[1/\ell]$*;
(2) *the boundary $\overline{M} \setminus M$ is a relative normal crossing divisor over $\mathbb{Z}[1/\ell]$*;
(3) *the ramification index of the covering $\overline{M} \to \overline{\mathcal{M}}_{g,n}$ along any irreducible component of the boundary is a power of ℓ*;
(4) *the natural map $\Gamma_{g,n}^{(\ell)} \to \mathrm{Sp}_g(\mathbb{Z}/\ell)$ factors through the quotient of $\Gamma_{g,n}^{(\ell)}$ corresponding to the Galois covering $M \otimes \overline{\mathbb{Q}} \to \mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}$ for odd ℓ; for $\ell=2$, the same statement with* $\mathrm{Sp}_g(\mathbb{Z}/4)$.

We shall explain how this statement follows from their results. For G a finite quotient of $\Pi_{g,n}$ by a characteristic subgroup, Deligne and Mumford [8] introduced the moduli stack ${}_G\mathcal{M}_{g,n}$ over $\mathbb{Z}[1/|G|]$ of smooth projective curves of type (g,n) with a Teichmüller structure of level G. It is

the normalization over $\mathbb{Z}[1/|G|]$ of the finite Galois covering ${}_G\mathcal{M}_{g,n/\mathbb{Q}} \to \mathcal{M}_{g,n/\mathbb{Q}}$ corresponding to the kernel of $\pi_1(\mathcal{M}_{g,n/\mathbb{Q}}) \to \operatorname{Out} G$. The construction over $\operatorname{Spec}\mathbb{Z}[1/|G|]$ works equally well when G is a quotient of $\Pi_{g,n}$ by a finite index subgroup stabilized by $\pi_1(\mathcal{M}_{g,n/\mathbb{Q}})$.

Proposition 7.2 is proved by considering $M = {}_G\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$ for a suitable choice of G. Specifically take G to be:

(1) the quotient group of $\Pi_{g,0}$ by the normal subgroup generated by the fourth term of its lower central subgroup and all ℓth powers when ℓ is odd and $n = 0$;
(2) the quotient group of $\Pi_{g,0}$ by the normal subgroup generated by the fourth term of its lower central subgroup and all fourth powers when $\ell = 2$ and $n = 0$;
(3) the quotient $\Pi_{g,n}/W^3\Pi_{g,n}$, where W^3 denotes the third term of the weight filtration defined in [4]), by all ℓth powers when ℓ is odd and $n > 0$;
(4) the quotient $\Pi_{g,n}/W^4\Pi_{g,n}$, where W^4 denotes the fourth term of the weight filtration defined in [4]), by all fourth powers when $\ell = 2$ and $n > 0$.

For details, see de Jong and Pikaart [27, Theorem 3.1.1(iii) and Proposition 2.3.6] for the first two cases, Boggi and Pikaart [4, Proposition 2.6(ii) and Proposition 2.8] in the third case, and Pikaart [35, Theorem 3.3.1(2) and Theorem 3.3.3(7)] in the fourth case. The extra condition for the case $\ell = 2$ in (4) in Proposition 7.2 is to assure that M is a scheme, not just a stack.

Let $p \neq \ell$ be a prime number. For a subring A of the field Ω, $\mathcal{M}_{g,n/A}$ denotes the base change to A. Denote by $\mathbb{Z}_p^{\mathrm{ur}}$ the maximal étale cover of $\mathbb{Z}_p$, and by $\mathbb{Q}_p^{\mathrm{ur}}$ its fraction field. There is a natural morphism

$$\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p} \xrightarrow{f} \mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}.$$

Denote the Galois categories of geometrically relative-ℓ coverings of $\mathcal{M}_{g,n/A}$ by $\mathcal{C}(\mathcal{M}_{g,n/A})$. Note that the fundamental group of $\mathcal{C}(\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p})$ is isomorphic to $\Gamma_{g,n}^{(\ell)}$.

Proposition 7.3. *The above f induces an equivalence of categories*

$$f^* : \mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}) \to \mathcal{C}(\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p}).$$

Proof. Take a geometrically relative ℓ-covering M as in Proposition 7.2. Choose a connected component of $M \otimes \mathbb{Z}_p^{\mathrm{ur}}$. It is a connected object of $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}})$, which we denote by $M_{\mathbb{Z}_p^{\mathrm{ur}}}$. Since $M \to \mathcal{M}_{g,n}[1/\ell]$ is etale and $p \neq \ell$, the base change $M_{\overline{\mathbb{Q}}_p}$ of $M_{\mathbb{Z}_p^{\mathrm{ur}}}$ is a connected object of $\mathcal{C}(\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p})$. Since the boundary of $\overline{M}$ is a relatively normal crossing divisor over $\mathbb{Z}[1/\ell]$, so is the boundary of the Zariski closure of $M_{\mathbb{Z}_p^{\mathrm{ur}}}$ in $\overline{M}_{\mathbb{Z}_p^{\mathrm{ur}}}$ over $\mathbb{Z}_p^{\mathrm{ur}}$. The Base Change Theorem [10, Exposé XII] implies that f induces an equivalence between the Galois categories of the finite étale ℓ-coverings of $M_{\mathbb{Z}_p^{\mathrm{ur}}}$ and $M_{\overline{\mathbb{Q}}_p}$. Since M is a geometrically relative-ℓ Galois cover of $\mathcal{M}_{g,n}$ defined over $\mathbb{Z}[1/\ell]$ that satisfies condition (4) of Proposition 7.2, the category $\mathcal{C}(M_A)$ of ℓ-coverings of M_A is the subcategory of $\mathcal{C}(\mathcal{M}_{g,n/A})$ consisting of all the objects (and the morphisms) over M_A ($A = \overline{\mathbb{Q}}_p$ or $\mathbb{Z}_p^{\mathrm{ur}}$).

Choose a geometric point $x_{\overline{\mathbb{Q}}_p}$ of $M_{\overline{\mathbb{Q}}_p}$. Let $x_{\mathbb{Z}_p^{\mathrm{ur}}}$ be its image in $M_{\mathbb{Z}_p^{\mathrm{ur}}}$. Let x'_A denote their image in $\mathcal{M}_{g,n/A}$, for $A = \overline{\mathbb{Q}}_p$ or $\mathbb{Z}_p^{\mathrm{ur}}$. Then $\pi_1(\mathcal{C}(M_A), x_A)$ is a finite index subgroup of $\pi_1(\mathcal{C}(\mathcal{M}_{g,n/A}), x'_A)$, and the quotient set is identified with the fiber $F(x'_A)$ of M_A over x'_A pointed

at x_A. The cardinality of $F(x'_A)$ is the covering degree of $M_A \to \mathcal{M}_{g,n/A}$, hence is the same for $A = \overline{\mathbb{Q}}_p$ and $\mathbb{Z}_p^{\mathrm{ur}}$. We have the following commutative diagram

$$\begin{array}{ccccc}
\pi_1(\mathcal{C}(M_{\overline{\mathbb{Q}}_p}), x_{\overline{\mathbb{Q}}_p}) & \hookrightarrow & \pi_1(\mathcal{C}(\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p}), x'_{\overline{\mathbb{Q}}_p}) & \longrightarrow & F(x'_{\overline{\mathbb{Q}}_p}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}}), x_{\mathbb{Z}_p^{\mathrm{ur}}}) & \hookrightarrow & \pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}), x'_{\mathbb{Z}_p^{\mathrm{ur}}}) & \longrightarrow & F(x'_{\mathbb{Z}_p^{\mathrm{ur}}}),
\end{array}$$

where, in each row, the right-hand set is the quotient of the middle group by the left-hand group. The left-hand vertical arrow has been proved to be an isomorphism. Since the right-hand mapping is bijective, the middle vertical arrow is an isomorphism. The result follows. □

7.3. *Proof of Theorem 7.1*

It suffices to show that for every $p \neq \ell$, the inertia group $\pi_1(\mathbb{Q}_p^{\mathrm{ur}})$ at p acts trivially on the relative pro-ℓ completion of the mapping class group.

We use the notation of Section 7.1. Take M over $\mathbb{Z}[1/\ell]$ as in Proposition 7.2 and $M_{\mathbb{Z}_p^{\mathrm{ur}}}$ as in the proof of Proposition 7.3. A tangential base point of M over $[C]$ induces an exact functor from $\mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}})$ to the category $\mathcal{C}(\mathbb{Z}_p^{\mathrm{ur}})$ of finite étale coverings of $\mathbb{Z}_p^{\mathrm{ur}}$ as follows.

Let $\widetilde{M}$ be the normalization of $\operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1, \dots, q_{3g-3+n}]\!]$ with respect to

$$\operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1, \dots, q_{3g-3+n}]\!]\big[(q_1 \dots q_{3g-3+n})^{-1}\big] \to \mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}} \leftarrow M_{\mathbb{Z}_p^{\mathrm{ur}}}.$$

The second and third assertions of Proposition 7.2 concerning ramification imply that there are ℓ-power integers $k_1, \dots, k_{3g-3+n}$ (possibly 1) such that

$$\widetilde{M} \cong \operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1^{1/k_1}, \dots, q_{3g-3+n}^{1/k_{3g-3+n}}]\!].$$

This gives a $\mathbb{Z}_p^{\mathrm{ur}}$-rational tangential base point on $M_{\mathbb{Z}_p^{\mathrm{ur}}}$. Abhyankar's lemma [10, Exposé XIII] implies that, for each finite étale ℓ-covering N of $M_{\mathbb{Z}_p^{\mathrm{ur}}}$, there is an ℓ-power integer j such that

$$\widetilde{N} \to \operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1^{1/jk_1}, \dots, q_{3g-3+n}^{1/jk_{3g-3+n}}]\!]$$

is a finite étale morphism, where $\widetilde{N}$ is the normalization of

$$\operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1^{1/jk_1}, \dots, q_{3g-3+n}^{1/jk_{3g-3+n}}]\!]$$

with respect to

$$\operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}[\![q_1^{1/jk_1}, \dots, q_{3g-3+n}^{1/jk_{3g-3+n}}]\!]\big[(q_1 \dots q_{3g-3+n})^{-1}\big] \to M_{\mathbb{Z}_p^{\mathrm{ur}}} \leftarrow N.$$

By specializing the q_i^{1/jk_i} to zero, we obtain a finite étale covering of $\mathbb{Z}_p^{\mathrm{ur}}$. This gives an exact functor $\mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}}) \to \mathcal{C}(\mathbb{Z}_p^{\mathrm{ur}})$. The geometric point $\xi : \operatorname{Spec} \overline{\mathbb{Q}}_p \to \operatorname{Spec} \mathbb{Z}_p^{\mathrm{ur}}$ gives a fiber functor F_ξ

of $\mathcal{C}(\mathbb{Z}_p^{\mathrm{ur}})$, whose composition with the above functor is the fiber functor F'_ξ of $\mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}})$ associated to $q_1^{1/k_1}, \dots, q_{3g-3+n}^{1/k_{3g-3+n}}$. Composing with the functor $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}) \to \mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}})$ gives a fiber functor F''_ξ of $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}})$:

$$\begin{tikzcd}
\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}) \arrow[r] \arrow[dr, "F''_\xi"'] & \mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}}) \arrow[r] \arrow[d, "F'_\xi"'] & \mathcal{C}(\mathbb{Z}_p^{\mathrm{ur}}) \arrow[dl, "F_\xi"] \\
& \mathsf{Sets} &
\end{tikzcd}$$

Note that the fiber functor $[C]_*(\xi)$ equals F''_ξ. Define $[C]_{\mathbb{Z}_p^{\mathrm{ur}}}$ to be the composite

$$\pi_1\big(\mathbb{Z}_p^{\mathrm{ur}}, F_\xi\big) \to \pi_1\big(\mathcal{C}(M_{\mathbb{Z}_p^{\mathrm{ur}}}), F'_\xi\big) \to \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}), F''_\xi\big).$$

This and the similar construction over $\mathbb{Q}_p^{\mathrm{ur}}$ gives a commutative diagram

$$\begin{tikzcd}
& \pi_1(\mathcal{M}_{g,n/\overline{\mathbb{Q}}_p}, F''_\xi)^{(\ell)} \arrow[d, hook] \\
\pi_1(\mathbb{Q}_p^{\mathrm{ur}}, F_\xi) \arrow[r, "{[C]_{\mathbb{Q}_p^{\mathrm{ur}}}}"] \arrow[d] & \pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}}), F''_\xi) \arrow[d] \\
\pi_1(\mathbb{Z}_p^{\mathrm{ur}}, F_\xi) \arrow[r, "{[C]_{\mathbb{Z}_p^{\mathrm{ur}}}}"] & \pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}}), F''_\xi).
\end{tikzcd}$$

where the top right-hand arrow is the inclusion of a normal subgroup. The top right-hand group is isomorphic to $\Gamma_{g,n}^{(\ell)}$. It is a normal subgroup of $\pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}}), F''_\xi)$. Proposition 7.3 implies that the composite of the two right-hand vertical maps is an isomorphism. The action of the inertia $\pi_1(\mathbb{Q}_p^{\mathrm{ur}})$ at p on $\Gamma_{g,n}^{(\ell)}$ is conjugation by an element of $\pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}}), F''_\xi)$. The commutativity of the diagram implies that the inertia action factors through $\pi_1(\mathbb{Z}_p^{\mathrm{ur}}, F_\xi)$, which is trivial. Theorem 7.1 follows.

Denote the kernel of $G_{\mathbb{Q}} \to \pi_1(\mathbb{Z}[1/\ell])$ by J_ℓ. Define $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ to be the quotient of $\pi_1(\mathcal{M}_{g,n/\mathbb{Q}}, [C])$ by the normal subgroup generated by $[C]_*(J_\ell)$ and the kernel of $\hat{\Gamma}_{g,n} \to \Gamma_{g,n}^{(\ell)}$. Taking the quotient of the middle and the right-hand groups in the short exact sequence (10), we obtain the following result.

Corollary 7.4. *For all prime numbers ℓ and all (g,n) satisfying $2g-2+n>0$, there is a split exact sequence*

$$1 \to \Gamma_{g,n}^{(\ell)} \to \Gamma_{g,n}^{\mathrm{arith},(\ell)} \to \pi_1\big(\mathbb{Z}[1/\ell]\big) \to 1.$$

Proof. Consider the section of the short exact sequence (10) given by the tangential base point $[C]$. Theorem 7.1 implies that the image of the kernel of $G_{\mathbb{Q}} \to \pi_1(\mathbb{Z}[1/\ell])$ under the section induced by $[C]$ centralizes $\Gamma_{g,n}^{(\ell)}$, hence is a normal subgroup of the middle group. We

can divide the middle and the right-hand groups by this kernel to obtain the short exact sequence. The splitting is induced by $[C]$. $\square$

The following corollary was suggested by the referee.

Corollary 7.5. *The group $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ is canonically isomorphic to the fundamental group of the Galois category $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]})$ with respect to the tangential base point $[C]$. In particular, although the definition of $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ depends on the choice of $[C]$, two different choices yield two isomorphic groups. The isomorphism is canonical up to an inner automorphism.*

Proof. Denote the kernel of $G_{\mathbb{Q}} \to \pi_1(\mathbb{Z}[1/\ell])$ by J_ℓ. By the definition of $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ given above, we have an identification

$$\Gamma_{g,n}^{\mathrm{arith},(\ell)} = \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}})\big)\big/\big([C]_*(J_\ell)\big).$$

On the other hand, by the "purity of the branched locus", we have

$$\pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]})\big) \cong \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}})\big)/\langle T_p \mid p \neq \ell\rangle,$$

where T_p denotes the inertia subgroup of $\pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}}))$ along the divisor $\mathcal{M}_{g,n/\mathbb{F}_p}$ of $\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$, and $\langle T_p \mid p \neq \ell\rangle$ denotes the normal subgroup topologically generated by the union of the T_p for all primes $p \neq \ell$. The result will follow from the statement that these two quotients are identical as it implies that $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ is canonically isomorphic to the fundamental group of the Galois category $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]})$ with fiber functor $[C]$, and another choice of $[C]$ yields an isomorphic fundamental group, with isomorphism determined up to an inner automorphism. The rest of the proof consists in proving this equality.

When a covering in $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]})$ is pulled back along the tangential section $[C]$ over $\mathbb{Z}[1/\ell]$, it gives a finite étale covering of $\mathbb{Z}[1/\ell]$. This implies that $[C]_*(J_\ell)$ acts trivially on the corresponding fiber, and hence that $[C]_*(J_\ell) \subseteq \langle T_p \mid p \neq \ell\rangle$. To establish the reverse inclusion, it suffices to show that the image of T_p in $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ is trivial. Consider the exact sequence

$$1 \to \langle \widetilde{T}_p\rangle \to \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}})\big) \to \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}})\big) \to 1,$$

where $\widetilde{T}_p$ is the inertia subgroup of $\pi_1(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}}))$, and $\langle \widetilde{T}_p\rangle$ is the normal subgroup topologically generated by $\widetilde{T}_p$. The construction in Corollary 7.4, when $\mathbb{Z}[1/\ell]$ is replaced with $\mathbb{Z}_p^{\mathrm{ur}}$, gives a short exact sequence

$$1 \to \Gamma_{g,n}^{(\ell)} \to \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}})\big)/[C]_*\big(\pi_1(\mathbb{Q}_p^{\mathrm{ur}})\big) \to \pi_1\big(\mathbb{Z}_p^{\mathrm{ur}}\big) \to 1.$$

Since $\pi_1(\mathbb{Z}_p^{\mathrm{ur}})$ is trivial, the left group is isomorphic to the middle group. Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{T}_p \arrow[r] \arrow[d, two heads] & \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}})\big)/[C]_*(\pi_1(\mathbb{Q}_p^{\mathrm{ur}})) \arrow[d] & \\
T_p \arrow[r] & \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}})\big)/[C]_*(J_\ell) \arrow[r, equal] & \Gamma_{g,n}^{\mathrm{arith},(\ell)}.
\end{tikzcd}$$

Since the left-hand vertical map is surjective, the triviality of the bottom horizontal map will follow from the triviality of the upper horizontal map. Since

$$\begin{array}{ccc} \pi_1(\mathbb{Q}_p^{\mathrm{ur}}) & \longrightarrow & \pi_1(\mathbb{Z}_p^{\mathrm{ur}}) \\ {\scriptstyle [C]_*}\downarrow & & \downarrow{\scriptstyle [C]_*} \\ \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}})\big) & \longrightarrow & \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}})\big) \end{array}$$

commutes and the top right group is trivial, the bottom horizontal map factors through a homomorphism

$$\pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Q}_p^{\mathrm{ur}}})\big)/[C]_*\big(\pi_1(\mathbb{Q}_p^{\mathrm{ur}})\big) \to \pi_1\big(\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}_p^{\mathrm{ur}}})\big),$$

which is an isomorphism because the both groups are isomorphic to $\Gamma_{g,n}^{(\ell)}$ by the above argument and Proposition 7.3. Since the image of $\widetilde{T}_p$ is trivial in the right-hand group, it is also trivial in the left-group. This establishes the isomorphism of $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$ with the fundamental group of the Galois category $\mathcal{C}(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]})$ and with it, the result. □

Theorem 7.6. *For all prime numbers ℓ and all (g,n) satisfying $2g-2+n>0$, the universal monodromy representation* (9)

$$\pi_1\big(\mathcal{M}_{g,n/\mathbb{Q}},[C]\big) \to \operatorname{Out}\pi_1^{(\ell)}(C_\Omega)$$

factors through $\Gamma_{g,n}^{\mathrm{arith},(\ell)}$.

Proof. Since the universal monodromy representation is the outer representation associated to the extension (8), it suffices to show that

$$1 \to \pi_1(C_\Omega)^{(\ell)} \to \Gamma_{g,n+1}^{\mathrm{arith},(\ell)} \to \Gamma_{g,n}^{\mathrm{arith},(\ell)} \to 1$$

is exact. This is equivalent to the exactness of

$$1 \to \pi_1(C_\Omega)^{(\ell)} \to \Gamma_{g,n+1}^{(\ell)} \to \Gamma_{g,n}^{(\ell)} \to 1,$$

which was proved in Proposition 3.1(2). □

Remark 7.7. The proof of Theorem 7.1 was communicated to us by Tamagawa. Mochizuki gave a more sophisticated proof, sketched below, which uses log geometry and avoids the constructions of Boggi, de Jong, and Pikaart in Proposition 7.2.

In the above proof, the problem of proving the result is reduced to studying the ℓ-coverings of M in Proposition 7.2, where $M \subset \overline{M}$ is the complement of a normal crossing divisor. This allows the use of [10, Exposé XII]. Mochizuki, on the other hand, takes $N \to \mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}$ to be the covering corresponding to $\pi_1(\mathcal{M}_{g,n/\mathbb{Z}[1/\ell]}) \to \mathrm{GSp}_g(\mathbb{Z}/\ell)$. He equips $\overline{\mathcal{M}}_{g,n}$ with the standard log structure. The normalization $\overline{N}$ of $\overline{\mathcal{M}}_{g,n/\mathbb{Z}[1/\ell]}$ is then log-regular since the ramification is tame. Mochizuki uses the log-purity theorem of Kato and Fujiwara (unpublished; see [33] for a proof) in place of [10, Exposé XII] to complete the proof.

7.4. The case $r > 0$

Theorem 7.1 establishes Theorem 3 in the case $r = 0$. The case $r > 0$ reduces to the case $r = 0$.

We first discuss the Deligne–Mumford compactification of $\mathcal{M}_{g,n,\vec{r}}$. Suppose that $2g - 2 + n + r > 0$. Define a stable curve of type $(g, n, \vec{r})$ to be a stable curve of type $(g, n + r)$ together with the choice of a non-zero tangent vector at each of the last r points. The moduli space of stable curves of type $(g, n, \vec{r})$ is a $(\mathbb{G}_m)^r$-bundle over $\overline{\mathcal{M}}_{g,n+r}$, which is not complete if $r > 0$. Denote the line bundle associated to the jth factor of this $(\mathbb{G}_m)^r$ bundle by L_j. Its fiber over the stable curve $[C, x_1, \dots, x_n, y_1, \dots, y_r]$ is $T_{y_j}C$, the tangent space of C at y_j. The jth factor of the $(\mathbb{G}_m)^r$ bundle over $\overline{\mathcal{M}}_{g,n+r}$ has fiber $T_{y_j}C - \{0\}$ over $[C, x_1, \dots, x_n, y_1, \dots, y_r]$ and is therefore defined over $\operatorname{Spec}\mathbb{Z}$. We compactify the jth factor of the $(\mathbb{G}_m)^r$ bundle in the usual way; namely:

$$\mathbb{P}(L_j \oplus \mathcal{O}_{\overline{\mathcal{M}}_{g,n+r}}).$$

Define $\overline{\mathcal{M}}_{g,n,\vec{r}}$ to be the fibered product over $\overline{\mathcal{M}}_{g,n+r}$ of the bundles $\mathbb{P}(L_j \oplus \mathcal{O}_{\overline{\mathcal{M}}_{g,n+r}})$. This is defined over $\operatorname{Spec}\mathbb{Z}$ as each of its factors is.

The construction of the deformation of maximally degenerate curves over $\mathbb{Z}$, and the construction of $\mathbb{Q}$-rational tangential base points are all similar to constructions in the case $r = 0$ explained above. We start with a maximally degenerate stable curve of type $(g, n + r)$ and then construct the universal deformation over $\mathbb{Z}$,

$$[C] : \operatorname{Spec}\mathbb{Z}[\![q_1, q_2, \dots, q_{3g-3+n+r}]\!] \to \overline{\mathcal{M}}_{g,n+r/\mathbb{Z}}.$$

The fiber product $\operatorname{Spec}\mathbb{Z}[\![q_1, q_2, \dots, q_{3g-3+n+r}]\!] \times_{\overline{\mathcal{M}}_{g,n+r/\mathbb{Z}}} \overline{\mathcal{M}}_{g,n,\vec{r}}$ is $(\mathbb{P}^1)^r$-bundle over $\operatorname{Spec}\mathbb{Z}[\![q_1, q_2, \dots, q_{3g-3+n+r}]\!]$. Take $q'_1, \dots, q'_r$ to be the standard coordinate of each copy of $\mathbb{P}^1$ so that $q'_i = 0$ gives the zero of the ith copy of $\mathbb{P}^1 \supset \mathbb{G}_m$. Now

$$[C'] : \operatorname{Spec}\mathbb{Z}\big[\!\big[q_1, q_2, \dots, q_{3g-3+n+r}, q'_1, \dots, q'_r\big]\!\big] \to \overline{\mathcal{M}}_{g,n,\vec{r}/\mathbb{Z}}$$

gives a tangential base point compatible with $[C]$. In the following, we use the Galois action given by these tangential base points.

Suppose that $r > 0$. We have to establish the triviality of the action on $\Gamma^{(\ell)}_{g,n,\vec{r}}$ of the inertia subgroup I_p of $G_{\mathbb{Q}}$ at p. By Proposition 3.3, the sequence

$$0 \to \mathbb{Z}_\ell(1)^r \to \Gamma^{(\ell)}_{g,n,\vec{r}} \to \Gamma^{(\ell)}_{g,n+r} \to 1$$

is exact. Fix $\sigma \in I_p$ and take $g \in \Gamma^{(\ell)}_{g,n,\vec{r}}$. Then, since I_p acts trivially on $\Gamma^{(\ell)}_{g,n+r}$ (this is the case $r = 0$), we can define a function

$$\varphi : \Gamma^{(\ell)}_{g,n,\vec{r}} \to \mathbb{Z}_\ell(1)^r$$

by $\varphi : h \mapsto \sigma(h)h^{-1}$. Since $\mathbb{Z}_\ell(1)^r$ is central in $\Gamma^{(\ell)}_{g,n,\vec{r}}$, φ is a group homomorphism. Since I_p acts trivially on $\mathbb{Z}_\ell(1)^r$, φ induces a homomorphism

$$\overline{\varphi} : \Gamma^{(\ell)}_{g,n+r} \to \mathbb{Z}_\ell(1)^r.$$

Since $\Gamma_{g,n+r}$ is finitely generated, $H^1(\Gamma_{g,n+r}, \mathbb{Z})$ is a finitely generated torsion free abelian group. Since $H^1(\Gamma_{g,n+r}, \mathbb{Q})$ vanishes for all $g \geqslant 1$,[11] $H^1(\Gamma_{g,n+r}, \mathbb{Z})$ vanishes whenever $g \geqslant 1$. This implies that $\overline{\varphi}$ is trivial whenever $g \geqslant 1$ as $\Gamma_{g,n+r}$ is dense in $\Gamma^{(\ell)}_{g,n+r}$. It follows that I_p acts trivially on $\Gamma^{(\ell)}_{g,n,\vec{r}}$. When $g = 0$, relative pro-ℓ completion coincides with pro-ℓ completion, so that we can appeal to [10, Exposé XII].

Acknowledgments

The authors are grateful to S. Mochizuki and A. Tamagawa for allowing us to reproduce the proof of Theorem 3. They are also grateful to Tamagawa for very helpful comments on an earlier draft of this manuscript. Last, but not least, the authors owe an enormous debt of gratitude the referee who read two drafts of the paper diligently. His/her detailed and constructive comments and corrections have resulted in stronger results, fewer errors and better exposition.

References

[1] M. Anderson, Exactness properties of profinite completion functors, Topology 13 (1974) 229–239.
[2] H. Bass, J. Milnor, J.-P. Serre, Solution of the congruence subgroup problem for SL_n, ($n \geqslant 3$) and Sp_{2n}, ($n \geqslant 2$), Publ. Math. Inst. Hautes Études Sci. 33 (1967) 59–137.
[3] A. Beilinson, A. Levin, The elliptic polylogarithm, in: Motives, Seattle, WA, 1991, in: Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., 1994, pp. 123–190, Part 2.
[4] M. Boggi, M. Pikaart, Galois covers of moduli of curves, Compos. Math. 120 (2000) 171–191.
[5] P. Deligne, Le groupe fondamental de la droite projective moins trois points, in: Galois Groups Over $\mathbb{Q}$, Berkeley, CA, 1987, in: Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York–Berlin, 1989, pp. 79–297.
[6] P. Deligne, Extensions centrales non résiduellement finies de groupes arithmétiques, C. R. Acad. Sci. Paris Sér. A–B 287 (1978) A203–A208.
[7] P. Deligne, J. Milne, Tannakian categories, in: P. Deligne, J. Milne, A. Ogus, K.-Y. Shih (Eds.), Hodge Cycles, Motives, and Shimura Varieties, in: Lecture Notes in Math., vol. 900, Springer-Verlag, 1982.
[8] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. Inst. Hautes Études Sci. 36 (1969) 75–109.
[9] J. Dieudonné, Sur les groupes classiques, Publ. Inst. Math. Univ. Strasbourg (N.S.) no. 1 (1945), Hermann et Cie., Paris, 1948.
[10] A. Grothendieck, Revêtement Étales et Groupe Fondamental (SGA 1), Lecture Notes in Math., vol. 224, Springer-Verlag, 1971.
[11] R. Hain, Completions of mapping class groups and the cycle $C - C^-$, in: Contemp. Math., vol. 150, Amer. Math. Soc., 1993, pp. 75–105.
[12] R. Hain, Infinitesimal presentations of Torelli groups, J. Amer. Math. Soc. 10 (1997) 597–651.
[13] R. Hain, Torelli groups and Geometry of Moduli Spaces of Curves, in: C.H. Clemens, J. Kollár (Eds.), Current Topics in Complex Algebraic Geometry, in: Math. Sci. Res. Inst. Publ., vol. 28, Cambridge Univ. Press, 1995, pp. 97–143.

[11] This follows from Harer's work [22]. He proved that $H_1(\Gamma_g)$ vanishes when $g \geqslant 3$ and is cyclic of order 10 when $g = 2$. It is classical that $H_1(\Gamma_{1,1}) = \mathbb{Z}/12$. The result when $n + r > 0$ follows by induction and a standard and straightforward spectral sequence argument. It is also proved in [13, Prop. 5.1] for $g \geqslant 3$ by different means.

[14] R. Hain, Finiteness and Torelli spaces, in: Problems on Mapping Class Groups and Related Topics, in: Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., 2006, pp. 57–70.
[15] R. Hain, Relative weight filtrations on completions of mapping class groups, in: Groups of Diffeomorphisms, in: Adv. Stud. Pure Math., vol. 52, 2008, pp. 309–368.
[16] R. Hain, M. Matsumoto, Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, Compos. Math. 139 (2003) 119–167.
[17] R. Hain, M. Matsumoto, Tannakian fundamental groups associated to Galois groups, in: Galois Groups and Fundamental Groups, in: Math. Sci. Res. Inst. Publ., vol. 41, Cambridge Univ. Press, Cambridge, 2003, pp. 183–216.
[18] R. Hain, M. Matsumoto, Weighted completion of arithmetic mapping class groups, preprint, 2007.
[19] R. Hain, D. Reed, Geometric proofs of some results of Morita, J. Algebraic Geom. 10 (2001) 199–217.
[20] R. Hain, D. Reed, On the Arakelov geometry of moduli spaces of curves, J. Differential Geom. 67 (2004) 195–228.
[21] R. Hain, S. Zucker, Unipotent variations of mixed Hodge structure, Invent. Math. 88 (1987) 83–124.
[22] J. Harer, The second homology group of the mapping class group of an orientable surface, Invent. Math. 72 (1983) 221–239.
[23] Y. Ihara, Automorphisms of pure sphere braid groups and Galois representations, in: The Grothendieck Festschrift II, Birkhäuser, 1991, pp. 353–373.
[24] Y. Ihara, H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda's problem, J. Reine Angew. Math. 487 (1997) 125–151.
[25] D. Johnson, The structure of the Torelli group, I, A finite set of generators for $\mathcal{I}$, Ann. of Math. (2) 118 (1983) 423–442.
[26] D. Johnson, The structure of the Torelli group, III, The abelianization of $\mathcal{I}$, Topology 24 (1985) 127–144.
[27] A.J. de Jong, M. Pikaart, Moduli of curves with non-abelian level structure, in: The Moduli Space of Curves, Texel Island, 1994, in: Progr. Math., vol. 129, Birkhäuser, 1995, pp. 483–509.
[28] F. Knudsen, The projectivity of the moduli space of stable curves, II. The stacks $M_{g,n}$, Math. Scand. 52 (1983) 161–199.
[29] F. Knudsen, The projectivity of the moduli space of stable curves, III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic 0, Math. Scand. 52 (1983) 200–212.
[30] J. Labute, On the descending central series of groups with a single defining relation, J. Algebra 14 (1970) 16–23.
[31] G. Mess, The Torelli groups for genus 2 and 3 surfaces, Topology 31 (1992) 775–790.
[32] S. Morita, A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles, in: S. Kojima, et al. (Eds.), Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, July 1995, World Scientific, 1996, 159–186.
[33] S. Mochizuki, Extending families of curves over log regular schemes, J. Reine Angew. Math. 511 (1999) 43–71.
[34] D. Passman, The Algebraic Structure of Group Rings, Pure Appl. Math., Wiley–Interscience, 1977.
[35] M. Pikaart, Moduli spaces of curves, stable cohomology and Galois covers, Thesis, University of Utrecht, 1997.
[36] D. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969) 205–295.
[37] M.S. Raghunathan, Cohomology of arithmetic subgroups of algebraic groups, I & II, Ann. of Math. (2) 86 (1967) 409–424, Ann. of Math. (2) 87 (1967) 279–304.