Tridiagonal pairs of q-Racah type

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ABSTRACT

Let $\mathbb{F}$ denote an algebraically closed field and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leqslant i \leqslant d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that $A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leqslant i \leqslant \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a *tridiagonal pair* on V. It is known that $d = \delta$. For $0 \leqslant i \leqslant d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*). The pair A, A^* is said to have q-*Racah type* whenever $\theta_i = a + bq^{2i-d} + cq^{d-2i}$ and $\theta_i^* = a^* + b^* q^{2i-d} + c^* q^{d-2i}$ for $0 \leqslant i \leqslant d$, where $q, a, b, c, a^*, b^*, c^*$ are scalars in $\mathbb{F}$ with q, b, c, b^*, c^* nonzero and $q^2 \notin \{1, -1\}$. This type is the most general one. We classify up to isomorphism the tridiagonal pairs over $\mathbb{F}$ that have q-Racah type. Our proof involves the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

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1. Tridiagonal pairs

Throughout this paper $\mathbb{F}$ denotes a field and $\bar{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. (See [22, Definition 1.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

  (i) Each of A, A^* is diagonalizable.
 (ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d), \tag{1}$$

    where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta), \tag{2}$$

    where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{F}$. We call V the *underlying vector space*.

Note 1.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

We now give some background on TD pairs; for more information we refer the reader to [22–27,60]. The concept of a TD pair originated in the study of the (P and Q)-polynomial association schemes [3] and their relationship to the Askey scheme of orthogonal polynomials [2,32]. The concept is implicit in [3, p. 263], [36] and more explicit in [53, Theorem 2.1]. A systematic study began in [22]. As research progressed, connections were found to representation theory [1,4,17,21,24,26,28,29,33–35, 49,51,52,56,61], partially ordered sets [57], the bispectral problem [18–20,63], statistical mechanical models [4–10,13–15,50], and classical mechanics [64]. We now recall some basic facts about TD pairs. Let A, A^* denote a TD pair on V, as in Definition 1.1. By [22, Lemma 4.5] the integers d and δ from (ii), (iii) are equal; we call this common value the *diameter* of the pair. By [22, Theorem 10.1] the pair A, A^* satisfies two polynomial equations called the tridiagonal relations; these generalize the q-Serre relations [56, Example 3.6] and the Dolan–Grady relations [56, Example 3.2]. See [9,37,54,56,61] for results on the tridiagonal relations. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [22, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). By [22, Corollary 5.7], for $0 \leqslant i \leqslant d$ the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. By [22, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leqslant i \leqslant d$ and $\rho_{i-1} \leqslant \rho_i$ for $1 \leqslant i \leqslant d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, A^*. The *shape conjecture* [22, Conjecture 13.5] states that

if $\mathbb{F}$ is algebraically closed then $\rho_i \leqslant \binom{d}{i}$ for $0 \leqslant i \leqslant d$. The shape conjecture has been proven for a number of special cases [23,25,28,29]. The TD pair A, A^* is called *sharp* whenever $\rho_0 = 1$. By [47, Theorem 1.3], if $\mathbb{F}$ is algebraically closed then A, A^* is sharp. It is an open problem to classify the sharp TD pairs up to isomorphism, but progress is being made [25,28,29,46,48,62]. The TD pairs of shape $(1, 1, \ldots, 1)$ are called *Leonard pairs* [55, Definition 1.1], and these are classified up to isomorphism [55,58]. This classification yields a correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and their relatives [2,59]. This family coincides with the terminating branch of the Askey scheme [32]. See [38–44,60] and the references therein for results on Leonard pairs.

We now summarize the present paper. For the above TD pair A, A^* let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \leqslant i \leqslant d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for V_i (resp. V_i^*). By [22, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{3}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$. For this constraint the "most general" solution is

$$\theta_i = a + bq^{2i-d} + cq^{d-2i} \quad (0 \leqslant i \leqslant d), \tag{4}$$

$$\theta_i^* = a^* + b^*q^{2i-d} + c^*q^{d-2i} \quad (0 \leqslant i \leqslant d), \tag{5}$$

$$q, a, b, c, a^*, b^*, c^* \in \overline{\overline{\mathbb{F}}}, \tag{6}$$

$$q \neq 0, \qquad q^2 \neq 1, \qquad q^2 \neq -1, \qquad bb^*cc^* \neq 0. \tag{7}$$

For this solution $q^2 + q^{-2} + 1$ is the common value of (3). The TD pair A, A^* is said to have *q-Racah type* whenever (4)–(7) hold. By [58, Theorem 5.16] the Leonard pairs of q-Racah type correspond to the q-Racah polynomials.

In this paper we classify up to isomorphism the TD pairs over an algebraically closed field that have q-Racah type. Our main result is Theorem 3.3. The proof involves the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

2. Tridiagonal systems

When working with a TD pair, it is often convenient to consider a closely related object called a TD system. To define a TD system, we recall a few concepts from linear algebra. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\text{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. Let A denote a diagonalizable element of $\text{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. For $0 \leqslant i \leqslant d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leqslant j \leqslant d)$. Here I denotes the identity of $\text{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $\sum_{i=0}^d E_i = I$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leqslant i, j \leqslant d)$; (iii) $V_i = E_i V$ $(0 \leqslant i \leqslant d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}. \tag{8}$$

Note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{F}$-subalgebra of $\text{End}(V)$ generated by A. Moreover $\prod_{i=0}^d (A - \theta_i I) = 0$. Now let A, A^* denote a TD pair on V. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1. (See [22, Definition 2.1].) Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal system* (or *TD system*) on V we mean a sequence

$$\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \left\{E_i^*\right\}_{i=0}^d\right)$$

that satisfies (i)–(iii) below.

 (i) A, A^* is a TD pair on V.
 (ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We say Φ is *over* $\mathbb{F}$. We call V the *underlying vector space*.

The notion of isomorphism for TD systems is defined in [45, Section 3].

Definition 2.2. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V. For $0 \leqslant i \leqslant d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace $E_i V$ (resp. $E_i^* V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{F}$. We say Φ is *sharp* whenever the TD pair A, A^* is sharp.

The following notation will be useful.

Definition 2.3. Let λ denote an indeterminate and let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{F}$. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ denote scalars in $\mathbb{F}$. Then for $0 \leqslant i \leqslant d$ we define the following polynomials in $\mathbb{F}[\lambda]$:

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),$$

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}),$$

$$\tau_i^* = \left(\lambda - \theta_0^*\right)\left(\lambda - \theta_1^*\right) \cdots \left(\lambda - \theta_{i-1}^*\right),$$

$$\eta_i^* = \left(\lambda - \theta_d^*\right)\left(\lambda - \theta_{d-1}^*\right) \cdots \left(\lambda - \theta_{d-i+1}^*\right).$$

Note that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

We now recall the split sequence of a sharp TD system. This sequence was originally defined in [28, Section 5] using the split decomposition [22, Section 4], but in [48] an alternate definition was introduced that is more convenient to our purpose.

Definition 2.4. (See [48, Definition 2.5].) Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp TD system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By [47, Lemma 5.4], for $0 \leqslant i \leqslant d$ there exists a unique $\zeta_i \in \mathbb{F}$ such that

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}.$$

Note that $\zeta_0 = 1$. We call $\{\zeta_i\}_{i=0}^d$ the *split sequence* of the TD system.

Definition 2.5. Let Φ denote a sharp TD system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ.

The following result shows the significance of the parameter array.

Proposition 2.6. *(See [29], [47, Theorem 1.6].) Two sharp TD systems over $\mathbb{F}$ are isomorphic if and only if they have the same parameter array.*

3. The classification

In this section we state our main result and discuss its significance.

Definition 3.1. Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$. We call this sequence *q-Racah* whenever the following (i), (ii) hold.

 (i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leqslant i, j \leqslant d)$.
(ii) There exist $q, a, b, c, a^*, b^*, c^*$ that satisfy (4)–(7).

Referring to Definition 3.1, condition (i) implies a restriction on the scalars in condition (ii). We now clarify this restriction.

Lemma 3.2. *The following are equivalent for all integers $d \geqslant 0$, nonzero $q \in \overline{\mathbb{F}}$, and $a, b, c \in \overline{\mathbb{F}}$:*

 (i) *The scalars $\{a + bq^{2i-d} + cq^{d-2i}\}_{i=0}^d$ are mutually distinct;*
(ii) *$q^{2i} \neq 1$ for $1 \leqslant i \leqslant d$ and $b \neq cq^{2d-2i}$ for $1 \leqslant i \leqslant 2d - 1$.*

Proof. Routine. $\square$

The following is our main result.

Theorem 3.3. *Assume the field $\mathbb{F}$ is algebraically closed and let d denote a nonnegative integer. Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a q-Racah sequence of scalars in $\mathbb{F}$ and let $\{\zeta_i\}_{i=0}^d$ denote any sequence of scalars in $\mathbb{F}$. Then the following are equivalent:*

 (i) *There exists a TD system Φ over $\mathbb{F}$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$;*
(ii) *$\zeta_0 = 1$, $\zeta_d \neq 0$, and*

$$0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0)\eta_{d-i}^*\big(\theta_0^*\big)\zeta_i. \tag{9}$$

Suppose (i)*,* (ii) *hold. Then Φ is unique up to isomorphism of TD systems.*

Our proof of Theorem 3.3 is contained in Section 10.

We now discuss the significance of Theorem 3.3. The following conjectured classification of TD pairs was introduced in [28, Conjecture 14.6]; see also [45, Conjecture 6.3] and [48, Conjecture 11.1].

Conjecture 3.4. *(See [28, Conjecture 14.6].) Assume the field $\mathbb{F}$ is algebraically closed. Let d denote a nonnegative integer and let*

$$\big(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\big) \tag{10}$$

denote a sequence of scalars taken from $\mathbb{F}$. Then there exists a TD system Φ over $\mathbb{F}$ with parameter array (10) if and only if (i)–(iii) hold below:

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leqslant i, j \leqslant d$).
(ii) $\zeta_0 = 1$, $\zeta_d \neq 0$, and

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}^*(\theta_0^*)\zeta_i.$$

(iii) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$.

Suppose (i)–(iii) *hold. Then Φ is unique up to isomorphism of TD systems.*

The "only if" direction of Conjecture 3.4 was proved in [45, Section 8]. The last assertion of Conjecture 3.4 follows from Proposition 2.6. The "if" direction of Conjecture 3.4 was proved for $d \leqslant 5$ in [48, Theorems 11.2 and 12.1]. Theorem 3.3 establishes the "if" direction of Conjecture 3.4 for the case in which $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ has q-Racah type. We remark that our forthcoming paper [29] contains a comprehensive treatment of the TD pairs for which q is not a root of unity, where $q^2 + q^{-2} + 1$ is the common value of (3). The treatment establishes the "if" direction of Conjecture 3.4 assuming that restriction on q.

4. An outline of the proof for Theorem 3.3

In the proof of Theorem 3.3 the main part is to demonstrate that (ii) implies (i). This demonstration will consume most of the paper from Section 6 through Section 10. Here we summarize the argument.

Assuming $\mathbb{F}$ is algebraically closed, we fix a q-Racah sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ of scalars in $\mathbb{F}$, and a sequence $\{\zeta_i\}_{i=0}^d$ of scalars in $\mathbb{F}$ that satisfy condition (ii) of Theorem 3.3. Our goal is to display a TD system over $\mathbb{F}$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$. To this end we fix $q, a, b, c, a^*, b^*, c^*$ that satisfy (4)–(7). Associated with q is the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ over $\mathbb{F}$ with Chevalley generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$. We consider the $U_q(\widehat{\mathfrak{sl}}_2)$-modules V of the form

$$V = V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_d),$$

where each $V(\alpha_i)$ is a 2-dimensional evaluation module with evaluation parameter α_i. The module V decomposes into a direct sum of weight spaces $\{U_i\}_{i=0}^d$, with $K_0 - q^{2i-d}1$ and $K_1 - q^{d-2i}1$ vanishing on U_i for $0 \leqslant i \leqslant d$. The dimension of U_i is $\binom{d}{i}$ for $0 \leqslant i \leqslant d$. Using b, c, b^*, c^* we obtain certain elements $R, L \in U_q(\widehat{\mathfrak{sl}}_2)$ such that

$$RU_i \subseteq U_{i+1}, \qquad LU_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d),$$

where $U_{-1} = 0$ and $U_{d+1} = 0$. The R, L satisfy some attractive equations and act nicely with respect to the Hopf algebra structure. Using R, L we associate with V a certain monic univariate polynomial P_V of degree d, called the (nonstandard) Drinfel'd polynomial [30]. Using some properties of P_V we show that the parameters $\{\alpha_i\}_{i=1}^d$ can be chosen so that ζ_i is the eigenvalue of $L^i R^i$ on U_0 for $0 \leqslant i \leqslant d$; for the rest of this section we work with this choice. Define

$$A = a1 + bK_0 + cK_1 + R,$$

$$A^* = a^*1 + b^*K_0 + c^*K_1 + L.$$

Using the equations satisfied by R, L we show that A, A^* satisfy a pair of tridiagonal relations. From the construction

$$(A - \theta_i 1)U_i \subseteq U_{i+1}, \qquad (A^* - \theta_i^* 1)U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d).$$

Using this we argue that A (resp. A^*) is diagonalizable on V with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). For $0 \leqslant i \leqslant d$ let E_i (resp. E_i^*) denote the element of $U_q(\widehat{\mathfrak{sl}}_2)$ that acts on V as the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*). Using the tridiagonal relations and a few other facts we show that on V,

$$E_i A^* E_j = 0, \qquad E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1 \ (0 \leqslant i, j \leqslant d). \tag{11}$$

Using $\zeta_d \neq 0$ and (9) we show that on V,

$$E_0^* E_d E_0^* \neq 0, \qquad E_0^* E_0 E_0^* \neq 0. \tag{12}$$

Let T denote the subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by A, A^* and let TE_0^*V denote the T-submodule of V generated by $E_0^*V = U_0$. We show that TE_0^*V contains a unique maximal proper T-submodule; denote this by M and consider the quotient T-module $L = TE_0^*V/M$. By construction the T-module L is irreducible. Using this and (11), (12) we show that the elements $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ act on L as a TD system which we denote by Φ. By the construction Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. We argue using the choice of V that Φ has split sequence $\{\zeta_i\}_{i=0}^d$. Therefore Φ has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ and we have accomplished our goal.

5. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$

The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ belongs to a family of algebras discovered independently by Drinfel'd [16] and Jimbo [31]. In this section we recall some facts about $U_q(\widehat{\mathfrak{sl}}_2)$ that will be useful in the proof of Theorem 3.3. For convenience we follow the notational conventions of Chari and Pressley [11,12].

Throughout this section assume $\mathbb{F}$ is algebraically closed. We fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$, and adopt the following notation:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, 2, \ldots. \tag{13}$$

Definition 5.1. (See [11, p. 262].) The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the associative $\mathbb{F}$-algebra with 1, defined by generators $e_i^{\pm}, K_i^{\pm 1}, i \in \{0, 1\}$ and the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{14}$$

$$K_0 K_1 = K_1 K_0, \tag{15}$$

$$K_i e_i^{\pm} K_i^{-1} = q^{\pm 2} e_i^{\pm}, \tag{16}$$

$$K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm}, \quad i \neq j, \tag{17}$$

$$[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{18}$$

$$[e_0^\pm, e_1^\mp] = 0, \tag{19}$$

$$(e_i^\pm)^3 e_j^\pm - [3]_q (e_i^\pm)^2 e_j^\pm e_i^\pm + [3]_q e_i^\pm e_j^\pm (e_i^\pm)^2 - e_j^\pm (e_i^\pm)^3 = 0, \quad i \neq j. \tag{20}$$

In (18), (19) the expression $[r, s]$ means $rs - sr$. We call $e_i^\pm$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ the *Chevalley generators* for $U_q(\widehat{\mathfrak{sl}}_2)$.

In [11, Section 4] Chari and Pressley consider some finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules $V_n(\alpha)$, where α is a nonzero scalar in $\mathbb{F}$ and n is a positive integer. These modules are called evaluation modules. The scalar α is the evaluation parameter and $n + 1$ is the dimension. We will make use of $V_1(\alpha)$; for notational convenience we denote this module by $V(\alpha)$.

Definition 5.2. (See [11, Section 4].) For all nonzero $\alpha \in \mathbb{F}$ the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\alpha)$ has a basis x, y on which the Chevalley generators act as follows:

$$
\begin{aligned}
K_1 x &= qx, & K_1 y &= q^{-1} y, \\
e_1^- x &= y, & e_1^- y &= 0, \\
e_1^+ x &= 0, & e_1^+ y &= x, \\
K_0 x &= q^{-1} x, & K_0 y &= qy, \\
e_0^- x &= 0, & e_0^- y &= q\alpha^{-1} x, \\
e_0^+ x &= q^{-1}\alpha y, & e_0^+ y &= 0.
\end{aligned}
$$

We now recall how the tensor product of two $U_q(\widehat{\mathfrak{sl}}_2)$-modules becomes a $U_q(\widehat{\mathfrak{sl}}_2)$-module. In what follows all unadorned tensor products are meant to be over $\mathbb{F}$.

Lemma 5.3. *(See [11, p. 263].)* $U_q(\widehat{\mathfrak{sl}}_2)$ *has the following Hopf algebra structure. The comultiplication Δ satisfies*

$$
\begin{aligned}
\Delta(e_i^+) &= e_i^+ \otimes K_i + 1 \otimes e_i^+, \\
\Delta(e_i^-) &= e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-, \\
\Delta(K_i) &= K_i \otimes K_i.
\end{aligned}
$$

The counit ε satisfies

$$\varepsilon(e_i^\pm) = 0, \qquad \varepsilon(K_i) = 1.$$

The antipode S satisfies

$$S(K_i) = K_i^{-1}, \qquad S(e_i^+) = -e_i^+ K_i^{-1}, \qquad S(e_i^-) = -K_i e_i^-.$$

Combining Lemma 5.3 with [12, p. 110] we routinely obtain the following.

Lemma 5.4. *Let V, W denote $U_q(\widehat{\mathfrak{sl}}_2)$-modules. Then the tensor product $V \otimes W$ has the following $U_q(\widehat{\mathfrak{sl}}_2)$-module structure. For $v \in V$, for $w \in W$ and for $i \in \{0, 1\}$,*

$$e_i^+(v \otimes w) = e_i^+ v \otimes K_i w + v \otimes e_i^+ w,$$

$$e_i^-(v \otimes w) = e_i^- v \otimes w + K_i^{-1} v \otimes e_i^- w,$$

$$K_i(v \otimes w) = K_i v \otimes K_i w.$$

Definition 5.5. (See [12, p. 110].) There exists a one-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module on which each element $z \in U_q(\widehat{\mathfrak{sl}}_2)$ acts as $\varepsilon(z)I$, where ε is from Lemma 5.3 and I is the identity map. In particular on this module each of $e_0^\pm$, $e_1^\pm$ vanishes and each of $K_0^{\pm 1}$, $K_1^{\pm 1}$ acts as I. This module is irreducible and unique up to isomorphism. We call this module the *trivial $U_q(\widehat{\mathfrak{sl}}_2)$-module*.

Definition 5.6. Let d denote a nonnegative integer. By a *standard $U_q(\widehat{\mathfrak{sl}}_2)$-module of diameter d* we mean

$$V(\alpha_1) \otimes V(\alpha_2) \otimes \cdots \otimes V(\alpha_d), \tag{21}$$

where $0 \neq \alpha_i \in \mathbb{F}$ for $1 \leqslant i \leqslant d$. For $d = 0$ we interpret (21) to be the trivial $U_q(\widehat{\mathfrak{sl}}_2)$-module.

For our purpose we only need those standard $U_q(\widehat{\mathfrak{sl}}_2)$-modules of diameter d such that $q^{2i} \neq 1$ for $1 \leqslant i \leqslant d$. The following definition will facilitate our discussion of these modules.

Definition 5.7. An integer d will be called *feasible* (with respect to q) whenever $d \geqslant 0$ and $q^{2i} \neq 1$ for $1 \leqslant i \leqslant d$. Note that 0 and 1 are feasible.

The following result is immediate from Definition 5.7.

Lemma 5.8. *For all feasible integers d the scalars $\{q^{d-2i}\}_{i=0}^d$ are mutually distinct.*

Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter d. The $\mathbb{F}$-vector space V has a basis

$$v_1 \otimes v_2 \otimes \cdots \otimes v_d, \quad v_i \in \{x, y\} \ (1 \leqslant i \leqslant d). \tag{22}$$

Note that V has dimension 2^d. For notational convenience we abbreviate the basis (22) as follows. For all subsets $s \subseteq \{1, 2, \ldots, d\}$ define $u_s = v_1 \otimes v_2 \otimes \cdots \otimes v_d$, where $v_i = x$ if $i \notin s$ and $v_i = y$ if $i \in s$ $(1 \leqslant i \leqslant d)$. For example $u_\emptyset = x \otimes x \otimes \cdots \otimes x$. Pick a subset $s \subseteq \{1, 2, \ldots, d\}$. By Lemma 5.4 we have

$$K_0 u_s = q^{2|s|-d} u_s, \qquad K_1 u_s = q^{d-2|s|} u_s, \tag{23}$$

where $|s|$ denotes the cardinality of s. Thus each of K_0, K_1 is diagonalizable on V with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. Moreover K_0, K_1 are inverses of one another on V. For $0 \leqslant i \leqslant d$ define

$$U_i = \text{Span}\{u_s \mid s \subseteq \{1, 2, \ldots, d\}, \ |s| = i\}.$$

Note that $V = \sum_{i=0}^d U_i$ (direct sum), and that U_i has dimension $\binom{d}{i}$ for $0 \leqslant i \leqslant d$. Moreover

$$\left(K_0 - q^{2i-d}1\right)U_i = 0, \qquad \left(K_1 - q^{d-2i}1\right)U_i = 0 \tag{24}$$

for $0 \leqslant i \leqslant d$. Combining (16), (17) with (24) we find that for $0 \leqslant i \leqslant d$,

$$e_0^+ U_i \subseteq U_{i+1}, \qquad e_1^- U_i \subseteq U_{i+1}, \tag{25}$$

$$e_0^- U_i \subseteq U_{i-1}, \qquad e_1^+ U_i \subseteq U_{i-1}, \tag{26}$$

where $U_{-1} = 0$ and $U_{d+1} = 0$. We call the sequence $\{U_i\}_{i=0}^d$ the *weight space decomposition* of V. We call $u_\emptyset$ the *highest weight vector* of V. The action of $e_0^\pm$, $e_1^\pm$ on the basis (22) can be obtained using Lemma 5.4 but the answer is slightly complicated. For all subsets $s \subseteq \{1, \ldots, d\}$ let $\bar{s}$ denote the complement of s in $\{1, \ldots, d\}$. By Lemma 5.4,

$$e_1^- u_s = \sum_{i \in \bar{s}} u_{s \cup i} q^{|\{1,2,\ldots,i-1\} \cap s| - |\{1,2,\ldots,i-1\} \cap \bar{s}|}, \tag{27}$$

$$e_1^+ u_s = \sum_{i \in s} u_{s \backslash i} q^{|\{i+1,i+2,\ldots,d\} \cap \bar{s}| - |\{i+1,i+2,\ldots,d\} \cap s|}, \tag{28}$$

$$e_0^- u_s = \sum_{i \in s} u_{s \backslash i} \alpha_i^{-1} q^{|\{1,2,\ldots,i-1\} \cap \bar{s}| - |\{1,2,\ldots,i-1\} \cap s| + 1}, \tag{29}$$

$$e_0^+ u_s = \sum_{i \in \bar{s}} u_{s \cup i} \alpha_i q^{|\{i+1,i+2,\ldots,d\} \cap s| - |\{i+1,i+2,\ldots,d\} \cap \bar{s}| - 1}. \tag{30}$$

6. The elements R, L of $U_q(\widehat{\mathfrak{sl}}_2)$

From now until the end of Section 7 we adopt the following assumption.

Assumption 6.1. We assume the field $\mathbb{F}$ is algebraically closed. We fix nonzero scalars q, b, c, b^*, c^* in $\mathbb{F}$ such that $q^2 \neq 1$.

In this section we define the elements R, L of $U_q(\widehat{\mathfrak{sl}}_2)$ and discuss their basic properties.

Definition 6.2. We define

$$R = u e_0^+ + v e_1^- K_1, \tag{31}$$

$$L = u^* e_1^+ + v^* e_0^- K_0, \tag{32}$$

where u, v, u^*, v^* are any scalars in $\mathbb{F}$ such that

$$u v^* = -b b^* q^{-1} (q - q^{-1})^2, \tag{33}$$

$$v u^* = -c c^* q^{-1} (q - q^{-1})^2. \tag{34}$$

Note that u, v, u^*, v^* are nonzero.

Note 6.3. Referring to Definition 6.2, the choice of u, v, u^*, v^* is immaterial and we could fix specific values for the duration of the paper. But doing so tends to obscure the essential relationships (33), (34).

Lemma 6.4. *We have*

$$K_0 R K_0^{-1} = q^2 R, \qquad K_1 R K_1^{-1} = q^{-2} R, \tag{35}$$

$$K_0 L K_0^{-1} = q^{-2} L, \qquad K_1 L K_1^{-1} = q^2 L. \tag{36}$$

Proof. Use (14)–(17) and Definition 6.2. □

Let $U_q(L(\mathfrak{sl}_2))$ denote the quotient of $U_q(\widehat{\mathfrak{sl}}_2)$ by the two sided ideal generated by $K_0 K_1 - 1$. The name $U_q(L(\mathfrak{sl}_2))$ is motivated by the fact that this algebra is a q-deformation of the universal enveloping algebra of the loop algebra $L(\mathfrak{sl}_2) = \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$. This is discussed in [11, Section 3.3]. In what follows, we will use the same notation for an element of $U_q(\widehat{\mathfrak{sl}}_2)$ and its image in $U_q(L(\mathfrak{sl}_2))$.

Lemma 6.5. *The following hold in* $U_q(L(\mathfrak{sl}_2))$:

$$R^3 L - [3]_q R^2 L R + [3]_q R L R^2 - L R^3$$
$$= (q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})(cc^* K_1 R^2 K_1 - bb^* K_0 R^2 K_0), \tag{37}$$
$$L^3 R - [3]_q L^2 R L + [3]_q L R L^2 - R L^3$$
$$= (q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})(bb^* K_0 L^2 K_0 - cc^* K_1 L^2 K_1). \tag{38}$$

Proof. To verify (37), eliminate R, L using Definition 6.2 and simplify the result using the relations in Definition 5.1, together with the fact in $U_q(L(\mathfrak{sl}_2))$ the elements K_0, K_1 become inverses of one another. Eq. (38) is similarly verified. □

Lemma 6.6. *For all integers* $n \geqslant 0$ *the element* $L^n R^n$ *commutes with each of* K_0, K_1.

Proof. Use Lemma 6.4. □

Lemma 6.7. *Let* V *denote a standard* $U_q(\widehat{\mathfrak{sl}}_2)$-*module with feasible diameter* d *and let* $\{U_i\}_{i=0}^d$ *denote the corresponding weight space decomposition. Then*

$$R U_i \subseteq U_{i+1}, \qquad L U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d). \tag{39}$$

Proof. Use (24)–(26) and Definition 6.2. □

7. The split sequence of a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module

Throughout this section Assumption 6.1 remains in effect. We fix elements $R, L \in U_q(\widehat{\mathfrak{sl}}_2)$ as in Definition 6.2.

Definition 7.1. Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter d and let $\{U_i\}_{i=0}^d$ denote the corresponding weight space decomposition. By Lemma 6.7, for $0 \leqslant i \leqslant d$ the space U_0 is invariant under $L^i R^i$; let ζ_i denote the corresponding eigenvalue. Note that $\zeta_0 = 1$. We call the sequence $\{\zeta_i\}_{i=0}^d$ the *split sequence* of V.

Our goal in this section is to obtain the following result.

Proposition 7.2. *Let* d *denote a feasible integer and let* $\{\zeta_i\}_{i=0}^d$ *denote a sequence of scalars in* $\mathbb{F}$ *such that* $\zeta_0 = 1$. *Then there exists a standard* $U_q(\widehat{\mathfrak{sl}}_2)$-*module* V *of diameter* d *that has split sequence* $\{\zeta_i\}_{i=0}^d$.

In order to prove Proposition 7.2 we will consider a generating function involving the split sequence called the (*nonstandard*) *Drinfel'd polynomial*. We will define this polynomial shortly.

Definition 7.3. Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter d. For $0 \leqslant i \leqslant d$ define

$$\sigma_i = \frac{\zeta_i}{(q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^i - q^{-i})^2}, \qquad (40)$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of V. The denominator in (40) is nonzero by Definition 5.7. Observe that $\sigma_0 = 1$. We call $\{\sigma_i\}_{i=0}^d$ the *normalized split sequence* of V.

Definition 7.4. For all integers $i \geqslant 0$ define $f_i \in \mathbb{F}[\lambda]$ by

$$f_i = bb^* q^{-2i} + cc^* q^{2i} - \lambda,$$

where q, b, b^*, c, c^* are from Assumption 6.1.

Definition 7.5. Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter d. We define a polynomial $P_V \in \mathbb{F}[\lambda]$ by

$$P_V = (-1)^d \sum_{i=0}^d \sigma_{d-i} f_0 f_1 \cdots f_{i-1}, \qquad (41)$$

where $\{\sigma_i\}_{i=0}^d$ is the normalized split sequence of V. We observe that P_V is monic with degree d. We call P_V the (*nonstandard*) *Drinfel'd polynomial* of V.

Note 7.6. The Drinfel'd polynomial from [30, Definition 9.3] is equal to the polynomial P_V from Definition 7.5, times $(-1)^d (q - q^{-1})^2 (q^2 - q^{-2})^2 \cdots (q^d - q^{-d})^2$.

From now on, when we refer to the Drinfel'd polynomial we mean the nonstandard Drinfel'd polynomial from Definition 7.5.

We now compute the Drinfel'd polynomial for a few easy cases.

Lemma 7.7. *Let V denote the trivial $U_q(\widehat{\mathfrak{sl}}_2)$-module from Definition 5.5. Then $P_V = 1$.*

Proof. Routine. $\square$

Lemma 7.8. *Pick a nonzero $\alpha \in \mathbb{F}$ and consider the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V = V(\alpha)$ from Definition 5.2. The corresponding Drinfel'd polynomial satisfies*

$$P_V = \lambda - \frac{\alpha uu^* q^{-2} + \alpha^{-1} vv^* q^2}{q^{-1}(q - q^{-1})^2}, \qquad (42)$$

where u, v, u^*, v^* are from Definition 6.2.

Proof. Let σ_1 denote term one of the normalized split sequence for V. We show

$$\sigma_1 = \frac{\alpha uu^* q^{-2} + \alpha^{-1} vv^* q^2}{q^{-1}(q - q^{-1})^2} - bb^* - cc^*. \qquad (43)$$

By Definitions 7.1 and 7.3, $\sigma_1 = \zeta_1 (q - q^{-1})^{-2}$ where ζ_1 is the eigenvalue of LR on the weight space U_0. Let x, y denote the basis for V from Definition 5.2. By construction x is a basis for U_0, so x is an eigenvector for LR with eigenvalue ζ_1. Using Definitions 5.2 and 6.2 we find $Rx = (uq^{-1}\alpha + vq)y$ and $Ly = (u^* + v^*q^2\alpha^{-1})x$; therefore $\zeta_1 = (uq^{-1}\alpha + vq)(u^* + v^*q^2\alpha^{-1})$. Evaluating this using (33), (34) and $\sigma_1 = \zeta_1 (q - q^{-1})^{-2}$ we obtain (43). Setting $d = 1$ and $\sigma_0 = 1$ in (41) we find $P_V = -\sigma_1 - f_0$. Evaluating this using (43) and $f_0 = bb^* + cc^* - \lambda$ we obtain (42). □

Consider a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module $V = \bigotimes_{i=1}^{d} V(\alpha_i)$ with feasible diameter d. We will now show that the Drinfel'd polynomial of V is equal to the product of the Drinfel'd polynomials $\prod_{i=1}^{d} P_{V(\alpha_i)}$.

Lemma 7.9. *The comultiplication Δ from Lemma 5.3 acts on the elements R, L as follows.*

$$\Delta(R) = 1 \otimes R + u e_0^+ \otimes K_0 + v e_1^- K_1 \otimes K_1, \tag{44}$$

$$\Delta(L) = 1 \otimes L + u^* e_1^+ \otimes K_1 + v^* e_0^- K_0 \otimes K_0. \tag{45}$$

Proof. Use Lemma 5.3, Definition 6.2, and the fact that Δ is an algebra homomorphism. □

Lemma 7.10. *Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with diameter 1. Let W denote any $U_q(\widehat{\mathfrak{sl}}_2)$-module. Then for all integers $n \geqslant 1$ the following* (i), (ii) *hold on $V \otimes W$:*

(i) $\Delta(R^n) = 1 \otimes R^n + [n]_q R_n$ *where*

$$R_n = u q^{n-1} e_0^+ \otimes R^{n-1} K_0 + v q^{1-n} e_1^- K_1 \otimes R^{n-1} K_1. \tag{46}$$

(ii) $\Delta(L^n) = 1 \otimes L^n + [n]_q L_n$ *where*

$$L_n = u^* q^{1-n} e_1^+ \otimes K_1 L^{n-1} + v^* q^{n-1} e_0^- K_0 \otimes K_0 L^{n-1}. \tag{47}$$

Proof. (i) The proof is by induction on n. First assume $n = 1$. Then the result is immediate from (44). Next assume $n \geqslant 2$. By (44) and since $\Delta(R^n) = \Delta(R^{n-1})\Delta(R)$, the expression $\Delta(R^n) - 1 \otimes R^n$ is equal to

$$\left(\Delta(R^{n-1}) - 1 \otimes R^{n-1}\right)(1 \otimes R) \tag{48}$$

plus u times

$$\Delta(R^{n-1})(e_0^+ \otimes K_0) \tag{49}$$

plus v times

$$\Delta(R^{n-1})(e_1^- K_1 \otimes K_1). \tag{50}$$

We now find the action of (48)–(50) on $V \otimes W$. By induction, on $V \otimes W$ the expression $\Delta(R^{n-1}) - 1 \otimes R^{n-1}$ is equal to $[n-1]_q$ times

$$u q^{n-2} e_0^+ \otimes R^{n-2} K_0 + v q^{2-n} e_1^- K_1 \otimes R^{n-2} K_1. \tag{51}$$

By this and Lemma 6.4, on $V \otimes W$ the expression (48) is equal to $[n-1]_q$ times

$$uq^n e_0^+ \otimes R^{n-1}K_0 + vq^{-n}e_1^- K_1 \otimes R^{n-1}K_1.$$

By (25) and since V has diameter 1, the elements $(e_0^+)^2$ and $e_1^- e_0^+$ are zero on V. Therefore (51) times $e_0^+ \otimes K_0$ is zero on $V \otimes W$, so (49) is equal to $e_0^+ \otimes R^{n-1}K_0$ on $V \otimes W$. Similarly (50) is equal to $e_1^- K_1 \otimes R^{n-1}K_1$ on $V \otimes W$. By these comments we routinely obtain $\Delta(R^n) = 1 \otimes R^n + [n]_q R_n$.

(ii) Similar to the proof of (i) above. $\square$

Proposition 7.11. *Pick a feasible integer $d \geqslant 1$. Let V denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with diameter 1, and let W denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with diameter $d-1$. Note that the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V \otimes W$ is standard with diameter d. The normalized split sequence for $V \otimes W$ is described as follows:*

$$\sigma_0(V \otimes W) = 1,$$
$$\sigma_n(V \otimes W) = \left(q^{d-n} - q^{n-d}\right)\left(bb^* q^{n-d} - cc^* q^{d-n}\right)\sigma_{n-1}(W) + \sigma_n(W) + \sigma_1(V)\sigma_{n-1}(W)$$
$$(1 \leqslant n \leqslant d-1),$$
$$\sigma_d(V \otimes W) = \sigma_1(V)\sigma_{d-1}(W).$$

Proof. Let x denote the highest weight vector in V, and observe

$$\left(K_0 - q^{-1}1\right)x = 0, \qquad (K_1 - q1)x = 0. \tag{52}$$

Let ξ denote the highest weight vector for W, and observe

$$\left(K_0 - q^{1-d}1\right)\xi = 0, \qquad \left(K_1 - q^{d-1}1\right)\xi = 0. \tag{53}$$

Note that $x \otimes \xi$ is the highest weight vector for $V \otimes W$. We claim that for all integers $n \geqslant 1$,

$$L^n R^n(x \otimes \xi) - x \otimes L^n R^n \xi - [n]_q^2 LRx \otimes L^{n-1}R^{n-1}\xi$$
$$= \left(q^n - q^{-n}\right)^2 \left(q^{d-n} - q^{n-d}\right)\left(bb^* q^{n-d} - cc^* q^{d-n}\right)x \otimes L^{n-1}R^{n-1}\xi. \tag{54}$$

To prove the claim we evaluate the left-hand side of (54). The term $L^n R^n(x \otimes \xi)$ coincides with the image of $\Delta(L^n)\Delta(R^n)$ on $x \otimes \xi$. Computing this image using Lemma 7.10 one encounters the terms $L_n(1 \otimes R^n)$ and $(1 \otimes L^n)R_n$. Using (52) and $e_1^+ x = 0$, $e_0^- x = 0$ we find $L_n(1 \otimes R^n)$ is zero on $x \otimes \xi$. Using (53) and $L^n R^{n-1}\xi = 0$ we find $(1 \otimes L^n)R_n$ is zero on $x \otimes \xi$. By these comments and Lemma 7.10 the left-hand side of (54) is equal to $[n]_q^2$ times

$$L_n R_n(x \otimes \xi) - LRx \otimes L^{n-1}R^{n-1}\xi. \tag{55}$$

The two terms in (55) are evaluated as follows. To evaluate the first term use (46) and (47). To evaluate the second term expand LRx using Definition 6.2. Now reduce further using (16), (52), (53) along with Definition 6.2, Lemma 6.6 and

$$\left(e_0^- e_0^+ - 1\right)x = 0, \qquad \left(e_1^+ e_1^- - 1\right)x = 0.$$

The reduction shows that the left-hand side of (54) is equal to the right-hand side of (54). The claim is now proved. As we examine the terms in (54), we note the following from Definitions 7.1, 7.3. For

$0 \leqslant n \leqslant d$ the vector $x \otimes \xi$ is an eigenvector for $L^n R^n$ with eigenvalue

$$\left(q - q^{-1}\right)^2 \left(q^2 - q^{-2}\right)^2 \cdots \left(q^n - q^{-n}\right)^2 \sigma_n(V \otimes W).$$

The vector x is an eigenvector for LR with eigenvalue $(q - q^{-1})^2 \sigma_1(V)$. For $0 \leqslant n \leqslant d-1$ the vector ξ is an eigenvector for $L^n R^n$ with eigenvalue

$$\left(q - q^{-1}\right)^2 \left(q^2 - q^{-2}\right)^2 \cdots \left(q^n - q^{-n}\right)^2 \sigma_n(W).$$

Also $L^d R^d \xi = 0$ by Lemma 6.7 and since W has diameter $d - 1$. Evaluating (54) using these facts we obtain the result. $\quad\square$

Proposition 7.12. *With the notation and assumptions of Proposition 7.11, we have $P_{V \otimes W} = P_V P_W$.*

Proof. By Definition 7.5 we have $P_V = -\sigma_1(V) - f_0$. Again using Definition 7.5,

$$P_W = (-1)^{d-1} \sum_{j=0}^{d-1} \sigma_{d-1-j}(W) f_0 f_1 \cdots f_{j-1}, \qquad (56)$$

$$P_{V \otimes W} = (-1)^d \sum_{i=0}^{d} \sigma_{d-i}(V \otimes W) f_0 f_1 \cdots f_{i-1}. \qquad (57)$$

Using Definition 7.4,

$$f_0 = f_j + \left(q^j - q^{-j}\right)\left(bb^* q^{-j} - cc^* q^j\right) \quad (0 \leqslant j \leqslant d-1). \qquad (58)$$

In Eq. (56) we multiply both sides by P_V and use (58) to get

$$P_V P_W = (-1)^d \sum_{j=0}^{d-1} \sigma_{d-1-j}(W) f_0 f_1 \cdots f_{j-1}\left(f_j + \left(q^j - q^{-j}\right)\left(bb^* q^{-j} - cc^* q^j\right) + \sigma_1(V)\right). \qquad (59)$$

In (59) the sum is a linear combination of $\{f_0 f_1 \cdots f_{i-1}\}_{i=0}^{d}$. In this linear combination, for $0 \leqslant i \leqslant d$ let γ_i denote the coefficient of $f_0 f_1 \cdots f_{i-1}$. We show

$$\gamma_i = \sigma_{d-i}(V \otimes W). \qquad (60)$$

First assume $i = 0$. Then (60) holds since both sides equal $\sigma_1(V)\sigma_{d-1}(W)$. Next assume $1 \leqslant i \leqslant d-1$. By construction

$$\gamma_i = \left(q^i - q^{-i}\right)\left(bb^* q^{-i} - cc^* q^i\right)\sigma_{d-1-i}(W) + \sigma_1(V)\sigma_{d-1-i}(W) + \sigma_{d-i}(W).$$

Evaluating this using Proposition 7.11 we routinely obtain (60). Next assume $i = d$. Then (60) holds since both sides equal 1. We have verified (60) for $0 \leqslant i \leqslant d$. Therefore

$$P_V P_W = (-1)^d \sum_{i=0}^{d} \sigma_{d-i}(V \otimes W) f_0 f_1 \cdots f_{i-1}.$$

Comparing this with (57) we obtain $P_{V \otimes W} = P_V P_W$. $\quad\square$

Proposition 7.13. *Let $V = \bigotimes_{i=1}^{d} V(\alpha_i)$ denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter d. Then the Drinfel'd polynomial P_V is given by*

$$P_V = \prod_{i=1}^{d} P_{V(\alpha_i)}. \tag{61}$$

Proof. Use Proposition 7.12 and induction on *d*. □

We will make a few more comments on the Drinfel'd polynomial and then prove Proposition 7.2.

Lemma 7.14. *For all $r \in \mathbb{F}$ there exists a nonzero $\alpha \in \mathbb{F}$ such that $\lambda - r$ is the Drinfel'd polynomial for $V(\alpha)$.*

Proof. Since $\mathbb{F}$ is algebraically closed and $uu^*vv^* \neq 0$, there exists a nonzero $\alpha \in \mathbb{F}$ such that the fraction on the right in (42) is equal to *r*. The result follows in view of Lemma 7.8. □

Note 7.15. Referring to Lemma 7.14, for a given *r* the scalar α is not uniquely determined in general. If α is a solution then $q^4 vv^* u^{-1} (u^*)^{-1} \alpha^{-1}$ is also a solution and there is no further solution.

Proposition 7.16. *Let d denote a feasible integer and let $P \in \mathbb{F}[\lambda]$ denote a monic polynomial of degree d. Then there exists a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module V of diameter d such that $P_V = P$.*

Proof. Since $\mathbb{F}$ is algebraically closed there exist scalars $\{r_i\}_{i=1}^{d}$ in $\mathbb{F}$ such that $P = \prod_{i=1}^{d}(\lambda - r_i)$. By Lemma 7.14, for $1 \leqslant i \leqslant d$ there exists a nonzero $\alpha_i \in \mathbb{F}$ such that $\lambda - r_i$ is the Drinfel'd polynomial for $V(\alpha_i)$. Define the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V = \bigotimes_{i=1}^{d} V(\alpha_i)$. By construction *V* is standard with diameter *d*. Also $P_V = \prod_{i=1}^{d}(\lambda - r_i)$ by Proposition 7.13, so $P_V = P$. The result follows. □

Proof of Proposition 7.2. For $0 \leqslant i \leqslant d$ define

$$\sigma_i = \frac{\zeta_i}{(q-q^{-1})^2(q^2-q^{-2})^2 \cdots (q^i-q^{-i})^2}. \tag{62}$$

Define a polynomial $P \in \mathbb{F}[\lambda]$ by

$$P = (-1)^d \sum_{i=0}^{d} \sigma_{d-i} f_0 f_1 \cdots f_{i-1} \tag{63}$$

where the f_j are from Definition 7.4. Observe that *P* is monic with degree *d*. By Proposition 7.16 there exists a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module of diameter *d* such that $P_V = P$. Comparing (41), (63) we find the sequence $\{\sigma_i\}_{i=0}^{d}$ from (62) is the normalized split sequence for *V*. Comparing (40), (62) we find $\{\zeta_i\}_{i=0}^{d}$ is the split sequence for *V*. □

We finish this section with some formulae for later use.

Let *V* denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter $d \geqslant 1$, and let $\{U_i\}_{i=0}^{d}$ denote the corresponding weight space decomposition. Let ζ_1 denote term one of the split sequence for *V* and recall from Definition 7.1 that ζ_1 is the eigenvalue of *LR* on U_0. By Lemma 6.7 the space U_d is invariant under *RL*; let $\zeta_1^{\times}$ denote the corresponding eigenvalue.

Lemma 7.17. *Let $V = \bigotimes_{i=1}^{d} V(\alpha_i)$ denote a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module with feasible diameter $d \geqslant 1$. Then the following* (i)–(iii) *hold.*

(i) $\zeta_1 = uu^*q^{-1}\sum_{i=1}^{d}\alpha_i + vv^*q^3\sum_{i=1}^{d}\alpha_i^{-1} - (q-q^{-1})(q^d - q^{-d})(bb^*q^{1-d} + cc^*q^{d-1})$;

(ii) $\zeta_1^\times = uu^*q^{-1}\sum_{i=1}^{d}\alpha_i + vv^*q^3\sum_{i=1}^{d}\alpha_i^{-1} - (q-q^{-1})(q^d - q^{-d})(bb^*q^{d-1} + cc^*q^{1-d})$;

(iii) $\zeta_1 - \zeta_1^\times = (q-q^{-1})(q^{d-1} - q^{1-d})(q^d - q^{-d})(bb^* - cc^*)$.

Proof. Parts (i), (ii) are routinely obtained using (23), (27)–(30) and Definition 6.2. Part (iii) follows from (i), (ii). $\square$

8. The elements A, A^* of $U_q(\widehat{\mathfrak{sl}}_2)$

From now until the end of Section 9 we adopt the following assumption.

Assumption 8.1. Assume the field $\mathbb{F}$ is algebraically closed. We fix a q-Racah sequence $(\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})$ of scalars in $\mathbb{F}$. We fix $q, a, b, c, a^*, b^*, c^*$ that satisfy (4)–(7) and fix $R, L \in U_q(\widehat{\mathfrak{sl}}_2)$ as in Definition 6.2.

In this section we define the elements $A, A^* \in U_q(\widehat{\mathfrak{sl}}_2)$ and investigate their properties.

Definition 8.2. With reference to Assumption 8.1 we define

$$A = a1 + bK_0 + cK_1 + R, \tag{64}$$

$$A^* = a^*1 + b^*K_0 + c^*K_1 + L. \tag{65}$$

We are going to show that A, A^* satisfy a pair of tridiagonal relations. We now introduce the parameters involved in those relations.

Definition 8.3. Define $\beta = q^2 + q^{-2}$ and

$$\gamma = -a(q - q^{-1})^2, \qquad \varrho = a^2(q - q^{-1})^2 - bc(q^2 - q^{-2})^2, \tag{66}$$

$$\gamma^* = -a^*(q - q^{-1})^2, \qquad \varrho^* = a^{*2}(q - q^{-1})^2 - b^*c^*(q^2 - q^{-2})^2. \tag{67}$$

We mention one significance of the parameters in Definition 8.3.

Lemma 8.4. *The following* (i)–(iv) *hold*:

(i) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ $(1 \leqslant i \leqslant d-1)$;

(ii) $\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$ $(1 \leqslant i \leqslant d-1)$;

(iii) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i)$ $(1 \leqslant i \leqslant d)$;

(iv) $\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*)$ $(1 \leqslant i \leqslant d)$.

Proof. Routine verification using (4), (5) and Definition 8.3. $\square$

Proposition 8.5. *In* $U_q(L(\mathfrak{sl}_2))$,

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A),$$

$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \varrho^*(A^*A - AA^*),$$

where $\gamma, \gamma^*, \varrho, \varrho^*$ *are from Definition* 8.3.

Proof. Routine verification using Lemmas 6.4, 6.5, and Definition 8.2. $\square$

Note 8.6. The above equations are called the *tridiagonal relations* [56].

9. The action of A, A^* on a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module

Throughout this section Assumption 8.1 remains in effect. We fix a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module V with diameter d and let $\{U_i\}_{i=0}^d$ denote the corresponding weight space decomposition. Let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of V.

In this section we describe the action of A, A^* on V.

Lemma 9.1. *We have $q^{2i} \neq 1$ for $1 \leqslant i \leqslant d$. In other words d is feasible with respect to q.*

Proof. Immediate from Lemma 3.2. $\quad\square$

Lemma 9.2. *The following hold for $0 \leqslant i \leqslant d$:*

(i) *The element R acts on U_i as $A - \theta_i 1$.*
(ii) *The element L acts on U_i as $A^* - \theta_i^* 1$.*

Proof. Immediate from Definition 8.2 and (24). $\quad\square$

Lemma 9.3. *The following hold for $0 \leqslant i \leqslant d$:*

(i) $(A - \theta_i 1)U_i \subseteq U_{i+1}$,
(ii) $(A^* - \theta_i^* 1)U_i \subseteq U_{i-1}$.

Proof. (i) Combine Lemma 9.2(i) with the inclusion on the left in (39).
(ii) Combine Lemma 9.2(ii) with the inclusion on the right in (39). $\quad\square$

Lemma 9.4. *The element A (resp. A^*) is diagonalizable on V with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). Moreover for $0 \leqslant i \leqslant d$ the dimension of the eigenspace for A (resp. A^*) associated with θ_i (resp. θ_i^*) is equal to $\binom{d}{i}$.*

Proof. We first display the eigenvalues of A. Recall that $\{\theta_i\}_{i=0}^d$ are mutually distinct, and U_i has dimension $\binom{d}{i}$ for $0 \leqslant i \leqslant d$. By Lemma 9.3(i) we see that, with respect to an appropriate basis for V, A is represented by a lower triangular matrix that has diagonal entries $\{\theta_i\}_{i=0}^d$, with θ_i appearing $\binom{d}{i}$ times for $0 \leqslant i \leqslant d$. Hence for $0 \leqslant i \leqslant d$ the scalar θ_i is a root of the characteristic polynomial of A with multiplicity $\binom{d}{i}$. We now show A is diagonalizable. To do this we show that the minimal polynomial of A has distinct roots. By Lemma 9.3(i) we find $\prod_{i=0}^d (A - \theta_i 1)$ vanishes on V. By this and since $\{\theta_i\}_{i=0}^d$ are distinct we see that the minimal polynomial of A has distinct roots. Therefore A is diagonalizable. We have now proved our assertions concerning A; our assertions concerning A^* are similarly proved. $\quad\square$

At this point it is convenient to introduce the primitive idempotents for A and A^*.

Definition 9.5. For $0 \leqslant i \leqslant d$ we define the following elements in $U_q(\widehat{\mathfrak{sl}}_2)$:

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j 1}{\theta_i - \theta_j}, \qquad E_i^* = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A^* - \theta_j^* 1}{\theta_i^* - \theta_j^*}. \tag{68}$$

We observe that E_i (resp. E_i^*) acts on V as the primitive idempotent of A (resp. A^*) associated with the eigenvalue θ_i (resp. θ_i^*). In particular $E_i V$ (resp. $E_i^* V$) is the eigenspace of A (resp. A^*) on V associated with the eigenvalue θ_i (resp. θ_i^*).

Lemma 9.6. *Each of $E_i V$ and $E_i^* V$ has dimension $\binom{d}{i}$ for $0 \leqslant i \leqslant d$.*

Proof. Immediate from Definition 9.5 and the last sentence in Lemma 9.4. $\quad\square$

Lemma 9.7. *The following hold for $0 \leqslant i \leqslant d$:*

(i) $E_i V + \cdots + E_d V = U_i + \cdots + U_d$,
(ii) $E_0^* V + \cdots + E_i^* V = U_0 + \cdots + U_i$.

Proof. (i) Let $X_i = \sum_{j=i}^{d} U_j$ and $X_i' = \sum_{j=i}^{d} E_j V$. We show $X_i = X_i'$. Define $T_i = \prod_{j=i}^{d}(A - \theta_j 1)$. Then $X_i' = \{v \in V \mid T_i v = 0\}$, and $T_i X_i = 0$ by Lemma 9.3(i), so $X_i \subseteq X_i'$. Now define $S_i = \prod_{j=0}^{i-1}(A - \theta_j 1)$. Observe that $S_i V = X_i'$, and $S_i V \subseteq X_i$ by Lemma 9.3(i), so $X_i' \subseteq X_i$. By these comments $X_i = X_i'$.

(ii) Similar to the proof of (i) above. $\quad\square$

Lemma 9.8. *The following* (i), (ii) *hold on V provided that $d \geqslant 1$.*

(i) $E_0^* A E_0^* = a_0 E_0^*$ where

$$a_0 = \theta_0 + \zeta_1 \left(\theta_0^* - \theta_1^*\right)^{-1}. \tag{69}$$

(ii) $E_d A^* E_d = a_d^* E_d$ where

$$a_d^* = \theta_1^* - \frac{\zeta_1 + (\theta_0^* - \theta_1^*)(\theta_0 - \theta_{d-1})}{\theta_{d-1} - \theta_d}. \tag{70}$$

Proof. (i) The expression $E_0^* - 1$ is zero on $E_0^* V$ by Definition 9.5, and $E_0^* V = U_0$ by Lemma 9.7(ii). Therefore it suffices to show that $E_0^*(A - a_0 1)$ is zero on U_0. By Definition 7.1 $LR - \zeta_1 1$ is zero on U_0. By Lemmas 9.2 and 9.3, $LR = (A^* - \theta_1^* 1)(A - \theta_0 1)$ on U_0. By Definition 9.5, $E_0^* A^* = \theta_0^* E_0^*$ on V. We may now argue that on U_0,

$$
\begin{aligned}
E_0^* \zeta_1 &= E_0^* LR \\
&= E_0^* (A^* - \theta_1^* 1)(A - \theta_0 1) \\
&= (\theta_0^* - \theta_1^*) E_0^* (A - \theta_0 1).
\end{aligned}
$$

By this and (69) the expression $E_0^*(A - a_0 1)$ is zero on U_0.

(ii) The expression $E_d - 1$ is zero on $E_d V$ by Definition 9.5, and $E_d V = U_d$ by Lemma 9.7(i). Therefore it suffices to show that $E_d(A^* - a_d^* 1)$ is zero on U_d. By the paragraph above Lemma 7.17, the expression $RL - \zeta_1^\times 1$ is zero on U_d. Evaluating Lemma 7.17(iii) using (4), (5) we find

$$\zeta_1 - \zeta_1^\times = \left(\theta_1^* - \theta_d^*\right)(\theta_{d-1} - \theta_d) - \left(\theta_0^* - \theta_1^*\right)(\theta_0 - \theta_{d-1}). \tag{71}$$

By Lemmas 9.2 and 9.3, $RL = (A - \theta_{d-1} 1)(A^* - \theta_d^* 1)$ on U_d. By Definition 9.5, $E_d A = \theta_d E_d$ on V. We may now argue that on U_d,

$$E_d \zeta_1^\times = E_d RL$$
$$= E_d(A - \theta_{d-1}1)(A^* - \theta_d^*1)$$
$$= (\theta_d - \theta_{d-1})E_d(A^* - \theta_d^*1).$$

By this and (70), (71) the expression $E_d(A^* - a_d^*1)$ is zero on U_d. $\quad\square$

Lemma 9.9. *For* $0 \leqslant i, j \leqslant d$ *the following* (i), (ii) *hold on* V.

(i) $E_i A^* E_j = 0$ *if* $|i - j| > 1$;
(ii) $E_i^* A E_j^* = 0$ *if* $|i - j| > 1$.

Proof. Assume $d \geqslant 2$; otherwise there is nothing to prove. We first show that $E_i A^* E_j = 0$ on V for $1 < |i - j| < d$. For notational convenience define a two variable polynomial

$$p(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho, \tag{72}$$

where β, γ, ϱ are from Definition 8.3. In the first equation of Proposition 8.5 we multiply each term on the left by E_i and the right by E_j. We simplify the result using the fact that $E_i A = \theta_i E_i$ and $A E_j = \theta_j E_j$ on V. After a brief calculation this shows

$$0 = E_i A^* E_j (\theta_i - \theta_j) p(\theta_i, \theta_j) \tag{73}$$

on V. We claim that in (73) the coefficient of $E_i A^* E_j$ is nonzero. Of course $\theta_i - \theta_j \neq 0$ since $\theta_0, \ldots, \theta_d$ are mutually distinct. We now show $p(\theta_i, \theta_j) \neq 0$. Since $|i - j| < d$ we have $1 \leqslant i \leqslant d - 1$ or $1 \leqslant j \leqslant d - 1$. We may assume $1 \leqslant i \leqslant d - 1$ since $p(\theta_i, \theta_j) = p(\theta_j, \theta_i)$. By Lemma 8.4(iii) we have $p(\theta_i, \theta_{i-1}) = 0$ and $p(\theta_i, \theta_{i+1}) = 0$. The expression $p(\theta_i, \mu)$ is a quadratic polynomial in μ with roots θ_{i-1}, θ_{i+1}. Since $|i - j| > 1$ we have $\theta_j \neq \theta_{i-1}$ and $\theta_j \neq \theta_{i+1}$. Therefore $p(\theta_i, \theta_j) \neq 0$ as desired. Now in (73) the coefficient of $E_i A^* E_j$ is nonzero, so $E_i A^* E_j = 0$ on V. Swapping the roles of A, A^* in the above argument, we similarly find $E_i^* A E_j^* = 0$ on V for $1 < |i - j| < d$. Next we show that $E_0 A^* E_d = 0$ on V. Observe $(A^* - \theta_d^*1)U_d \subseteq U_{d-1}$ by Lemma 9.3(ii), so $A^* U_d \subseteq U_{d-1} + U_d$. By Lemma 9.7(i), $E_d V = U_d$ and $E_{d-1}V + E_d V = U_{d-1} + U_d$. By these comments $A^* E_d V \subseteq E_{d-1}V + E_d V$. We assume $d \geqslant 2$ so E_0 vanishes on $E_{d-1}V + E_d V$ and therefore $E_0 A^* E_d = 0$ on V. Next we show that $E_d^* A E_0^* = 0$ on V. Observe $(A - \theta_0 1)U_0 \subseteq U_1$ by Lemma 9.3(i), so $A U_0 \subseteq U_0 + U_1$. By Lemma 9.7(ii), $E_0^* V = U_0$ and $E_0^* V + E_1^* V = U_0 + U_1$. By these comments $A E_0^* V \subseteq E_0^* V + E_1^* V$. We assume $d \geqslant 2$ so E_d^* vanishes on $E_0^* V + E_1^* V$ and therefore $E_d^* A E_0^* = 0$ on V. Next we show that $E_d A^* E_0 = 0$ on V. Since $V = \sum_{i=0}^d U_i$ it suffices to show that $E_d A^* E_0 = 0$ on U_i for $0 \leqslant i \leqslant d$. Observe $E_d A^* E_0 = 0$ on U_i for $1 \leqslant i \leqslant d$, since $\sum_{i=1}^d U_i = \sum_{i=1}^d E_i V$ by Lemma 9.7(i) and since E_0 vanishes on $E_i V$ for $1 \leqslant i \leqslant d$. To show $E_d A^* E_0 = 0$ on U_0, recall $U_0 = E_0^* V$ so it suffices to show that $E_d A^* E_0 E_0^* = 0$ on V. By Definition 9.5, $1 = \sum_{i=0}^d E_i^*$ on V. In this equation we multiply each term on the right by $A E_0^*$. We evaluate the result using Lemma 9.8(i) and the fact that $E_i^* A E_0^* = 0$ on V for $2 \leqslant i \leqslant d$. This yields

$$A E_0^* = a_0 E_0^* + E_1^* A E_0^* \tag{74}$$

on V. In (74) we multiply each term on the left by A^* to find

$$A^* A E_0^* = a_0 \theta_0^* E_0^* + \theta_1^* E_1^* A E_0^* \tag{75}$$

on V. By Definition 9.5, $1 = \sum_{i=0}^d E_i$ on V. In this equation we multiply each term on the left by $E_d A^*$. We evaluate the result using Lemma 9.8(ii) and the fact that $E_d A^* E_i = 0$ on V for $1 \leqslant i \leqslant d - 2$; this yields

$$E_d A^* = E_d A^* E_0 + E_d A^* E_{d-1} + a_d^* E_d \tag{76}$$

on V. In (76) we multiply each term on the right by A to find

$$E_d A^* A = \theta_0 E_d A^* E_0 + \theta_{d-1} E_d A^* E_{d-1} + \theta_d a_d^* E_d \tag{77}$$

on V. Consider the equation which is $\theta_1^* E_d$ times (74) minus E_d times (75) minus (76) times $\theta_{d-1} E_0^*$ plus (77) times E_0^*. We simplify this equation using the fact that $A^* E_0^* = \theta_0^* E_0^*$ and $E_d A = \theta_d E_d$ on V. The calculation shows that on V the expression $(\theta_0 - \theta_{d-1}) E_d A^* E_0 E_0^*$ coincides with $E_d E_0^*$ times

$$\left(\theta_0^* - \theta_1^*\right) a_0 + (\theta_{d-1} - \theta_d) a_d^* + \theta_d \theta_1^* - \theta_{d-1}\theta_0^*. \tag{78}$$

Note that $\theta_0 - \theta_{d-1}$ is nonzero since $d \geqslant 2$. By (69), (70) the expression (78) is zero. Therefore $E_d A^* E_0 E_0^* = 0$ on V and hence $E_d A^* E_0 = 0$ on V. Next we show that $E_0^* A E_d^* = 0$ on V. Since $V = \sum_{i=0}^d U_i$ it suffices to show that $E_0^* A E_d^* = 0$ on U_i for $0 \leqslant i \leqslant d$. Observe $E_0^* A E_d^* = 0$ on U_i for $0 \leqslant i \leqslant d-1$, since $\sum_{i=0}^{d-1} U_i = \sum_{i=0}^{d-1} E_i^* V$ by Lemma 9.7(ii) and since E_d^* vanishes on $E_i^* V$ for $0 \leqslant i \leqslant d-1$. To show $E_0^* A E_d^* = 0$ on U_d, recall $U_d = E_d V$ so it suffices to show that $E_0^* A E_d^* E_d = 0$ on V. We mentioned earlier that $1 = \sum_{i=0}^d E_i$ on V. In this equation we multiply each term on the right by $A^* E_d$. We evaluate the result using Lemma 9.8(ii) and the fact that $E_i A^* E_d = 0$ on V for $0 \leqslant i \leqslant d-2$. This yields

$$A^* E_d = E_{d-1} A^* E_d + a_d^* E_d \tag{79}$$

on V. In (79) we multiply each term on the left by A to find

$$A A^* E_d = \theta_{d-1} E_{d-1} A^* E_d + \theta_d a_d^* E_d \tag{80}$$

on V. We mentioned earlier that $1 = \sum_{i=0}^d E_i^*$ on V. In this equation we multiply each term on the left by $E_0^* A$. We evaluate the result using Lemma 9.8(i) and the fact that $E_0^* A E_i^* = 0$ on V for $2 \leqslant i \leqslant d-1$; this yields

$$E_0^* A = a_0 E_0^* + E_0^* A E_1^* + E_0^* A E_d^* \tag{81}$$

on V. In (81) we multiply each term on the right by A^* to find

$$E_0^* A A^* = \theta_0^* a_0 E_0^* + \theta_1^* E_0^* A E_1^* + \theta_d^* E_0^* A E_d^* \tag{82}$$

on V. Consider the equation which is $\theta_{d-1} E_0^*$ times (79) minus E_0^* times (80) minus (81) times $\theta_1^* E_d$ plus (82) times E_d. We simplify this equation using the fact that $A E_d = \theta_d E_d$ and $E_0^* A^* = \theta_0^* E_0^*$ on V. The calculation shows that on V the expression $(\theta_1^* - \theta_d^*) E_0^* A E_d^* E_d$ coincides with $E_0^* E_d$ times (78). The scalar $\theta_1^* - \theta_d^*$ is nonzero and we already showed that (78) is zero, so $E_0^* A E_d^* E_d = 0$ on V and hence $E_0^* A E_d^* = 0$ on V. $\square$

Lemma 9.10. *For $0 \leqslant i \leqslant d$ the following holds on V:*

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}. \tag{83}$$

Proof. It suffices to show that

$$\left(\theta_0^* - \theta_1^*\right)\left(\theta_0^* - \theta_2^*\right)\cdots\left(\theta_0^* - \theta_i^*\right)E_0^*\tau_i(A) - \zeta_i 1 \tag{84}$$

is zero on E_0^*V. We pick $w \in E_0^*V$ and show (84) is zero at w. Setting $i = 0$ in Lemma 9.7(ii) we find $E_0^*V = U_0$. By Definition 7.1 we find $L^i R^i - \zeta_i 1$ is zero on U_0. By these comments $L^i R^i w = \zeta_i w$. Using Lemmas 9.2 and 9.3,

$$L^i R^i = \left(A^* - \theta_1^* 1\right)\left(A^* - \theta_2^* 1\right)\cdots\left(A^* - \theta_i^* 1\right)\tau_i(A)$$

on U_0. Therefore

$$\left(A^* - \theta_1^* 1\right)\left(A^* - \theta_2^* 1\right)\cdots\left(A^* - \theta_i^* 1\right)\tau_i(A)w = \zeta_i w.$$

In this equation we apply E_0^* to both sides and use $E_0^*A^* = \theta_0^*E_0^*$, $E_0^*w = w$ to find (84) is zero at w. The result follows. □

10. The proof of Theorem 3.3

Throughout this section we adopt the following assumption.

Assumption 10.1. Assume the field $\mathbb{F}$ is algebraically closed. We fix a q-Racah sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ of scalars in $\mathbb{F}$, and a sequence $\{\zeta_i\}_{i=0}^d$ of scalars in $\mathbb{F}$ that satisfy condition (ii) of Theorem 3.3.

With reference to Assumption 10.1, our goal in this section is to display a TD system over $\mathbb{F}$ that has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$. To this end we fix $q, a, b, c, a^*, b^*, c^*$ that satisfy (4)–(7). Using this data we define $R, L \in U_q(\widehat{\mathfrak{sl}}_2)$ as in Definition 6.2, and then $A, A^* \in U_q(\widehat{\mathfrak{sl}}_2)$ as in Definition 8.2. Let $\{E_i\}_{i=0}^d$, $\{E_i^*\}_{i=0}^d$ be as in Definition 9.5. In view of Proposition 7.2 and Lemma 9.1 we fix a standard $U_q(\widehat{\mathfrak{sl}}_2)$-module V with diameter d that has split sequence $\{\zeta_i\}_{i=0}^d$.

Lemma 10.2. *The elements $E_0^*E_0E_0^*$, $E_0^*E_dE_0^*$ are nonzero on V.*

Proof. By construction $E_0^*V \neq 0$. Concerning $E_0^*E_dE_0^*$, by the equation on the left in (68) we have $E_d = \tau_d(A)\tau_d(\theta_d)^{-1}$. By Lemma 9.10 (with $i = d$) $E_0^*\tau_d(A)E_0^* = \eta_d^*(\theta_0^*)^{-1}\zeta_d E_0^*$ on V. Therefore $E_0^*E_dE_0^* = \tau_d(\theta_d)^{-1}\eta_d^*(\theta_0^*)^{-1}\zeta_d E_0^*$ on V. By this and since $\zeta_d \neq 0$ we find $E_0^*E_dE_0^*$ is nonzero on V. Concerning $E_0^*E_0E_0^*$, by the equation on the left in (68) we have $E_0 = \eta_d(A)\eta_d(\theta_0)^{-1}$. By [41, Proposition 5.5], $\eta_d = \sum_{i=0}^d \eta_{d-i}(\theta_0)\tau_i$. By these comments and Lemma 9.10,

$$E_0^*E_0E_0^* = E_0^*\eta_d(\theta_0)^{-1}\eta_d^*\left(\theta_0^*\right)^{-1}\sum_{i=0}^d \eta_{d-i}(\theta_0)\eta_{d-i}^*\left(\theta_0^*\right)\zeta_i$$

on V. In the above line the sum is nonzero by (9) so $E_0^*E_0E_0^*$ is nonzero on V. □

Definition 10.3. Let T denote the subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ generated by A, A^*. We observe that T contains E_i, E_i^* for $0 \leqslant i \leqslant d$.

Observe that TE_0^*V is the T-submodule of V generated by E_0^*V. We now examine this module.

Lemma 10.4. *Let W denote a proper T-submodule of TE_0^*V. Then $E_0^*W = 0$.*

Proof. Suppose $E_0^*W \neq 0$. The space E_0^*V contains E_0^*W and has dimension 1, so $E_0^*V = E_0^*W$. The space W is T-invariant and $E_0^* \in T$ so $E_0^*W \subseteq W$. Therefore $E_0^*V \subseteq W$, which yields $TE_0^*V \subseteq W$. This contradicts the fact that W is properly contained in TE_0^*V. Therefore $E_0^*W = 0$. □

Lemma 10.5. *Let W and W' denote proper T-submodules of TE_0^*V. Then $W + W'$ is a proper T-submodule of TE_0^*V.*

Proof. We show $W + W' \neq TE_0^*V$. The kernel of E_0^* on TE_0^*V is properly contained in TE_0^*V, since $0 \neq E_0^*V \subseteq TE_0^*V$. This kernel contains each of W, W' by Lemma 10.4, so this kernel contains $W + W'$. Therefore $W + W' \neq TE_0^*V$ and the result follows. □

Definition 10.6. Let W denote a proper T-submodule of TE_0^*V. Then W is called *maximal* whenever W is not contained in any proper T-submodule of TE_0^*V, besides itself.

Lemma 10.7. *There exists a unique maximal proper T-submodule in TE_0^*V.*

Proof. Concerning existence, consider

$$\sum_W W, \tag{85}$$

where the sum is over all proper T-submodules W of TE_0^*V. The space (85) is a proper T-submodule of TE_0^*V by Lemma 10.5, and since TE_0^*V has finite dimension. The T-submodule (85) is maximal by the construction. Concerning uniqueness, suppose W and W' are maximal proper T-submodules of TE_0^*V. By Lemma 10.5 $W + W'$ is a proper T-submodule of TE_0^*V. The space $W + W'$ contains each of W, W' so $W + W'$ is equal to each of W, W' by the maximality of W and W'. Therefore $W = W'$ and the result follows. □

Definition 10.8. Let M denote the maximal proper T-submodule of TE_0^*V. Let L denote the quotient T-module TE_0^*V/M. By construction the T-module L is nonzero, finite-dimensional and irreducible.

Proposition 10.9. *The sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a TD system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.*

Proof. We first show that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a TD system. We start with a few statements that follow from the construction. The space L is a direct sum of the nonzero spaces among $\{E_iL\}_{i=0}^d$ and a direct sum of the nonzero spaces among $\{E_i^*L\}_{i=0}^d$. For $0 \leqslant i \leqslant d$, $(A - \theta_i 1)E_iL = 0$ and $(A^* - \theta_i^* 1)E_i^*L = 0$. Using Lemma 9.9,

$$A^*E_iL \subseteq E_{i-1}L + E_iL + E_{i+1}L \quad (0 \leqslant i \leqslant d), \tag{86}$$

where $E_{-1} = 0$ and $E_{d+1} = 0$. Moreover

$$AE_i^*L \subseteq E_{i-1}^*L + E_i^*L + E_{i+1}^*L \quad (0 \leqslant i \leqslant d), \tag{87}$$

where $E_{-1}^* = 0$ and $E_{d+1}^* = 0$. Observe that $E_0^*L \neq 0$ since M does not contain E_0^*V. We now show $E_0L \neq 0$. Suppose $E_0L = 0$. Then $E_0TE_0^*V \subseteq M$ so $E_0E_0^*V \subseteq M$. In this containment we apply E_0^* to both sides and use $E_0^*M = 0$ to get $E_0^*E_0E_0^*V = 0$. This contradicts Lemma 10.2 so $E_0L \neq 0$. Next we show $E_dL \neq 0$. Suppose $E_dL = 0$. Then $E_dTE_0^*V \subseteq M$ so $E_dE_0^*V \subseteq M$. In this containment we apply E_0^*

to both sides and use $E_0^* M = 0$ to get $E_0^* E_d E_0^* V = 0$. This contradicts Lemma 10.2 so $E_d L \neq 0$. We now show $E_i L \neq 0$ for $1 \leqslant i \leqslant d-1$. Let i be given and suppose $E_i L = 0$. Then $E_0 L + \cdots + E_{i-1} L$ is a nonzero proper T-submodule of L in view of (86). This contradicts the irreducibility of the T-module L. Therefore $E_i L \neq 0$ for $1 \leqslant i \leqslant d-1$. There exists an integer δ ($0 \leqslant \delta \leqslant d$) such that $E_i^* L \neq 0$ for $0 \leqslant i \leqslant \delta$ and $E_{\delta+1}^* L = 0$. By the above comments the sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta)$ acts on L as a TD system. Now $d = \delta$ by the third sentence below Note 1.2. We have shown $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a TD system which we denote by Φ. By construction Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Lemma 9.10 and since the canonical map $T E_0^* V \to L$ is a T-module homomorphism, we have

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)} \quad (0 \leqslant i \leqslant d)$$

on L. By this and Definition 2.4 the sequence $\{\zeta_i\}_{i=0}^d$ is the split sequence for Φ. By these comments Φ has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ and the result follows. $\square$

It is now a simple matter to prove Theorem 3.3.

Proof of Theorem 3.3. The implication (i) $\Rightarrow$ (ii) is proved in [45, Corollary 8.3]. The implication (ii) $\Rightarrow$ (i) follows from Proposition 10.9. Now assume (i), (ii) hold. Then the last assertion of the theorem follows from Proposition 2.6. $\square$

11. Remarks

In this section we prove the shape conjecture for the TD pairs over an algebraically closed field that have q-Racah type.

Proposition 11.1. *Assume the field $\mathbb{F}$ is algebraically closed, and let $\{\rho_i\}_{i=0}^d$ denote the shape of a TD pair over $\mathbb{F}$ that has q-Racah type. Then $\rho_i \leqslant \binom{d}{i}$ for $0 \leqslant i \leqslant d$.*

Proof. For the TD pair in question we pick a standard ordering of their primitive idempotents to obtain a TD system. Without loss we may identify this TD system with the one in Proposition 10.9. Referring to the TD system in Proposition 10.9, we show that each of $E_i L$ and $E_i^* L$ has dimension at most $\binom{d}{i}$. The space $E_i L$ is the image of $E_i T E_0^* V$ under the canonical homomorphism $T E_0^* V \to L$. Therefore the dimension of $E_i L$ is at most the dimension of $E_i T E_0^* V$. The space $E_i T E_0^* V$ is contained in $E_i V$ so the dimension of $E_i T E_0^* V$ is at most the dimension of $E_i V$. The dimension of $E_i V$ is $\binom{d}{i}$ by Lemma 9.6. Our conclusion for $E_i L$ follows from the above comments. Our conclusion for $E_i^* L$ are similarly obtained. $\square$

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