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# Ring of subquotients of a finite group I: Linearization

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## ABSTRACT

We introduce the ring  $\Lambda(G)$  of subquotients of a finite group  $G$ . As an abelian group, it is free on the set of conjugacy classes of subquotients of the group  $G$ . The ring  $\Lambda(G)$  integrally extends the Burnside ring and there is a linearization map with range the Grothendieck ring of the Mackey algebra.

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## 1. Introduction

The Burnside ring  $B(G)$  of a finite group  $G$  is the Grothendieck ring of the category of finite  $G$ -sets. It is well known that as an abelian group, the Burnside ring is free on a set of representatives of conjugacy classes of subgroups of  $G$ . In other words, as an abelian group, it has a basis consisting of conjugacy classes of subgroups of  $G$ . The ring structure is given by the linear extension of the following product:

$$[G/H] \cdot [G/K] = \sum_{x \in H \backslash G / K} [G/H \cap {}^x K].$$

Here  $H$  and  $K$  are subgroups of  $G$  and the sum is over a set of representatives of double cosets of  $H$  and  $K$  in  $G$ . Finally,  $[G/H]$  is the isomorphism class of the transitive  $G$ -set with point stabilizer  $H$ . The ring structure of the Burnside ring is well known. As well, it is known that the Burnside ring is a Mackey functor, even a biset functor. The functorial structure is also well understood, see [4] for a summary of properties of  $B(G)$ .

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In this paper, we introduce a generalization of the Burnside ring by considering subquotients instead of subgroups. We denote by  $\Lambda(G)$  the free abelian group on a set of representatives of conjugacy classes of subquotients of  $G$ , that is, we set

$$\Lambda(G) := \bigoplus_{H/N \leqslant_G G} \mathbb{Z}[H/N]_G.$$

Here by a subquotient of  $G$ , we mean a quotient of a subgroup of  $G$ , and denote by  $\leqslant_G$  the subquotient relation, up to conjugation, and by  $[H/N]_G$  we denote the conjugacy class containing the subquotient  $H/N$ . See Section 2.1 for a more precise definition. We make  $\Lambda(G)$  a ring by extending linearly the following multiplication.

$$[H/N]_G \cdot [K/M]_G = \sum_{\substack{x \in H \backslash G/K \\ {}^xMN \leqslant H \cap {}^xK}} [H \cap {}^xK / {}^xMN]_G$$

where  $H/N$  and  $K/M$  are subquotients of  $G$ . With this definitions,  $\Lambda(G)$  becomes an associative commutative ring with unity  $[G/1]_G$ . We call  $\Lambda(G)$  the *ring of subquotients* of  $G$ . It is clear that the ring  $\Lambda(G)$  of subquotients extends the Burnside ring and the extension  $\Lambda(G)/B(G)$  is integral.

Aim of this paper is to show that, for the correct settings, the ring of subquotients plays the role of the Burnside ring where the correct settings for the Burnside ring is the category of Mackey functors. In particular, we show that there are natural actions of bisets on the groups  $\Lambda(H)$  and the composition rule for these actions turns out to be a restricted product  $\circ^*$  of bisets. This product is opposite to the restricted product introduced by Bouc in [2]. As usual one can consider the category  $\mathcal{C}_G^*$  with objects equal to the set of subquotients of  $G$  and morphisms given by finite bisets together with the composition rule induced by the composition of the above actions. Now the functor associating a subquotient of  $G$  to its ring of subquotients is a functor  $\mathcal{C}_G^* \rightarrow \mathbb{Z}\text{-mod}$ . We name these kind of functors as  $\circ$ -biset functors,  $\circ$  referring to the above composition rule.

Note that the category of  $\circ$ -biset functors is a variant of the category of functors introduced by Bouc in [2] and they share similar properties. In Section 3, we make this connection clear and omit the proofs of properties that our category has in common with Bouc's functors.

The main results of this paper is about the version of the linearization map. Recall that the linearization map is the map  $B(G) \rightarrow \mathcal{R}_k(G)$  associating a  $G$ -set  $X$  to the permutation representation  $kX$  with permutation basis  $X$ . Here  $\mathcal{R}_k(G)$  is the representation ring of  $G$  over  $k$ . It is well known that the linearization map is a ring homomorphism and if  $k = \mathbb{Q}$  is the field of rational numbers, after extending the coefficients to a field of characteristic zero, it is surjective. It is also well known that the representation ring is a Mackey functor and the linearization map is a morphism of Mackey functors.

It is easy, with certain results of Bouc, to show that the representation ring  $R_{\mu_k}(G)$  of the category of Mackey functors over  $k$  is a  $\circ$ -biset functor. Here the ring structure of  $R_{\mu_k}(G)$  is given by the tensor product of Mackey functors introduced by Sasaki [9]. In Section 5, we introduce a linearization map  $\Lambda(G) \rightarrow \mathcal{R}_{\mu_k}(G)$  which associates to a subquotient  $H/N$  of  $G$  the Mackey functor

$$B_{H/N}^G := \text{Ind}_H^G \text{Inf}_{H/N}^H B$$

where  $B$  denotes the Burnside ring Mackey functor and the induction and inflation functors are the usual ones defined for Mackey functors, see Section 4. Our linearization map also has the versions of the above mentioned properties of the usual linearization map. In particular, we see that any (virtual) Mackey functor over a field of characteristic zero is a rational combination of the Mackey functors  $B_{H/N}^G$ .

The ring structure of  $\Lambda(G)$  is also interesting. In a sequel to this paper [8], we shall study its fundamental structure by introducing the ghost ring, the table of generalized marks of subquotients and a mark homomorphism.

## 2. Ring of subquotients

### 2.1. Poset of subquotients

Let  $G$  be a finite group. A *subquotient* of  $G$  is a pair  $(H^*, H_*)$  where  $H_* \trianglelefteq H^* \leq G$ . We write the pair  $(H^*, H_*)$  as  $H$  and denote the subquotient relation by  $H \preceq G$ . Here, and afterwards, we regard the group  $G$  as the subquotient  $(G, 1)$ . When it is more convenient, we write  $(H, N)$  and  $H/N$  instead of  $(H^*, H_*)$  and  $H$ .

The group  $G$  acts on the set of its subquotients by conjugation. We write  $H \preceq_G G$  to mean that  $H$  is taken up to  $G$ -conjugacy. Note that we always consider  $H$  as the quotient group  $H^*/H_*$ . Therefore, for example, what we mean by up to  $G$ -conjugation is that the subgroup  $H^*$  is taken up to  $G$ -conjugacy and the normal subgroup  $H_*$  of  $H^*$  is taken up to  $N_G(H^*)$ -conjugacy.

The relation  $\preceq$  extends to a partial order on the set of all subquotients of  $G$  in the following way. Let  $J$  and  $H$  be two subquotients of  $G$ . Then we write  $J \preceq H$  if and only if  $H_* \leq J_*$  and  $H^* \geq J^*$ . In this case the pair  $(J^*/H_*, J_*/H_*)$  is a subquotient of  $H$ . The poset structure is compatible with the  $G$ -action, that is, the set of subquotients of  $G$  is a  $G$ -poset.

Also we say that two subquotients  $H$  and  $K$  of  $G$  are isomorphic if and only if they are isomorphic as groups, that is, if  $H^*/H_* \cong K^*/K_*$ . In this case, write  $H \preceq_* G$  to mean that  $H$  runs over a set of representatives of isomorphism classes of subquotients of  $G$ , or we simply say that  $H$  is taken up to isomorphism.

Finally note that the pair  $(H^*, H_*)$  is sometimes called a section and the name subquotient is only used for the quotient group  $H$ . As we identify the pair with the group, we do not use two different names.

**2.2. Definition.** Consider the free abelian group  $\Lambda(G)$  on a set

$$\{H/N \mid H/N \preceq_G G\}$$

of representatives of the conjugacy classes of subquotients of  $G$ . Write  $[H/N]_G$  for the image of the subquotient  $H/N$  in the group  $\Lambda(G)$ . Then as an abelian group,

$$\Lambda(G) = \bigoplus_{H/N \preceq_G G} \mathbb{Z}[H/N]_G$$

where the sum is over a set of representatives of conjugacy classes of subquotients of  $G$ . We make the group  $\Lambda(G)$  a ring with the following multiplication. Let  $H/N$  and  $K/M$  be two subquotients of  $G$ . Then the *direct product* of these subquotients is given by

$$[H/N]_G \cdot [K/M]_G = \sum_{\substack{x \in H \backslash G / K \\ {}^x M N \leq H \cap {}^x K}} [H \cap {}^x K / {}^x M N]_G.$$

The product in  $\Lambda(G)$  is then defined as the linear extension of the above product of the basis elements. It is straightforward to show that this multiplication is well defined and together with the above abelian group structure, they give  $\Lambda(G)$  a commutative associative ring structure with the unity  $[G/1]_G$ . We call  $\Lambda(G)$  the *ring of subquotients* of  $G$ . By extending the coefficients to  $k$ , we also obtain the  $k$ -module  $k\Lambda(G) := k \otimes \Lambda(G)$ .

**2.3. Remark.** There are other possible rules for multiplying two subquotients, for example,

$$[H/N]_G \cdot_1 [K/M]_G = \sum_{x \in H \backslash G / K} [H \cap {}^x K / (H \cap {}^x M)(N \cap {}^x K)]_G.$$

or

$$[H/N]_G \cdot [K/M]_G = \sum_{x \in H \backslash G / K} [H \cap {}^x K / N \cap {}^x M]_G.$$

They also induce well-defined multiplications in  $\Lambda(G)$ . The reason for choosing the above rule as the multiplication will become clear in the next section when we introduce the linearization morphism.

**2.4. Remark.** The ring  $\Lambda(G)$  of subquotients of  $G$  is an integral ring extension of the Burnside ring, via the natural inclusion

$$\iota_G : B(G) \rightarrow \Lambda(G)$$

defined by  $\iota_G([G/K]) = [K/1]_G$ .

## 2.5. Relations with subquotients

In this subsection, we will consider the maps between the rings of subquotients for the subquotients of  $G$ . There are five classical maps to consider, namely induction map, restriction map, inflation map, deflation map and transport of structure by a group isomorphism. We rename the transport of structure map as the isogation map.

### 2.5.1. Induction map

Let  $X \leq Y$  be subgroups of  $G$ . Let  $H/N$  be a subquotient of  $X$ . Then clearly the subquotient  $H/N$  embeds into the over group  $Y$  in a natural way. We still denote the embedding of  $H/N$  to  $Y$  by  $H/N$ . This induces a map on the set of conjugacy classes of subquotients and by linear extension, we obtain a map

$$\text{Ind}_X^Y : \Lambda(X) \rightarrow \Lambda(Y)$$

which we call *induction*. More precisely we have

$$\text{Ind}_X^Y([H/N]_X) = [H/N]_Y.$$

Note that, clearly, induction is a map of abelian groups and not a homomorphism of rings. It is also clear that the induction map is transitive.

### 2.5.2. Restriction map

Let  $X \leq Y$  still be subgroups of  $G$ . Let  $K/M$  be a subquotient of  $Y$ . As in the case of the product, there are several options for the map going from the ring  $\Lambda(Y)$  to  $\Lambda(X)$  and we shall choose the one that is appropriate for the rest of the paper. We define the *restriction* map

$$\text{Res}_X^Y : \Lambda(Y) \rightarrow \Lambda(X)$$

by the formula

$$\text{Res}_X^Y([K/M]_Y) = \sum_{\substack{x \in X \backslash Y / K \\ {}^x M \leq X}} [X \cap {}^x K / {}^x M].$$

The restriction map, as defined above, is a homomorphism of rings and is transitive. The transitivity is straightforward. One can directly check the other claim. We do not need this fact in this paper, so we postpone the proof to the sequel.

### 2.5.3. Isogation map

Let  $X$  and  $Y$  be two isomorphic finite groups and  $\phi : X \rightarrow Y$  be a group isomorphism. The isogation map

$$c_{X,Y}^{\phi} : \Lambda(Y) \rightarrow \Lambda(X)$$

is defined as the map

$$c_{X,Y}^{\phi}([\phi(K)/\phi(M)]_Y) := [K/M]_X.$$

In other words, this map is induced by the action of the isomorphism  $\phi$  on the set of subquotients. This map is also transitive and is a homomorphism of rings.

### 2.5.4. Inflation map

Let  $N$  be a normal subgroup of  $G$ . Given a subquotient  $I/J$  of  $G/N$ . Note that we can identify the subgroups  $I$  and  $J$  of  $G/N$  by the subquotients  $I'/N$  and  $J'/N$  of  $G$ . Here  $I'$  and  $J'$  are the respective preimages of  $I$  and  $J$  in  $G$ . Now for each subquotient  $I/J$  of  $G/N$ , there is a canonical isomorphism

$$\lambda_{I/J} : \frac{I'/N}{J'/N} \rightarrow \frac{I'}{J'}.$$

This correspondence induces a map from the set of subquotients of  $G/N$  to the set of subquotients of  $G$ . It is also clear that the above canonical map is compatible with the conjugation action of  $G$ . Therefore the correspondence induces the *inflation map*

$$\text{Inf}_{G/N}^G : \Lambda(G/N) \rightarrow \Lambda(G)$$

given explicitly by

$$\text{Inf}_{G/N}^G([I/J]_{G/N}) = [I'/J']_G.$$

Clearly this map is transitive and is a homomorphism of rings.

### 2.5.5. Deflation map

Let  $N$  still be a normal subgroup of  $G$ . We can construct a left inverse of the correspondence of the previous item as follows. Given a subquotient  $K/M$  of  $G$ , construct the subquotient  $KN/MN$  of  $G$ . Now the above canonical isomorphism  $\lambda$  becomes an isomorphism

$$\lambda_{KN/MN} : \frac{KN/N}{MN/N} \rightarrow \frac{KN}{MN}.$$

This correspondence induces a map from the set of subquotients of  $G$  to that of  $G/N$ . Clearly this correspondence is compatible with the conjugation action of  $G$  and hence induces the *deflation map*

$$\text{Def}_{G/N}^G : \Lambda(G) \rightarrow \Lambda(G/N).$$

The deflation map is given explicitly by

$$\text{Def}_{G/N}^G([K/M]_G) = [(KN/N)/(MN/N)]_{G/N}.$$

This map is also transitive and is not a homomorphism of rings.

### 3. $\circ$ -biset functor structure

#### 3.1. Bisets

Let  $H$  and  $K$  be two finite groups and  $X$  be a finite set. The set  $X$  is called an  $(H, K)$ -biset if it admits a left  $H$ -action and a right  $K$ -action such that

$$h \cdot (x \cdot k) = (h \cdot x) \cdot k$$

for all elements  $h \in H$  and  $k \in K$ .

An  $(H, K)$ -biset  $X$  is called *transitive* if for any  $x \in X$ , we have  $X = H \cdot x \cdot K$ , in other words, for any elements  $x, y \in X$ , there exist an element  $h \in H$  and an element  $k \in K$  such that  $h \cdot x \cdot k$  is equal to  $y$ .

We can regard any  $(H, K)$ -biset as a left  $H \times K$ -set with the action given by

$$(h, k) \cdot x = h \cdot x \cdot k^{-1}$$

for all  $(h, k) \in H \times K$  and  $x \in X$ . Clearly,  $X$  is a transitive  $(H, K)$ -biset if and only if  $X$  is a transitive  $H \times K$ -set. Hence there is a bijective correspondence between

- (i) the isomorphism classes  $[X]$  of transitive  $(H, K)$ -bisets, and
- (ii) the conjugacy classes  $[L]$  of subgroups of  $H \times K$

where the correspondence is given by associating the class  $[X]$  to the class  $[L]$  if and only if the stabilizer of some point  $x \in X$  is in the class  $[L]$ .

Hereafter we denote a transitive  $(H, K)$ -biset with point stabilizer  $L$  by  $(\frac{H \times K}{L})$ . This is the set of (left) cosets of  $L$  in  $H \times K$ . We define a composition product of bisets as follows. Given finite groups  $H, K, M$  and an  $H \times K$ -biset  $X$  and a  $K \times M$ -biset  $Y$ . We define the *Mackey product*  $X \times_K Y$  of  $X$  and  $Y$  as the set

$$X \times_K Y := X \times Y / K$$

of  $K$ -orbits of the Cartesian product  $X \times Y$ . Here  $K$  acts via  $k \cdot (x, y) := (x \cdot k^{-1}, k \cdot y)$ . The set  $X \times_K Y$  is an  $(H, M)$ -biset via

$$h \cdot (x, {}_K y) \cdot m := (h \cdot x, {}_K y \cdot m)$$

where  $h \in H$ ,  $m \in M$  and  $(x, {}_K y)$  denotes the image of  $(x, y)$  in  $X \times_K Y$ . For the transitive bisets  $(H \times K)/L$  and  $(K \times M)/N$ , the Mackey product is explicitly given by

$$\left( \frac{H \times K}{L} \right) \times_K \left( \frac{K \times M}{N} \right) = \sum_{x \in p_2(L) \backslash K / p_1(N)} \left( \frac{H \times M}{L * (x, 1) N} \right)$$

where the subgroup  $L * N$  of  $H \times M$  is defined by

$$L * N = \{ (h, m) \in H \times M : (h, k) \in L \text{ and } (k, m) \in N \text{ for some } k \in K \}$$

and the subgroups  $p_1(N)$  and  $p_2(L)$  of  $K$  are projections of  $N$  and  $L$  to  $K$ , respectively, that is,

$$p_1(L) = \{ h \in H : (h, k) \in L \text{ for some } k \in K \}$$

and

$$p_2(L) = \{k \in K : (h, k) \in L \text{ for some } h \in H\}.$$

In [3], Bouc proved that any transitive biset is a Mackey product of the following five types of *basic* bisets: Let  $H$  be a finite group and  $N \trianglelefteq J$  be subgroups of  $H$  and let  $L, M$  be two isomorphic finite groups with a fixed isomorphism  $\phi : L \rightarrow M$ , then the basic bisets are given as follows.

- (1) *Induction*  $(H, J)$ -biset:  $\text{Ind}_J^H := (\frac{H \times J}{T})$  where  $T = \{(j, j) : j \in J\}$ .
- (2) *Inflation*  $(J, J/N)$ -biset:  $\text{Inf}_{J/N}^J := (\frac{J \times J/N}{I})$  where  $I = \{(j, jN) : j \in J\}$ .
- (3) *Isomorphism*  $(M, L)$ -biset:  $c_{M,L}^\phi = (\frac{M \times L}{C^\phi})$  where  $C^\phi = \{(\phi(l), l) : l \in L\}$ .
- (4) *Deflation*  $(J/N, J)$ -biset:  $\text{Def}_{J/N}^J = (\frac{J/N \times J}{D})$  where  $D = \{(jN, j) : j \in J\}$ .
- (5) *Restriction*  $(J, H)$ -biset:  $\text{Res}_J^H = (\frac{J \times H}{R})$  where  $R = \{(j, j) : j \in J\}$ .

Throughout the paper, we use notations  $\text{Ind}_J^H \text{Res}_J^G, \text{Inf}_J^K \text{Res}_J^G$  etc. instead of  $\text{Ind}_J^H \times_J \text{Res}_J^G, \text{Inf}_J^K \times_J \text{Res}_J^G$  etc. We also use two amalgamations to simplify the notation. We write  $\text{Tin}$  for the composition of induction (or transfer) and inflation, which is called tinflation, and write  $\text{Des}$  for the composition of deflation and restriction, which is called destruction. The following theorem explicitly shows the decomposition of any transitive biset in terms of these basic bisets.

**3.2. Theorem (Bouc).** Let  $L$  be any subgroup of  $H \times K$ . Then

$$\left( \frac{H \times K}{L} \right) = \text{Tin}_{p_1(L)/k_1(L)}^H c_{p_1(L)/k_1(L), p_2(L)/k_2(L)}^\phi \text{Des}_{p_2(L)/k_2(L)}^K$$

where the subgroup  $k_1(L)$  of  $H$  and the subgroup  $k_2(L)$  of  $K$  are given by

$$k_1(L) = \{h \in H : (h, 1) \in L\} \quad \text{and} \quad k_2(L) = \{x \in K : (1, x) \in L\}.$$

The isomorphism

$$\phi : p_2(L)/k_2(L) \rightarrow p_1(L)/k_1(L)$$

is the one given by associating  $xk_2(L)$  to  $mk_1(L)$  where for a given element  $x \in p_2(L)$ , we let  $mk_1(L)$  be the unique element in  $p_1(L)/k_1(L)$  such that  $(m, l) \in L$ .

### 3.3. $\circ$ -product of bisets

Let  $G, H$  and  $K$  be finite groups and  $k$  be a field. In [2], Bouc introduced a restricted product of finite bisets to study the functors between categories of  $G$ -sets. The product is given on the transitive bisets as follows. Let  $U$  be a subgroup of  $G \times H$  and  $V$  be a subgroup of  $H \times K$ . As in the previous section, denote by  $(\frac{G \times H}{U})$  (resp.  $(\frac{H \times K}{V})$ ) the set of left cosets of  $U$  in  $G \times H$  (resp.  $V$  in  $H \times K$ ). Then a  $(G, K)$ -biset  $(\frac{G \times H}{U}) \circ_H (\frac{H \times K}{V})$  is defined as follows:

$$\left( \frac{G \times H}{U} \right) \circ_H \left( \frac{H \times K}{V} \right) = \sum_{\substack{x \in p_2(U) \setminus H/p_1(V) \\ k_2(U)^x \leq p_1(V)}} \left( \frac{G \times K}{U * (x, 1)V} \right).$$

Note that this product is a subset of the previous one and the decomposition of Theorem 3.2 still holds with this product.

3.4. Having this product, Bouc considered the category  $\mathcal{C}$  whose objects are the finite groups, morphisms between two finite groups  $G$  and  $H$  are finite  $(H \times G)$ -bisets and composition of morphisms is the above restricted product. Then he studied the structure of the category  $\mathcal{F}_k$  of functors from this category to the category of  $k$ -modules. In particular, the category  $\mathcal{F}_k$  is abelian and he showed that if  $k$  is a field of characteristic zero, it is semisimple. He also described the simple functors in  $\mathcal{F}_k$ . Further details can be found in [2].

### 3.5. $\circ^*$ -product of bisets

In this paper, we consider an opposite construction. Namely we consider the above category yet with another composition rule, and denote this category by  $\mathcal{C}^*$ . Our composition rule is the opposite of the above one, given on the transitive bisets by

$$\left( \frac{G \times H}{U} \right) \circ_H^* \left( \frac{H \times K}{V} \right) = \sum_{\substack{x \in p_2(U) \setminus H/p_1(V) \\ x_{k_1(V)} \leq p_2(U)}} \left( \frac{G \times K}{U *_{(x,1)} V} \right).$$

The product  $\circ_H^*$  is the opposite of the product  $\circ_H$  in the following sense: Given a  $(G, H)$ -biset  $X$ , denote by  $X^*$  the  $(H, G)$ -biset which is the same as  $X$  as a set and the action is the opposite action. Then we have

$$X \circ_H Y = (Y^* \circ_H^* X^*)^*.$$

This also shows that the above decomposition of transitive bisets into basic bisets still holds.

3.6. Now we define a  $\circ$ -biset functor as a  $k$ -linear functor  $\mathcal{C}^* \rightarrow k\text{-mod}$ . The category  $\mathcal{F}_k^*$  of  $\circ$ -biset functors is also abelian. Moreover if we denote by  $\Gamma_\circ$  the algebra whose (left) module category is equivalent to the category  $\mathcal{F}_k$ , then the category  $\mathcal{F}_k^*$  is just equivalent to the category of right  $\Gamma_\circ$ -modules. Classification and description of the simple  $\circ$ -biset functors are similar to those of the simple objects in the category  $\mathcal{F}_k$ . In particular,  $\mathcal{F}_k^*$  is also semisimple if  $k$  is of characteristic zero.

3.7. In the following, we will write  $\mathcal{C}_G^*$  for the full subcategory of  $\mathcal{C}^*$  consisting only of the subquotients of the group  $G$ . We will also denote by  $\mathcal{F}_{k,G}^*$  the corresponding functor category and in this case an object of  $\mathcal{F}_{k,G}^*$  is called a  $\circ$ -biset functor for  $G$ . Note that if we denote by  $1_G$  the sum of the identity morphisms over the subquotients of  $G$ , then  $\mathcal{F}_{k,G}^*$  is equivalent to the category of right  $1_G \Gamma_\circ 1_G$ -modules. For simplicity, we will write  $\Gamma_\circ(G)$  for the truncated algebra  $1_G \Gamma_\circ 1_G$  and when there is no ambiguity, we will write  $\Gamma_\circ$ .

### 3.8. Description of the algebra $\Gamma_\circ(G)$

It is standard to show that the functor category of  $\circ$ -biset functors is equivalent to the module category of the modules over the algebra generated by the morphisms in the category  $\mathcal{C}_G^*$  with the product given by composition of the morphisms. By Bouc's decomposition theorem, this algebra is generated by the families of the three basic bisets, tinflation bisets, dstriction bisets and isogation bisets. More precisely, the algebra  $\Gamma_\circ(G)$  is generated by the families of variables:

- V1.** the family of tinflations,  $\text{Tin}_J^H$  defined for each  $J \preccurlyeq H \preccurlyeq G$ ,
- V2.** the family of dstrictions,  $\text{Des}_J^H$  defined for each  $J \preccurlyeq H \preccurlyeq G$ ,
- V3.** the family of isogations,  $c_{M,L}^\phi$  defined for each  $M, L \preccurlyeq G$  such that  $M \cong L$  and for each isomorphism  $\phi: M \rightarrow L$ ,

subject to the following set of relations. Note that we use the notation  $H = H^*/H_*$  below, to simplify the notation.



- R1.** Let  $h : H \rightarrow H$  denote the inner automorphism of  $H$  induced by conjugation by  $h \in H$ . Then  $c_{H,H}^h = \text{Tin}_H^H = \text{Des}_H^H$ .
- R2.** Let  $L \leq J$  and  $\psi : M \rightarrow S$  be an isomorphism. Then
- (i)  $c_{S,M}^\psi c_{M,L}^\phi = c_{S,L}^{\psi \circ \phi}$ ,
  - (ii)  $\text{Tin}_J^H \text{Tin}_L^J = \text{Tin}_L^H$ ,
  - (iii)  $\text{Des}_L^J \text{Des}_J^H = \text{Des}_L^H$ .
- R3.** Let  $K \leq G$  and let  $\alpha : H \rightarrow K$  be an isomorphism and let  ${}^\alpha J$  denote  $\alpha(J^*)/\alpha(J_*)$ . Then
- (i)  $c_{K,H}^\alpha \text{Tin}_J^H = \text{Tin}_{\alpha J}^K c_{\alpha J,J}^\alpha$ ,
  - (ii)  $\text{Des}_I^K c_{K,H}^\alpha = c_{I, \alpha^{-1}I}^\alpha \text{Des}_{\alpha^{-1}I}^H$ .
- R4.** (Mackey relation). Let  $I \leq H$ . Then

$$\text{Des}_I^H \text{Tin}_J^H = \sum_{\substack{x \in I^* \setminus H/J^* \\ x J_* \leq I^*}} \text{Tin}_{I \cap^x J}^I c^{x \circ \lambda} \text{Des}_{J \cap I^x}^J.$$

Here  $c^{x \circ \lambda} := c_{I \cap^x J, J \cap I^x}^{x \circ \lambda}$  and  $\lambda$  is the canonical isomorphism introduced in the previous section.

**R5.**  $1 = \sum_{H \leq G} c_H$  where  $c_H := c_{H,H}^1$ .

**R6.** All other products of the generators are zero.

Note that the above relations are the same as the relations for the alchemic algebra, introduced in [7], except for the Mackey relation. The proof of the claim that the above algebra is isomorphic to the algebra generated by the morphisms in  $\mathcal{C}_G^*$  is easy and similar to the proof of Theorem 3.2 in [7].

### 3.9. Simple functors

It follows from the constructions in [2] and Section 3.5 that there is a bijective correspondence between the isomorphism classes of simple  $\circ$ -biset functors for  $G$  and pairs  $(H, V)$  where  $H$  is a subquotient of  $G$  and  $V$  is a simple  $k\text{Out}(H)$ -module, both taken up to isomorphism. We denote by  $S_{H,V}^G$  the simple functor corresponding to the pair  $(H, V)$ . It can be shown that over a field of characteristic zero, the simple functor  $S_{H,V}^G$  can be identified with the coinduced functor  $\text{Coind}_{\mathcal{E}(H)}^{\Gamma_\circ} V$  where  $\mathcal{E}(H) := \text{End}_{\mathcal{C}^*}(H)$  is the algebra of endomorphisms of  $H$  in  $\mathcal{C}^*$ , that is,

$$S_{H,V}^G = \text{Coind}_{\mathcal{E}(H)}^{\Gamma_\circ} V.$$

Note the coinduction functor is right adjoint to the restriction functor which is equivalent to the ‘evaluation at  $H$ ’ functor.

### 3.10. Functorial properties of $\Lambda$

It is straightforward to check that the functor associating a finite group  $G$  to the abelian group  $\Lambda(G)$  and any finite biset to the corresponding composition of the maps of the previous section is a  $\circ$ -biset functor. Indeed one needs to check the relation R1 – R6 of Section 3.8 and all except the Mackey relation are trivial. On the other hand it is a complicated calculation to show that the Mackey relation holds, as an example we check the Mackey relation when  $I^* = H$  and  $J_* = 1$ , that is, we evaluate  $\text{Def}_{H/N}^H \text{Ind}_J^H$  on  $\Lambda(J)$  in two ways. Let  $[K/M]_J \in \Lambda(J)$ , then using the definitions of the involved maps, we get

$$\text{Def}_{H/N}^H \text{Ind}_J^H [K/M]_J = \text{Def}_{H/N}^H [K/M]_J = [(KN/N)/(MN/N)]_{H/N}.$$

Now using the Mackey relation,

$$\begin{aligned}\mathrm{Def}_{H/N}^H \mathrm{Ind}_J^H [K/M]_J &= \mathrm{Ind}_{H/N \cap J} c^\lambda \mathrm{Def}_{J \cap H/N}^J [K/M]_J \\ &= \mathrm{Ind}_{H/N \cap J} c^\lambda [K(J \cap N)/M(J \cap N)]_{J/J \cap N}.\end{aligned}$$

Here we have

$$H/N \cap J = JN/N, \quad J \cap H/N = J/J \cap N$$

and  $\lambda : J/J \cap N \rightarrow JN/N$  is the canonical isomorphism between these groups. Now clearly  $\lambda$  maps the subgroup  $K(J \cap N)/J \cap N$  to  $KN/N$  and  $M(J \cap N)/J \cap N$  to  $MN/N$ . Now this special case of the Mackey relation is satisfied by the definition of the induction map.

We denote this functor by  $\Lambda$ . For the rest of the section, we discuss certain properties of this  $\circ$ -biset functor most of which will be used in the next section.

An important property of the functor  $\Lambda$  is that when considered as an object in  $\mathcal{F}_{k,G}^*$ , it is cyclic as proved in the next proposition. This also leads to the determination of the set of morphisms from the functor of subquotients to an arbitrary  $\circ$ -biset functor.

**3.11. Proposition.** *The  $\circ$ -biset functor  $\Lambda$  for  $G$  is generated as a  $\circ$ -biset functor by the unity  $[G/1]_G$  of its evaluation at  $G$ .*

**Proof.** Given a subquotient  $H/N$  of  $K/M$ . Then straightforward calculations show that we have

$$[H/N]_{K/M} = \mathrm{Tin}_{H/N}^{K/M} \mathrm{Des}_{H/N}^G [G/1]_G = \mathrm{Tin}_{H/N}^{K/M} c_{H/N,L}^\alpha \mathrm{Des}_L^G [G/1]_G$$

for any subquotient  $L$  of  $G$  which is isomorphic to  $H/N$  and for any isomorphism  $\alpha : L \rightarrow H/N$ .  $\square$

**3.12. Corollary.** *Let  $F$  be a  $\circ$ -biset functor. Any morphism of  $\circ$ -biset functors  $\Lambda \rightarrow F$  is determined by its value at  $[G/1]_G$ . Moreover the assignment  $[G/1]_G \mapsto f \in F(G)$  induces a morphism biset functors if, and only if,  $f$  satisfies*

$$\mathrm{Des}_H^G f = c_{H,K}^\phi \mathrm{Des}_K^G f$$

for any isomorphic subquotients  $H$  and  $K$  of  $G$  and for any choice of the isomorphism  $\phi : H \rightarrow K$ .

**Proof.** It is clear that any morphism is determined by the value at  $[G/1]_G$ , since  $\Lambda$  is generated by  $[G/1]_G$ . Now let  $\phi : \Lambda \rightarrow F$  be the morphism induced by associating  $[G/1]_G$  to  $f$ , that is, for any  $[H/N]_{K/M} \in \Lambda(K/M)$ , we define

$$\phi_{K/M}([H/N]_{K/M}) := \mathrm{Tin}_{H/N}^{K/M} \mathrm{Des}_{H/N}^G f.$$

By the proof of the previous proposition, we have

$$\begin{aligned}\phi_{K/M}([H/N]_{K/M}) &= \phi_{K/M}(\mathrm{Tin}_{H/N}^{K/M} c_{H/N,L}^\alpha \mathrm{Des}_L^G [G/1]_G) \\ &= \mathrm{Tin}_{H/N}^{K/M} c_{H/N,L}^\alpha \mathrm{Des}_L^G f\end{aligned}$$

for any isomorphism  $\alpha : L \rightarrow H/N$ . Now it is clear that this morphism is well defined if and only if the element  $f$  has the stated property. Indeed to obtain the property, just take  $H/N = K/M$ . Finally when it is well defined, the morphism is a morphism of  $\circ$ -biset functors by its definition.  $\square$

The following result, in analogy with [11, Corollary 8.9], shows that the functor of the ring of subquotients in the category  $\mathcal{F}_{k,G}^*$  is a good substitute for the functor of the Burnside ring in the category of Mackey functors for  $G$ .

**3.13. Theorem.** *Let  $k$  be a field of characteristic zero. Then there is an isomorphism of  $\circ$ -biset functors for  $G$*

$$k\Lambda \cong \bigoplus_{H \triangleleft^* G} S_{H,1}^G.$$

**Proof.** The multiplicity  $m_{H,V}$  of a simple functor  $S_{H,V}^G$  in  $k\Lambda$  is given by

$$m_{H,V} = \frac{\dim_k \operatorname{Hom}_{\mathcal{F}_{k,G}^*}(k\Lambda, S_{H,V}^G)}{\dim_k \operatorname{End}_{\mathcal{F}_{k,G}^*}(S_{H,V}^G)}$$

since  $\mathcal{F}_{k,G}^*$  is semisimple. But we have

$$\operatorname{Hom}_{\mathcal{F}_{k,G}^*}(k\Lambda, S_{H,V}^G) = \operatorname{Hom}_{\mathcal{E}(H)}(k\Lambda(H), V)$$

where  $\mathcal{E}(H) = \operatorname{End}_{\mathcal{C}_{k,G}^*}^*(H)$  is the algebra of endomorphisms of  $H$  in  $\mathcal{C}_{k,G}^*$ . Indeed the simple functor  $S_{H,V}^G$  is coinduced from the algebra  $\mathcal{E}(H)$  and its adjoint is given by the evaluation functor (cf. [2]). Now it follows from Proposition 3.11 that the  $\mathcal{E}(H)$ -module  $k\Lambda(H)$  is generated by  $[H/1]_H$  and hence any homomorphism  $k\Lambda(H) \rightarrow V$  is determined by its value at  $[H/1]_H$ .

Let  $f: k\Lambda(H) \rightarrow V$  be a morphism of  $\mathcal{E}(H)$ -modules. Since the element  $[H/1]_H$  is fixed by any group automorphism of  $H$ , its image  $f([H/1]_H)$  should have the same property. But  $V$  is a simple  $\mathcal{E}(H)$ -module inflated from the group algebra  $k\operatorname{Out}(H)$  and hence it has an  $\operatorname{Out}(H)$ -fixed point only if it is the inflation of the trivial  $k\operatorname{Out}(H)$ -module. Now the result follows since when  $V = k$  is the trivial module, the endomorphism ring of the corresponding simple functor is 1-dimensional (as it is isomorphic to  $\operatorname{Coind}_{\mathcal{E}(H)}^{\Gamma_{\circ}} k$  where  $k$  denotes the 1-dimensional  $\mathcal{E}(H)$ -module inflated from the trivial  $k\operatorname{Out}(H)$ -module) and it is clear that there is a unique, up to scalar multiple, morphism  $f$ . Here to see that  $\operatorname{End}_{\mathcal{F}_{k,G}^*}(S_{H,1}^G)$  is 1-dimensional, note that since coinduction is adjoint to the restriction functor,

$$\operatorname{Hom}_{\mathcal{F}_{k,G}^*}(S_{H,1}^G, S_{H,1}^G) = \operatorname{Hom}_{\mathcal{E}(H)}(\operatorname{Res}_{\mathcal{E}(H)}^{\Gamma_{\circ}} S_{H,1}^G, k) = \operatorname{Hom}_{\mathcal{E}(H)}(k, k) = k. \quad \square$$

#### 4. Character ring of Mackey functors

In this section, we will show that the character ring of Mackey functors is a  $\circ$ -biset functor. This will follow from Bouc's classification of functors between categories of  $G$ -sets. In [2], Bouc proved that disjoint union and Cartesian product preserving functors from the category  $G\text{-set}$  of finite  $G$ -sets to the category  $H\text{-set}$  of finite  $H$ -sets are in one-to-one correspondence with finite  $(H, G)$ -bisets with the composition determined by the restricted product  $\circ$ . Hence by composition on the right, he obtained functors between the corresponding categories of Mackey functors. Before making these explicit, we recall two equivalent definitions of a Mackey functor without giving details which can be found in [11].

For the rest of the paper, we assume that  $k$  is a field of characteristic zero.

##### 4.1. Degression on Mackey functors

A Mackey functor for the finite group  $G$  over a commutative ring  $k$  is a quadruple  $(M, t, c, r)$  where  $M$  is a family of  $k$ -modules consisting of a  $k$ -module  $M(H)$  for each subgroup  $H$  of  $G$ . The triple  $(t, c, r)$  is a triple of families of maps between these  $k$ -modules and are subject to a number of

relations, including the Mackey relation, see [11]. More precisely, for each pair  $K \leq H$  of subgroups of  $G$ , we have a transfer map  $t_K^H : M(K) \rightarrow M(H)$  and a restriction map  $r_K^H : M(H) \rightarrow M(K)$  and for an element  $g$  and a subgroup  $H$  of  $G$ , a conjugation map  $c_H^g : M(H) \rightarrow M({}^gH)$ . It is well known that Mackey functors are modules over the Mackey algebra  $\mu_k(G)$ , see [11]. The Mackey algebra is generated by the above families of maps together with the relations they satisfy.

The other definition is in terms of  $G$ -sets. Denote by  $G\text{-set}$  the category of finite  $G$ -sets. Then a Mackey functor for  $G$  over  $k$  is a bifunctor  $M : G\text{-set} \rightarrow k\text{-mod}$  having certain properties, again see [11]. Here a bifunctor  $M$  is a pair  $(M^*, M_*)$  of functors  $G\text{-set} \rightarrow k\text{-mod}$  such that  $M_*$  is covariant and  $M^*$  is contravariant and such that they agree on the objects, that is, for any  $G$ -set  $X$ , one has  $M_*(X) = M^*(X)$ .

The passage between the two definitions is done, on the objects, by associating a subgroup  $H$  of  $G$  to the transitive  $G$ -set  $G/H$ . The correspondence of the morphisms is a bit more complicated, which can be found in [11].

4.2. Given two finite groups  $G$  and  $H$  and a finite  $(H, G)$ -biset  $U$  and a Mackey functor  $M$  for  $H$ . There is an associated Mackey functor  $M \circ U$  for  $G$ , given for a  $G$ -set  $Y$  by

$$(M \circ U)(Y) = M(U \circ_G Y).$$

Here the  $G$ -set  $Y$  is regarded as a  $(G, 1)$ -biset in the trivial way. Moreover if  $K$  is another finite group and  $V$  is a  $(G, K)$ -biset then, it is shown in [2] that

$$M \circ (U \circ_G V) = (M \circ U) \circ V.$$

4.3. Denote by  $\mathcal{R}_{\mu_k}(G)$  the Grothendieck group of the category  $\text{Mack}_k(G)$  of Mackey functors for  $G$  over  $k$  where  $k$  is a field of characteristic zero. More precisely, let  $\mathcal{R}_{\mu_k}(G)$  be the free abelian group on the isomorphism classes of simple Mackey functors for  $G$  over  $k$ , i.e.

$$\mathcal{R}_{\mu_k}(G) = \bigoplus_{S \in \text{Irr}(\mu_k(G))} \mathbb{Z}[S]$$

where  $[S]$  denotes the isomorphism class of the simple Mackey functor  $S$  and  $\text{Irr}(\mu_k(G))$  is a set of representatives of the isomorphism classes of simple Mackey functors for  $G$  over  $k$ . Recall that simple Mackey functors are parameterized by pairs  $(H, V)$  where  $H$  is a subgroup of  $G$  and  $V$  is a simple  $kW_G(H)$ -module and the simple Mackey functor corresponding to the pair  $(H, V)$  is denoted by  $S_{H,V}^G$ , see [10]. Here  $W_G(H) = N_G(H)/H$  is the Weyl group of  $H$  in  $G$ .

4.4. Let  $U$  be an  $(H, G)$ -biset. Then the above construction of Bouc induces a map  $U \cdot - : \mathcal{R}_{\mu_k}(G) \rightarrow \mathcal{R}_{\mu_k}(H)$  given for a Mackey functor  $M$  for  $G$  by

$$U \cdot [M] = [M \circ U^*].$$

Now it is clear that the functor associating a finite group  $G$  to  $\mathcal{R}_{\mu_k}(G)$  and a finite  $(H, G)$ -biset to the map  $U \cdot -$  is a  $\circ$ -biset functor.

4.5. In the special cases where  $U$  is one of induction, restriction, inflation, deflation or transport of structure by an isomorphism, the corresponding morphism is the usual induction, restriction, inflation or the transport of structure of modules of the Mackey algebra. Explicit descriptions of induction, restriction and inflation maps can be found in [2]. Next we introduce a deflation map and describe the effect of the five maps on the isomorphism classes of the simple Mackey functors, when  $k$  is of characteristic zero, which we assume for the rest of the section. It will be clear from the construction that the deflation map is the one that is induced by the deflation biset.

**4.6. Definition.** The inflation map has a left adjoint  $?^+$  and a right adjoint  $?^-$ , all of which are described in [11, Section 2]. First we recall the definitions of these adjoints. Let  $M$  be a Mackey functor for  $G$  and let  $N$  be a normal subgroup of  $G$ . We define the Mackey functors  $M^+$  and  $M^-$  for the quotient  $G/N$  by the following equalities. Let  $H$  be a subgroup of  $G$  containing  $N$ , then

$$M^+(H/N) = M(H) / \sum_{K \leq H, N \not\leq K} \text{Ind}_K^H M(K)$$

and

$$M^-(H/N) = \bigcap_{K \leq H, N \not\leq K} \text{Ker Res}_K^H$$

with the induced induction, conjugation and restriction maps. In general the above constructions are non-isomorphic. But Dress' Induction Theorem implies that over a field of characteristic zero, there is an isomorphism

$$M^+ \cong M^-$$

of Mackey functors for  $G/N$ . Now we define the *deflation map*

$$\text{Def}_{G/N}^G : \mathcal{R}_{\mu_k}(G) \rightarrow \mathcal{R}_{\mu_k}(G/N)$$

by  $\text{Def}_{G/N}^G([M]) := [M^+] = [M^-]$ .

#### 4.7. Notation

Given a subgroup  $H$  of  $G$  and a  $kW_G(H)$ -module  $W$ , not necessarily simple. We denote by  $D_{H,W}^G$  the direct sum of simple Mackey functors corresponding to the pairs  $(H, V)$  where  $V$  runs over the distinct direct summands of  $W$ . By [6, Corollary 6.9], we have

$$D_{H,W}^G \cong \text{Ind}_\rho^\mu D_{H,W}^\rho$$

where  $D_{H,W}^\rho$  is the  $\rho$ -module which is generated by its evaluation at  $H$ , which is  $W$ , and on which any non-trivial restriction acts as the zero map.

**4.8. Proposition.** Let  $H$  and  $K$  be two subgroups, and  $N$  be a normal subgroup of  $G$ , and  $V$  be a simple  $kW_G(H)$ -module. Then the following isomorphisms of Mackey functors hold.

- (1)  $\text{Res}_K^G S_{H,V}^G \cong \bigoplus_{(H', V') : (H', V') =_G (H, V)} D_{H', \text{Res}_{W_K(H)}^{W_G(H')} V'}^{G/N}$ , provided that  $H \leq_G K$  (and it is zero otherwise).  
Here the sum is over the  $K$ -classes of pairs  $(H', V')$  which are  $G$ -conjugate to the pair  $(H, V)$ .
- (2) If  $N \leq H$ , then  $\text{Inf}_{G/N}^G S_{H/N, V}^{G/N} \cong S_{H, V}^G$ .
- (3)  $\text{Def}_{G/N}^G S_{H, V}^{G/N} \cong S_{H/N, V}^{G/N}$  provided that  $N \leq H$  (it is zero otherwise).
- (4) Assume that  $H \leq K$  and let  $W$  be a simple  $kW_K(H)$ -module, then  $\text{Ind}_K^G S_{H, W}^K \cong D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} W}^G$ .

**Proof.** All except the last one is either trivial or follows from the well-known results. We only prove the last one. First, note that it is proved in [6, Corollary 6.9] that there is an isomorphism of Mackey functors

$$S_{H,W}^K \cong \text{Ind}_{\rho(K)}^{\mu(K)} \text{Inf}_{\gamma(K)}^{\rho(K)} S_{H,W}^{\gamma(K)}.$$

Here  $\rho(K) = \rho_k(K)$  is the restriction algebra for  $K$  and  $\gamma(K) = \gamma_k(K)$  is the conjugation algebra for  $K$  and  $S_{H,W}^{\gamma(K)}$  is a simple  $\gamma(K)$ -module. For details of this construction, see [6]. It is also easy to see that the group algebra  $kW_G(H)$  is a subalgebra of  $\gamma(H)$  and there is an isomorphism of  $\gamma(K)$ -modules  $S_{H,W}^{\gamma(K)} \cong \text{Ind}_{kW_K(H)}^{\gamma(K)} W$ . So we are to determine the Mackey functor

$$\text{Ind}_K^G \text{Ind}_{\rho(K)}^{\mu(K)} \text{Inf}_{\gamma(K)}^{\rho(K)} \text{Ind}_{kW_K(H)}^{\gamma(K)} W.$$

Now we claim that the following diagram commutes:

$$\begin{array}{ccccccc} \mu(G)\text{-mod} & \xrightarrow{\text{Res}_{\rho(K)}^{\mu(K)}} & \rho(G)\text{-mod} & \xrightarrow{\text{Codef}_{\gamma(G)}^{\rho(G)}} & \gamma(G)\text{-mod} & \xrightarrow{\text{Res}_{W_G(H)}^{\gamma(G)}} & kW_G(H)\text{-mod} \\ \downarrow \text{Res}_K^G & & \downarrow \text{Res}_K^G & & \downarrow \text{Res}_K^G & & \downarrow \text{Res}_{W_K(H)}^{W_G(H)} \\ \mu(K)\text{-mod} & \xrightarrow{\text{Res}_{\rho(K)}^{\mu(K)}} & \rho(K)\text{-mod} & \xrightarrow{\text{Codef}_{\gamma(K)}^{\rho(K)}} & \gamma(K)\text{-mod} & \xrightarrow{\text{Res}_{W_K(H)}^{\gamma(K)}} & kW_K(H)\text{-mod} \end{array}$$

Here we write  $\text{Codef}_{\gamma}^{\rho}$  for the left adjoint of the inflation  $\text{Inf}_{\gamma}^{\rho}$ . Explicitly, for a  $\rho$ -module  $D$  and for  $L \leq G$ , we have

$$\text{Codef}_{\gamma}^{\rho} D(L) = \bigcap_{J < K} \text{Ker}(\text{Res}_J^L : D(L) \rightarrow D(J))$$

and the conjugation maps are obtained from those for the  $\rho$ -module  $D$ .

Indeed, the first and the third squares commute trivially and it is straightforward to check that the middle square is also commutative. By the uniqueness of (left) adjoint, the following adjoint diagram is also commutative:

$$\begin{array}{ccccccc} kW_K(H)\text{-mod} & \xrightarrow{\text{Ind}_{W_K(H)}^{\gamma(K)}} & \gamma(K)\text{-mod} & \xrightarrow{\text{Inf}_{\gamma(K)}^{\rho(K)}} & \rho(K)\text{-mod} & \xrightarrow{\text{Ind}_{\rho(K)}^{\mu(K)}} & \mu(K)\text{-mod} \\ \downarrow \text{Ind}_{W_K(H)}^{W_G(H)} & & \downarrow \text{Ind}_K^G & & \downarrow \text{Ind}_K^G & & \downarrow \text{Ind}_K^G \\ kW_G(H)\text{-mod} & \xrightarrow{\text{Ind}_{W_G(H)}^{\gamma(G)}} & \gamma(G)\text{-mod} & \xrightarrow{\text{Inf}_{\gamma(G)}^{\rho(G)}} & \rho(G)\text{-mod} & \xrightarrow{\text{Ind}_{\rho(G)}^{\mu(G)}} & \mu(G)\text{-mod} \end{array}$$

In particular there is an isomorphism of Mackey functors

$$\text{Ind}_K^G \text{Ind}_{\rho(K)}^{\mu(K)} \text{Inf}_{\gamma(K)}^{\rho(K)} \text{Ind}_{W_K(H)}^{\gamma(K)} W \cong \text{Ind}_{\rho(G)}^{\mu(G)} \text{Inf}_{\gamma(G)}^{\rho(G)} \text{Ind}_{W_G(H)}^{\gamma(G)} \text{Ind}_{W_K(H)}^{W_G(H)} W.$$

Moreover it is clear from [6, Corollary 6.9] that the right-hand side is isomorphic to  $D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} W}^G$ , as required.  $\square$

4.9. The abelian group  $\mathcal{R}_{\mu_k}(G)$  becomes a ring under the tensor product of Mackey functors, introduced by Sasaki [9], see also [5]. Given two Mackey functors  $M$  and  $N$  for  $G$  over  $k$ , the tensor product of  $M$  and  $N$  is denoted by  $M \otimes N$ . For a precise definition of the tensor product, see [5, Chapter 1]. Here we will only need the following result of Bouc [5, Proposition 11.6.4].

**4.10. Proposition (Bouc).** *Let  $S_{H,V}^G$  and  $S_{K,W}^G$  be two simple Mackey functors for  $G$  over  $k$ . Then there is an isomorphism*

$$S_{H,V}^G \widehat{\otimes} S_{K,W}^G \cong \begin{cases} \text{FQ}_{H,V \otimes_k W}^G, & \text{if } H =_G K, \text{ (hence } H = K); \\ 0, & \text{otherwise,} \end{cases}$$

of Mackey functors for  $G$ .

Here, for a  $kN_G(H)/H$ -module  $M$ , the functor  $\text{FQ}_{H,M}^G$  is defined by

$$\text{FQ}_{H,M}^G = \text{Ind}_{N_G(H)}^G \text{Inf}_{N_G(H)/H}^{N_G(H)} \text{FQ}_M.$$

Now by [6, Proposition 5.4], we get

$$\text{FQ}_{H,M}^G = \text{Ind}_{N_G(H)}^G \text{Inf}_{W_G(H)}^{N_G(H)} \text{Ind}_{\rho(W_G(H))}^{\mu(W_G(H))} D_M$$

where  $D_M$  is the  $\rho(W_G(H))$ -module whose evaluation at the trivial subgroup is  $M$  and any other evaluation is zero. Now we claim that there is an isomorphism of Mackey functors

$$\text{FQ}_{H,M}^G \cong D_{H,M}^G = \text{Ind}_{\rho}^{\mu} D_{H,M}^{\rho}.$$

This follows from commutativity of the following diagram:

$$\begin{array}{ccccc} & & \xrightarrow{\text{Def}_{W_G(H)}^{N_G(H)}} & & \\ \mu(N_G(H))\text{-mod} & \xrightarrow{\text{Res}_{N_G(H)}^G} & \mu(W_G(H))\text{-mod} & \xrightarrow{\text{Res}_{\rho(W_G(H))}^{\mu(W_G(H))}} & \rho(W_G(H))\text{-mod} \\ \uparrow \text{Res}_{\rho(G)}^{\mu(G)} & & & & \uparrow \text{Def}_{W_G(H)}^{N_G(H)} \\ \mu(G)\text{-mod} & \xrightarrow{\text{Res}_{N_G(H)}^G} & \rho(N_G(H))\text{-mod} & & \\ \downarrow \text{Res}_{\rho(G)}^{\mu(G)} & & & & \downarrow \text{Res}_{\rho(G)}^{\mu(G)} \\ \rho(G)\text{-mod} & \xrightarrow{\text{Res}_{N_G(H)}^G} & \rho(N_G(H))\text{-mod} & & \end{array}$$

The proof of the commutativity is similar to the above proof of Proposition 4.8 and is left to the reader. Note that left adjoint of the composition of maps in the upper part gives the functor  $\text{FQ}_{H,M}^G$  whereas that of the composition of the maps in the lower part of the diagram gives the functor  $D_{H,M}^G$ .

Therefore we obtain, with our notation that

$$S_{H,V}^G \widehat{\otimes} S_{H,W}^G \cong D_{H,V \otimes_k W}^G.$$

4.11. In particular in  $\mathcal{R}_{\mu_k}(G)$ , the product  $[S_{H,V}^G] \cdot [S_{K,W}^G]$  is non-zero only if  $H =_G K$  and in this case we have

$$[S_{H,V}^G] \cdot [S_{H,W}^G] = \sum_{V' | V \otimes_k W} [S_{H,V'}]$$

and hence  $\mathcal{R}_{\mu_k}(G)$  is a commutative associative ring with unity  $\sum_{H \leqslant G} [S_{H,k}^G]$ . Note that by [11, Corollary 8.9], we have

$$[B^G] = \sum_{H \leqslant G} [S_{H,1}^G]$$

where  $B^G$  denotes the functor of the Burnside ring.

## 5. Linearization map

In this section we introduce the linearization map that connects the two  $\circ$ -biset functors of the previous two sections. We shall prove that as in the case of the Burnside ring and the rational representation ring, our version of the linearization map is a ring homomorphism when evaluated at a fixed group. Moreover if we extend the coefficients to a field of characteristic zero, and if we consider the Mackey functors for  $G$  over  $\mathbb{Q}$ , the linearization map is surjective. In particular, over a field of characteristic zero, any virtual Mackey functor is a virtual sum of certain Mackey functors, which we discuss in the next section.

**5.1. Definition.** Let  $G$  be a finite group and  $k$  be a field. We define the *linearization map*

$$\text{lin}_{k,G} : \Lambda(G) \rightarrow \mathcal{R}_{\mu_k}(G)$$

as the map induced by associating the unit  $[G/1]_G$  of  $\Lambda(G)$  to the unit  $[B^G]$  of  $\mathcal{R}_{\mu_k}(G)$ .

5.2. First we want to justify that the linearization map induces a morphism of  $\circ$ -biset functors. It suffices to check that  $[B^G]$  satisfies the condition of Corollary 3.12. But this is trivial since restriction and deflation of the Burnside ring Mackey functor is again the Burnside ring Mackey functor and it is sent to the Burnside ring Mackey functor with any group isomorphism. Hence our definition indeed induces a morphism of  $\circ$ -biset functors. Now it is trivial by the above remark that for an arbitrary subquotient  $H/N$  of  $G$ , we have

$$\text{lin}_{k,G}([H/N]_G) = [\text{Tin}_{H/N}^G \text{Des}_{H/N}^G B^G] = [\text{Tin}_{H/N}^G B^{H/N}] = [B_{H/N}^G].$$

The fundamental property of the classical linearization map is that it is a homomorphism of rings. The next results show that our version is also a ring homomorphism.

**5.3. Theorem.**  $\text{lin}_{k,G} : \Lambda(G) \rightarrow \mathcal{R}_{\mu_k}(G)$  is a ring homomorphism.

**Proof.** Recall the product in  $\Lambda(G)$ . Let  $H/N$  and  $K/M$  be two subquotients of  $G$ . Then

$$[H/N]_G \cdot [K/M]_G = \sum_{\substack{x \in H \setminus G/K \\ {}^xMN \leqslant H \cap {}^xK}} [H \cap {}^xK / {}^xMN]_G.$$

Hence we are to show that after applying the linearization map, a similar equality holds in  $\mathcal{R}_{\mu_k}(G)$ . Explicitly we are to show that the following isomorphism holds:

$$\begin{aligned} & (\text{Ind}_H^G \text{Inf}_{H/N}^H B^{H/N}) \hat{\otimes} (\text{Ind}_K^G \text{Inf}_{K/M}^K B^{K/M}) \\ & \cong \bigoplus_{\substack{x \in H \setminus G/K \\ {}^xMN \leqslant H \cap {}^xK}} \text{Ind}_{H \cap {}^xK}^G \text{Inf}_{H \cap {}^xK / N \cap {}^xM}^{H \cap {}^xK} B^{H \cap {}^xK / N \cap {}^xM}. \end{aligned}$$



By Proposition 5.4, below, the functor  $\mathcal{R}_{\mu_k}$  satisfies the Frobenius relations. Hence applying this proposition together with the Mackey relation we obtain the following series of isomorphism:

$$\begin{aligned}
 & (\text{Ind}_H^G \text{Inf}_{H/N}^H B^{H/N}) \widehat{\otimes} (\text{Ind}_K^G \text{Inf}_{K/M}^K B^{K/M}) \\
 & \cong \text{Ind}_H^G (\text{Inf}_{H/N}^H B^{H/N} \widehat{\otimes} (\text{Res}_H^G \text{Ind}_K^G \text{Inf}_{K/M}^K B^{K/M})) \\
 & \cong \bigoplus_{x \in H \backslash G/K} \text{Ind}_H^G (\text{Inf}_{H/N}^H B^{H/N} \widehat{\otimes} \text{Ind}_{H \cap {}^x K}^H \text{Res}_{H \cap {}^x K}^{xK} {}^x \text{Inf}_{K/M}^K B^{K/M}) \\
 & \cong \bigoplus_{x \in H \backslash G/K} \text{Ind}_{H \cap {}^x K}^G (\text{Res}_{H \cap {}^x K}^H \text{Inf}_{H/N}^H B^{H/N} \widehat{\otimes} \text{Res}_{H \cap {}^x K}^{xK} {}^x \text{Inf}_{K/M}^K B^{K/M}).
 \end{aligned}$$

Next we use Proposition 4.8 to interchange inflation and restriction and obtain the following isomorphism

$$\begin{aligned}
 & \bigoplus_{x \in H \backslash G/K} \text{Ind}_{H \cap {}^x K}^G (\text{Res}_{H \cap {}^x K}^H \text{Inf}_{H/N}^H B^{H/N} \widehat{\otimes} \text{Res}_{H \cap {}^x K}^{xK} {}^x \text{Inf}_{K/M}^K B^{K/M}) \\
 & \cong \bigoplus_{\substack{x \in H \backslash G/K \\ N, {}^x M \leq H \cap {}^x K}} \text{Ind}_{H \cap {}^x K}^G (\text{Inf}_{(H \cap {}^x K)/N}^{H \cap {}^x K} B \widehat{\otimes} \text{Inf}_{(H \cap {}^x K)/{}^x M}^{H \cap {}^x K} B).
 \end{aligned}$$

Now the result follows since  $\text{inf}_{H/N}^H B \cong \bigoplus_{N \leq {}_H L \leq {}_H H} S_{L,1}^H$ .  $\square$

**5.4. Proposition.** Let  $K \leq G$  be finite groups. Let  $M$  be a Mackey functor for  $G$  over  $k$  and  $N$  be a Mackey functor for  $K$  over  $k$ . Then the Frobenius relations

$$M \widehat{\otimes} \text{Ind}_K^G(N) \cong \text{Ind}_K^G(\text{Res}_K^G(M) \widehat{\otimes} N)$$

and

$$\text{Ind}_K^G(N) \widehat{\otimes} M \cong \text{Ind}_K^G(N \widehat{\otimes} \text{Res}_K^G(M))$$

are satisfied.

**Proof.** Since both restriction and induction are additive and the Mackey algebra is semisimple over  $k$ , it suffices to prove the claim for simple Mackey functors.

Let  $H \leq K$  be two subgroups of  $G$ . Also let  $V$  be a simple  $kW_K(H)$ -module and  $W$  be a simple  $kW_G(H)$ -module. Then we are to show that

$$\text{Ind}_K^G(S_{H,V}^K) \widehat{\otimes} S_{H,W}^G \cong \text{Ind}_K^G(S_{H,V}^K \widehat{\otimes} \text{Res}_K^G(S_{H,W}^G))$$

and

$$S_{H,W}^G \widehat{\otimes} \text{Ind}_K^G(S_{H,V}^K) \cong \text{Ind}_K^G(\text{Res}_K^G(S_{H,W}^G) \widehat{\otimes} S_{H,V}^K)$$

as Mackey functors for  $G$ . Next we prove the first relation. The second one can be proved similarly. Using Propositions 4.8 and 4.10, we evaluate the left-hand side as

$$\begin{aligned} \text{Ind}_K^G(S_{H,V}^K) \widehat{\otimes} S_{H,W}^G &\cong D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} V}^G \widehat{\otimes} S_{H,W}^G \\ &\cong D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} V \otimes W}^G. \end{aligned}$$

The right-hand side can also be evaluated using the same results as

$$\begin{aligned} \text{Ind}_K^G(S_{H,V}^K \widehat{\otimes} \text{Res}_K^G(S_{H,W}^G)) &\cong \text{Ind}_K^G(D_{H, V \otimes \text{Res}_{W_K(H)}^{W_G(H)} W}^K) \\ &\cong D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} (V \otimes \text{Res}_{W_K(H)}^{W_G(H)} W)}^G. \end{aligned}$$

Note that although restriction of the simple Mackey functor  $S_{H,W}^G$  to  $K$  gives a sum of simple Mackey functors for  $K$ , taking the product with  $S_{H,V}^K$  annihilates all summands except the one indexed by the  $K$ -conjugacy class of  $H$ . Now the result follows since the Frobenius relations are satisfied by the group algebras.  $\square$

After extending the coefficients to a field of characteristic zero, we find that the image of the linearization map coincides with the rational character ring of the Mackey algebra.

**5.5. Theorem.** *The linearization map  $\mathbb{Q} \text{lin}_{\mathbb{Q},G} : \mathbb{Q} \Lambda(G) \rightarrow \mathbb{Q} \mathcal{R}_{\mu_{\mathbb{Q}}}(G)$  is surjective.*

**Proof.** We prove the theorem in two steps. In the first step we observe that the  $\mathbb{Q}$ -vector space  $\mathbb{Q} \mathcal{R}_{\mu_{\mathbb{Q}}}(G)$  is generated by the set of Mackey functors of the form  $\text{Ind}_K^G S_{H,1}^K$  as  $H \leq K$  runs over all subgroups of  $G$ . Then in the second step, we construct each  $S_{H,1}^K$  for any pair  $H \leq K$  of subgroups of  $G$  using a suitable virtual  $(K, G)$ -biset  $X$ .

Since the simple Mackey functors form a basis for  $\mathbb{Q} \mathcal{R}_{\mu_{\mathbb{Q}}}(G)$ , it suffices to show that  $[S_{H,V}^G]$  is a  $\mathbb{Q}$ -linear combination, as stated above. Now by Proposition 4.8, we have

$$\text{Ind}_K^G S_{H,1}^K \cong D_{H, \text{Ind}_{W_K(H)}^{W_G(H)} 1}^G.$$

Now the result follows from the surjectivity of the linearization map  $\text{lin}_T : \mathbb{Q} B(T) \rightarrow \mathbb{Q} \mathcal{R}_{\mathbb{Q}}(T)$ .

Next we are aiming to construct the simple functors  $S_{H,1}^K$ . Since  $\text{Des}_K^G B^G \cong B^K$ , we can assume, without loss of generality, that  $K = G$ .

To prove the claim, argue by induction on the order  $|G|$  of  $G$ . If  $G = 1$  is the trivial group, there is a unique simple Mackey functor for  $G$  and it is isomorphic to the Burnside ring Mackey functor. Now let  $G$  be non-trivial and assume the claim for all groups of order less than the order  $|G|$  of  $G$ .

First note that we can reduce to the case where  $H$  is a normal subgroup of  $G$ . Indeed if  $H$  is not normal, then by the induction hypothesis, we have already constructed  $S_{H,1}^{N_G(H)}$  and

$$\text{Ind}_{N_G(H)}^G S_{H,1}^{N_G(H)} \cong D_{H, \text{Ind}_{W_{N_G(H)}(H)}^{W_G(H)} 1}^G$$

but  $W_{N_G(H)}(H) = W_G(H)$ . So  $\text{Ind}_{N_G(H)}^G S_{H,1}^{N_G(H)} = S_{H,1}^G$ .

Therefore assume that  $H$  is a normal subgroup of  $G$ . Now if  $H$  is not trivial, then it is easy to calculate using Proposition 4.8 that

$$S_{H,1}^G = \text{Inf}_{G/H}^G S_{H/H,1}^{G/H}$$

and hence we are done by induction. As a result we have constructed all  $S_{H,1}^G$  except when  $H = 1$  is the trivial group. But recall the following equality to finish the proof:

$$[B^G] = \sum_{H \leq G} [S_{H,1}^G]. \quad \square$$

**5.6. Corollary.** *The  $\circ$ -biset functor  $\mathbb{Q}\mathcal{R}_{\mu_{\mathbb{Q}}}$  for  $G$  is cyclic generated by the class of Burnside ring Mackey functor  $[B^G]$  for  $G$ .*

## 6. The functors $B_{H/N}^G$

Theorem 5.5 asserts that over  $\mathbb{Q}$ , any (virtual) Mackey functor is a linear combination of functors  $\mathbb{Q}B_{H/N}^G$ . In this section, we explicitly describe these functors. Note that the case  $N = 1$  is well known as the functor  $B_H^G$  coincides with the Dress construction  $B_{G/H}^G$  of the Burnside ring since by [4, Section 5.5], we have

$$B_{G/H}^G = \text{Tin}_H^G \text{Des}_H^G B^G.$$

Recall that by definition,

$$B_{H/N}^G = \text{Tin}_{H/N}^G B^{H/N}.$$

By Corollary 8.9 in [11] and Proposition 4.8, we have

$$\mathbb{Q}B_{H/N}^G \cong \bigoplus_{\substack{K \leq G \\ N \leq K \leq H}} \text{Ind}_H^G S_{K,\mathbb{Q}}^G \cong \bigoplus_{\substack{K \leq G \\ N \leq K \leq H}} D_{K, \text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}}^G.$$

Now we want to determine a basis for each evaluation  $\mathbb{Q}B_{H/N}^G(R)$  of  $\mathbb{Q}B_{H/N}^G$  at each  $R \leq G$ . It suffices to find one for each  $D_{K, \text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}}^G(R)$ . Note that the set  $N_G(K)/N_H(K)$  is a basis for  $\text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}$  and it is stable in the sense of [1, Section 7]. Hence identifying  $D_{K, \text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}}^G$  with  $\text{Ind}_{\rho}^{\mu} D_{K, \text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}}^{\rho}$  and applying [1, Lemma 7.2], we obtain that the set

$$\{t_L^R \otimes x \mid L \leq_R R, L = {}^g K, x^g \in N_G(K)/N_H(K)\}$$

is a basis for  $D_{K, \text{Ind}_{W_H(K)}^{W_G(K)} \mathbb{Q}}^G(R)$ .

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