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Extensions with overlap

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ABSTRACT

Let N be a normal subgroup of the group G . By a result of P. Hall, G is nilpotent if N and G/N' are nilpotent. We are looking for group classes \mathcal{C} and characteristic subgroups $f(N)$ of N such that the statement above, with $(\text{belongs to } \mathcal{C}, f(N))$ substituted for $(\text{is nilpotent}, N')$, remains true.

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Introduction

It is well known that extensions of nilpotent groups by nilpotent groups need not be nilpotent; the nonabelian group of order 6 is a counterexample. A famous result of P. Hall (see [9]) states that a group G with normal subgroup N is nilpotent whenever N and G/N' is nilpotent. This is an example of an *extension with overlap*: Only by “covering” the quotient N/N' twice it is possible to reach to a conclusion. Another example of an overlap is the following: If N and G/N'' are supersolvable then G is supersolvable. This follows from the work of Robinson [15] together with the fact that the commutator subgroup of the supersolvable group N is nilpotent.

We want to consider this situation in the class of finite groups for specific classes of groups different from nilpotent groups, in other words, we are looking for word subgroups $w(N)$ of normal subgroups N of G such that $G \in \mathbf{C}$ whenever $N, G/w(N) \in \mathbf{C}$, for a given class \mathbf{C} of groups. In this case we will say that the word w is \mathbf{C} -sufficient or sufficient with respect to \mathbf{C} . So for the overlap $N/w(N)$ we will restrict ourselves to varieties, $w(N)$ being the smallest normal subgroup such that $N/w(N)$ belongs to the variety defined by the identity $w = 1$.

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In the case of formations we obtain also that the statement remains true whenever $w(N)$ is substituted with a G -invariant subgroup $K \subseteq w(N)$. In the case of saturated formations it is almost inevitable that the Frattini subgroup or some derivation comes into play (see Lemma 1), and for each class **C** a word subgroup has to be found which is contained in this subgroup. We will take, in particular, subclasses of the class of supersolvable groups as examples for this procedure.

All groups considered in this article are finite unless otherwise stated.

1. The role of the Frattini subgroup

For the Frattini subgroup, the intersection $\Phi(G)$ of all maximal subgroups of a group G , we have the following general statement: If N is a normal subgroup of G , then $\Phi(N) \subseteq \Phi(G)$, see for instance Huppert [11, Hilfssatz 3.3(b), p. 269]. For the main result of this section we will need a slightly stronger result, which may be considered as a localization. We begin with a definition.

Definition 1. Let Λ be a set of primes.

- (1) We say that the group G is Λ -solvable, if the following holds: every chief factor of G that is of order divisible by some $p \in \Lambda$ is of order a power of p .
- (2) The group G is Λ -supersolvable, if in addition to (1) all these chief factors are cyclic.
- (3) The group G is Λ -nilpotent, if in addition to (1) all these chief factors are central.

Lemma 1. Assume that Λ is a set of primes and G is a Λ -solvable group with normal subgroup N . Denote by $\Phi_\Lambda(K)$ the intersection of all maximal subgroups of the group K that have index a power of a prime belonging to Λ . Then $\Phi_\Lambda(N) \subseteq \Phi_\Lambda(G)$.

For the proof remember that any maximal subgroup M of G with $N \not\subseteq M$ satisfies $MN = G$ and so $|G : N| = |MN : N| = |M : M \cap N|$, also maximal subgroups of N containing $N \cap M$ have index dividing $|G : M|$, a power of p .

Let Λ be a set of primes and consider a Λ -solvable group G , and a saturated formation with local definition by formations \mathbf{f}_p for all primes $p \in \Lambda$. We consider the subset Γ of primes p for which \mathbf{f}_p is neither empty nor the class of all finite groups. For us at the moment, this is the *relevant* set of primes, and so we will name it. Now we are able to formulate the main result of this section.

Proposition 1. Assume that we have a saturated formation \mathcal{F} of Λ -solvable groups with relevant subset Γ of primes. If N is a normal subgroup of G such that N and $G/\Phi_\Gamma(N)$ belong to \mathcal{F} , then also G belongs to \mathcal{F} .

Proof. G is Λ -solvable since N and G/N are. If the local definition for a prime p is empty, then the orders of all groups belonging to \mathcal{F} are prime to p , so in our case again $|G/N|$ and $|N|$ are prime to p and so is $|G|$. If the local definition is the class of all finite groups, nothing has to be checked. So the primes belonging to Γ are the only ones to be tested, in particular, $G \in \mathcal{F}$ if and only if $G/\Phi_\Gamma(G) \in \mathcal{F}$. Now the result follows from Lemma 1. \square

2. First applications

We use now for brevity the notation of sufficient word introduced in the introduction.

Proposition 2.

- (a) Let \mathcal{F} be the class of Σ -nilpotent groups with finite set of primes Σ , and let m be the product of all primes in Σ . Then $x^m[x, y]$ is an \mathcal{F} -sufficient word.
- (b) A sufficient word for the class of p -supersolvable groups is $v = (x^{p-1}[x, y])^p[x^{p-1}[x, y], z^{p-1}[z, w]]$.

Proof. For a Σ -nilpotent group we have that all maximal subgroups with index a power of a prime $p \in \Sigma$ are in fact normal, the quotient group is cyclic of order p . So $N/\Phi_\Sigma(N)$ is abelian and of exponent m , $\Phi_\Sigma(N) = N'N^m$, and (a) is shown by Lemma 1.

For (b) we remember that all maximal subgroups $M \subset N$ of index a power of p are (not necessarily normal but) of index p . It follows that $N/\bigcap_{x \in N} M^x$ is isomorphic to a subgroup of the extension of C_p by C_{p-1} . The corresponding variety is that of elementary abelian p -groups by abelian groups of exponent $p-1$. The word v given in (b) corresponds to this variety, so $v(N) \subseteq \Phi_p(N) \subseteq \Phi_p(G)$, and (b) follows. \square

Remark 1.

- (a) If Σ in Proposition 2(a) is infinite, the power x^m has to be removed and the result of P. Hall appears. Notice however, that the proof, here designed for finite groups, is unsuitable in the infinite case since $\Phi(N) = N$ is true for divisible nilpotent groups N .
- (b) Since a supersolvable group is just p -supersolvable for all primes, we deduce $[[x, y], [z, w]]$ as a sufficient word for this class. If we have the class of Σ -supersolvable groups with finite set Σ and $m = \prod\{p \mid p \in \Sigma\}$, if also n is the lowest common multiple of all $p-1$ with $p \in \Sigma$, then $(x^n[x, y])^m[x^n[x, y], z^n[z, w]]$ is a sufficient word for Σ -supersolvable groups.

3. Classes of p -solvable groups with certain permutability conditions

A subgroup H of a group G is said to permute with a subgroup K of G if $KH = HK$. H is said to be *permutable* (*S-permutable*) if H permutes with all subgroups (Sylow subgroups) of G . Kegel [12] proved that an S-permutable subgroup is subnormal in G .

We say S-permutability is a transitive relation in a group G if a subgroup H which is S-permutable in an S-permutable subgroup K of G is S-permutable in G . A group G is called a *PST-group* if S-permutability is a transitive relation in G . By Kegel's result PST-groups are exactly those groups in which all subnormal subgroups are S-permutable. Likewise permutability is a transitive relation if every subnormal subgroup of G is permutable. Groups with this property are called *PT-groups*. Examples of PT-groups are the *Iwasawa* groups, the groups all of whose subgroups are permutable. A group is called a *T-group* if normality is a transitive relation in G . *Dedekind* groups, that is, groups all of whose subgroups are normal, are examples of T-groups. These classes have been investigated in [1–8, 13, 14, 18].

We note that the class of T-groups is a proper subclass of the class of PT-groups which is a proper subclass of the class of PST-groups. Also, every nilpotent group is a PST-group. The classes of solvable *T* (resp. *PT* and *PST*)-groups are closed with respect to forming subgroups and quotient groups.

Agrawal [1] showed that a group G is a solvable PST-group if and only if it has an abelian normal Hall subgroup N on which G acts by conjugation as group of power automorphisms and G/N is nilpotent; G is a solvable *T* (resp. *PT*)-group if and only if it is a solvable PST-group whose Sylow subgroups are Dedekind (resp. Iwasawa). The subgroup N mentioned above may be taken to be the nilpotent residual of G (see [1, 8, 18]).

In [7] it is established that G is a solvable PST-group if it has a normal subgroup N such that N and G/N'' are solvable PST-groups. Thus we have

Theorem A. *The word $w = [[x, y], [z, t]]$ is sufficient for the class of solvable PST-groups.*

This theorem is another example of an extension with overlap. We now develop a local approach to provide a different proof of Theorem A.

Let p be a prime. A group G is called a C_p -group if every subgroup of a Sylow p -subgroup P of G is normal in $N_G(P)$. Robinson [14] showed that a group G is a solvable T-group if and only if it is a C_p -group for all primes p . A group is called an X_p -group if every subgroup of a Sylow p -subgroup P of G is permutable in $N_G(P)$. In [4] the first author, Brewster and Robinson showed that a group G is

a solvable PT -group if and only if it is an X_p -group for all primes p . A group G is called a Y_p -group provided that whenever K is a p -subgroup of G , every subgroup of K is S -permutable in $N_G(K)$. Ballester-Bolínches and Esteban-Romero showed that a group G is a solvable PST -group if and only if it is a Y_p -group for all primes p (see [3]).

Each of the classes C_p , X_p and Y_p are subgroup and quotient closed. Moreover, $C_p \subset X_p \subset Y_p$. The next two results show how closely related these classes are; they are used in the proofs of the main theorems of this article.

Theorem 1. (See [3].) *Let p be a prime and let G be a Y_p -group. Then*

- (1) G is an X_p -group if and only if the Sylow p -subgroups of G are Iwasawa groups.
- (2) G is a C_p -group if and only if the Sylow p -subgroups of G are Dedekind groups.

Theorem 2. (See [3,4].) *Let G be a prime and G a group. Then*

- (1) G is an X_p -group if and only if G has Iwasawa Sylow p -subgroups and is p -nilpotent or a Sylow p -subgroup of G is abelian and G is a C_p -group.
- (2) G is a Y_p -group if and only if a Sylow p -subgroup of G is abelian and satisfies C_p or G is p -nilpotent.

Now we formulate our statement.

Theorem B. *Let p be a prime and put $w = [x^{p-1}[x, y], z^{p-1}[z, t]]$.*

- (1) *The word w is sufficient for the class of p -supersolvable groups with abelian Sylow p -subgroups.*
- (2) *The word w is sufficient for p -solvable C_p -groups, $p > 2$.*
- (3) *The word w is sufficient for p -solvable Y_p -groups.*
- (4) *The word $v = [x, y]^2[[x, y], z]$ is sufficient for solvable C_2 -groups.*
- (5) *The word $u = [[x, y], [z, t]]$ is sufficient for p -solvable X_p -groups.*

The following two lemmas are useful for the proof.

Lemma 2. (See [13].) *Let p be a prime and let N be a normal p' -subgroup of a group G . Let \mathcal{P} be one of the properties C_p , X_p or Y_p . Then if G/N is a \mathcal{P} -group, also G is a \mathcal{P} -group.*

This follows directly from the definitions.

Lemma 3. *Let G be a p -solvable Y_p -group. Then G is p -supersolvable.*

Proof. Let G be a minimal counterexample. Then a minimal normal subgroup N of G must be an elementary abelian p -subgroup. Let $g \neq 1$ be an element of $N \cap Z(P)$ where P is a Sylow p -subgroup of G . Let Q be a Sylow q -subgroup of G , where $q \neq p$. Since G is a Y_p -group, $\langle g \rangle Q$ is a subgroup of G and so Q normalizes $\langle g \rangle = N \cap \langle g \rangle Q$. Thus $\langle g \rangle = N$. By minimality of G we also have that G/N is p -supersolvable, and so G is p -supersolvable. Lemma 3 is proved by this contradiction. \square

Proof of Theorem B. (1) Let N be a normal subgroup of a group G such that N and $G/w(N)$ are p -supersolvable with abelian Sylow p -subgroups. G is p -supersolvable by Proposition 2. Assume $p > 2$. Note that $d(N) = N'N^{p-1}$ is p -nilpotent where $d = x^{p-1}[x, y]$. Put $K = N'N^{p-1}$ and denote the normal Hall p' -subgroup of K by H . Let P be a Sylow p -subgroup of K . Since P is abelian, we note that $K/H \subseteq PH/H \cong P$ is also abelian and K' is a p' -subgroup. Recall $w(N) = (N'N^{p-1})' = K'$ so that G/K' is p -supersolvable with abelian Sylow p -subgroups. This means that G is p -supersolvable with abelian Sylow p -subgroups. Now assume $p = 2$. In this case the 2-supersolvable group G is 2-nilpotent. Put $K = N'$ and the same argument as above shows that K' is a 2'-group and the Sylow 2-subgroups of G are abelian.

(2) Assume $p > 2$. Let N be a normal subgroup of a group G such that N and $G/w(N)$ are p -solvable C_p -groups. G is p -supersolvable by Proposition 2 and Lemma 3, and (since $p > 2$) the Sylow p -subgroups of $G/w(N)$ and of N are abelian; by (1) the same is true for G . Let $K = N'N^{p-1}$ as in the proof of (1). Then K is p -nilpotent and $K' = w(N)$ is a p' -group. Now G is a C_p -group by Lemma 2 since $G/w(N)$ is a C_p -group.

(3) Let N be a normal subgroup of a group G such that N and $G/w(N)$ are p -solvable Y_p -groups. N and $G/w(N)$ are p -supersolvable by Lemma 3, hence G is p -supersolvable by Proposition 2. We use Theorem 2(2) to show that G is a Y_p -group. If N and $G/w(N)$ are p -nilpotent, then G is p -nilpotent by Proposition 1. G is a Y_p -group by Theorem 2(2). Also if $p = 2$, then G is 2-nilpotent and G is a Y_2 -group.

We may assume $p > 2$. Assume that N is not p -nilpotent. Then N has abelian Sylow p -subgroups and is a C_p -group by Theorem 2(2). Again, as in the proof of (1), $w(N)$ is a p' -group. Hence if $G/w(N)$ is p -nilpotent, then G and N are both p -nilpotent. Thus $G/w(N)$ is not p -nilpotent. Using Theorem 2(2) again we note that $G/w(N)$ has abelian Sylow p -subgroups and is a C_p -group. Now G has abelian Sylow p -subgroups and is a C_p -group by (2). We conclude that G is a Y_p -group by Theorem 2(2).

(4) Let N be a normal subgroup of G such that N and $G/v(N)$ are 2-solvable C_2 -groups. So N and $G/v(N)$ are 2-supersolvable and 2-nilpotent and their Sylow 2-subgroups are Dedekind groups. So if S is a Sylow 2-subgroup of N , then $v(S) = 1$ and consequently $v(N)$ is a $2'$ -group. Now with $G/v(N)$ also G is a C_2 -group by Lemma 2.

(5) Let N be a normal subgroup of a group G such that N and $G/u(N)$ are p -solvable X_p -groups. N and $G/u(N)$ are p -supersolvable by Lemma 3 and so G is p -supersolvable. Consider a Sylow p -subgroup S of G . We distinguish two cases: $SC_G(S) = N_G(S)$ and $SC_G(S) \neq N_G(S)$. First assume $SC_G(S) = N_G(S)$. Then $N_G(S)$ is p -nilpotent. We now show that $u(N)$ is a p' -group so that $G/u(N)$ being an X_p -group yields that G is an X_p -group. By Theorem 2(1), N is p -nilpotent with Iwasawa Sylow p -subgroups or N is a C_p -group with abelian Sylow p -subgroups. Assume that N is p -nilpotent with Iwasawa Sylow p -subgroups. Denote the normal Hall p' -subgroup of N by H . Then $N = H(S \cap N)$. $S \cap N$ is metabelian (see [17, Theorem 2.3.1, p. 55]). Note that $u(N) = N''$ so that $N'' \subseteq H$ and $u(N)$ is a p' -group. Next assume that N is a C_p -group with abelian Sylow p -subgroups. Since N is p -supersolvable, N' is p -nilpotent and by a similar argument as before we have that N'' is contained in the normal Hall p' -subgroup of N' . So $u(N)$ is a p' -group. In both alternatives we have that $u(N)$ is a p' -group, now G is an X_p -group since $G/u(N)$ is an X_p -group, by Lemma 2.

Finally, consider the case $SC_G(S) \neq N_G(S)$. Thus $p > 2$ since 2-supersolvable groups are 2-nilpotent. As in part (1), let $w = [x^{p-1}[x, y], z^{p-1}[z, t]]$ and note that $N'' = u(N) \subseteq w(N) = (N'N^{p-1})'$. Also note that N and $G/u(N)$ are X_p -groups and likewise Y_p -groups. As in the proof of (2) all the subgroups of S are normal in $N_G(S)$ so that G is an X_p -group. \square

Remark 2. The pair $(G, N) = (Q_8 \times C_4, Q_8)$ shows that Theorem B(2) cannot be extended to $p = 2$; here we would have $w = [x, z]$ which is not sufficient.

Proof of Theorem A. This follows from Theorem 1(1) and Theorem B(3).

Since solvable T - and PT -groups are metabelian, the word $w = [[x, y], [z, t]]$ is (trivially) sufficient for these classes. We will have a look at two other classes containing the class of T -groups. \square

Definition 2.

- (i) G is a T_0 -group if and only if $G/\Phi(G)$ is a T -group, where $\Phi(G)$ is the Frattini subgroup of G .
- (ii) G is a T_1 -group if and only if $G/H(G)$ is a T -group, where $H(G)$ is the hypercenter of G .

The groups considered by Huppert in [10] are T_0 -groups. For T_0 -groups we have

Theorem C. The word $w = [[x, y], [z, t]]$ is sufficient for the class of solvable T_0 -groups.

For a proof notice that $G'' \subseteq \Phi(G)$ for T_0 -groups G since T -groups are metabelian.

The following counterexamples show that the word $[[x, y], [z, w]]$ is not sufficient for solvable T_1 -groups.

Example 1. Let $q > 3$ be a prime and $k > 1$ such that $2k|q-1$. Choose r such that $r^{2k} \equiv 1 \pmod{q}$ and $r^s \not\equiv 1 \pmod{q}$ for $1 \leq s < 2k$. Consider

$$G = \langle a, b, c \mid a^{2k} = b^q = c^q = [b, c], c = [b, c], b = 1; a^{-1}ba = b^r; a^{-1}ca = c^r \rangle.$$

Then $N = \langle a^k, b, c \rangle$ is a T_1 -group and G/N'' is even a T -group, but G is not a T_1 -group since $G'' = \langle [b, c] \rangle = [G'', G]$.

Instead, we find

Theorem D. The word $[[[[x, y], [z, t]], [a, b]], [c, d]]$ is sufficient for solvable T_1 -groups.

We will need the following statement, which may be of independent interest. *This is the only theorem in this article where finiteness is not required.*

Theorem E. Assume that the following conditions are satisfied for the group G and its nilpotent normal subgroup N :

- (i) $[G, k, N]N' = [G, k+1, N]N'$ for some integer k ;
- (ii) $[G, m, N'] \subseteq N_4$ for some integer m .

Then there is some integer t such that $[G, t, N'] = 1$.

The reader will notice that the replacement of N_4 by N'' in Theorem E is proved quickly by a theorem of Robinson (see [16, p. 129]). Later we will need the following form.

Corollary. Assume that G is a finite group and N is a normal p -subgroup of G . Then $N' \subseteq H(G)$ if and only if $N'/N_4 \subseteq H(G/N_4)$.

Proof of Theorem E. Let A and B normal subgroups of G with $B \subseteq A$. To avoid arithmetic we say that the quotient A/B is *polycentral* if and only if there is an integer r such that $[G, r, A] \subseteq B$ (this corresponds to the expression polytrivial used by Robinson [16, p. 129] and [15] for modules). Let $K = [G, k, N]N'$. Then by hypothesis $[G, K]N' = K$, and N/K and N'/N_4 are polycentral.

As a first step we consider commutators formed with K . Assume that N_m/N_{m+2} is polycentral for some $m \geq 2$. Then there is a number t such that $[[G, t+1, N_m], K] \subseteq N_{m+2}$. Assume that t is minimal with this property and that $t \geq 0$. Then

$$\begin{aligned} [[G, t, N_m], K]N_{m+2} &= [[G, t, N_m], [G, K]]N_{m+2} \subseteq [G, [[G, t, N_m], K]][[G, t+1, N_m], K]N_{m+2} \\ &= [G, [[G, t, N_m], K]]N_{m+2} \end{aligned}$$

by the Three Subgroup Lemma (see [16, 5.1.10, p. 122]) applied for G, K and $[G, t, N_m]$. Comparing the first and the last expression, we deduce

- (i) $[N_m, K] \subseteq N_{m+2}$ if N_m/N_{m+2} is polycentral.

Assume that A/B and C/D are polycentral. Then, by repeated application of the Three Subgroup Lemma, we obtain that $[A, C]/[A, D][C, B]$ is polycentral. This yields directly for $(A, B, C, D) =$

(N_m, N_{m+1}, N, K) :

(ii) $N_{m+1}/[N_m, K]N_{m+2}$ is polycentral if N_m/N_{m+1} is polycentral.

Now using the Three Subgroup Lemma for N, K , and N_{m-1} we have

$$[N_m, K]N_{m+2} \subseteq [[N_{m-1}, K], N][N_{m-1}, [K, N]]N_{m+2} \subseteq [N_{m-1}, N']N_{m+2}.$$

Now $[N_{m-1}, N_3][N_m, N'] \subseteq N_{m+2}$, and so

(iii) $[N_m, K]N_{m+2}/N_{m+2}$ is polycentral if N_{m-1}/N_{m+1} is polycentral.

We have assembled the tools and proceed by induction on the nilpotency class c of N . By hypothesis the theorem is true for $c = 3$. Assume that the theorem is true for all pairs G, N satisfying the conditions such that N is of nilpotency class $d \geq 3$, we want to show it for pairs G, N with N of nilpotency class $d + 1$. So N'/N_{d+1} is polycentral. We obtain from (ii) that $N_{d+1}/[N_d, K]N_{d+2}$ is polycentral, further $[N_d, K]N_{d+2}/N_{d+2}$ is polycentral by (iii). These three statements yield that N'/N_{d+2} is polycentral; the induction is complete and Theorem E is proved. \square

Remark 3. Let N in Theorem E be of nilpotency class c and $[G, r, N'] \subseteq N_3$. Then $[G, t, N'] = 1$ where $t \leq \binom{c-1}{2}k + (c-1)(r-1)$. This follows from statement (i) of the proof of Theorem E.

Proof of Theorem D. If w is the word given in Theorem D, then $w(K) = (K')_4$.

By hypothesis, N is a solvable T_1 -group and hence it is supersolvable. So, by definition, $N'' \subseteq H(N)$ and $N/H(N)$ is a T -group. Further $G/w(N)$ is a solvable T_1 -group and $(G''/w(N)) = (G/w(N))'' \subseteq H(G/w(N))$. Note that G is supersolvable. From $N''/w(N) \subseteq G''/w(N)$ we obtain $N''/w(N) \subseteq H(G/w(N))$. We apply Theorem E for the nilpotent normal subgroup N' : we have that $(N')'/(N')_4$ is contained in the hypercenter of $G/(N')_4$ and so $N'' \subseteq H(G)$. Now G/N'' is a T_1 -group and $H(G/N'') = H(G)/N''$. Finally we have that $G/H(G) \cong (G/N'')/H(G/N'')$ is a T -group; Theorem D is shown. \square

Remark 4. We consider the corresponding local property: A p -solvable group G is a \bar{C}_p -group if G/Z_p is a C_p -group, where Z_p is the Sylow p -subgroup of the hypercenter of G . A group G is a solvable T_1 -group if and only if it is a \bar{C}_p -group for all primes p , see [6].

For this class \bar{C}_p of groups there is no suitable word for an extension with overlap: Assume to the contrary that the word w is suitable. Construct a group G with normal subgroup N satisfying the following conditions:

- (i) G/N is a nonabelian p -group,
- (ii) N is a p' -group and a split extension of $w(N)$ by some subgroup U ,
- (iii) $C(U)w(N) = G$,
- (iv) $C(w(N)) \subseteq N$.

A construction like this is always possible. In this construction, N is a \bar{C}_p -group by (ii), $G/w(N)$ is a \bar{C}_p -group by (iii) and (ii), but G is no \bar{C}_p -group since the commutator subgroup of any Sylow p -subgroup of G does not centralize the p' -subgroup $w(N)$, according to (iv). So this localization cannot be used to find a suitable word for T_1 -groups.

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