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On the theory of $\mathfrak{F}$-centrality of chief factors and $\mathfrak{F}$-hypercentre for Fitting classes

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ABSTRACT

In this paper, we establish the theory of $\mathfrak{F}$-centrality of chief factors and $\mathfrak{F}$-hypercentre for Fitting classes. Based on this, we prove that every $\mathfrak{F}$-injector of G covers each $\mathfrak{F}$-central chief factor of G and avoids each $\mathfrak{F}$-eccentric chief factor of G if the Fitting class $\mathfrak{F}$ has an integrated and invariant H-function. As an application, we also give a new method to define local Fitting classes, that is, we use local functions but not H-functions to define local Fitting classes.

1. Introduction

Throughout this paper, all groups are finite and soluble and G denotes a finite soluble group.

It is well known that a group G is nilpotent if and only if every chief factor H/K of G is central, that is, $H/K \subseteq Z(G/K)$. Developing this result, Gaschütz [6] firstly introduced the general theory of central factors for the class of all soluble groups. Let $f : \mathbb{P} \to \{\text{formation of groups}\}$ be a formation function. Then a p-chief factor H/K of G is said to be f-central in G if $G/C_G(H/K) \in f(p)$. Following [4,6], a class $\mathfrak{F}$ of groups is called a formation if it is closed under homomorphic images and subdirect products. Let $LF(f) = \{G \mid G/C_G(H/K) \in f(p),$ for every p-chief factor H/K and every $p \mid |H/K|\}$. A formation is said to be a local formation if there exists a formation function f such that $\mathfrak{F} = LF(f)$.

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Note that if a formation function f is such that $f(p) = (1)$, where (1) is the class of identity groups, for all primes $p \in \mathbb{P}$, then $LF(f) = \mathfrak{N}$ is the class of all nilpotent groups.

For a local formation $\mathfrak{F} = LF(f)$, Huppert [11] introduced the concept of $\mathfrak{F}$-hypercentre of a group G: a normal subgroup N of G is called $\mathfrak{F}$-hypercentral if every G-chief factor H/K of N is f-central in G. It is easy to know that the product of any two $\mathfrak{F}$-hypercentral normal subgroups is an $\mathfrak{F}$-hypercentral normal subgroup of G. Therefore, every group G has a unique maximal $\mathfrak{F}$-hypercentral normal subgroup, which is called $\mathfrak{F}$-hypercentre of G and denoted by $Z_{\mathfrak{F}}(G)$. When $\mathfrak{F}$ is the class $\mathfrak{N}$ of all nilpotent groups, the $\mathfrak{F}$-hypercentre is $Z_\infty(G)$.

Recall that a class $\mathfrak{F}$ of groups is said to be a Fitting class if the following two conditions hold: (i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $N \in \mathfrak{F}$; (ii) if $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 \in \mathfrak{F}$, then $N_1N_2 \in \mathfrak{F}$.

The concept of Fitting class is actually the dual concept of formation.

In 1967, Fischer, Gaschütz and Hartley found an important generalization of Sylow theorem and Hall theorem in the theory of Fitting classes (see [5]). Based on this, Hartley [9] and D'Arcy [3] established the theory of local Fitting classes, as follows. Let f be a function $\mathbb{P} \to \{\text{Fitting classes}\}$, which was later called a Hartley function or, for brevity, an H-function (see, for example, [15]), and let $Supp(f) = \{p \in \mathbb{P} \mid f(p) \neq \emptyset\}$, which is called the support of f. Let $\pi = Supp(f)$ and $LR(f) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'})$. Then a Fitting class $\mathfrak{F}$ is said to be local if there exists an H-function f such that $\mathfrak{F} = LR(f)$.

It is well known that the f-hypercentre in the theory of local formations plays an important role in the research in finite groups, and a large number of interesting results have been obtained by using it (see, for example, [1,4,7,10–12]). In this connection, naturally, the following problem arises.

Problem 1.1. Would we establish the theory of $\mathfrak{F}$-centrality of chief factors and $\mathfrak{F}$-hypercentre of a group for local Fitting classes $\mathfrak{F}$?

This problem was proposed at Gomel seminar many years ago. But, up to now, this problem has not been resolved. If the answer to this problem is positive, then the new idea would play a positive role in the research in group theory.

Recall that for a class $\mathfrak{X}$ of groups, a subgroup V of G is said to be an $\mathfrak{X}$-injector of G if $V \cap K$ is an $\mathfrak{X}$-maximal subgroup of K for any subnormal subgroup K of G. It is well known that for a Fitting class $\mathfrak{F}$, every $\mathfrak{F}$-injector V of G possesses the cover–avoidance property, that is, V either covers or avoids each chief factor of G (see [4, VIII.2.14(c)]). For any local formation $\mathfrak{F}$, Carter and Hawkes [2] proved that every $\mathfrak{F}$-normalizer covers each $\mathfrak{F}$-central chief factor of G and avoids each $\mathfrak{F}$-eccentric chief factor of G. Note that a subgroup R of G covers (avoids) a chief factor H/K if $H \subseteq RK$ (correspondingly, $H \cap R \subseteq K$). This leads to the following problem.

Problem 1.2. Let $\mathfrak{F} = LR(f)$ be a local Fitting class. Is it true that every $\mathfrak{F}$-injector of G covers each $\mathfrak{F}$-central chief factor of G and avoids each $\mathfrak{F}$-eccentric chief factor of G?

The purpose of this paper is to give answers to the above two problems.

In Sections 3 and 4, we will establish the theory of $\mathfrak{F}$-central factors and $\mathfrak{F}$-hypercentre for any local Fitting class $\mathfrak{F} = LR(f)$ and so give the positive answer to Problem 1.1.

In Section 5, we give the answer to Problem 1.2 in the case where $\mathfrak{F}$ has an integrated and invariable H-function f.

As an application of our results, in Section 6, we give a new method to define local Fitting classes. In fact, we use local functions but not H-functions to define local Fitting classes.

All unexplained notation and terminology are standard. The reader is referred to [4] and [12] if necessary.

2. Preliminaries

From the definition of formations and Fitting classes, we see that

(1) for a formation $\mathfrak{F}$, every group G has the smallest normal subgroup whose quotient is in $\mathfrak{F}$, which is called the $\mathfrak{F}$-residual of G and denoted by $G^{\mathfrak{F}}$;

(2) for a Fitting class $\mathfrak{F}$, every group G has the largest normal $\mathfrak{F}$-subgroup, which is called the $\mathfrak{F}$-radical of G and denoted by $G_{\mathfrak{F}}$.

The product $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is the class $(G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies associative law (see [4, IX.1.12]).

We denote by $\mathfrak{S}$ the class of all finite soluble groups; $\mathfrak{S}_\pi$ denotes the class of all finite soluble π-groups; $\mathfrak{N}$ denotes the class of all finite nilpotent groups; $\mathfrak{N}_\pi$ denotes the class of all finite nilpotent π-groups. In particular, $\mathfrak{N}_p$ is the class of all finite p-groups. It is well known that $\mathfrak{S}$, $\mathfrak{S}_\pi$, $\mathfrak{N}$, $\mathfrak{N}_\pi$ and $\mathfrak{N}_p$ are all local Fitting classes.

Note that if φ and ψ are any two H-functions of a local Fitting class $\mathfrak{F}$, then $Supp(\varphi) = Supp(\psi) = \pi(\mathfrak{F})$ (see [8, Lemma 2.3]).

Let $\mathfrak{F} = LR(f)$ for some H-function f. Then f is called

(i) integrated if $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, and
(ii) full if $f(p) = f(p)\mathfrak{N}_p$ for all $p \in \mathbb{P}$.

Let $\Omega = \{f_i \mid i \in I\}$ be some nonempty set of H-functions. Suppose that $f_i,\ f_j \in \Omega$. Then we write $f_i \leqslant f_j$ if $f_i(p) \subseteq f_j(p)$ for all $p \in \mathbb{P}$.

The following known results will be frequently used in this paper.

Lemma 2.1. *(See [13].) Let $\mathfrak{F} = LR(f)$ for some H-function f. Then*

(1) *$\mathfrak{F} = LR(\varphi)$ for some integrated H-function φ;*
(2) *$\mathfrak{F} = LR(\psi)$ for some full and integrated H-function ψ.*

Lemma 2.2. *(See [4, VIII.2.4(d)].) If $\mathfrak{F}$ is a nonempty Fitting class and N is a subnormal subgroup of G, then $N_{\mathfrak{F}} = G_{\mathfrak{F}} \cap N$.*

Lemma 2.3. *(See [14].) The product of any two local Fitting classes is a local Fitting class.*

It is also well known that for any Fitting class $\mathfrak{F}$, every soluble group has a unique conjugate class of $\mathfrak{F}$-injectors (see [5] and [7, Theorem 2.5.2]).

3. $\mathfrak{F}$-central factors

In this section, we establish the theory of $\mathfrak{F}$-central factors for local Fitting classes.

Let $\mathfrak{F}$ be a local Fitting class, that is, $\mathfrak{F} = LR(f) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'})$ for some H-function f, where $\pi = Supp(f)$.

Definition 3.1. A p-chief factor H/K of G is said to be f-central if $p \in \pi$ and $G_{f(p)\mathfrak{N}_p}$ covers H/K; otherwise, it is said to be f-eccentric.

Lemma 3.2. *Let $\mathfrak{F} = LR(f)$ and $p \in \pi = Supp(f)$. Then a p-chief factor H/K is f-central if and only if $H = KH_{f(p)\mathfrak{N}_p}$.*

Proof. Assume that the p-chief factor H/K of G is f-central. Then $H \subseteq KG_{f(p)\mathfrak{N}_p}$. By Dedekind modular law, we have

$$H = H \cap KG_{f(p)\mathfrak{N}_p} = K(H \cap G_{f(p)\mathfrak{N}_p}).$$

It follows from Lemma 2.2 that $H = KH_{f(p)\mathfrak{N}_p}$. The converse is clear.

Remark 3.3. The definition of f-central chief factors depends on the choice of H-functions of $\mathfrak{F}$. In order to show it, we give the following example.

Example 3.4. Let $\mathfrak{F} = \mathfrak{N}_3\mathfrak{S}_{3'}$. Since $\mathfrak{N}_3$ and $\mathfrak{S}_{3'}$ are local Fitting classes, by Lemma 2.3, $\mathfrak{F}$ is also a local Fitting class. By a direct check, we can see that $\mathfrak{F}$ can be defined by each of the following two H-functions φ and ψ, where

$$\varphi(p) = \begin{cases} \mathfrak{N}_3 & \text{if } p = 3, \\ \mathfrak{F} & \text{if } p \neq 3, \end{cases}$$
$$\psi(p) = \begin{cases} \mathfrak{N}_3 & \text{if } p = 3, \\ \mathfrak{S} & \text{if } p \neq 3. \end{cases}$$

for every prime p.

Let $G = S_4$ be the symmetric group of degree 4 and $A = A_4$ the alternating group of degree 4. Then G/A is a chief factor of G, $G_{\varphi(2)\mathfrak{N}_2} = K$ is Klein 4-group and $G_{\psi(2)\mathfrak{N}_2} = G$. Clearly, G has a unique chief series $1 \lhd K \lhd A \lhd G$. Since $AG_{\varphi(2)\mathfrak{N}_2} = AK = A < G$, the 2-factor G/A is not φ-central by Lemma 3.2. But, since $AG_{\psi(2)\mathfrak{N}_2} = AG = G$, G/A is a ψ-central 2-chief factor of G.

However, the following theorem shows that the concept of f-central chief factor does not depend on the choice of H-functions f of $\mathfrak{F}$ if the H-functions f are integrated.

Theorem 3.5. *Suppose that $\mathfrak{F} = LR(\varphi) = LR(\psi)$ for some integrated H-functions φ and ψ, and H/K be a p-chief factor of G. Then the following two statements are equivalent.*

(i) *H/K is φ-central in G;*
(ii) *H/K is ψ-central in G.*

Proof. By Lemma 2.1, every local Fitting class can be defined by an integrated H-function. Hence the integrated H-function of $\mathfrak{F}$ exists. Without loss of generality, we may assume that $\varphi \leqslant \psi$. In fact, since $\mathfrak{F} = LR(\varphi \cap \psi)$ and $\varphi \cap \psi$ is an integrated H-function of $\mathfrak{F}$, we can always choose two integrated H-functions φ and ψ such that $\varphi \leqslant \psi$. Let $\pi = Supp(\varphi) = Supp(\psi)$. Then we have $G_{\varphi(p)} \subseteq G_{\psi(p)}$ for all $p \in \pi$ and so every φ-central chief factor is a ψ-central p-chief factor.

Now we only need to show that (ii) $\Rightarrow$ (i). Let H/K be a ψ-central p-chief factor of G. Then by Lemma 3.2, we have

$$H/K = (H \cap G_{\psi(p)\mathfrak{N}_p})K/K. \tag{3.1}$$

We fist prove that $\psi(p)\mathfrak{N}_p \subseteq \mathfrak{F}$. For the purpose, we construct an H-function f of $\mathfrak{F}$ such that $f(r) = \psi(r)\mathfrak{N}_r$ if $r \in \pi$ and $f(r) = \emptyset$ if $r \notin \pi$. Obviously, $Supp(\psi) = Supp(f) = \pi$. Since $\mathfrak{F} = LR(\psi) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} \psi(p)\mathfrak{N}_p\mathfrak{S}_{p'})$, by the multiplicative associative law of Fitting classes, we have $LR(f) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'}) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} \psi(p)\mathfrak{N}_p\mathfrak{N}_p\mathfrak{S}_{p'}) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} \psi(p)\mathfrak{N}_p\mathfrak{S}_{p'}) = \mathfrak{F}$. This means that f is also an H-function of $\mathfrak{F}$. Let $L \in f(p)$. Since $f(p) \subseteq f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$, we have $L \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Assume that q is an arbitrary prime in π different from p. Then $\mathfrak{N}_p \subseteq \mathfrak{S}_{q'}$ and so $L^{\mathfrak{S}_{q'}} \subseteq L^{\mathfrak{N}_p}$. But since $L \in f(p) = \psi(p)\mathfrak{N}_p$, $L/L_{\psi(p)} \in \mathfrak{N}_p$. Hence $L^{\mathfrak{N}_p} \subseteq L_{\psi(p)}$. Since ψ is an integrated H-function of $\mathfrak{F}$, $L^{\mathfrak{N}_p} \in \mathfrak{F}$. Hence $L^{\mathfrak{S}_{q'}} \in \mathfrak{F}$. Then by $\mathfrak{F} = LR(f)$, we obtain that $L^{\mathfrak{S}_{q'}}/(L^{\mathfrak{S}_{q'}})_{f(q)} \in \mathfrak{N}_q\mathfrak{S}_{q'}$. It follows that $(L^{\mathfrak{S}_{q'}})^{\mathfrak{N}_q\mathfrak{S}_{q'}} \in f(q)$. Since $(L^{\mathfrak{S}_{q'}})^{\mathfrak{N}_q\mathfrak{S}_{q'}} = L^{\mathfrak{N}_q\mathfrak{S}_{q'}\mathfrak{S}_{q'}}$ (see [4, IV.1.8(b)], we have $L^{\mathfrak{N}_q\mathfrak{S}_{q'}} \in f(q)$. Hence

$$L/L_{f(q)} \simeq \big(L/L^{\mathfrak{N}_q\mathfrak{S}_{q'}}\big)\big/\big(L_{f(q)}/L^{\mathfrak{N}_q\mathfrak{S}_{q'}}\big) \in \mathfrak{N}_q\mathfrak{S}_{q'}$$

and so $L \in f(q)\mathfrak{N}_q\mathfrak{S}_{q'}$ for $q \neq p$ and $q \in \pi$. This implies that $L \in \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'})$. Thus $\psi(p)\mathfrak{N}_p \subseteq \mathfrak{F}$. It follows that

$$G_{\varphi(p)\mathfrak{N}_p} \subseteq G_{\psi(p)\mathfrak{N}_p} \subseteq G_{\mathfrak{F}}. \tag{3.2}$$

Consequently, $G_{\psi(p)\mathfrak{N}_p} \in \mathfrak{F}$ and thereby $G_{\psi(p)\mathfrak{N}_p}/(G_{\psi(p)\mathfrak{N}_p})_{\varphi(p)\mathfrak{N}_p} \in \mathfrak{S}_{p'}$. But since $(G_{\psi(p)\mathfrak{N}_p})_{\varphi(p)\mathfrak{N}_p} = G_{\psi(p)\mathfrak{N}_p\cap\varphi(p)\mathfrak{N}_p}$ and $\varphi \leqslant \psi$, we have $G_{\psi(p)\mathfrak{N}_p}/G_{\varphi(p)\mathfrak{N}_p} \in \mathfrak{S}_{p'}$. This means that every Sylow p-subgroup P of $G_{\varphi(p)\mathfrak{N}_p}$ is a Sylow p-subgroup of $G_{\psi(p)\mathfrak{N}_p}$. Now because H/K is a p-factor of G, by comparing the orders of $K(H \cap G_{\varphi(p)\mathfrak{N}_p})$ and $K(H \cap P)$, we see that

$$K(H \cap G_{\varphi(p)\mathfrak{N}_p}) = K(H \cap P). \tag{3.3}$$

Thus by (3.1) and (3.2), we obtain that

$$H/K = (G_{\psi(p)\mathfrak{N}_p} \cap H)K/K = (P \cap H)K/K = (G_{\varphi(p)\mathfrak{N}_p} \cap H)K/K.$$

This induces that $KH_{\varphi(p)\mathfrak{N}_p} \supseteq H$. Therefore, H/K is φ-central in G. This completes the proof. □

This theorem shows that the f-centrality of chief factors in the theory of Fitting classes does not depend on the choice of integrated H-functions of the Fitting class $\mathfrak{F}$. In connection with this, we call an f-central chief factor an $\mathfrak{F}$-central chief factor when f is an integrated H-function of $\mathfrak{F}$.

In the theory of formations, there exists an analogue of Theorem 3.4. In fact, Carter and Hawkes [2] proved that if f_1 and f_2 are two integrated formation functions of a formation $\mathfrak{F}$, then a chief factor is f_1-central in G if and only if it is f_2-central in G. Moreover, Carter and Hawkes [2] proved that for a local formation $\mathfrak{F} = LF(f)$, a group $G \in \mathfrak{F}$ if and only if every chief factor of G is f-central in G (see also [4, Theorem IV.3.2]). Now we give the following theorem which is an analogue of the result of Carter and Hawkes.

Theorem 3.6. *Let $\mathfrak{F} = LR(f)$ be a local Fitting class, where f is an H-function of $\mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if every chief factor of G is f-central.*

Proof. Assume that $G \in \mathfrak{F}$ and let $\pi = Supp(f)$. Then $G \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$ for all primes $p \in \pi$. Let H/K be a p-chief factor of G and $H_{f(p)\mathfrak{N}_p}$ be an $f(p)\mathfrak{N}_p$-radical of H. Since $H \in \mathfrak{F}$, $H/H_{f(p)\mathfrak{N}_{p'}} \in \mathfrak{S}_{p'}$. Let P be a Sylow p-subgroup of H. Then $PH_{f(p)\mathfrak{N}_p}/H_{f(p)\mathfrak{N}_p}$ is a Sylow subgroup of $H/H_{f(p)\mathfrak{N}_p}$. Hence $PH_{f(p)\mathfrak{N}_p}/H_{f(p)\mathfrak{N}_p} \in \mathfrak{S}_{p'} \cap \mathfrak{N}_p = (1)$. It follows that $PH_{f(p)\mathfrak{N}_P} = H_{f(p)\mathfrak{N}_p}$ and so $P \subseteq H_{f(p)\mathfrak{N}_p}$. This implies that P is also a Sylow p-subgroup of $H_{f(p)\mathfrak{N}_p}$. But since $H/K \in \mathfrak{N}_p$, $H/K = PK/K \leqslant H_{f(p)\mathfrak{N}_p}K/K \leqslant H/K$. It follows that $H = H_{f(p)\mathfrak{N}_p}K$. Therefore, by Lemma 3.2, the p-chief factor H/K is f-central in G.

Now assume that $G \notin \mathfrak{F}$. Then there exists some prime $p \in \pi$ such that $G \notin f(p)\mathfrak{N}_p$. Hence there exists a p-chief factor H/K above $G_{f(p)\mathfrak{N}_p}$ such that $G_{f(p)\mathfrak{N}_p}$ does not cover H/K. This shows H/K is not f-central. The theorem is proved. □

Corollary 3.7. *Suppose that $\mathfrak{F} = LR(f)$ is a local Fitting class, where f is an integrated H-function of $\mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if every chief factor of G is $\mathfrak{F}$-central.*

Obviously, the description of $\mathfrak{F}$-central chief factors in the theory of formations is different from the description of $\mathfrak{F}$-central chief factors in the theory of Fitting classes. Recall that for the formation $\mathfrak{N}$ of all nilpotent groups, a chief factor H/K of G is $\mathfrak{N}$-central in G if and only if $H/K \subseteq Z(G/K)$, that is, it is central in G. Hence, a group G is nilpotent if and only if every chief factor of G is $\mathfrak{N}$-central in the theory of formations. The following Corollary 3.8 shows that though the method of the description of $\mathfrak{N}$-central chief factors in the theory of formations is different from the method of the description of $\mathfrak{N}$-central chief factors in the theory of Fitting classes, they both can give the characterization of a group belonging to the class $\mathfrak{N}$ of all nilpotent groups by using each method.

Corollary 3.8. *Every chief factor of G is $\mathfrak{N}$-central in the theory of Fitting classes if and only if every Sylow subgroup of G is normal.*

Proof. Obviously, $\mathfrak{N} = LR(f)$ is the local Fitting class for the integrated H-function f such that $f(p) = \mathfrak{N}_p$ for every prime $p \in \pi = Supp(f)$. Hence, by Theorem 3.6, every p-chief factor of G is $\mathfrak{N}$-central in G in the theory of Fitting classes if and only if $G \in \mathfrak{N}_p\mathfrak{S}_{p'}$. It follows that every Sylow p-subgroup of G is normal in G. □

Corollary 3.9. *Every chief factor of G is $\mathfrak{N}^2$-central in the theory of Fitting classes if and only if all Sylow subgroups of $G/F(G)$ are normal in $G/F(G)$.*

Proof. It is easy to see that the class $\mathfrak{N}^2$ of all metanilpotent groups is a local Fitting class such that $\mathfrak{F} = LR(f)$, where f is the H-function such that $f(p) = \mathfrak{N}\mathfrak{N}_p$ for all primes p. Hence by Theorem 3.6, every chief factor of G is $\mathfrak{N}^2$-central if and only if $G \in \mathfrak{N}\mathfrak{N}_p\mathfrak{S}_{p'}$ for all $p \in \pi$. But it is possible only in the case that every Sylow p-subgroup of $G/F(G)$ is normal in $G/F(G)$. □

4. $\mathfrak{F}$-hypercentre

In this section, we establish the theory of $\mathfrak{F}$-hypercentre for local Fitting classes $\mathfrak{F}$.

By Theorem 3.5, we see that the concept of $\mathfrak{F}$-central chief factor for a local Fitting class $\mathfrak{F}$ does not depend on the choice of integrated H-functions of $\mathfrak{F}$. Hence, in this section, we always assume that any local Fitting class $\mathfrak{F} = LR(f)$ is defined by an integrated H-function f.

Definition 4.1. Let $\mathfrak{F}$ be a local Fitting class.

(i) Suppose that L is a normal subgroup of G. Then the factor group G/L is said to be $\mathfrak{F}$-hypercentral in G if there exists a series

$$L = L_0 \lhd L_1 \lhd \cdots \lhd L_k = G \tag{4.1}$$

such that every factor L_{i+1}/L_i is a G-chief factor and it is $\mathfrak{F}$-central in G, for all $i \in \{0, 1, \ldots, k-1\}$.

(ii) The intersection of all normal subgroups K of G such that G/K is $\mathfrak{F}$-hypercentral is called the $\mathfrak{F}$-hypercentre of G and denoted by $Z^{\mathfrak{F}}(G)$.

In general, $G/Z^{\mathfrak{F}}(G)$ is not $\mathfrak{F}$-hypercentral (see Example 4.3).

In the theory of formations, it is well known that if $\mathfrak{F}$ is a local formation with full and integrated local function F and $\mathfrak{F}$ is closed with respect to normal subgroups, then $Z_{\mathfrak{F}}(G) \in \mathfrak{F}$ (see [4, Theorem IV, 6.15]). We now establish an analogous result in the theory of Fitting classes.

Theorem 4.2. *Let $\mathfrak{F} = LR(f)$ be a local Fitting class, where f is an integrated H-function. Then for any group G and any normal subgroup L of G, the following statements hold*:

(a) *If G/L is $\mathfrak{F}$-hypercentral in G, then $G = LG_{\mathfrak{F}}$;*
(b) *If $\mathfrak{F}$ is closed under homomorphic images and G/L is $\mathfrak{F}$-hypercentral in G, then $G/L \in \mathfrak{F}$;*
(c) *If $\mathfrak{F}$ is a formation as well, then $G/Z^{\mathfrak{F}}(G) \in \mathfrak{F}$;*
(d) *Assume that the integrated H-function is also full and $Supp(f) = \mathbb{P}$. If G/L is $\mathfrak{F}$-hypercentral in G, then every chief factor above L is $\mathfrak{F}$-central.*

Proof. (a) Since G/L is $\mathfrak{F}$-hypercentral in G, there exists a G-chief series between L and G such that every p-chief factor in this series is $\mathfrak{F}$-central in G. Hence $G_{f(p)\mathfrak{N}_p}$ covers every p-chief factor in this series, for all primes $p \in \pi = Supp(f)$. By the proof of "(ii) ⇒ (i)" in Theorem 3.5, we see that $G_{f(p)\mathfrak{N}_p} \subseteq G_{\mathfrak{F}}$ for every $p \in \pi$. Therefore, $G_{\mathfrak{F}}$ covers every p-chief factor of G in this series and so it also covers G/L. Thus $G = LG_{\mathfrak{F}}$.

(b) Assume that $\mathfrak{F}$ is closed under homomorphic images and G/L is $\mathfrak{F}$-hypercentral in G. Then by (a), $G = LG_{\mathfrak{F}}$ and so $G/L = LG_{\mathfrak{F}}/L \simeq G_{\mathfrak{F}}/L \cap G_{\mathfrak{F}}$. Since $G_{\mathfrak{F}} \in \mathfrak{F}$ and $\mathfrak{F}$ is closed under homomorphic images, $G_{\mathfrak{F}}/L \cap G_{\mathfrak{F}} \in \mathfrak{F}$ and consequently $G/L \in \mathfrak{F}$.

(c) Assume that $\mathfrak{F}$ is a formation. Then, since $Z^{\mathfrak{F}}(G) = \bigcap\{L \trianglelefteq G \mid G/L$ is $\mathfrak{F}$-hypercentral in $G\}$, we obtain that $G/Z^{\mathfrak{F}}(G) \in \mathfrak{F}$ by (b).

(d) Supposed that G/L is $\mathfrak{F}$-hypercentral in G and H/K is a p-chief factor of G between L and G. Then by (a), the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G covers G/L. Hence $G_{\mathfrak{F}}$ covers H/K. Since $H/K \in \mathfrak{N}_p$, a Sylow p-subgroup P of $G_{\mathfrak{F}}$ covers H/K. Because f is an integrated and full H-function, we have $f(p) \subseteq \mathfrak{F} \subseteq f(p)\mathfrak{N}_p\mathfrak{S}_{p'} = f(p)\mathfrak{S}_{p'}$ for every $p \in \mathbb{P}$. Hence $G_{\mathfrak{F}}/(G_{\mathfrak{F}})_{f(p)} \in \mathfrak{S}_{p'}$. But since $(G_{\mathfrak{F}})_{f(p)} = G_{\mathfrak{F} \cap f(p)} = G_{f(p)}$, the Sylow p-subgroup P of $G_{\mathfrak{F}}$ is also a Sylow p-subgroup of $G_{f(p)}$. Since P covers H/K, $G_{f(p)}$ covers H/K. Thus H/K is $\mathfrak{F}$-central. This completes the proof. $\square$

The following example shows that $G/Z^{\mathfrak{F}}(G)$ is not $\mathfrak{F}$-hypercentral, in general.

Example 4.3. Let $\mathfrak{F} = \mathfrak{N}$. Then $\mathfrak{N} = LR(f)$, where f is the H-function such that $f(p) = \mathfrak{N}_p$ for every prime $p \in \mathbb{P}$. Put $G = S_3 \times Z_2$, in which the symmetric group $S_3 = \langle s, t\rangle$, where $s^3 = 1$ and $t^2 = 1$. Let $Z_2 = \langle z\rangle$ and α be the map from S_3 to Z_2 such that $\alpha(s) = 1$ and $\alpha(t) = 2$. Let $H = \{(k, \alpha(k)) \mid k \in S_3\}$. Then $|G : H| = 2$ and so $H \lhd G$.

Consider all possible chief series of G:

$$1 \lhd A_3 \times 1 \lhd S_3 \times 1 \lhd G,$$

$$1 \lhd A_3 \times 1 \lhd H \lhd G,$$

$$1 \lhd A_3 \times 1 \lhd A_3 \times Z_2 \lhd G,$$

$$1 \lhd 1 \times Z_2 \lhd A_3 \times Z_2 \lhd G.$$

It is easy to see that G/H and $G/S_3 \times 1$ are $\mathfrak{N}$-hypercentral in G since they are covered by $G_{f(p)} = G_{\mathfrak{N}_2} = 1 \times Z_2$. Besides, the chief factor $G/A_3 \times Z_2$ is $\mathfrak{N}$-eccentric since $G_{f(2)}$ avoids it. Analogously, $S_3 \times 1/A_3 \times 1$ and $H/A_3 \times 1$ are not $\mathfrak{N}$-hypercentral in G. This shows that G/H and $G/S_3 \times 1$ are the only two $\mathfrak{N}$-hypercentral factors of G. Therefore, $Z^{\mathfrak{N}}(G) = A_3 \times 1$. Obviously, $G/Z^{\mathfrak{N}}(G)$ is not $\mathfrak{N}$-hypercentral in G.

Remark 4.4. If a class $\mathfrak{F}$ of finite groups is both a local formation and a local Fitting class, then the $\mathfrak{F}$-hypercentre of a group G in the theory of formations is different from the $\mathfrak{F}$-hypercentre of G in the theory of Fitting classes, in general. For instance, let $\mathfrak{F} = \mathfrak{N}$ be the class of all finite nilpotent groups. Then $\mathfrak{N}$ is both a local formation and a local Fitting class. Let $G = S_3 \times Z_2$. Then, obviously, the $\mathfrak{N}$-hypercentre $Z_{\mathfrak{N}}(G)$ of G is $1 \times Z_2$ if $\mathfrak{N}$ is regarded as a formation. However, by the above Example 4.3, we see that the $\mathfrak{N}$-hypercentre $Z^{\mathfrak{N}}(G)$ of G is $A_3 \times 1$ if we regard $\mathfrak{N}$ as a Fitting class.

5. On the problem of cover–avoidance properties for $\mathfrak{F}$-injectors

Suppose that f is an H-function of a local Fitting class $\mathfrak{F}$, that is, $\mathfrak{F} = LR(f)$, and let $\pi = Supp(f)$. If $f(p) = f(q)$ for all primes $p, q \in \pi$, then f is said to be invariable.

In this section, we give the answer to Problem 1.2 in the case where $\mathfrak{F}$ has an invariable H-function (see the following Theorem 5.4(3)).

Definition 5.1. Let $\mathfrak{F} = LR(f)$ for some H-function f with $Supp(f) = \pi$ and $p \in \pi$. We say that a p-chief factor H/K of G is $f(p)$-covered in G if $H = K(V_{f(p)} \cap H)$, and $f(p)$-avoided if $K = K(V_{f(p)} \cap H)$ for some $\mathfrak{F}$-injector V of G.

Lemma 5.2. *Let $\mathfrak{F} = LR(f)$ for some H-function f and $\pi = Supp(f)$. If $\mathfrak{X}$ is a non-empty Fitting class and $\mathfrak{X} \subseteq \bigcap_{p\in\pi} f(p)$, then $C_G(G_{\mathfrak{F}}/G_{\mathfrak{X}}) \subseteq G_{\mathfrak{F}}$ for any group G.*

Proof. The proof is analogous to the proof of [8, Theorem 3.2] and we omit the details. $\square$

Lemma 5.3. *Let $\mathfrak{F} = LR(f)$ for some invariable H-function f and $Supp(f) = \pi$. If V is an $\mathfrak{F}$-injector of G, then $V_{f(p)} = G_{f(p)}$ for all $p \in \pi$.*

Proof. Since $G_{\mathfrak{F}} \trianglelefteq V$ and $f(p) \subseteq \mathfrak{F}$, by Lemma 2.2, $G_{f(p)} = (G_{\mathfrak{F}})_{f(p)} = G_{\mathfrak{F}} \cap V_{f(p)}$ for all $p \in \pi$. Hence $[V_{f(p)}, G_{\mathfrak{F}}] \leqslant G_{f(p)}$ and thereby $V_{f(p)} \leqslant C_G(G_{\mathfrak{F}}/G_{f(p)})$. Since the H-function f is invariable, $C_G(G_{\mathfrak{F}}/G_{f(p)}) \leqslant G_{\mathfrak{F}}$ by Lemma 5.2. Therefore, $V_{f(p)} = G_{f(p)}$, for every $p \in \pi$. $\square$

Theorem 5.4. *Let $\mathfrak{F} = LR(f)$ with $Supp(f) = \pi$ and f be an integrated H-function of $\mathfrak{F}$. Then the following statements hold:*

(1) *If $p \in \pi$, then every p-chief factor of G is either $f(p)$-covered or $f(p)$-avoided in G.*
(2) *Assume that f is invariable and $p \in \pi$. Then*
  (i) *A p-chief factor of G is $f(p)$-covered if and only if it is f-central in G;*
  (ii) *An $\mathfrak{F}$-injector of G covers every $f(p)$-covered chief factor of G.*
(3) *If f is a full and invariable H-function, then an $\mathfrak{F}$-injector of G covers each $\mathfrak{F}$-central chief factor of G and avoids each $\mathfrak{F}$-eccentric chief factor of G.*

Proof. (1) Suppose that H/K is a p-chief factor of G and let V be an $\mathfrak{F}$-injector of G. Since $V \cap H \trianglelefteq V$, by Lemma 2.2, we have

$$(V \cap H)_{f(p)} = V_{f(p)} \cap (V \cap H) = V_{f(p)} \cap H.$$

But by [4, Theorem VIII.2.13], $V \cap H$ is an $\mathfrak{F}$-injector of H. Hence, by [4, Theorem VIII.2.9], every conjugate subgroup of $(V \cap H)_{f(p)}$ in G is a conjugate subgroup of $(V \cap H)_{f(p)}$ in H. Hence by Frattini argument, $G = HN_G((V \cap H)_{f(p)})$ and so $K(V \cap H)_{f(p)} \trianglelefteq G$. This implies that the p-chief factor H/K is either $f(p)$-covered or $f(p)$-avoided.

(2) The statement (i) can be directly obtained by Lemma 5.3 and the definition of f-central chief factor. We now prove (ii). Assume that V is an $\mathfrak{F}$-injector of G and H/K is an $f(p)$-covered chief factor of G, that is, $H = K(V_{f(p)} \cap H)$. Since the H-function f is invariable, by Lemma 5.3, $V_{f(p)} = G_{f(p)}$ for all primes $p \in \pi$. But since $G_{f(p)} \subseteq G_{\mathfrak{F}} \subseteq F$ for every $\mathfrak{F}$-injector F of G, F covers H/K and so (ii) holds.

(3) Let H/K be an $\mathfrak{F}$-central p-chief factor of G, where $p \in \pi$, and V an $\mathfrak{F}$-injector of G. Then $H \leqslant KG_{f(p)\mathfrak{N}_p}$. Since f is integrated, by the proof of (ii) $\Rightarrow$ (i) in Theorem 3.5, we see that $G_{f(p)\mathfrak{N}_p} \leqslant G_{\mathfrak{F}}$. Besides, by the definition of $\mathfrak{F}$-injector, $G_{\mathfrak{F}} \leqslant V$. Hence $H \leqslant KV$ and so V covers H/K.

Now assume that an $\mathfrak{F}$-injector V of G covers some p-chief factor H/K of G. Obviously, a Sylow p-subgroup P of V covers H/K. If $p \notin \pi$, then $\mathfrak{N}_p \nsubseteq \mathfrak{F}$ (see [3, Lemma 2.3] or [4, Theorem IX.1.9]) and so the Sylow p-subgroup P of V is trivial. It follows that $H/K = 1$, which contradicts the choice of H/K. Hence, we can assume that $p \in \pi$. Since $V \in \mathfrak{F}$, we have $V \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'}$. Since f is full, we have $f(p)\mathfrak{N}_p = f(p)$. This implies that $V/V_{f(p)} \in \mathfrak{S}_{p'}$ and hence every Sylow p-subgroup P of V is a Sylow p-subgroup of $V_{f(p)}$. Consequently, $P \leqslant V_{f(p)}$. By Lemma 5.3, $V_{f(p)} = G_{f(p)}$ for every $p \in \pi$ and obviously $G_{f(p)} \subseteq G_{f(p)\mathfrak{N}_p}$. Hence H/K is covered by $G_{f(p)\mathfrak{N}_p}$. This shows that H/K is $\mathfrak{F}$-central in G. This completes the proof. $\square$

6. Applications

Formerly, in the definition of local Fitting classes $\mathfrak{F} = LR(f)$, $f(p)$ is always a Fitting class for all $p \in Supp(f)$, that is, f is an H-function. In this connection, the following problem naturally arises.

Problem 6.1. Could a local Fitting class be defined by a local function f such that the value $f(p)$ is not necessarily a Fitting class, that is, f is not an H-function?

In this section, we use the theory of $\mathfrak{F}$-centrality in the theory of Fitting classes given in the above sections to resolve this problem. In fact, we will construct local Fitting classes by using some local function f which is not an H-function.

Following [4], a map $f : \mathbb{P} \to \{\text{class of groups}\}$ is said to be a local function and let $\pi = Supp(f) = \{p \in \mathbb{P} \mid f(p) \neq \emptyset\}$. For the local function f, we let $LR(f) = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'})$. Analogously, a local function f is said to be: (a) integrated if $f(p) \subseteq \mathfrak{F}$ for all $p \in \pi$; (b) full if $f(p) = f(p)\mathfrak{N}_p$ for all $p \in \pi$.

Recall that a class $\mathfrak{X}$ of groups is said to be S_n-closed if every normal subgroup of every $\mathfrak{X}$-group is also a $\mathfrak{X}$-group. We say a local function f is S_n-closed if $f(p)$ is S_n-closed for all $p \in \mathbb{P}$.

Theorem 6.2. *Suppose that $\mathfrak{F} = LR(F)$ is a local Fitting class with a full and integrated H-function F and $Supp(F) = \pi$. Then $\mathfrak{F}$ can be defined by a full S_n-closed local function f such that f is neither integrated nor an H-function in general, where*

$$f(p) = \begin{cases} (G \mid G_{F(p)} \text{ covers every } \mathfrak{F}\text{-central chief factor of } G) & \text{if } p \in \pi, \\ \emptyset & \text{if } p \notin \pi'. \end{cases}$$

Proof. We proceed via the following steps.

(1) *Let $\varphi(p) = (G \mid G_{\mathfrak{F}} \in F(p))$ for all $p \in \mathbb{P}$. Then $f(p) = \varphi(p)$.*

Obviously, if $p \notin \pi$, then $\varphi(p) = f(p) = \emptyset$. Assume that $G \in \varphi(p)$, where $p \in \pi$. Since F is full, by Lemma 3.2, we see that every $\mathfrak{F}$-central r-chief factor of G is covered by $G_{F(r)}$, for all $r \in \pi$. Since F is integrated, $F(r) \subseteq \mathfrak{F}$ for all $r \in \pi$. Hence $G_{F(r)} \leqslant G_{\mathfrak{F}}$ and so the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G covers every $\mathfrak{F}$-central r-chief factor of G. Besides, because $G_{\mathfrak{F}} \in F(p)$, we have $G_{\mathfrak{F}} \leqslant G_{F(p)}$. Hence $G_{\mathfrak{F}} = G_{F(p)}$ and thereby every $\mathfrak{F}$-central r-chief factor is covered by $G_{F(p)}$. This shows that $G \in f(p)$ and so $\varphi \leqslant f$.

Assume that $f(p) \neq \varphi(p)$ for some $p \in \pi$ and let G be a group in $f(p) \setminus \varphi(p)$ of minimal order. Since $G_{F(p)} \leqslant G_{\mathfrak{F}}$ and $G \notin \varphi(p)$, $G_{F(p)} < G_{\mathfrak{F}}$. Then there exists a q-chief factor H/K of G between $G_{F(p)}$ and $G_{\mathfrak{F}}$ for some prime q. Obviously, $G_{\mathfrak{F}}$ covers H/K. Since $G_{\mathfrak{F}} \in F(q)\mathfrak{N}_q\mathfrak{S}_{q'} = F(q)\mathfrak{S}_{q'}$ and $F(q) \subseteq \mathfrak{F}$, $G_{\mathfrak{F}}/(G_{\mathfrak{F}})_{F(q)} = G_{\mathfrak{F}}/G_{F(q)} \in \mathfrak{S}_{q'}$. Hence every Sylow q-subgroup Q of $G_{\mathfrak{F}}$ is a Sylow q-subgroup of $G_{F(q)}$. Consequently, Q covers H/K and so $G_{F(q)} = G_{F(q)\mathfrak{N}_q}$ covers H/K. Hence H/K is $\mathfrak{F}$-central. But since H/K is a chief factor between $G_{F(p)}$ and $G_{\mathfrak{F}}$, obviously, $G_{F(q)}$ does not cover H/K. This contradiction shows that $f = \varphi$.

(2) *The local function f is full and S_n-closed.*

Since $f = \varphi$ by (1), we only need to prove that φ is full and S_n-closed.

We first show that φ is S_n-closed. Assume that $X \in \varphi(p)$ and $V \trianglelefteq X$. Then $X_{\mathfrak{F}} \in F(p)$. By Lemma 2.2, $N_{\mathfrak{F}} = N \cap X_{\mathfrak{F}} \in F(p)$ and so $N \in \varphi(p)$. Hence φ is S_n-closed.

Now we prove that $\varphi(p)\mathfrak{N}_p = \varphi(p)$ for all $p \in \mathbb{P}$. Note that $\pi = Supp(F) = Supp(f) = Supp(\varphi)$. If $\varphi(p) = \emptyset$, then it is clear. Obviously, $\varphi(p) \subseteq \varphi(p)\mathfrak{N}_p$ for all $p \in \pi$. Assume that $\varphi(p)\mathfrak{N}_p \neq \varphi(p)$ and let G be a group in $\varphi(p)\mathfrak{N}_p \setminus \varphi(p)$ of minimal order. Now by the choice of G, we see that G has a unique maximal normal subgroup M and $|G : M| = p$.

If $G \in \mathfrak{F}$, then $G \in F(p)\mathfrak{S}_{p'}$ and so $G/G_{F(p)} \in \mathfrak{S}_{p'}$. Since M is the unique maximal normal subgroup, $G_{F(p)} \subseteq M$. Hence $(G/G_{F(p)})/(M/G_{F(p)}) \simeq G/M \in \mathfrak{N}_p \cap \mathfrak{S}_{p'} = (1)$, which contradicts $|G : M| = p$. Thus $G \notin \mathfrak{F}$. It follows that $G_{F(p)} \subseteq G_{\mathfrak{F}} \leqslant M$. This induces that $G_{\mathfrak{F}} = M_{\mathfrak{F}} = M_{F(p)} = G_{F(p)}$ and so $G \in \varphi(p)$. This contradiction shows that φ is full.

(3) *$\mathfrak{F}$ is locally defined by f.*

We only need to prove that $\mathfrak{F} = \mathfrak{M}$, where $\mathfrak{M} = \mathfrak{S}_\pi \cap (\bigcap_{p\in\pi} f(p)\mathfrak{N}_p\mathfrak{S}_{p'})$ and $\pi = Supp(f)$. Obviously, $F(p) \subseteq (G \mid G_{\mathfrak{F}} \in F(p)) = \varphi(p) = f(p)$ for all $p \in \pi$. Hence $F \leqslant \varphi = f$ and so $\mathfrak{F} \subseteq \mathfrak{M}$. Now we prove that $\mathfrak{M} \subseteq \mathfrak{F}$. If not, we let G be a group in $\mathfrak{M} \setminus \mathfrak{F}$ of minimal order. Then $G_{\mathfrak{F}}$ is the unique maximal normal subgroup of G and $|G : G_{\mathfrak{F}}| = p$ for some $p \in \pi$. Since $G \in \mathfrak{M}$ and the local function f is full by (2), $G \in f(p)\mathfrak{N}_p\mathfrak{S}_{p'} = f(p)\mathfrak{S}_{p'}$. Then there exists a normal subgroup K of G such that $K \in f(p)$ and $G/K \in \mathfrak{S}_{p'}$. If $K = G$, then $G \in f(p)$. Hence by (1), $G \in \varphi(p)$. It follows that $G_{\mathfrak{F}} \in F(p)$ and so $G \in F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$. This contradiction shows that $K \neq G$. Then, since $G_{\mathfrak{F}}$ is the unique maximal normal subgroup of G, $K \subseteq G_{\mathfrak{F}}$ and so $(G/K)/(G_{\mathfrak{F}}/K) \simeq G/G_{\mathfrak{F}} \in \mathfrak{S}_{p'}$. This implies that $G/G_{\mathfrak{F}} \in \mathfrak{N}_p \cap \mathfrak{S}_{p'} = (1)$. Hence $G = G_{\mathfrak{F}} \in \mathfrak{F}$. This contradiction shows that $\mathfrak{F} = \mathfrak{M}$.

(4) *f is neither integrated nor an H-function, in general.*

Let $\mathfrak{F} = \mathfrak{N}_\pi$, where π is a set of primes such that $\{2,3\} \subseteq \pi$. It is easy to see that $\mathfrak{F} = LR(F)$, where F is an H-function such that

$$F(p) = \begin{cases} \mathfrak{N}_p & \text{if } p \in \pi, \\ \emptyset & \text{if } p \notin \pi'. \end{cases}$$

Then by (1) and (3), we have that $\mathfrak{F}$ is defined by the local function f such that

$$f(p) = \begin{cases} (G \mid G_{\mathfrak{N}_\pi} \in \mathfrak{N}_p) & \text{if } p \in \pi, \\ \emptyset & \text{if } p \notin \pi'. \end{cases}$$

Let $p = 3$ and S_3 be the symmetric group of degree 3. Then $S_3 \in f(3)$ and $S_3 \notin \mathfrak{N}_\pi$. Hence the local function f is not integrated in general.

We now prove that $f(3)$ is not a Fitting class.

Let $Y = SL(2,3)$ be the special linear group of degree 2 over the field $GF(3)$. Then the order of Y is 24 and Y contains the unique minimal normal group $D = Z(Y)$ of order 2. Let $H = Y_1 \times Y_2$, where $Y_i \simeq Y$ for $i = 1, 2$ and $\widetilde{D} = \{(d,d) \mid d \in D\}$. Then $\widetilde{D}$ is a minimal normal subgroup of H with order 2.

By [4, B.9.16], there exists a faithful irreducible representation Φ of H/D over $GF(3)$. Assume that V is the space of the representation Φ. Then V is a 3-group. Hence the map $\Phi\colon H/\widetilde{D} \to GL(V)$ is a isomorphism from H/D to some subgroup of the group $GL(V)$ of all automorphisms of the space V. Let

$$\varphi : H \to GL(V)$$

be such that $\varphi(h) = \Phi(h\widetilde{D})$ for all $h \in H$. Since

$$\varphi(h_1h_2) = \Phi\big((h_1\widetilde{D})(h_2\widetilde{D})\big) = \Phi(h_1\widetilde{D})\Phi(h_2\widetilde{D}) = \varphi(h_1)\varphi(h_2)$$

for any $h_1, h_2 \in H$, we see that φ is a homomorphism. Let $G = [V]H$ be a semidirect product, where V is a normal subgroup of G and H acts on V by φ.

Let $\operatorname{Ker}\varphi = \{h \in H \mid \varphi(h) = 1\}$ be the kernel of φ. We now prove that

$$\operatorname{Ker}\varphi = \widetilde{D}.$$

If $h \in \widetilde{D}$, then $\varphi(h) = \Phi(h\widetilde{D}) = \Phi(\widetilde{D}) = 1$ and hence $\widetilde{D} \subseteq \operatorname{Ker}\varphi$. Conversely, assume that $h \in \operatorname{Ker}\varphi$. Then $\Phi(h\widetilde{D}) = \varphi(h) = 1$. Since Φ is a monomorphism, $h\widetilde{D} = \widetilde{D}$ and so $h \in \widetilde{D}$. This means that $\operatorname{Ker}\varphi \subseteq \widetilde{D}$. Thus, $\operatorname{Ker}\varphi = \widetilde{D}$.

Because $\pi(Y) = \{2,3\}$ and $\pi(V) = \{3\}$, we have that $\pi(G) = \{2,3\}$ and $F(G) = O_3(G)O_2(G)$.

We now prove that $O_3(G) = V$ and $O_2(G) = \widetilde{D}$. In fact, since $\pi(Y) = \{2,3\}$ and $F(Y)$ is a Sylow 2-subgroup of Y, $F(H) = F(Y_1)F(Y_2) = O_2(H)$. Hence $O_3(H) = 1$. Since $V \trianglelefteq G$ and $\pi(V) = \{3\}$, $V \leqslant O_3(G) \leqslant G$. Then by Dedekind modular law, $O_3(G) \cap VH = V(O_3(G) \cap H)$. Since $O_3(G) \cap H \trianglelefteq H$ and $O_3(G) = 1$, we obtain that $O_3(G) \cap H = 1$. This implies that $O_3(G) = V$.

Since $\pi(V) = \{3\}$, $O_2(G) \leqslant H$. By the properties of semidirect product, we see that if $N \trianglelefteq H$, then N is a normal subgroup of G if and only if $N \leqslant \operatorname{Ker}\varphi$. Hence, by $O_2(G) \trianglelefteq G$ and $\operatorname{Ker}\varphi = \widetilde{D}$, we have $O_2(G) \leqslant \widetilde{D}$. It follows from $|\widetilde{D}| = 2$ that $O_2(G) = \widetilde{D}$. Hence $F(G) = O_3(G)O_2(G) = V\widetilde{D}$.

Obviously, $G = VY_1VY_2$ and $VY_i \trianglelefteq G$ for $i = 1, 2$. By Lemma 2.2,

$$F(VY_i) = F(G) \cap VY_i = V\widetilde{D} \cap VY_i = V \in \mathfrak{S}_3.$$

Then since $\{2,3\} \subseteq \pi$ and $\pi(VY_i) = \{2,3\}$, we have

$$(VY_i)_{\mathfrak{N}_\pi} = (VY_i)_{\mathfrak{N}_{\{2,3\}}} = O_2(VY_i)O_3(VY_i) = F(VY_i).$$

Analogously, $G_{\mathfrak{N}_\pi} = F(G)$. Hence $(VY_i)_{\mathfrak{N}_\pi} \in \mathfrak{N}_3$ and $G_{\mathfrak{N}_\pi} = V\widetilde{D} \notin \mathfrak{S}_3$. Therefore, by the definition of f, we obtain that $VY_i \in f(3)$ for $i = 1, 2$ and $G \notin f(3)$. This shows that $f(3)$ is not a Fitting class and thereby f is not an H-function in general. This completes the proof. □

Note that the class $\mathfrak{N}$ of all nilpotent groups is a local Fitting class with a full and integrated H-function F such that $F(p) = \mathfrak{N}_p$ for all prime $p \in \mathbb{P}$. Hence, as an immediate corollary of Theorem 6.2 and its proof, we have the following

Corollary 6.3. *The class $\mathfrak{N}$ of all nilpotent groups can be defined by a local function f such that $f(p) = (G \mid O_p(G)$ covers all $\mathfrak{N}$-central chief factors of $G)$, and $f(p) = (G \mid F(G) = O_p(G))$ for all $p \in \mathbb{P}$.*

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