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# Lie identities for skew and symmetric elements of semiprime superalgebras with superinvolution

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ABSTRACT

Let $A$ be a non-trivial semiprime associative superalgebra with superinvolution. In the present note we investigate when the subspaces of symmetric elements or skew elements of $A$ are Lie nilpotent or Lie solvable. We show that these conditions determine the algebraic structure of $A$.



## 1. Introduction

Let $A = A_0 \oplus A_1$ be an associative superalgebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$. An element $a$ of $A$ is said to be *homogeneous* (of degree $i$) if $a \in A_i$ (and we write $\bar{a} = i$). Let us denote by $A^-$ the Lie superalgebra obtained from $A$ via the Lie superbracket $[a,b]_s := ab - (-1)^{\bar{a}\bar{b}}ba$, for all homogeneous elements $a, b \in A$ (the expression extends over the rest of the elements by linearity). If $A$ has a superinvolution $*$, let $K$ be the subalgebra of the Lie superalgebra $A^-$ consisting of *skew elements* of $A$ with respect to $*$, namely $K := \{a \mid a \in A,\ a^* = -a\}$. When $A$ is *trivial*, i.e. $A_1 = 0$, $A$ is nothing but an associative algebra with involution and the Lie superbracket $[\,,\,]_s$

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coincides with the usual Lie bracket [ , ]. In this setting, an interesting question is to decide if crucial information on the algebraic structure of $A$ can be deduced from properties of $A^-$ or $K$. This interplay has been the subject of a good deal of attention over the decades.

In the last years the relation between $A$, $A^-$ and $K$ has been profusely investigated by several authors for *non-trivial* superalgebras as well. The motivations for this line of research mainly come from the classification of the finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero given by Kac [7]. In fact, we can find in it examples that are superalgebras of skew elements with respect to a superinvolution in a simple associative superalgebra or Lie superalgebras associated to a simple associative superalgebra. This result suggests that the structure of $A$ as associative superalgebra and $A^-$ and $K$ as Lie superalgebras could be related. In this direction, Gómez-Ambrosi and Shestakov [5] studied the Lie structure of $K$ when $A$ is simple. Later these results were extended to the context of prime and semiprime associative superalgebras in [3] and [10], respectively. More recently, Laliena and Sacristán explored the structure of semiprime associative superalgebras with superinvolution under certain additional regularity condition on symmetric and skew elements [12] and when $[K^2, K^2]_s = 0$ [11]. We notice that the Lie structure of simple and prime associative superalgebras without superinvolution was previously studied by Montgomery [15] and Montaner [14], respectively.

On the other hand, one can consider the Jordan superalgebra $A^+$ obtained from $A$ via the circle operation $a \circ_s b := ab + (-1)^{\bar{a}\bar{b}} ba$ for all homogeneous elements $a, b$ of $A$ (also in this case, the expression extends over the rest of the elements by linearity). When $A$ is equipped with a superinvolution $*$, let $H := \{a \mid a \in A,\ a^* = a\}$ be the subalgebra of the Jordan superalgebra $A^+$ consisting of *symmetric elements* of $A$ with respect to $*$. In the Kac's classification [8] of finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero we find examples of simple Jordan superalgebras of the form $A^+$ and of the form $H(A, *)$, where $A$ is a simple associative superalgebra (with a superinvolution $*$ in the latter case). This is one of the reasons for which $A^+$ and $H$ have been the subject of a good deal of attention as well (we refer, for instance, to [2] and [4]).

The goal of this paper is to investigate semiprime associative superalgebras with superinvolution whose subspaces of skew elements or symmetric elements are *Lie nilpotent* or *Lie solvable*. We recall that a graded subspace $S$ of a superalgebra $A$ is said to be Lie nilpotent if, set $[x_1, \dots, x_n]_s := [[x_1, \dots x_{n-1}]_s, x_n]_s$ for all $n \geqslant 2$, there exists an integer $m$ such that

$$[x_1, \dots, x_m]_s = 0$$

for all $x_1, \dots, x_m \in S$, and *Lie solvable* if, set $[x_1, x_2]_s^\circ := [x_1, x_2]_s$ and inductively

$$[x_1, \dots, x_{2^{n+1}}]_s^\circ := \big[[x_1, \dots, x_{2^n}]_s^\circ, [x_{2^n+1}, \dots, x_{2^{n+1}}]_s^\circ\big]_s,$$

there exists an integer $m$ such that

$$[x_1, \dots, x_{2^{m+1}}]_s^\circ = 0$$

for all $x_1, \dots, x_{2^{m+1}} \in S$.

We notice that in every semiprime associative superalgebra $A$ the intersection of all the prime ideals of $A$ is zero. Consequently $A$ is a subdirect product of its prime images. If each prime image of $A$ is a central order in a simple superalgebra at most $n^2$-dimensional over its centre, we say that $A$ is $S(n)$. This definition is required to state the main result on Lie solvability condition.

**Theorem 1.1.** *Let $A$ be a non-trivial semiprime associative superalgebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$ endowed with a superinvolution. If $H$ is Lie solvable, then $A$ is $S(2)$.*

We stress that if $K$ is Lie solvable, so is $H$. Thus the result still holds by replacing $H$ with $K$. Furthermore it is true if $H$ is Lie nilpotent as well: indeed, the latter fact implies that $H$ is Lie solvable (obviously, the same holds also for $K$, which has the structure of Lie superalgebra).

In the case in which $H$ or $K$ are Lie nilpotent, we are able to provide a characterization in terms of identities satisfied by the symmetric or skew elements of $A$.

**Theorem 1.2.** *Let $A$ be a non-trivial semiprime associative superalgebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$ endowed with a superinvolution. Then*

(a) *$H$ is Lie nilpotent if, and only if, the elements of $H$ commute*;
(b) *$K$ is Lie nilpotent if, and only if, the elements of $K$ commute.*

Classically, this situation has been studied in the context of semiprime algebras with involution by Giambruno and Sehgal (Theorem 1 of [1]) and Lee, Sehgal and Spinelli (Propositions 2.4 and 2.6 of [13], although there the authors consider algebras over fields, the statements still hold for algebras over rings). Their results can be summarized in the following

**Theorem 1.3.** *Let $A$ be a semiprime associative algebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$ endowed with an involution. The following statements are equivalent*:

(i) *$K$ is Lie nilpotent*;
(ii) *$K$ is Lie solvable*;
(iii) *$K$ is commutative.*

**Theorem 1.4.** *Let $A$ be a semiprime associative algebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$ endowed with an involution. Then*

(a) *$H$ is Lie nilpotent if, and only if, $H$ is commutative*;
(b) *$H$ is Lie solvable if, and only if, $H$ is Lie metabelian.*

*In particular, if $H$ is Lie solvable, then $A$ is $S(2)$.*

We notice that only a partial superanalogous of Theorem 1.3 is obtained. In fact, in non-trivial superalgebras setting the Lie solvability of $K$ does not imply the Lie nilpotency of $K$, not even if the superalgebra is simple. An easy example is provided by the superalgebra of $(2 \times 2)$-matrices $M_{1,1}(F)$ over a field $F$ of characteristic not 2 equipped with the transposition superinvolution. Furthermore, we cannot expect that the Lie nilpotency of $K$ or $H$ forces them to be supercommutative (namely, $[a,b]_s = 0$ for all $a, b$ in $K$ or $H$).

## 2. Preliminaries and notations

Throughout the sequel, unless otherwise stated, $A = A_0 \oplus A_1$ will denote a non-trivial associative superalgebra over a commutative unital ring of scalars $R$ such that $\frac{1}{2} \in R$. A subspace $V \subseteq A$ is called *graded* if $V = (V \cap A_0) \oplus (V \cap A_1)$. For instance, the centre $Z(A)$ is a graded subalgebra of $A$ and we will use $Z$ to denote $Z(A)_0$. By an *ideal* of $A$ we mean a graded ideal of $A$. The superalgebra $A$ is said to be *simple* if it has no non-zero ideals and the multiplication is non-trivial, *prime* if $IJ = 0$ for $I$, $J$ ideals of $A$ implies that either $I = 0$ or $J = 0$, and *semiprime* if it has no non-zero nilpotent ideals.

A superinvolution of $A$ is a $\mathbb{Z}_2$-graded linear map $*: A \to A$ such that, for all homogeneous elements $a, b \in A$, $(a^*)^* = a$ and $(ab)^* = (-1)^{\bar{a}\bar{b}} b^* a^*$. When $A$ is equipped with a superinvolution $*$, let $H$ denote the Jordan superalgebra (with respect to the circle product $\circ_s$) of symmetric elements of $A$ and let $K$ be the Lie superalgebra (with respect to the usual superbracket $[\,,\,]_s$) of skew elements of $A$. In particular, we recall that, if $a, b$ are homogeneous elements of $A$, $[a,b]_s$ coincides with the classical Lie bracket $[a,b] := ab - ba$ if at least one of the arguments is in $A_0$. Obviously, $H$ and $K$ are graded subspaces and $A = H \oplus K$. If $P$ is any subset of $A$, set $P_H := P \cap H$ and $P_K := P \cap K$. The superinvolution $*$ is said to be of the *first kind* if $Z_H = Z$ and of the *second kind* otherwise.

In case $Z \neq 0$, we can consider the localization $Z^{-1}A := \{z^{-1}a \mid 0 \neq z \in Z,\ a \in A\}$. If $A$ is prime, it is a central prime associative superalgebra over the field $Z^{-1}Z$, which we call *the central closure* of $A$ (it should be pointed out that this terminology is not the standard one, which involves the extended centroid). We also say that $A$ is a *central order* in $Z^{-1}A$. In particular $A$ is called a *central order in* $C(n)$ if $Z \neq 0$ and $Z^{-1}A$ is isomorphic to the Clifford superalgebra $C(W,q)$ of a non-degenerate quadratic space $(W,q)$ of dimension $n$ over $Z^{-1}Z$ (see Example 1.5 of [5]). Set $V := Z_H \setminus \{0\}$ (which is non-zero if $Z \neq 0$), it is well-known that $Z^{-1}A = V^{-1}A$. For our aims it will be more convenient to represent the central closure of $A$ in the form $V^{-1}A$. In fact, if $A$ has a superinvolution $*$, $V^{-1}A$ is an associative superalgebra over the field $V^{-1}Z_H$ that can be endowed with a superinvolution $\hat{*}$ of the same kind of the superinvolution $*$ of $A$ via $(v^{-1}a)^{\hat{*}} := v^{-1}a^*$ for all $a \in A$. It is then easy to check that $H(V^{-1}A, \hat{*}) = V^{-1}H$ and $K(V^{-1}A, \hat{*}) = V^{-1}K$. The superalgebra $V^{-1}A$ over the field $V^{-1}Z_H$ is called *the $*$-central closure of* $A$. Directly from the structure of $V^{-1}A$ and its skew and symmetric elements one deduces that

(a) $A$ is Lie nilpotent (solvable, respectively) if, and only if, $V^{-1}A$ is Lie nilpotent (solvable, respectively);
(b) $K$ is Lie nilpotent (solvable, respectively) if, and only if, $K(V^{-1}A, \hat{*})$ is Lie nilpotent (solvable, respectively).
(b′) $H$ is Lie nilpotent (solvable, respectively) if, and only if, $H(V^{-1}A, \hat{*})$ is Lie nilpotent (solvable, respectively).

We shall use these facts without further reference in the sequel.

For any $C$, $D$ subspaces of $A$ let us denote by $[C, D]_s$ the subspace generated by all the elements of the form $[c, d]_s$, with $c \in C$ and $d \in D$. If $S$ is a graded subspace of $A$, set $\gamma_1(S) := S$ and $\delta^{[0]}(S) := S$, for all $i \geqslant 1$ we define by induction

$$\gamma_{i+1}(S) := \left[\gamma_i(S), S\right]_s$$

and

$$\delta^{[i]}(S) := \left[\delta^{[i-1]}(S), \delta^{[i-1]}(S)\right]_s.$$

Obviously, the $\gamma_i(S)$'s and the $\delta^{[j]}(S)$'s are graded subspaces of $A$ and $S$ is Lie nilpotent (Lie solvable) if, and only if, there exists an integer $n$ such that $\gamma_n(S) = 0$ ($\delta^{[n]}(S) = 0$). In the case in which $S$ is Lie solvable, the smallest integer $m$ such that $\delta^{[m]}(S) = 0$ will be denoted by $dl_L(S)$. When $S = A$ or, if $A$ has a superinvolution, $S = K$ the corresponding $\gamma_i(S)$'s and $\delta^{[j]}(S)$'s are *Lie ideals* of $A$ and $K$ respectively, namely graded ideals of the Lie superalgebras $A^-$ and $K$. Obviously, if $A$ ($K$, respectively) is Lie nilpotent, then $A$ ($K$, respectively) is Lie solvable. For what concerns $H$ and its possible link with the Lie structure of $K$, it is well-known that $[H, K]_s \subseteq H$ and $[H, H]_s \subseteq K$. Therefore, for any integer $i$, $\gamma_{2i+1}(H) \subseteq H$ and $\gamma_{2i}(H) \subseteq K$, whereas $\delta^{[i]}(H) \subseteq K$, except for $i = 0$. Easy consequence of the last relations is that *the Lie solvability of $K$ implies that of $H$*.

On the other hand, even if there is no obvious inclusion among the consecutive terms of the sequence of the $\gamma_i(H)$'s, a standard induction argument allows to conclude (also in the case in which $A$ is trivial) that, for any $i, j \geqslant 1$,

$$\left[\gamma_i(H), \gamma_j(H)\right]_s \subseteq \gamma_{i+j}(H).$$

Consequently, one has that, for all $i \geqslant 0$,

$$\delta^{[i]}(H) \subseteq \gamma_{2^i}(H),$$

which proves the following

**Lemma 2.1.** *Suppose that $A$ is non-necessarily non-trivial and has a superinvolution $*$. If $H$ is Lie nilpotent, then $H$ is Lie solvable.*

An interesting case to deal with, which will be the main obstruction to a superanalogous of Theorem 1.3, is that of the *(generalized) quaternion superalgebra* $Q(\alpha,\beta)$ over a field $F$ of characteristic not 2, with basis $\{1,u,v,w\}$ such that $u^2=\alpha$, $v^2=\beta$ and $uv=-vu$ and $\mathbb{Z}_2$-grading given by $Q(\alpha,\beta)_0 = F.1 + F.uv$ and $Q(\alpha,\beta)_1 = F.u + F.v$. It represents the special case of the Clifford superalgebra $C(W,q)$ with $\dim W = 2$ (see Example 1.5 of [5]). Set $Q := Q(\alpha,\beta)$, it is easily seen that $[Q,Q]_s = F.1 + Q_1$. Hence $[Q_0,Q_1]_s = Q_1$ and therefore

$$\delta^{[2]}(Q) = [F.1+Q_1, F.1+Q_1]_s = [Q_1,Q_1]_s \subseteq F.1.$$

Obviously, this forces to be $\delta^{[3]}(Q)=0$. On the other hand, an easy induction argument shows that $Q_1 \subseteq \gamma_n(Q)$, for all $n$. We have so proved the following

**Lemma 2.2.** *The quaternion superalgebra over a field of characteristic not 2 is Lie solvable, but not Lie nilpotent.*

When $Q$ is split, it is isomorphic to $M_{1,1}(F)$, the superalgebra of the $(2\times 2)$-matrices over the field $F$ endowed with the grading $M_{1,1}(F)_0 := \left(\begin{smallmatrix} F & 0 \\ 0 & F \end{smallmatrix}\right)$ and $M_{1,1}(F)_1 := \left(\begin{smallmatrix} 0 & F \\ F & 0 \end{smallmatrix}\right)$. The only two superinvolutions of this superalgebra have been described in Theorem 3.2 of [5] and are the maps $*$ and $\star$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.$$

Let us consider the skew elements $K(M_{1,1}(F),*)$ and $K(M_{1,1}(F),\star)$. Now, $K(M_{1,1}(F),*)$ is the subspace generated by the elements $x := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $y := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $[y,x]_s = -2y$, for any $n \geqslant 1$ we get

$$[y, \underbrace{x,\ldots,x}_{n \text{ times}}]_s \neq 0.$$

On the other hand, $K(M_{1,1}(F),\star)$ is generated by the elements $x$ and $k := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. In this case $[k,x]_s = 2k$ and, consequently, for any $n$, $[k, \underbrace{x,\ldots,x}_{n \text{ times}}]_s \neq 0$. Furthermore, set $h := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $h$ and $k$ generate $H(M_{1,1}(F),*)$, whereas $h$ and $y$ generate $H(M_{1,1}(F),\star)$. Consequently, in any case the symmetric elements of $M_{1,1}(F)$ commute (both in graduate and in non-graduate sense). As $M_{1,1}(F)$ is Lie solvable, we have proved

**Lemma 2.3.** *Let $F$ be a field of characteristic not 2. Then, with respect to any superinvolution of $M_{1,1}(F)$,*

(a) *the skew elements of $M_{1,1}(F)$ are Lie solvable, but not Lie nilpotent*;
(b) *the symmetric elements of $M_{1,1}(F)$ are Lie nilpotent (hence Lie solvable as well). In particular, they commute and supercommute each other.*

Another useful result concerning with the subspaces $\gamma_2(H)$ and $\gamma_2(K)$ when $A$ is not a quaternion superalgebra is quoted from [2].

**Lemma 2.4.** *(See 2 of [2].) Suppose that $A$ is unital simple over a field of characteristic not 2. If $A$ has a superinvolution $*$ and is not a quaternion superalgebra, then $[K,K]_s \subseteq [H,H]_s$.*

Also in view of Lemma 2.4, as we have enough information on the structure of Lie ideals of $K$ and $[K,K]_s$, an obvious way of proceeding is to investigate the sequences of the $\gamma_i(K)$'s and $\delta^{[i]}(K)$'s, instead of working with the identities defining the Lie nilpotency or solvability of $A$, $K$ or $H$. We confine ourselves to report here the following result showing that, in general, $K$ cannot be supercommutative.

**Lemma 2.5.** *(See 4.1 of [3].) Suppose that $A$ is prime and has a superinvolution $*$. If $A$ is not a central order in $C(2)$, then $[K,K]_s \neq 0$.*

Finally, a rather simple, but very useful, remark is in order now. Assume that $A$ has a superinvolution and the set of its symmetric elements $H$ is Lie solvable. Then $A_0$ is an algebra with involution whose symmetric elements $A_0 \cap H$ are still Lie solvable. Hence $A_0$ satisfies a $*$-polynomial identity. By a Theorem of Amitsur (6.5.1 of [6]), we conclude that $A_0$ satisfies an ordinary polynomial identity. As $A$ is 2-torsion free, from a result of Kharchenko (Theorem 4 of [9]) it follows that $A$ is a PI-algebra. The structure of prime superalgebras (non-necessarily equipped with superinvolutions) satisfying a polynomial identity was described by Montaner.

**Lemma 2.6.** *(See 1.7 of [14].) Assume that $A$ is prime. If $A$ is a PI-algebra, then any non-zero ideal of $A$ intersects $Z$ non-trivially. Moreover, $A$ is a central order in its central closure $Z^{-1}A$ which is a finite-dimensional central simple superalgebra over $Z^{-1}Z$.*

In particular, from Lemma 2.6 we deduce that, if $A$ is prime and $K$ or $H$ is Lie solvable, then $Z \neq 0$. Obviously, we can apply Lemma 2.6 to prime superalgebras (without superinvolution) which are Lie solvable.

## 3. Prime superalgebras

The first part of this section is devoted to characterize non-trivial prime associative superalgebras with superinvolution whose skew elements or symmetric elements are Lie nilpotent or Lie solvable. We first deal with Lie solvability conditions.

**Theorem 3.1.** *Suppose that $A$ is prime and has a superinvolution $*$. The following statements are equivalent*:

(i) *$H$ is Lie solvable*;
(ii) *$K$ is Lie solvable*;
(iii) *one of the following conditions occurs*:
  (a) *$A$ is a central order in $C(2)$*;
  (b) *$*$ is of the second kind and $A$ is commutative* (*as algebra*).

**Proof.** Assume that $H$ is Lie solvable. If $A$ is a central order in a quaternion superalgebra, $A$ is Lie solvable and, consequently, so is $K$. Therefore suppose that $A$ is not a central order in $C(2)$. According to Lemma 2.6, $A$ is a central order in its central closure $Z^{-1}A$ which is a finite-dimensional central simple superalgebra over $Z^{-1}Z$. Hence also the $*$-central closure $V^{-1}A$ of $A$ is simple as superalgebra over $V^{-1}Z_H$. Moreover, the extension $\hat{*}$ of $*$ on $V^{-1}A$ is a superinvolution (of the same kind of $*$) of $V^{-1}A$ such that $H(V^{-1}A,\hat{*})$ is Lie solvable. But we may apply Lemma 2.4 to $V^{-1}A$ and conclude that $K(V^{-1}A,\hat{*})$ is Lie solvable. Consequently, $K$ must be Lie solvable.

In such an event, set $m := dl_L(K)$. Again under the assumption that $A$ is not a central order in $C(2)$, from Lemma 2.5 we know that $m \geqslant 2$. Suppose, if possible, that $*$ is of the first kind. As $[\delta^{[m-1]}(K), \delta^{[m-1]}(K)]_s = 0$, Lemma 4.5 of [3] forces to be $\delta^{[m-1]}(K) = 0$, which is a contradiction. Thus $*$ must be of the second kind. By using the same above arguments, we get that the extension $\hat{*}$ of $*$ to the $*$-central closure $V^{-1}A$ of $A$ is a superinvolution of the second kind of $V^{-1}A$ such that $K(V^{-1}A,\hat{*})$ is Lie solvable. This implies that $(V^{-1}A)_0$ is commutative, otherwise, by invoking Corollary 5.1 of [5], one has that $\delta^{[1]}(K(V^{-1}A,\hat{*})) \subseteq \delta^{[2]}(K(V^{-1}A,\hat{*}))$, which is in contradiction with the

Lie solvability of the skew elements of $V^{-1}A$. From Lemma 2.6 of [14] it follows that the elements of $V^{-1}A$ must commute, and we are done.

Finally, if $A$ is a central order in $C(2)$, by Lemma 2.2 $A$, and so $H$, is Lie solvable. Hence, assume that the elements of $A$ commute. Then $[A, A]_s = [A_1, A_1]_s \subseteq A_0$. But this forces to be $\gamma_3(A) = 0$. Therefore $A$ is Lie nilpotent and, consequently, Lie solvable, and this completes the proof. □

We stress that we cannot expect that the Lie solvability of $H$ is equivalent to that of $K$ when $A$ is trivial. To show this, it is sufficient to consider the algebra of $(2 \times 2)$-matrices $M_2(F)$ over a field $F$ of characteristic not 2 endowed with the symplectic involution. Indeed, the symmetric elements of this algebra commute, whereas its skew elements are not Lie solvable.

In the case in which $K$ is Lie nilpotent, $A$ cannot be a central order in $C(2)$, as proved in the following

**Theorem 3.2.** *Suppose that $A$ is prime and has a superinvolution $*$. Then $K$ is Lie nilpotent if, and only if, $*$ is of the second kind and $A$ is commutative (as algebra).*

**Proof.** By virtue of Theorem 3.1, it remains only to show that, if $K$ is Lie nilpotent, then $A$ cannot be a central order in a quaternion superalgebra. Suppose, if possible, the contrary. If $*$ is of the first kind, its $*$-central closure is split, thus isomorphic to $D := M_{1,1}(V^{-1}Z_H)$. Moreover the skew elements $K(D, \hat{*})$ of $D$ must be Lie nilpotent. But this is in contradiction with Lemma 2.3.

Therefore, assume that $*$ is of the second kind. According to Lemma 2.6, the central closure $Z^{-1}A$ of $A$ over the field $Z^{-1}Z$ is simple. Hence also the $*$-central closure $V^{-1}A$ of $A$ over $V^{-1}Z_H$ is simple. Let us call $B := V^{-1}A$ and consider the induced superinvolution $\hat{*}$ on $B$. By applying Lemma 3.1 of [5] we conclude that $Z(B)_0$ is an extension of degree 2 of $Z(B)_0 \cap H(B, \hat{*})$ and $H(B, \hat{*}) = qK(B, \hat{*})$, for a non-zero skew element $q$ of $Z(B)_0$. Now, $B = H(B, \hat{*}) \oplus K(B, \hat{*}) = qK(B, \hat{*}) \oplus K(B, \hat{*})$. But $K(B, \hat{*})$ is Lie nilpotent. Thus $B$ is Lie nilpotent. As $Z^{-1}A = B$, $Z^{-1}A$ is still Lie nilpotent, but this is in contradiction with Lemma 2.2. □

An extra condition is instead required when $H$ is Lie nilpotent.

**Theorem 3.3.** *Suppose that $A$ is prime and has a superinvolution $*$. Then $H$ is Lie nilpotent if, and only if, one of the following conditions occurs*:

(a) *$*$ is of the second kind and $A$ is commutative (as algebra)*;
(b) *$*$ is of the first kind and $A$ is a central order in $M_{1,1}(Z^{-1}Z)$.*

**Proof.** Suppose that $H$ is Lie nilpotent. According to Lemma 2.1, $H$ is Lie solvable. Hence, by invoking Theorem 3.1, either $A$ is central order in a quaternion superalgebra or $*$ is of the second kind and $A$ is commutative as algebra. In the latter case, we are done. Therefore, assume that it does not hold.

Thus $A$ is a central order in $C(2)$. If the superinvolution $*$ of $A$ is of the second kind, by using the same arguments of the proof of Theorem 3.2 one has that the $*$-central closure $B := V^{-1}A$ of $A$ is a simple superalgebra with induced superinvolution $\hat{*}$ (of the second kind) over $V^{-1}Z_H$. Again Lemma 3.1 of [5] says that $Z(B)_0$ is an extension of degree 2 of $Z(B)_0 \cap H(B, \hat{*})$ and $K(B, \hat{*}) = pH(B, \hat{*})$, for a non-zero skew element $p$ of $Z(B)_0$. Now, $B = H(B, \hat{*}) \oplus K(B, \hat{*}) = H(B, \hat{*}) \oplus pH(B, \hat{*})$. But $H(B, \hat{*})$ is Lie nilpotent. Thus $B$ is Lie nilpotent, which is in contradiction with Lemma 2.2.

We conclude that $*$ must be of the first kind. This implies that the $*$-central closure of $A$ is split. Hence $A$ is a central order in $M_{1,1}(Z^{-1}Z)$.

Conversely, if the conditions in (a) are satisfied, as observed in the proof of Theorem 3.1, $A$ is Lie nilpotent. On the other hand, when $A$ and $*$ are as in (b), we are done by virtue of Lemma 2.3. □

In the rest of the section we investigate the Lie structure of prime superalgebras without superinvolution.

**Theorem 3.4.** *Suppose that $A$ is prime. Then $A$ is Lie solvable if, and only if, one of the following conditions occurs*:

(a) *$A$ is a central order in $C(2)$*;
(b) *$A$ is commutative* (*as algebra*).

**Proof.** We have only to prove the necessary part of the statement (as the necessary conditions have been already discussed at the end of the proof of Theorem 3.1). Hence, assume that $A$ is Lie solvable and $A$ is not a central order in $C(2)$. If $A_0$ is not commutative, from Lemma 3 of [2] it follows that $A$ is not Lie solvable, but this cannot be the case. Therefore $[A_0, A_0] = 0$ and from Lemma 2.6 of [14] we deduce that the elements of $A$ must commute, and this concludes the proof. □

Assume now that $A$ is Lie nilpotent. Thus $A$ is Lie solvable. Hence we may apply the previous theorem and conclude that either $A$ is a central order in a quaternion superalgebra over a field or the elements of $A$ commute. But, according to Lemma 2.2, a quaternion superalgebra cannot be Lie nilpotent. Therefore we have the following

**Theorem 3.5.** *Suppose that $A$ is prime. Then $A$ is Lie nilpotent if, and only if, $A$ is commutative* (*as algebra*).

The following lemma extends to superalgebras a classical result in the setting of prime algebras. For the sake of completeness its easy proof is included.

**Lemma 3.6.** *Suppose that $A$ is prime and let $I$ be a non-zero ideal of $A$. If $I$ is commutative, then $A$ is commutative* (*as algebra*).

**Proof.** According to Lemma 1.2 of [14], $A$ is a semiprime algebra. If $A$ is prime as algebra, we are done (see the corollary to Lemma 1.1.5 of [6], p. 7). Thus, suppose that $A$ is not a prime algebra. The above cited result of [6] shows that $I \subseteq Z(A)$. Now, let $a$ be a homogeneous element of $I$ and $x, y$ homogeneous elements of $A$. Then, as $ax$ and $a$ are in $I$, hence in the centre of $A$, we get

$$0 = [y, ax] = a[y, x] + [y, a]x = a[y, x].$$

From the arbitrariness of $a$ it follows that $I[y, x] = 0$. Therefore, for any non-zero homogeneous element $z$ of $I$ one has that $zA[y, x] = 0$. Since $A$ is a prime superalgebra, it must be $[y, x] = 0$, and this concludes the proof. □

## 4. Proof of the main results and concluding remarks

We are now in position to prove our main results.

**Proof of Theorem 1.1.** Let $P$ be a prime ideal of $A$. If $P^* \nsubseteq P$, then $(P + P^*)/P$ is a non-zero (graded) ideal of the prime superalgebra $A/P$. Moreover $(P + P^*)/P$ is a prime superalgebra as well. Suppose that $\delta^{[n]}(H) = 0$. Working in $(P + P^*)/P$, we are clearly dealing with elements of the form $x + P$, with $x \in P^*$. But for any such $x$ we have $x + x^* \in H$, hence

$$\left[x_1 + x_1^*, \ldots, x_{2^n} + x_{2^n}^*\right]_s^o = 0$$

for all $x_i \in P^*$. That is,

$$[x_1 + P, \ldots, x_{2^n} + P]_s^o = P,$$

hence $\delta^{[n]}((P+P^*)/P)=0$. If $(P+P^*)/P$ has trivial grading, $(P+P^*)/P$ must be commutative. When $(P+P^*)/P$ is a non-trivial superalgebra, the conclusions of Theorem 3.4 apply and one has that either $(P+P^*)/P$ is commutative or it is a central order in a quaternion superalgebra. If the elements of $(P+P^*)/P$ commute, from Lemma 3.6 it follows that $A/P$ is commutative, and is therefore a central order in a field. Thus suppose that $(P+P^*)/P$ is a central order in $C(2)$. Now, $A/P$ is a PI-algebra since that $A$ is PI. By virtue of Lemma 2.6 we conclude that $Z(A/P)_0$ is non-zero and the central closure of $A/P$, we call $W$, is a finite-dimensional central simple superalgebra over the field $Z(A/P)_0^{-1}Z(A/P)_0$. But also $(P+P^*)/P$ is a PI-algebra, thus also $Z((P+P^*)/P)_0$ is non-zero. Furthermore, as $A/P$ is semiprime as algebra, Lemma 1.1.5 of [6] shows that $Z((P+P^*)/P)\subseteq Z(A/P)$. Let us consider the central closure $S$ of $(P+P^*)/P$. Pick non-zero elements $z\in Z(A/P)_0$ and $w\in Z((P+P^*)/P)_0$. If $a\in A/P$ and $x\in (P+P^*)/P$, then $(z^{-1}a)(w^{-1}x)=(zw)^{-1}ax$. Now, $wz\in Z((P+P^*)/P)_0$ and, consequently, $(wz)^{-1}ax\in S$. Since $S$ is an additive subgroup and a $Z(A/P)_0^{-1}Z(A/P)_0$-supermodule, $S$ is a graded ideal of $W$. But $W$ is simple, thus $S=W$ and therefore $A/P$ must be an order in a quaternion superalgebra.

If $P^*\subseteq P$, then $A/P$ is a prime superalgebra with the induced superinvolution, we call again $*$, such that $H(A/P,*)$ is Lie solvable. If the grading on $A/P$ is trivial, we are done by virtue of Theorem 1.4. Thus, assume that $A/P$ is non-trivial. But, in this case, from Theorem 3.1 it follows that either $A/P$ is a central order in a quaternion superalgebra over its centre or $A/P$ is commutative, therefore a central order in a field, and this concludes the proof. □

**Proof of Theorem 1.2.** (a) By following the same above strategy, let us consider a prime ideal $P$ of $A$. If $P^*\nsubseteq P$, then $(P+P^*)/P$ is a non-zero (graded) ideal of the prime superalgebra $A/P$. Moreover, it is a Lie nilpotent prime superalgebra. Hence, by applying Theorem 3.5 when $(P+P^*)/P$ has non-trivial grading, we conclude that $(P+P^*)/P$ is commutative and, by virtue of Lemma 3.6, so is $A/P$.

If $P^*\subseteq P$, then the symmetric elements of $A/P$ with respect to the induced superinvolution are Lie nilpotent. If the grading on $A/P$ is non-trivial, directly from Theorem 3.3 it follows that either $A/P$ is commutative or the induced superinvolution on $A/P$ is of the first kind and $A/P$ is a central order in $M_{1,1}(Z(A/P)_0^{-1}Z(A/P)_0)$. But the symmetric elements of this superalgebra commute, thus also the elements of $H(A/P)$ must commute each other.

When $A/P$ is a trivial superalgebra, from Theorem 1.4 one has that the symmetric elements $(H+P)/P$ of $A/P$ must commute.

In any case, $[H,H]$ is contained in the intersection of all the prime ideals of $A$, and we are done.

(b) It uses the same arguments of the proof of (a) replacing the reference to Theorems 3.3 and 1.4 with an appeal to Theorems 3.2 and 1.3. □

We conclude with a final remark. In [11] the authors determined the structure of a semiprime superalgebra with superinvolution satisfying the identity $[K^2,K^2]_s=0$. This condition implies that $[[K,K]_s,[K,K]_s]_s=0$, thus $K$ must be Lie solvable and our result in Theorem 1.1 applies. Obviously, the conditions $[K^2,K^2]_s=0$ and $\delta^{[2]}(K)=0$ are not equivalent. To show this, it is sufficient to consider the space $A:=\mathbb{Q}(i)\oplus\mathbb{Q}(i)u$ with $i^2=-1$ and $u^2=i$, grading induced by setting $\bar{u}:=1$ and superinvolution

$$*: A\longrightarrow A,\qquad (q+pi)+(r+ti)u\longmapsto (q-pi)+(r-ti)u,\quad q,p,r,t\in\mathbb{Q}.$$

Then $A$ is a simple superalgebra and the set of its skew elements $K=\mathbb{Q}i\oplus\mathbb{Q}iu$ is Lie nilpotent. But $u\in K^2$ and $[u,u]_s=2u^2\neq 0$.

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