



Existence of terminal resolutions of geometric Brauer pairs of arbitrary dimension

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ABSTRACT

A *geometric Brauer pair* is a pair (X, α) where X is a smooth quasi-projective variety over an algebraically closed field and α is an element in the 2-torsion part of the Brauer group of the function field of X . A geometric Brauer pair (Y, α) is a *terminal pair* if the Brauer discrepancy of (Y, α) is positive. We show that given a geometric Brauer pair (X, α) , there is a terminal pair (Y, α) with a birational morphism $Y \rightarrow X$. In short, any geometric Brauer pair admits a terminal resolution.

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1. Introduction

The problem considered in this article is central to research in the area of maximal orders on algebraic varieties. The works of Daniel Chan, Colin Ingalls and Rajesh Kulkarni [2–4,8] considered various questions regarding orders on algebraic surfaces. The main idea was to build a minimal model program for maximal orders on surfaces. This was successfully established in a seminal article in the area by Daniel Chan and Colin Ingalls [1]. In this article, they define the notion of terminal orders (which are analogs of smooth surfaces) and then prove the main theorem: any maximal order has a terminal resolution. This led to several articles in which terminal models of orders on surfaces were classified. This classification has been a significant achievement of the past decade.

In [11] it is proven that a geometric Brauer pair (X, α) where X is 3-dimensional admits a terminal resolution. The technique of the proof involves the computation of Brauer discrepancies (Definition 3.1) of the possible local models of a pair in order to show that each admits a terminal resolution. However, this method of proof is not amenable when the dimension of the variety X is 4 or higher, as the number of possible local models increase rapidly as the dimension increases. In any case, it would be impossible to prove the result in arbitrary dimension by analyzing local models, simply because

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there are infinitely many local models. In this paper, we circumvent this problem by realizing that it is enough to compute Brauer discrepancies of a pair (X, α) , where X is of arbitrary dimension, in two situations: (1) Blowing up of a subvariety of X of codimension ≥ 3 (see Section 4.1) and (2) blowing up of a subvariety of codimension 2 (see Section 4.2). In the first case, we show that the Brauer discrepancy along the exceptional divisor is positive. In the second case, the Brauer discrepancy is either positive or the variety can be blown up to obtain a terminal pair (Definition 4.4).

This paper is organized as follows: In Section 2, we explain how an element in the Brauer group of the function field of a smooth variety induces a boundary divisor, via a complex that appears in the coniveau spectral sequence of the variety. In Section 3, we describe birational geometry of geometric Brauer pairs after introducing the notion of Brauer discrepancy of a pair. Finally, in Section 4, we prove the main result that any geometric Brauer pair admits a terminal resolution.

2. Logarithmic pairs from Brauer pairs

Level 1 terms of the coniveau spectral sequence for a smooth algebraic variety X over an algebraically closed field k were written by Grothendieck [6]: One has

$$E_1^{i,j} = \bigoplus_{x \in X^{(i)}} H^{j-i}(k(x), \mu_n^{\otimes(1-i)})$$

where μ_n is the group of n th roots of unity in k , and $X^{(i)}$ is the set of all irreducible subvarieties of X of codimension i . The cohomology mentioned is Galois cohomology. The tensor product is over $\mathbf{Z}/n\mathbf{Z}$. By definition, $\mu_n^{-1} := \text{Hom}(\mu_n, \mathbf{Z}/n\mathbf{Z})$, and we write $\mu_n^{\otimes(-m)}$ for $(\mu_n^{-1})^{\otimes m}$ when m is positive. For more details, see Section 3.5 of [10].

Accordingly, if X is an irreducible smooth 3-fold, we get Fig. 1 on page 655 for the first quadrant of level 1. In that figure, D, C, pt are prime divisors, irreducible curves and points of X respectively. The row $j = 2$ has the same form for any irreducible n -fold where $n \geq 2$ with the interpretation that C and pt are irreducible subvarieties of codimension 2 and 3 respectively.

Now, from row $j = 2$, we obtain the complex

$$H^2(k(X), \mu_n) \longrightarrow \bigoplus_D H^1(k(D), \mathbf{Z}/n\mathbf{Z}) \longrightarrow \bigoplus_C H^0(k(C), \mu_n^{-1}) \longrightarrow 0.$$

We know that $H^2(k(X), \mu_n) \cong Br_n(k(X))$, where $Br_n(k(X))$ is the n -torsion part of the Brauer group $Br(k(X))$ of $k(X)$. (See Section 4.4 of [5].) Therefore, we get the complex

$$Br_n k(X) \xrightarrow{a} \bigoplus_D H^1(k(D), \mathbf{Z}/n\mathbf{Z}) \longrightarrow \bigoplus_C \mu_n^{-1} \longrightarrow 0. \quad (2.1)$$

This tells us, in particular, that any element α in $Br_2(k(X))$, induces a (possibly ramified) 2-sheeted cover or a 1-sheeted cover on each irreducible divisor D . Note that the ramifications must cancel on the irreducible subvarieties C , since the sequence above is a complex. This simple observation will play an important role in our study.

Definition 2.2. A *geometric Brauer pair* is a pair (X, α) where X is a smooth quasi-projective variety over an algebraically closed field and α is an element in $Br_2(k(X))$.

Let (X, α) be such a pair. Then α induces a boundary divisor $\Delta_{X, \alpha}$ on X as follows:

Consider the complex (2.1) that we obtained above, where D runs through all the irreducible divisors of X and C runs through all the irreducible subvarieties of codimension 2 of X . For a given irreducible divisor D , let $a(\alpha)_D$ be the image of α indexed by D . Since $H^1(k(D), \mathbf{Q}/\mathbf{Z})$ classifies cyclic

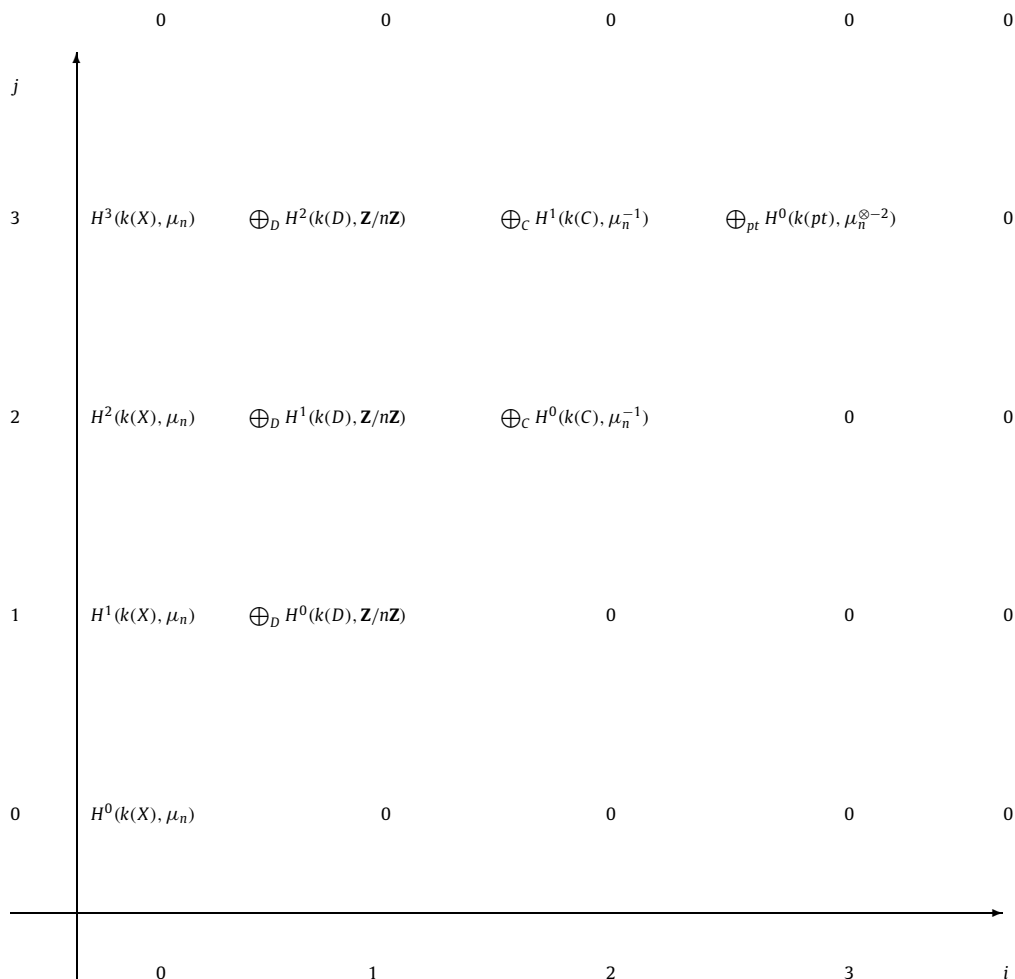


Fig. 1. Level 1 of the coniveau spectral sequence for an irreducible 3-fold X .

covers of D , $a(\alpha)_D$ determines a ramified cover of D . Let e_D be the degree of this cover. We define the boundary divisor $\Delta_{X,\alpha}$ to be

$$\Delta_{X,\alpha} := \sum_D \left(1 - \frac{1}{e_D}\right) D$$

where D runs through all the prime divisors of X such that $a(\alpha)_D \neq 0$. (See Section 3.3 of [1].)

3. Birational geometry of Brauer pairs

In the following, by a *divisor over X* we mean an irreducible divisor $E \subseteq Y$ where Y is a normal variety with a birational morphism $Y \rightarrow X$.

Two divisors D_1, D_2 of X are said to be *numerically equivalent* if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves $C \subseteq X$.

Throughout this paper, by $K_{Z,\alpha}$ we mean $K_Z + \Delta_{Z,\alpha}$, and \equiv denotes numerical equivalence.

Definition 3.1. Let E be an irreducible exceptional divisor over X and $\alpha \in \text{Brk}(X)$. The Brauer discrepancy of the pair (X, α) along E , denoted by $b(E, X, \alpha)$, is the coefficient of E in the formula

$$K_{Y, \alpha} \equiv f^* K_{X, \alpha} + \sum_i b(E_i, X, \alpha) E_i$$

where E is one of the exceptional divisors E_i of f where $f : Y \rightarrow X$ is a proper birational morphism.

Definition 3.2. Let $f : Y \rightarrow X$ and $g : Y' \rightarrow X$ be birational morphisms. Suppose $E \subseteq Y$ and $E' \subseteq Y'$ are f -exceptional and g -exceptional divisors respectively. We say that E and E' are *isomorphic as divisors* if there exist open sets U, U' containing the generic points of E and E' respectively and an isomorphism $\phi : U \rightarrow U'$ such that $\phi|_E : E \rightarrow E'$ is an isomorphism and $\phi = g^{-1} \circ f$ on $U \setminus \{f\text{-exceptional divisors}\}$.

The following lemma is the reason why we suppress the birational morphism f and the variety Y in the notation $b(E, X, \alpha)$ for the Brauer discrepancy of the pair (X, α) along E . The analogous remark for usual discrepancy (see below for the definition), is mentioned in Remark 2.23 of [9].

Lemma 3.3. $b(E, X, \alpha)$ does not depend on the particular birational morphism.

Proof. Suppose $f : Y \rightarrow X$ and $g : Y' \rightarrow X$ are birational morphisms, $E \subseteq Y$ and $E' \subseteq Y'$ are f -exceptional and g -exceptional divisors respectively, that are isomorphic as divisors. Then there is an isomorphism $\phi : U \rightarrow U'$ on a neighborhood U of E such that $\phi = g \circ f^{-1}$ almost everywhere on U , and U' is a neighborhood of E' . Now, we have

$$K_{Y, \alpha} \equiv f^* K_{X, \alpha} + \sum_i b(E_i, X, \alpha) E_i + b(E, X, \alpha) E$$

where E_i are f -exceptional divisors, $E_i \neq E$.

Similarly,

$$K_{Y', \alpha} \equiv g^* K_{X, \alpha} + \sum_j b(E'_j, X, \alpha) E'_j + b(E', X, \alpha) E'$$

where E'_j are g -exceptional divisors, $E'_j \neq E'$.

Now,

$$(K_{Y, \alpha} - f^*(K_{X, \alpha}))|_U = \phi^*((K_{Y', \alpha} - g^*(K_{X, \alpha}))|_{U'}).$$

Therefore,

$$\sum b(E_i, X, \alpha) E_i|_U + b(E, X, \alpha) E = \phi^*\left(\sum b(E'_j, X, \alpha) E'_j|_{U'}\right) + b(E', X, \alpha) E.$$

Hence $b(E, X, \alpha) = b(E', X, \alpha)$. \square

For comparison purposes, and also for later use, we give the definition of the (usual) discrepancy of a log pair (X, Δ) , which chronologically preceded the notion of Brauer discrepancy.

Definition 3.4. Let (X, Δ) be a logarithmic pair and E an exceptional divisor over X . The *discrepancy* of the pair (X, Δ) along E , denoted by $a(E, X, \Delta)$, is the coefficient of E in

$$K_Y + f_*^{-1} \Delta \equiv f^*(K_X + \Delta) + \sum_{E_i} a(E_i, X, \Delta) E_i$$

where E is one of the exceptional divisors E_i and $f: Y \rightarrow X$ is a proper birational morphism.

This is the usual notion of discrepancy in algebraic geometry.

The lemma below tells us how Brauer discrepancy and the (usual) discrepancy are related.

Lemma 3.5. *Let (X, α) be a geometric Brauer pair, E an exceptional divisor over X , and $\Delta_{X, \alpha}$ the boundary divisor induced by α . Then,*

$$b(E, X, \alpha) = a(E, X, \Delta_{X, \alpha}) + 1 - \frac{1}{e_E}$$

where e_E is the degree of the cover on E induced by α .

For a proof, see the proof of Proposition 3.15 of [1].

We define the *Brauer discrepancy* of the pair (X, α) to be

$$\text{bdiscrep}(X, \alpha) := \inf \{e_E \cdot b(E, X, \alpha) : E \text{ is an exceptional divisor over } X\}.$$

Definition 3.6. A pair (X, α) is a *terminal pair* if $\text{bdiscrep}(X, \alpha) > 0$.

Definition 3.7. A *terminal resolution* of (X, α) is a proper birational morphism $Y \rightarrow X$ from a smooth variety Y such that the pair (Y, α) is a terminal pair.

Now we can state our main theorem.

Theorem 3.8. *Any geometric Brauer pair (X, α) has a terminal resolution.*

We will see that we can arrive at a terminal pair by successively blowing up $(X, \Delta_{X, \alpha})$.

Using Hironaka's theorem on resolution of singularities [7], we can improve $(X, \Delta_{X, \alpha})$ such that $\Delta_{X, \alpha}$ is simple normal crossing. Thus, étale-locally, X has the form $X = \text{spec} k\{x_1, x_2, \dots, x_n\}$ and the boundary divisor is of the form

$$\Delta_{X, \alpha} = \sum_{i=1}^n \left(1 - \frac{1}{e_i}\right) V(x_i)$$

where $V(x_i)$ is the hyperplane defined by $x_i = 0$. I.e. $V(x_i) = \text{spec} k\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ with $V(x_i) \hookrightarrow X$ the dual of the map $k\{x_1, \dots, x_n\} \rightarrow k\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ arrived at by setting $x_i = 0$. The number e_i is the degree of the cover on $V(x_i)$ induced by α . Since α is in the 2-torsion part of $\text{Br}k(X)$, we have $e_i \in \{1, 2\}$, by complex (2.1) of Section 2.

Suppose D_1 and D_2 are prime divisors of X with covers on them. Let $C = D_1 \cap D_2$. If one of the covers ramifies on C then the other also must ramify on C , since the sequence (2.1) is a complex.

4. Proof of the main result

In this section we will prove the main result that given a geometric Brauer pair (X, α) , there is a smooth variety Y and a proper birational morphism $Y \rightarrow X$ such that (Y, α) is a terminal pair.

First, we present a few lemmas.

Lemma 4.1. Let X be a smooth variety, $Z \subseteq X$ a smooth subvariety of codimension c and $\alpha \in \text{Br } k(X)$. Suppose $p: B_Z X = Y \rightarrow X$ is the blow-up of X along Z , E is the exceptional divisor. Then,

$$b(E, X, \alpha) = c - \frac{1}{e_E} - \sum_i a_i \cdot \text{mult}_Z D_i$$

where e_E is the degree of the cover on E determined by α and $\sum_i a_i D_i$ is the boundary divisor on X determined by α .

Proof. By the definition of Brauer discrepancy,

$$\begin{aligned} b(E, X, \alpha)E &= K_Y + \Delta_{Y, \alpha} - p^*(K_X + \Delta_{X, \alpha}) \\ &= K_Y - p^*K_X + \Delta_{Y, \alpha} - p^*\Delta_{X, \alpha}. \end{aligned}$$

Now, $K_Y - p^*K_X = (c-1)E$ and

$$\begin{aligned} \Delta_{Y, \alpha} - p^*(\Delta_{X, \alpha}) &= \left(1 - \frac{1}{e}\right)E + p_*^{-1}(\Delta_{X, \alpha}) - p^*\Delta_{X, \alpha} \\ &= \left(1 - \frac{1}{e}\right)E + p_*^{-1}\left(\sum_i a_i D_i\right) - \sum_i a_i (p^*D_i) \\ &= \left(1 - \frac{1}{e}\right)E - \sum_i a_i (p^*D_i - p_*^{-1}D_i) \\ &= \left(1 - \frac{1}{e}\right)E - \sum_i a_i \cdot (\text{mult}_Z D_i)E. \end{aligned}$$

Thus,

$$b(E, X, \alpha)E = (c-1)E + \left(1 - \frac{1}{e}\right)E - \left(\sum_i a_i \cdot \text{mult}_Z D_i\right)E, \quad \text{and so}$$

$$b(E, X, \alpha) = c - \frac{1}{e} - \sum_i a_i \cdot \text{mult}_Z D_i. \quad \square$$

The following lemma, which is the analog of Lemma 4.1 for usual discrepancy, appears as Lemma 2.29 in [9] without proof.

Lemma 4.2 (Analogous to Lemma 4.1). Let X be a smooth variety and $\sum a_i D_i$ is a boundary divisor on X . Let $Z \subseteq X$ be a smooth subvariety of codimension c . Suppose $p: B_Z X = Y \rightarrow X$ is the blow-up of X along Z , and E denotes the exceptional divisor. Then,

$$a(E, X, \Delta) = c - 1 - \sum_i a_i \cdot \text{mult}_Z D_i.$$

Proof is similar to the proof of Lemma 4.1, and therefore omitted.

Lemma 2.45 of [9] tells us that any exceptional divisor over a variety X can be reached by finitely many blow-ups. This encourages the following definitions.

Definition 4.3. Let E be an exceptional divisor over a variety X . The divisor E is called a *level n exceptional divisor*, if

$$n = \inf\{m \in \mathbf{Z}^+ : E \text{ can be reached by } m \text{ successive blow-ups starting from } X\}.$$

Definition 4.4. A Brauer pair (X, α) is called *level n Brauer terminal* if $b(E, X, \alpha) > 0$ for all level m exceptional divisors E over X for $1 \leq m \leq n$.

4.1. Blowing up along a subvariety of codimension ≥ 3

In this section we show that if E is an exceptional divisor generated by a blow-up of a subvariety of codimension ≥ 3 , then $b(E, X, \alpha)$ is positive.

Proposition 4.5. Let (X, α) be a geometric Brauer pair with $\Delta_{X, \alpha}$ simple normal crossing and $Z \subseteq X$ be a subvariety of codimension c . Let E be the exceptional divisor generated in the blow-up of X along Z . If $c \geq 3$, then $b(E, X, \alpha)$ is positive.

Proof. As described towards the end of Section 3, étale-locally $(X, \Delta_{X, \alpha})$ is of the form $(\text{spec } k\{x_1, \dots, x_n\}, \sum_i (1 - \frac{1}{e_i})V(x_i))$ where $n = \dim X$. Thus, by Lemma 4.1,

$$b(E, X, \alpha) = c - \frac{1}{e_E} - \sum_i \left(1 - \frac{1}{e_i}\right) \cdot \text{mult}_Z V(x_i).$$

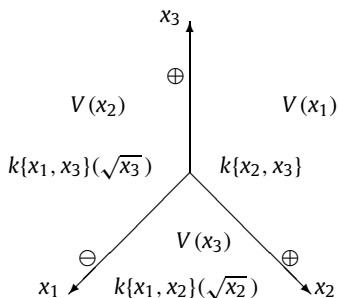
Since Z is of codimension c , it lies on at most c of $V(x_i)$. Thus,

$$\sum_i \left(1 - \frac{1}{e_i}\right) \cdot \text{mult}_Z V(x_i) \leq c - \left(\frac{1}{e_{i_1}} + \frac{1}{e_{i_2}} + \dots + \frac{1}{e_{i_c}}\right).$$

Therefore,

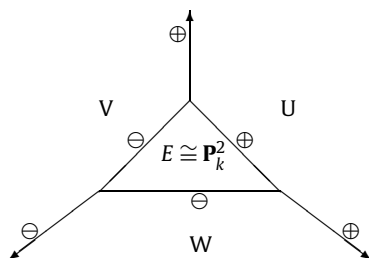
$$\begin{aligned} b(E, X, \alpha) &\geq c - \frac{1}{e_E} - \left\{c - \left(\frac{1}{e_{i_1}} + \dots + \frac{1}{e_{i_c}}\right)\right\} \\ &\geq \frac{1}{e_{i_1}} + \dots + \frac{1}{e_{i_c}} - \frac{1}{e_E} \\ &\geq c \cdot \frac{1}{2} - \frac{1}{e_E} \quad \text{since } e_{i_j} = 1 \text{ or } 2 \\ &> 0 \quad \text{since } e_E = 1 \text{ or } 2 \text{ and } c \geq 3. \quad \square \end{aligned}$$

Example 4.6. Let $X = \text{spec } k\{x_1, x_2, x_3\}$ and let $V(x_1), V(x_2), V(x_3)$ be the hyperplanes defined by $x_1 = 0, x_2 = 0$ and $x_3 = 0$ respectively. Suppose that the function fields of the covers on $V(x_1), V(x_2), V(x_3)$ induced by $\alpha \in \text{Br}_2 k(X)$ are $k\{x_2, x_3\}$, $k\{x_1, x_3\}(\sqrt{x_3})$ and $k\{x_1, x_2\}(\sqrt{x_2})$ respectively. I.e. the cover on $V(x_1)$ is an unramified degree 1 cover whereas the covers on $V(x_2)$ and $V(x_3)$ are 2-sheeted covers ramified on the x_1 -axis, but unramified elsewhere. This is depicted in the figure below.



The negative sign on the x_1 -axis indicates that the covers on $V(x_1)$ and $V(x_3)$ ramify on that axis, while the positive signs on x_2 - and x_3 -axes indicate that no cover ramifies on those axes. We will continue this notation in the succeeding diagrams.

Now suppose X is blown up centered at the origin. The resulting ramifications are as shown in the figure below. (For ramification computations see [11].)



Now we see that $e_1 = 1$, $e_2 = e_3 = 2$ and $e_E = 2$. Thus, by Lemma 4.1,

$$b(E, X, \alpha) = 3 - \frac{1}{2} - \left\{ (1 - 1) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) \right\} = \frac{3}{2} > 0.$$

4.2. Blowing up along a subvariety of codimension 2

Here we show that if X is blown up along a subvariety of codimension 2, then either the Brauer discrepancy of (X, α) along the resulting exceptional divisor is positive, or X can be blown up to obtain X' so that (X', α) is level 1 Brauer terminal.

Proposition 4.7. *Let (X, α) be a geometric Brauer pair with $\Delta_{X, \alpha}$ simple normal crossing and $Z \subseteq X$ be a subvariety of codimension 2. Let E be the f -exceptional divisor where $f : \text{Bl}_Z(X) \rightarrow X$ is the blow-up of X along Z . Then, either (1) $b(E, X, \alpha) > 0$, or (2) X can be blown up to obtain a level 1 Brauer terminal pair (X', α) .*

Proof. Locally, we have $(\text{speck}\{x_1, \dots, x_n\}, \sum_i (1 - \frac{1}{e_i})V(x_i))$ where $n = \dim X$. Since Z is of codimension 2, it lies on at most two of $V(x_i)$. If Z lies on none or exactly one of $V(x_i)$, then by Lemma 4.1,

$$b(E, X, \alpha) = 2 - \frac{1}{e_E}$$

or

$$b(E, X, \alpha) = 2 - \frac{1}{e_E} - \left(1 - \frac{1}{e_i}\right) = 1 - \frac{1}{e_E} + \frac{1}{e_i}$$

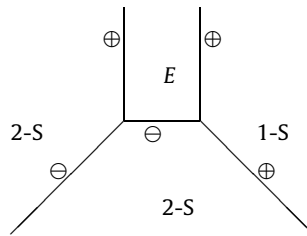
respectively. In either case, we have $b(E, X, \alpha) > 0$, because $e_E, e_i \in \{1, 2\}$.

Now, suppose Z lies on two of $V(x_i)$. Assume, without loss of generality, that Z lies on $V(x_1)$ and $V(x_2)$. Then, again by Lemma 4.1,

$$\begin{aligned} b(E, X, \alpha) &= 2 - \frac{1}{e_E} - \left(1 - \frac{1}{e_1} + 1 - \frac{1}{e_2}\right) \\ &= \frac{1}{e_1} + \frac{1}{e_2} - \frac{1}{e_E}. \end{aligned}$$

Thus, we see that $b(E, X, \alpha)$ is positive except when $e_1 = e_2 = 2$ and $e_E = 1$, in which case $b(E, X, \alpha) = 0$. But this latter situation can occur only when the covers on $V(x_1)$ and $V(x_2)$ are double covers that do not ramify on $V(x_1, x_2)$. (If the double covers ramify, then the induced cover on E must be a double cover which makes $e_E = 2$, a contradiction.) Whenever this situation occurs, we can obtain a new variety X' by blowing up along $V(x_1, x_2)$. The resulting variety X' will not have two prime divisors with double covers on them without the covers ramifying on the intersection of the prime divisors. Therefore, in the new variety this undesirable situation ($e_1 = e_2 = 2$ and $e_E = 1$) does not occur. Hence (X', α) is level 1 Brauer terminal. \square

Example 4.8. Consider the same model we considered in the previous example, but now blow up along the x_3 -axis. Then, we get the model shown below:



Here, by 2-S we mean a 2-sheeted cover and 1-S is a 1-sheeted cover.

Lemma 4.1 gives

$$b(E, X, \alpha) = 2 - \frac{1}{2} - \left(1 - \frac{1}{2}\right) = 1 > 0.$$

4.3. Completion of the proof

In this section we show that if (X, α) is level 1 Brauer terminal, then it is indeed Brauer terminal and complete the proof that any geometric Brauer pair admits a terminal resolution.

In the following lemma, $a(E, X, \Delta)$ denotes the (usual) discrepancy of the logarithmic pair (X, Δ) along an exceptional divisor E over X .

Lemma 4.9. *Let (X, α) be a geometric Brauer pair such that $\Delta_{X, \alpha}$ is simple normal crossing. Let $Z \subseteq X$ be an irreducible subvariety of codimension c , where $c \geq 2$. Suppose $p: B_Z X \rightarrow X$ is the blow-up of X along Z and $E \subseteq B_Z X$ the exceptional divisor. Then, $a(E, X, \Delta_{X, \alpha}) \geq 0$.*

Proof. Étale-locally, we have

$$\Delta_{X, \alpha} = \sum_{i=1}^n \left(1 - \frac{1}{e_i}\right) V(x_i).$$

Applying Lemma 4.2, we see that

$$a(E, X, \Delta_{X,\alpha}) = c - 1 - \sum_{i=1}^n \left(1 - \frac{1}{e_i}\right) \cdot \text{mult}_Z V(x_i).$$

Since Z is of codimension c , it lies on a maximum of c prime divisors $V(x_i)$. Thus,

$$a(E, X, \Delta_{X,\alpha}) \geq c - 1 - c \left(1 - \frac{1}{2}\right) = \frac{c}{2} - 1 \geq 0,$$

since $c \geq 2$. \square

The following lemma can be considered as a Brauer version of the composition of Lemmas 2.29 and 2.30 of [9].

Lemma 4.10. *Let $f : Y \rightarrow X$ be a birational morphism, E is an f -exceptional divisor in Y , E_0 an irreducible subvariety of E . Suppose $g : Z \rightarrow Y$ is the blow-up of Y along E_0 and F the g -exceptional divisor. Suppose $b(E', X, \alpha) \geq 0$ for all f -exceptional divisors E' and $a(F, Y, \Delta_{Y,\alpha}) \geq 0$. Then $b(F, X, \alpha) \geq b(E, X, \alpha)$.*

Proof. Let

$$\Delta_Y = f_*^{-1} \Delta_{X,\alpha} - \sum_{E'} a(E', X, \Delta_{X,\alpha}) E'$$

where the sum runs through all the f -exceptional divisors E' . Then $f_* \Delta_Y = \Delta_{X,\alpha}$ and $K_Y + \Delta_Y \equiv f^*(K_X + \Delta_{X,\alpha})$. Therefore, we can apply Lemma 2.30 of [9], which gives

$$a(F, X, \Delta_{X,\alpha}) = a(F, Y, \Delta_Y).$$

Now, by Lemma 4.2, we get

$$a(F, Y, \Delta_Y) = c - 1 - \left[\sum a_i \cdot \text{mult}_{E_0}(f_*^{-1} D_i) - \sum_{E'} a(E', X, \Delta_{X,\alpha}) \cdot \text{mult}_{E_0} E' \right]$$

where $c = \text{codim}_Y E_0$ and $\Delta_{X,\alpha} = \sum_i a_i D_i$. Now,

$$\Delta_{Y,\alpha} = \sum_i a_i (f_*^{-1} D_i) + \sum_{E'} \left(1 - \frac{1}{e_{E'}}\right) E'$$

where $e_{E'}$ is the degree of the cover on E' induced by $\alpha \in \text{Br}(k(X)) \cong \text{Br}(k(Y))$. Again, by Lemma 4.2,

$$a(F, Y, \Delta_{Y,\alpha}) = c - 1 - \left[\sum a_i \cdot \text{mult}_{E_0}(f_*^{-1} D_i) + \sum_{E'} \left(1 - \frac{1}{e_{E'}}\right) \cdot \text{mult}_{E_0} E' \right].$$

Thus, we get

$$\begin{aligned} a(F, Y, \Delta_Y) - a(F, Y, \Delta_{Y,\alpha}) &= \sum_{E'} \left[a(E', X, \Delta_{X,\alpha}) + \left(1 - \frac{1}{e_{E'}}\right) \right] \text{mult}_{E_0} E' \\ &= \sum_{E'} b(E', X, \alpha) \text{mult}_{E_0} E'. \end{aligned}$$

Since $b(E', X, \alpha) \geq 0$, $b(E, X, \alpha) \geq 0$ and $\text{mult}_{E_0} E = 1$, we have

$$a(F, Y, \Delta_Y) - a(F, Y, \Delta_{Y,\alpha}) \geq b(E, X, \alpha).$$

Since $a(F, Y, \Delta_{Y,\alpha}) \geq 0$ by hypothesis, we get $a(F, Y, \Delta_Y) \geq b(E, X, \alpha)$. But we proved earlier that $a(F, X, \Delta_{X,\alpha}) = a(F, Y, \Delta_Y)$. Thus, $a(F, X, \Delta_{X,\alpha}) \geq b(E, X, \alpha)$. This gives $b(F, X, \alpha) = a(F, X, \Delta_{X,\alpha}) + (1 - \frac{1}{e_F}) \geq b(E, X, \alpha)$. \square

Theorem 4.11. *Any geometric Brauer pair admits a terminal resolution.*

Proof. Let (X, α) be a geometric Brauer pair. Using Hironaka's desingularization theorem [7], we can assume that $\Delta_{X,\alpha}$ is simple normal crossing, where $\Delta_{X,\alpha}$ is the boundary divisor on X induced by α .

If (X, α) is level 1 Brauer terminal then Lemma 4.10 with Lemma 4.9 shows that (X, α) is Brauer terminal.

If (X, α) is not level 1 Brauer terminal, then X can be blown up using Proposition 4.7 to obtain a level 1 Brauer terminal pair. This pair is Brauer terminal, again by Lemmas 4.9 and 4.10. \square

In summary, we have shown that given a geometric Brauer pair (X, α) , we can associate to it a pair (Y, α) with the following properties:

1. Y is nonsingular.
2. There is a birational morphism $f : Y \rightarrow X$.
3. The boundary divisor $\Delta_{Y,\alpha}$ induced on Y by α is simple normal crossing.
4. The Brauer discrepancy of any exceptional divisor over Y is positive.

In short, any Brauer pair (X, α) admits a terminal resolution $(Y, \alpha) \rightarrow (X, \alpha)$.

Remark. Note that in our analysis, we restricted α to be in the 2-torsion part of the Brauer group $\text{Br}k(X)$. If we require α to be in $\text{Br}_3(k(X))$ instead, the analogous statement to the main result we proved here may not be true. For example, consider a 3-fold X with $\alpha \in \text{Br}_3 k(X)$ that induces a simple normal crossing divisor $\Delta_{X,\alpha}$ that has the local form,

$$\frac{1}{3}V(x_1) + \frac{1}{3}V(x_2) + \frac{1}{3}V(x_3).$$

Suppose the 3-sheeted covers on $V(x_1)$ and $V(x_2)$ ramify on $V(x_1, x_2)$, but there is no ramification on $V(x_2, x_3)$ and $V(x_1, x_3)$. This pair (X, α) cannot be improved by blowing up. Now, let E and Y be the exceptional divisor and the variety generated respectively, when X is blown up along $V(x_1, x_3)$. Then,

$$b(E, X, \alpha) = 2 - \frac{1}{3} - \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{1}{3},$$

by Lemma 4.1. Now blow up Y along $E \cap V(x_1)$, and let F be the exceptional divisor generated. Then,

$$b(F, Y, \alpha) = 2 - \frac{1}{e_F} - \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{2}{3} - \frac{1}{e_F}$$

where e_F is the degree of the cover on F induced by α . We can show, using Brauer versions of Lemmas 2.29 and 2.30 of [9], that

$$b(F, X, \alpha) = b(F, Y, \alpha) + b(E, X, \alpha).$$

Then, we get

$$b(F, X, \alpha) = \frac{2}{3} - \frac{1}{e_F} + \frac{1}{3} = 1 - \frac{1}{e_F}.$$

Note that it is not possible to determine the degree e_F of the cover on F induced by α without carrying out a detailed ramification computation involving roots of unity. If it happens that $e_F = 1$, then we get $b(F, X, \alpha) = 0$, indicating that the pair (X, α) may not admit a terminal resolution. However, to determine the degree e_F of the cover definitely, one must carry out ramification computations involving roots of unity.

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