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Dualizing complexes of seminormal affine semigroup rings and toric face rings

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ABSTRACT

We characterize the *seminormality* of an affine semigroup ring in terms of the dualizing complex, and the *normality* of a Cohen–Macaulay semigroup ring by the "shape" of the canonical module. We also characterize the seminormality of a toric face ring in terms of the dualizing complex. A toric face ring is a simultaneous generalization of Stanley–Reisner rings and affine semigroups.

1. Introduction

Let $\mathbf{M}$ be a finitely generated additive submonoid of $\mathbb{Z}^d$ (i.e., $\mathbf{M}$ is an affine semigroup) with $\mathbb{Z}\mathbf{M} \cong \mathbb{Z}^d$, and $\mathcal{C}(\mathbf{M}) := \mathbb{R}_{\geq 0}\mathbf{M} \subset \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ the polyhedral cone spanned by $\mathbf{M}$. Set $\overline{\mathbf{M}} := \mathbb{Z}\mathbf{M} \cap \mathcal{C}(\mathbf{M})$. Throughout the paper, we assume that $\mathbf{M}$ is *positive*, that is, $\mathbf{M}$ has no invertible element except 0.

In the former half of the present paper, we study the affine semigroup ring $\Bbbk[\mathbf{M}] = \bigoplus_{a \in \mathbf{M}} \Bbbk x^a$ of $\mathbf{M}$ over a field $\Bbbk$. Now we have $\dim \Bbbk[\mathbf{M}] = d$. It is a classical result by Hochster, Stanley and Danilov that if $R = \Bbbk[\mathbf{M}]$ is normal (equivalently, $\mathbf{M} = \overline{\mathbf{M}}$),

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then R is Cohen–Macaulay and the canonical module ω_R has an easy description (cf. [3, Theorem 6.3.5]). On the other hand, the behavior of non-normal affine semigroup rings is delicate and complicated, and many works have been done on this subject.

Definition 1.1. Let A be a reduced noetherian commutative ring, and $Q(A)$ its total quotient ring. We say A is *seminormal*, if $a \in Q(A)$ and $a^2, a^3 \in A$ imply $a \in A$.

This notion is much more natural than it seems. In fact, it is known that R is seminormal if and only if $\operatorname{Pic} R \cong \operatorname{Pic}(R[x])$. See [17] and the references cited therein.

The seminormality of an affine semigroup ring $R = \Bbbk[\mathbf{M}]$ is characterized in a combinatorial (resp. homological) way by Reid and Roberts [14] (resp. Bruns, Li and Römer [5]). In the present paper, we will give a new characterization using the dualizing complex. Our characterization is relatively closer to that in [5]. However, contrary to their result, ours does not use the $\mathbb{Z}^d$-grading of the local cohomology modules (or the dualizing complex). To introduce our result, we need some preparation.

For a face F of the cone $\mathcal{C}(\mathbf{M})$, $\mathbf{M}_F := \mathbf{M} \cap F$ is a submonoid of $\mathbf{M}$. The semigroup ring $\Bbbk[\mathbf{M}_F]$ can be seen as a quotient ring of R, and its normalization $\Bbbk[\overline{\mathbf{M}}_F]$ has the natural R-module structure. Then we have the following complex.

$$
\begin{aligned}
{}^+I_R^\bullet &: 0 \longrightarrow {}^+I_R^{-d} \longrightarrow {}^+I_R^{-d+1} \longrightarrow \cdots \longrightarrow {}^+I_R^0 \longrightarrow 0,\\
{}^+I_R^{-i} &= \bigoplus_{\substack{F\colon \text{a face of } \mathcal{C}(\mathbf{M}) \\ \dim F = i}} \Bbbk[\overline{\mathbf{M}}_F].
\end{aligned}
$$

The differential map $\partial : {}^+I_R^{-i} \to {}^+I_R^{-i+1}$ is the combination of the natural surjections $\Bbbk[\overline{\mathbf{M}}_F] \twoheadrightarrow \Bbbk[\overline{\mathbf{M}}_G]$ for faces F, G with $F \supset G$ and $\dim F = \dim G + 1$.

Proposition 2.3. *For a semigroup ring $R = \Bbbk[\mathbf{M}]$, it is seminormal if and only if ${}^+I_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$.*

We can characterize the normality of $\Bbbk[\mathbf{M}]$ using the dualizing complex in a similar way. As a byproduct of this observation, we have the following (unexpected) result.

Theorem 3.1. *For $R = \Bbbk[\mathbf{M}]$, the following are equivalent.*

(a) *R is normal.*
(b) *R is Cohen–Macaulay and the canonical module ω_R is isomorphic to the ideal $(x^a \mid a \in \mathbf{M} \cap \operatorname{int}(\mathcal{C}(\mathbf{M})))$ of R as (graded or nongraded) R-modules.*

The implication (a) $\Rightarrow$ (b) is a classical result (see above).

Stanley–Reisner rings and affine semigroup rings are important subjects of combinatorial commutative algebra. The notion of *toric face rings*, which originated in an earlier work of Stanley [16], generalizes both of them, and has been studied by Bruns, Römer, and their coauthors (e.g. [2,4,9]). Roughly speaking, to make a toric face ring $\Bbbk[\mathcal{M}]$ from a (locally) polyhedral CW complex $\mathcal{X}$, we assign each cell $\sigma \in \mathcal{X}$ an affine semigroup $\mathbf{M}_\sigma \subset \mathbb{Z}^{\dim \sigma+1}$, and "glue" their semigroup rings $\Bbbk[\mathbf{M}_\sigma]$ along with $\mathcal{X}$.

Recently, Nguyen [12] studied seminormal toric face rings mainly focusing on the local cohomology modules, but he also remarked that $\Bbbk[\mathcal{M}]$ is seminormal if and only if $\Bbbk[\mathbf{M}_\sigma]$ is seminormal for all σ. In this sense, the seminormality is a natural condition for toric face rings.

Generalizing the construction for affine semigroup rings, a toric face ring $\Bbbk[\mathcal{M}]$ of dimension d admits the cochain complex ${}^+I_R^\bullet$ of the form

$$0 \longrightarrow {}^+I_R^{-d} \longrightarrow {}^+I_R^{-d+1} \longrightarrow \cdots \longrightarrow {}^+I_R^0 \longrightarrow 0$$

with

$${}^+I_R^{-i} := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} \Bbbk[\overline{\mathbf{M}}_\sigma],$$

where $\Bbbk[\overline{\mathbf{M}}_\sigma]$ is the normalization of $\Bbbk[\mathbf{M}_\sigma]$.

Theorem 5.2. *If a toric face ring $R = \Bbbk[\mathcal{M}]$ is seminormal, then ${}^+I_R^\bullet$ is quasi-isomorphic to a dualizing complex $D_R^\bullet$. (The converse is also true. See Proposition 5.12.)*

Under the assumption that each $\Bbbk[\mathcal{M}_\sigma]$ is normal (of course, ${}^+I_R^{-i} = \bigoplus_{\dim \sigma = i-1} \Bbbk[\mathbf{M}_\sigma]$, in this case), the above theorem was proved by the present author and Okazaki [13, Theorem 5.2]. Even in this case, the proof requires quite technical argument, since R is not a graded ring in the usual sense. The proof of Theorem 5.2 heavily depends on [13, Theorem 5.2], but we have to make more effort.

Finally, for an arbitrary toric face ring $R = \Bbbk[\mathcal{M}]$, we study the local cohomology modules $H^i_\mathfrak{m}(R)$ at the "graded" maximal ideal $\mathfrak{m}$. Let ${}^+R$ (resp. $\widetilde{R}$) be the seminormalization (resp. cone-wise normalization) of R. Both of them are toric face rings supported by the same CW complex $\mathcal{X}$ as R, but the construction of the latter is not straightforward (see Example 5.3). In Section 6, we show that $H^i_\mathfrak{m}({}^+R) \subset H^i_\mathfrak{m}(R)$, and $H^i_\mathfrak{m}(\widetilde{R}) \neq 0$ implies $H^i_\mathfrak{m}(R) \neq 0$. Hence we have;

$$R \text{ is Cohen–Macaulay} \implies {}^+R \text{ is Cohen–Macaulay} \implies \widetilde{R} \text{ is Cohen–Macaulay.}$$

We remark that the Cohen–Macaulay property of $\widetilde{R}$ only depends on the topology of the underlying space of $\mathcal{X}$ (and $\mathrm{char}(\Bbbk)$).

Convention. In this paper, we use the following notation: For a commutative ring A, $\mathrm{Mod}\, A$ denotes the category of A-modules.

For cochain complexes $M^\bullet$ and $N^\bullet$, $M^\bullet \cong N^\bullet$ means that two complexes are isomorphic in the derived category, and $M^\bullet = N^\bullet$ means that these are isomorphic as (explicit) complexes. If $M^\bullet \cong N^\bullet$, we say these two complexes are *quasi-isomorphic* (especially when a direct quasi-isomorphism $M^\bullet \to N^\bullet$ or $N^\bullet \to M^\bullet$ exists).

While the word "dualizing complex" sometimes means its isomorphism class in the derived category, we use the convention that a dualizing complex $D_A^\bullet$ of a noetherian ring A is the one of the form

$$0 \longrightarrow D_A^{-\dim A} \longrightarrow \cdots \longrightarrow D_A^{-1} \longrightarrow D_A^0 \longrightarrow 0$$

with

$$D_A^{-i} = \bigoplus_{\substack{\mathfrak{p}\in \operatorname{Spec} A \\ \dim A/\mathfrak{p}=i}} E(A/\mathfrak{p}), \tag{1.1}$$

where $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$.

In this paper, we freely use the $\mathbb{Z}^d$-graded versions of Matlis duality and local duality. These are implicit in Chapters 5 and 6 of [3], but the detailed argument is found in [7].

2. Dualizing complexes of seminormal affine semigroup rings

For the convention and notation about an affine semigroup $\mathbf{M} \subset \mathbb{Z}^d$ and the cone $\mathcal{C}(\mathbf{M}) \subset \mathbb{R}^d$ spanned by $\mathbf{M}$, see the end of the previous section.

Let

$$\Bbbk[\mathbf{M}] := \bigoplus_{a\in\mathbf{M}} \Bbbk x^a \subset \Bbbk\left[x_1^{\pm 1}, \dots, x_d^{\pm 1}\right]$$

be the semigroup ring of $\mathbf{M}$ over a field $\Bbbk$. Here, for $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$, x^a denotes the monomial $\prod_{i=1}^d x_i^{a_i}$. Clearly, $R := \Bbbk[\mathbf{M}]$ is a $\mathbb{Z}^d$-graded ring, and ${}^*\mathrm{Mod}R$ denotes the category of $\mathbb{Z}^d$-graded R-modules.

For $M = \bigoplus_{a\in\mathbb{Z}^d} M_a \in {}^*\mathrm{Mod}R$, set

$$M_{\mathcal{C}(\mathbf{M})} := \bigoplus_{a\in\mathbb{Z}^d\cap\mathcal{C}(\mathbf{M})} M_a.$$

It is clear that $M_{\mathcal{C}(\mathbf{M})}$ is a $\mathbb{Z}^d$-graded R-submodule of M, and we call it the $\mathcal{C}(\mathbf{M})$-*graded part* of M. Similarly, for a cochain complex $M^\bullet$ in ${}^*\mathrm{Mod}R$, we can define a subcomplex $(M^\bullet)_{\mathcal{C}(\mathbf{M})}$.

For a face F of $\mathcal{C}(\mathbf{M})$,

$$\mathbf{M}_F := \mathbf{M} \cap F$$

is a submonoid of $\mathbf{M}$. Consider the monomial ideal (i.e., $\mathbb{Z}^d$-graded ideal)

$$\mathfrak{p}_F := \left(x^a \mid a \in \mathbf{M} \setminus \mathbf{M}_F\right)$$

of R. Since $R/\mathfrak{p}_F$ is isomorphic to the affine semigroup ring $\Bbbk[\mathbf{M}_F]$ of $\mathbf{M}_F$, $\mathfrak{p}_F$ is a prime ideal. Conversely, any monomial prime ideal coincide with $\mathfrak{p}_F$ for some F. We regard $\Bbbk[\mathbf{M}_F]$ as an R-module through $R/\mathfrak{p}_F \cong \Bbbk[\mathbf{M}_F]$.

For a face F of $\mathcal{C}(\mathbf{M})$, $T_F := \{x^a \mid a \in \mathbf{M}_F\} \subset R$ is a multiplicatively closed subset. So we have the localization $T_F^{-1}R$ of R by T_F. The *Cěch complex* $\check{C}_R^\bullet$ is defined as follows:

$$\check{C}_R^\bullet : 0 \longrightarrow \check{C}_R^0 \xrightarrow{\partial} \check{C}_R^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \check{C}_R^d \longrightarrow 0,$$

where

$$\check{C}_R^i := \bigoplus_{\substack{F\colon \text{a face of } \mathcal{C}(\mathbf{M}) \\ \dim F = i}} T_F^{-1}R.$$

The differential map $\partial : \check{C}_R^i \to \check{C}_R^{i+1}$ is given by

$$\partial(x) = \sum_{\substack{G \supset F \\ \dim G = i+1}} \varepsilon(G,F) \cdot \iota_{G,F}(x),$$

where $\iota_{G,F}$ is the natural injection $T_F^{-1}R \longrightarrow T_G^{-1}R$ for $G \supset F$, and $\varepsilon(G,F)$ is the incidence function of the regular CW complex given by a cross section of $\mathcal{C}(\mathbf{M})$. The precise information on $\varepsilon(G,F)$ is found in [3, §6.2], and we will use this function later in a more general situation. Here we just remark that $\varepsilon(G,F) = \pm 1$ for all F, G with $G \supset F$ and $\dim G = \dim F + 1$, and this sign makes $\check{C}_R^\bullet$ a cochain complex.

As shown in [3, Theorem 6.2.5], the local cohomology module $H_{\mathfrak{m}}^i(R)$ at the graded maximal ideal $\mathfrak{m} := (x^a \mid 0 \neq a \in \mathbf{M})$ is isomorphic to $H^i(\check{C}_R^\bullet)$ in ${}^*\mathrm{Mod}R$. Moreover, $\check{C}_R^\bullet$ is a ($\mathbb{Z}^d$-graded) flat resolution of $\mathbf{R}\Gamma_{\mathfrak{m}}R$.

The $\mathbb{Z}^d$-graded Matlis dual $(T_F^{-1}R)^\vee$ of $T_F^{-1}R$ is of the form

$$\left(T_F^{-1}R\right)^\vee = \bigoplus_{a \in \mathbf{M}_F - \mathbf{M}} \Bbbk e_a,$$

where e_a is a basis element with the degree a, and

$$\mathbf{M}_F - \mathbf{M} = \{b - c \mid b \in \mathbf{M}_F \text{ and } c \in \mathbf{M}\}.$$

The multiplication map $x^a \times (-) : [(T_F^{-1}R)^\vee]_b \longrightarrow [(T_F^{-1}R)^\vee]_{a+b}$ is surjective for all $a \in \mathbf{M}$ and $b \in \mathbb{Z}^d$. By the flatness of $T_F^{-1}R$ and [11, Lemma 11.16], $(T_F^{-1}R)^\vee$ is an injective object in ${}^*\mathrm{Mod}R$, moreover, it is the injective envelope ${}^*E(\Bbbk[\mathbf{M}_F])$ of $\Bbbk[\mathbf{M}_F] = R/\mathfrak{p}_F$ in ${}^*\mathrm{Mod}R$.

The $\mathbb{Z}^d$-graded Matlis dual $J_R^\bullet := (\check{C}_R^\bullet)^\vee$ of $\check{C}_R^\bullet$ is of the form

$$J_R^\bullet : 0 \longrightarrow J_R^{-d} \longrightarrow J_R^{-d+1} \longrightarrow \cdots \longrightarrow J_R^0 \longrightarrow 0,$$
$$J_R^{-i} = \bigoplus_{\substack{F:\text{ a face of } \mathcal{C}(\mathbf{M}) \\ \dim F = i}} {}^*E\bigl(\Bbbk[\mathbf{M}_F]\bigr).$$

The differential map $\partial : J_R^{-i} \to J_R^{-i+1}$ is given by

$$\partial(x) = \sum_{\substack{G \subset F \\ \dim G = i-1}} \varepsilon(F, G) \cdot p_{G,F}(x)$$

for $x \in {}^*E[\mathbf{M}_F] \subset J_R^{-i}$. Here $p_{G,F} : {}^*E(\Bbbk[\mathbf{M}_F]) \to {}^*E(\Bbbk[\mathbf{M}_G])$ is the Matlis dual of $\iota_{F,G}$, and also induced by the map $\Bbbk[\mathbf{M}_F] \to {}^*E(\Bbbk[\mathbf{M}_G])$ which is the composition of the natural surjection $\Bbbk[\mathbf{M}_F] \twoheadrightarrow \Bbbk[\mathbf{M}_G]$ and the inclusion $\Bbbk[\mathbf{M}_G] \hookrightarrow {}^*E(\Bbbk[\mathbf{M}_G])$.

As is well-known, $J_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$ of R, moreover, it is nothing other than the dualizing complex of R in the $\mathbb{Z}^d$-graded context (see [15, Proposition 4.4], also [10]).

For a face F of the polyhedral cone $\mathcal{C}(\mathbf{M})$, we regard

$$\Bbbk[\mathbb{Z}\mathbf{M}_F \cap F] := \bigoplus_{b \in \mathbb{Z}\mathbf{M}_F \cap F} \Bbbk x^b$$

as a $\mathbb{Z}^d$-graded R-module by

$$x^a x^b = \begin{cases} x^{a+b} & \text{if } a \in \mathbf{M}_F, \\ 0 & \text{otherwise,} \end{cases}$$

for $x^a \in R = \Bbbk[\mathbf{M}]$ and $x^b \in \Bbbk[\mathbb{Z}\mathbf{M}_F \cap F]$. Note that $\Bbbk[\mathbb{Z}\mathbf{M}_F \cap F]$ is the normalization of $\Bbbk[\mathbf{M}_F]$, and

$${}^*E\bigl(\Bbbk[\mathbf{M}_F]\bigr)_{\mathcal{C}(\mathbf{M})} \cong \Bbbk[\mathbb{Z}\mathbf{M}_F \cap F]$$

as R-modules. Let F, G be faces of $\mathcal{C}(\mathbf{M})$ with $F \supset G$. As R-modules, $\Bbbk[\mathbb{Z}\mathbf{M}_G \cap G]$ is a quotient module of $\Bbbk[\mathbb{Z}\mathbf{M}_F \cap F]$ (note that $\mathbb{Z}\mathbf{M}_G$ is a sublattice of $\mathbb{Z}\mathbf{M}_F \cap G$). Hence there is the $\mathbb{Z}^d$-graded surjection $\pi_{G,F} : \Bbbk[\mathbb{Z}\mathbf{M}_F \cap F] \longrightarrow \Bbbk[\mathbb{Z}\mathbf{M}_G \cap G]$, which is the $\mathcal{C}(\mathbf{M})$-graded part of $p_{G,F}$ (if $\dim G = \dim F - 1$).

Hence the $\mathcal{C}(\mathbf{M})$-graded part

$${}^+I_R^\bullet := \bigl(J_R^\bullet\bigr)_{\mathcal{C}(\mathbf{M})}$$

of the complex $J_R^\bullet$ is of the form

$$
\begin{aligned}
{}^{+}I_R^{\bullet} &: 0 \longrightarrow {}^{+}I_R^{-d} \longrightarrow {}^{+}I_R^{-d+1} \longrightarrow \cdots \longrightarrow {}^{+}I_R^{0} \longrightarrow 0, \\
{}^{+}I_R^{-i} &= \bigoplus_{\substack{F\colon \text{a face of } \mathcal{C}(\mathbf{M}) \\ \dim F = i}} \Bbbk[\mathbb{Z}\mathbf{M}_F \cap F].
\end{aligned}
$$

The differential map $\partial : {}^{+}I_R^{-i} \to {}^{+}I_R^{-i+1}$ is given by

$$
\partial(x) = \sum_{\substack{G \subset F \\ \dim G = i-1}} \varepsilon(F, G) \cdot \pi_{G,F}(x),
$$

for $x \in \Bbbk[\mathbb{Z}\mathbf{M}_F \cap F] \subset {}^{+}I_R^{-i}$.

As is well-known, $R = \Bbbk[\mathbf{M}]$ is normal if and only if $\mathbf{M} = \overline{\mathbf{M}} := \mathbb{Z}\mathbf{M} \cap \mathcal{C}(\mathbf{M})$. We can characterize the seminormality of R in a similar way. For a face F of $\mathcal{C}(\mathbf{M})$, $\operatorname{int}(F)$ denotes its relative interior. Clearly,

$$
\mathcal{C}(\mathbf{M}) = \bigsqcup_{F\colon \text{a face of } \mathcal{C}(\mathbf{M})} \operatorname{int}(F).
$$

Set

$$
{}^{+}\mathbf{M} := \bigsqcup_{F\colon \text{a face of } \mathcal{C}(\mathbf{M})} \mathbb{Z}\mathbf{M}_F \cap \operatorname{int}(F). \tag{2.1}
$$

Then ${}^{+}\mathbf{M}$ is an affine semigroup with $\mathbf{M} \subseteq {}^{+}\mathbf{M} \subseteq \overline{\mathbf{M}}$ and ${}^{+}({}^{+}\mathbf{M}) = {}^{+}\mathbf{M}$.

Theorem 2.1. *(See L. Reid and L.G. Roberts [14], Bruns, Li and Römer [5].) For an affine semigroup ring $R = \Bbbk[\mathbf{M}]$, the following are equivalent.*

- (i) *R is seminormal.*
- (ii) $\mathbf{M} = {}^{+}\mathbf{M}$.
- (iii) *$H^i_{\mathfrak{m}}(R)_a \neq 0$ for $a \in \mathbb{Z}^d$ implies $-a \in \mathcal{C}(\mathbf{M})$.*

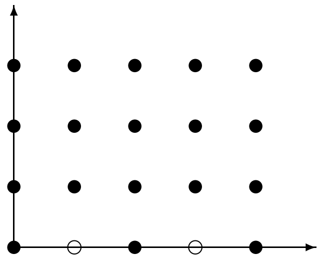
Hence ${}^{+}R := \Bbbk[{}^{+}\mathbf{M}]$ is the seminormalization of $R = \Bbbk[\mathbf{M}]$.

In the above theorem, the equivalence between (i) and (ii) (resp. (i) and (iii)) is [14, Theorem 4.3] (resp. [5, Theorem 4.7]).

Example 2.2. For the additive submonoid

$$
\mathbf{M} = \{(m, n) \mid m \geq 0, n \geq 1\} \cup \{(2m, 0) \mid m \geq 0\}
$$

of $\mathbb{N}^2$, $\Bbbk[\mathbf{M}]$ is seminormal, but not normal.



Proposition 2.3. *If $R = \Bbbk[\mathbf{M}]$ is seminormal, then ${}^+I_R^\bullet$ is isomorphic to the $\mathbb{Z}^d$-graded dualizing complex $J_R^\bullet$ in the derived category $\mathsf{D}^b({}^*\mathrm{Mod}R)$, hence ${}^+I_R^\bullet \cong D_R^\bullet$ in $\mathsf{D}^b(\mathrm{Mod}\, R)$. Conversely, if ${}^+I_R^\bullet \cong D_R^\bullet$ in $\mathsf{D}^b(\mathrm{Mod}\, R)$ then R is seminormal.*

Proof. We start from the proof of the first assertion. Since $H_{\mathfrak{m}}^i(R)^\vee \cong H^{-i}(J_R^\bullet)$ by the local duality theorem, $H^i(J_R^\bullet)_a \neq 0$ implies $a \in \mathcal{C}(\mathbf{M})$ by Theorem 2.1. Hence the $\mathcal{C}(\mathbf{M})$-graded part ${}^+I_R^\bullet$ of $J_R^\bullet$ is quasi-isomorphic to $J_R^\bullet$ itself.

Next, we show the last assertion. For the seminormalization ${}^+R$ of R, the explicit computation gives the isomorphism ${}^+I_R^\bullet = {}^+I_{{}^+R}^\bullet$ as cochain complexes of R-modules. We just shown that ${}^+I_{{}^+R}^\bullet \cong D_{{}^+R}^\bullet$ in $\mathsf{D}^b(\mathrm{Mod}\, {}^+R)$. Hence ${}^+I_{{}^+R}^\bullet \cong D_{{}^+R}^\bullet$ also in $\mathsf{D}^b(\mathrm{Mod}\, R)$. Since ${}^+R$ is a finitely generated R-module, $\mathrm{Hom}_R^\bullet({}^+I_{{}^+R}^\bullet, D_R^\bullet) \cong {}^+R$ in $\mathsf{D}^b(\mathrm{Mod}\, R)$. Clearly, we also have $\mathrm{Hom}_R^\bullet({}^+I_R^\bullet, D_R^\bullet) \cong R$. So taking the functor $\mathrm{Hom}_R^\bullet(-, D_R^\bullet)$ to ${}^+I_R^\bullet = {}^+I_{{}^+R}^\bullet$, we have $R \cong {}^+R$ as R-modules. It means that $R = {}^+R$, and hence R is seminormal. $\square$

3. The normality and the canonical module of an affine semigroup ring

Consider the following subcomplex of ${}^+I_R^\bullet$:

$$I_R^\bullet : 0 \longrightarrow I_R^{-d} \longrightarrow I_R^{-d+1} \longrightarrow \cdots \longrightarrow I_R^0 \longrightarrow 0,$$
$$I_R^{-i} = \bigoplus_{\substack{F:\text{ a face of } \mathcal{C}(\mathbf{M}) \\ \dim F = i}} \Bbbk[\mathbf{M}_F].$$

If R is normal, then $\Bbbk[\mathbf{M}_F]$ is normal for all F and $I_R^\bullet = {}^+I_R^\bullet$. Hence, in this case, $I_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$. This is a well-known result essentially appears in [3, §6.3]. The next result states that the converse also holds.

Theorem 3.1. *For an affine semigroup ring $R = \Bbbk[\mathbf{M}]$, the following are equivalent.*

(i) *R is normal.*
(ii) *The complex $I_R^\bullet$ is quasi-isomorphic to the dualizing complex $D_R^\bullet$.*
(iii) *R is Cohen–Macaulay and the canonical module ω_R is isomorphic to the ideal $W_R := (x^a \mid a \in \mathbf{M} \cap \mathrm{int}(\mathcal{C}(\mathbf{M})))$ of R in $\mathrm{Mod}\, R$.*

The implication (i) ⇒ (iii) is a classical result due to Hochster, Stanley and Danilov. Note that if R is normal then $\omega_R \cong W_R$ even in ${}^*\mathrm{Mod}R$.

Proof. (i) ⇒ (ii): We have mentioned above.

(ii) ⇒ (iii): The assertion follows form direct computation similar to the proof of [3, Theorem 6.3.4] (but we have to take the $\mathbb{Z}^d$-graded Matlis dual).

(iii) ⇒ (i): Since W_R and ω_R are $\mathbb{Z}^d$-graded modules, $\mathrm{Hom}_R(W_R, \omega_R)$ has the natural $\mathbb{Z}^d$-grading. On the other hand, since $W_R \cong \omega_R$ in $\mathrm{Mod}\,R$ now, we have $\mathrm{Hom}_R(W_R, \omega_R) \cong R$ in $\mathrm{Mod}\,R$. Since the unit group of R is $\Bbbk \setminus \{0\}$, the way to equip the (ungraded) module R with a $\mathbb{Z}^d$-grading is unique up to a shift. Hence there is $a \in \mathbb{Z}^d$ such that $\mathrm{Hom}_R(W_R, \omega_R) \cong R(-a)$ in ${}^*\mathrm{Mod}R$. We use a in this meaning throughout this proof.

By [3, Proposition 3.3.18], R/W_R is a Gorenstein ring of dimension $d-1$ and $\mathrm{Ext}^1_R(R/W_R, \omega_R) \cong R/W_R$ in $\mathrm{Mod}\,R$. By an argument similar to the above, these are isomorphic even in ${}^*\mathrm{Mod}R$ up to a degree shift. Since $\mathrm{Hom}_R(W_R, \omega_R) \cong R(-a)$ in ${}^*\mathrm{Mod}R$, the short exact sequence $0 \longrightarrow W_R \longrightarrow R \longrightarrow R/W_R \longrightarrow 0$ yields

$$\mathrm{Ext}^1_R(R/W_R, \omega_R) \cong (R/W_R)(-a). \tag{3.1}$$

Note that $J^\bullet_{R/W_R} := \mathrm{Hom}^\bullet_R(R/W_R, J^\bullet_R)$ is the $\mathbb{Z}^d$-graded dualizing complex of R/W_R, and

$$H^{-d+1}\big(J^\bullet_{R/W_R}\big) \cong \mathrm{Ext}^1_R(R/W_R, \omega_R) \tag{3.2}$$

in ${}^*\mathrm{Mod}R$. Since

$$\mathrm{Hom}_R\big(R/W_R, {}^*E\big(\Bbbk[\mathbf{M}_F]\big)\big) = \begin{cases} 0 & \text{if } F = \mathcal{C}(\mathbf{M}), \\ {}^*E(\Bbbk[\mathbf{M}_F]) & \text{if } F \text{ is a proper face of } \mathcal{C}(\mathbf{M}), \end{cases}$$

$J^\bullet_{R/W_R}$ coincides with the brutal truncation $J^{>-d}_R$ of $J^\bullet_R$ (for this assertion, we do not use any assumption on $R = \Bbbk[\mathbf{M}]$).

Let ${}^+R = \Bbbk[{}^+\mathbf{M}]$ be the seminormalization of R. Since

$$\big(J^i_{R/W_R}\big)_{\mathcal{C}(\mathbf{M})} = \big(J^i_R\big)_{\mathcal{C}(\mathbf{M})} = {}^+I^i_{{}^+R}$$

for all $i > -d$, we have

$$\big(J^\bullet_{{}^+R/W_{{}^+R}}\big)_{\mathcal{C}(\mathbf{M})} = {}^+I^{>-d}_{{}^+R} = \big(J^\bullet_{R/W_R}\big)_{\mathcal{C}(\mathbf{M})},$$

where $J^\bullet_{{}^+R/W_{{}^+R}}$ is the $\mathbb{Z}^d$-graded dualizing complex of ${}^+R/W_{{}^+R}$. Hence we have

$$\big[H^{-d+1}\big(J^\bullet_{R/W_R}\big)\big]_{\mathcal{C}(\mathbf{M})} \cong \big[H^{-d+1}\big(J^\bullet_{{}^+R/W_{{}^+R}}\big)\big]_{\mathcal{C}(\mathbf{M})} \cong \big[\mathrm{Ext}^1_R\big({}^+R/W_{{}^+R}, \omega_R\big)\big]_{\mathcal{C}(\mathbf{M})}.$$

If ${}^+R$ is *normal*, then $W_{{}^+R}$ is its canonical module, and

$$\big[H^{-d+1}\big(J^\bullet_{R/W_R}\big)\big]_{\mathcal{C}(\mathbf{M})} \cong \operatorname{Ext}^1_R({}^+R/W_{{}^+R}, \omega_R) \cong {}^+R/W_{{}^+R}.$$

In general, there might be gap between $[H^{-d+1}(J^\bullet_{R/W_R})]_{\mathcal{C}(\mathbf{M})}$ and ${}^+R/W_{{}^+R}$, but an easy computation shows that $H^{-d+1}(J^\bullet_{R/W_R})$ still contains a submodule which is isomorphic to ${}^+R/W_{{}^+R}$ in ${}^*\mathrm{Mod}R$. (Note that $[H^{-d+1}(J^\bullet_{R/W_R})]_{\mathcal{C}(\mathbf{M})}$ is isomorphic to the kernel of $\partial : {}^+I^{-d+1}_{{}^+R} \to {}^+I^{-d+2}_{{}^+R}$.) Combining this fact with (3.1) and (3.2), we have a $\mathbb{Z}^d$-graded injection

$${}^+R/W_{{}^+R} \hookrightarrow (R/W_R)(-a).$$

This implies that $a = 0$, and hence $W_R \cong \omega_R$ in ${}^*\mathrm{Mod}R$. Since $H^d_{\mathfrak{m}}(R)_b (= (\omega_R)_{-b} = (W_R)_{-b}) \neq 0$ implies $b \in -\mathcal{C}(M)$, R is seminormal by Theorem 2.1.

Since R is seminormal, we have

$$\mathbf{M} \cap \operatorname{int}\big(\mathcal{C}(\mathbf{M})\big) = \mathbb{Z}\mathbf{M} \cap \operatorname{int}\big(\mathcal{C}(\mathbf{M})\big) = \overline{\mathbf{M}} \cap \operatorname{int}\big(\mathcal{C}(\mathbf{M})\big),$$

and W_R coincides with the canonical module $\omega_{\overline{R}}$ $(= W_{\overline{R}})$ of $\overline{R}$, where $\overline{R} = \Bbbk[\overline{\mathbf{M}}]$ with $\overline{\mathbf{M}} = \mathbb{Z}\mathbf{M} \cap \mathcal{C}(\mathbf{M})$ is the normalization of R. Hence we have

$$\overline{R} \cong \operatorname{Hom}_R(\omega_{\overline{R}}, \omega_R) = \operatorname{Hom}_R(W_R, \omega_R) \cong \operatorname{Hom}_R(\omega_R, \omega_R) \cong R$$

in $\mathrm{Mod}\, R$. Hence $\overline{R} \cong R$ and R is normal. $\square$

Remark 3.2. Let $\overline{R} = \Bbbk[\overline{\mathbf{M}}]$ be the normalization of $R = \Bbbk[\mathbf{M}]$. For a face F of $\mathcal{C}(\mathbf{M})$, $\mathbb{Z}\mathbf{M}_F$ is a sublattice of $\mathbb{Z}\overline{\mathbf{M}}_F$, and hence $\Bbbk[\mathbb{Z}\mathbf{M}_F \cap F]$ is a direct summand of $\Bbbk[\overline{\mathbf{M}}_F]$ as an R-module. So ${}^+I^i_R$ is a submodule (actually, a direct summand) of $I^i_{\overline{R}}$ for each i, but it does *not* mean ${}^+I^\bullet_R$ is a subcomplex of $I^\bullet_{\overline{R}}$.

For example, consider the seminormal semigroup $\mathbf{M}$ given in Example 2.2. Then R is of the form $\Bbbk[x^2, y, xy]$. In this case, ${}^+I^{-2}_R = \Bbbk[x,y]$, ${}^+I^{-1}_R = \Bbbk[x^2] \oplus \Bbbk[y]$, and the degree $(1,0)$ component of $\partial : {}^+I^{-2}_R \to {}^+I^{-1}_R$ is the zero map. On the other hand, the normalization $\overline{R}$ of R is $\Bbbk[x,y]$. Hence ${}^+I^{-2}_{\overline{R}} = \Bbbk[x,y]$, ${}^+I^{-1}_{\overline{R}} = \Bbbk[x] \oplus \Bbbk[y]$, and the degree $(1,0)$ component of $\partial : {}^+I^{-2}_{\overline{R}} \to {}^+I^{-1}_{\overline{R}}$ is non-zero.

Anyway, this phenomena makes the proof of Theorem 5.2 below complicated.

4. Preliminaries on toric face rings

Let $\mathcal{X}$ be a finite regular CW complex with the intersection property, and X its underlying topological space. More precisely, the following conditions are satisfied.

(1) $\emptyset \in \mathcal{X}$ (for the convenience, we set $\dim \emptyset = -1$), $X = \bigcup_{\sigma \in \mathcal{X}} \sigma$, and the cells $\sigma \in \mathcal{X}$ are pairwise disjoint;
(2) If $\emptyset \neq \sigma \in \mathcal{X}$, then, for some $i \in \mathbb{N}$, there exists a homeomorphism from the i-dimensional ball $\{x \in \mathbb{R}^i \mid \|x\| \leq 1\}$ to the closure $\bar{\sigma}$ of σ which maps $\{x \in \mathbb{R}^i \mid \|x\| < 1\}$ onto σ;
(3) For $\sigma \in \mathcal{X}$, the closure $\bar{\sigma}$ is the union of some cells in $\mathcal{X}$;
(4) For $\sigma, \tau \in \mathcal{X}$, there is a cell $\upsilon \in \mathcal{X}$ such that $\bar{\upsilon} = \bar{\sigma} \cap \bar{\tau}$ (here υ can be $\emptyset$).

We regard $\mathcal{X}$ as a partially ordered set (*poset* for short) by $\sigma \geq \tau \overset{\text{def}}{\Longleftrightarrow} \bar{\sigma} \supset \tau$.

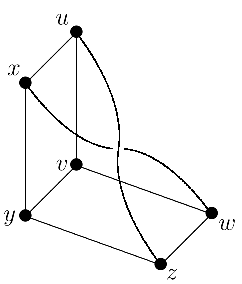
The following definitions of conical complexes and monoidal complexes are taken from [13], and equivalent to the original ones in Bruns, Koch and Römer [4] under the assumption that the cones C_σ contain no line (equivalently, the semigroups $\mathbf{M}_\sigma$ are all positive). However, the notation has been changed a little from that of [13] for the usages in the present paper.

Definition 4.1. A *conical complex* $(\Sigma, \mathcal{X}, \{\iota_{\sigma,\tau}\})$ on $\mathcal{X}$ consists of the following data.

(0) To each $\sigma \in \mathcal{X}$, we assign an Euclidean space $\mathbf{E}_\sigma = \mathbb{R}^{\dim \sigma + 1}$.
(1) $\Sigma = \{C_\sigma \mid \sigma \in \mathcal{X}\}$, where $C_\sigma \subset \mathbf{E}_\sigma = \mathbb{R}^{\dim \sigma + 1}$ is a polyhedral cone with $\dim C_\sigma = \dim \sigma + 1$. Here each cone C_σ contains no line.
(2) The injection $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$ satisfying the following.
   (a) $\iota_{\sigma,\tau}$ can be lifted to a linear map $\tilde{\iota}_{\sigma,\tau} : \mathbf{E}_\tau \to \mathbf{E}_\sigma$.
   (b) The image $\iota_{\sigma,\tau}(C_\tau)$ is a face of C_σ. Conversely, for a face C' of C_σ, there is a *sole* cell τ with $\tau \leq \sigma$ such that $\iota_{\sigma,\tau}(C_\tau) = C'$.
   (c) $\iota_{\sigma,\sigma} = \mathrm{Id}_{C_\sigma}$ and $\iota_{\sigma,\tau} \circ \iota_{\tau,\upsilon} = \iota_{\sigma,\upsilon}$ for $\sigma, \tau, \upsilon \in \mathcal{X}$ with $\sigma \geq \tau \geq \upsilon$.

A polyhedral fan Σ in $\mathbb{R}^n$ gives a conical complex. In this case, as an underlying CW complex, we can take $\{\mathrm{int}(C \cap \mathbb{S}^{n-1}) \mid C \in \Sigma\}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, and the injections $\iota_{\sigma,\tau}$ are inclusion maps.

Example 4.2. Consider the following cell decomposition of a Möbius strip. Regarding each rectangles as the cross-sections of 3-dimensional cones, we have a conical complex that is not a fan (see [2, Example 1.36]).



Let $\mathbf{L}_\sigma$ be the set of lattice points $\mathbb{Z}^{\dim\sigma+1}$ of $\mathbf{E}_\sigma = \mathbb{R}^{\dim\sigma+1}$. Assume that $\tilde{\iota}_{\sigma,\tau}(\mathbf{L}_\tau) = \tilde{\iota}_{\sigma,\tau}(\mathbf{E}_\tau) \cap \mathbf{L}_\sigma$ for all $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$.

Definition 4.3. A *monoidal complex* supported by a conical complex $(\Sigma, \mathcal{X}, \{\iota_{\sigma,\tau}\})$ is a set of monoids $\mathcal{M} = \{\mathbf{M}_\sigma\}_{\sigma\in\mathcal{X}}$ with the following conditions:

(1) $\mathbf{M}_\sigma \subset \mathbf{L}_\sigma = \mathbb{Z}^{\dim\sigma+1}$ for each $\sigma \in \mathcal{X}$, and it is a finitely generated additive submonoid (so $\mathbf{M}_\sigma$ is an affine semigroup);
(2) $\mathbf{M}_\sigma \subset C_\sigma$ and $\mathbb{R}_{\geq 0}\mathbf{M}_\sigma = C_\sigma$ for each $\sigma \in \mathcal{X}$;
(3) for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$, the map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ induces an isomorphism $\mathbf{M}_\tau \cong \mathbf{M}_\sigma \cap \iota_{\sigma,\tau}(C_\tau)$ of monoids.

If Σ is a rational fan in $\mathbb{R}^n$, then $\{C \cap \mathbb{Z}^n \mid C \in \Sigma\}$ gives a monoidal complex. More generally, taking submonoids of $C \cap \mathbb{Z}^n$ carefully, we can get a "non-normal" monoidal complex.

For a monoidal complex $\mathcal{M} = \{\mathbf{M}_\sigma\}_{\sigma\in\mathcal{X}}$, set

$$|\mathcal{M}| := \varinjlim_{\sigma\in\mathcal{X}} \mathbf{M}_\sigma,$$

where the direct limit is taken with respect to $\iota_{\sigma,\tau} : \mathbf{M}_\tau \to \mathbf{M}_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$. Note that $|\mathcal{M}|$ is just a set and no longer a monoid in general. Since all $\iota_{\sigma,\tau}$ are injective, we can regard $\mathbf{M}_\sigma$ as a subset of $|\mathcal{M}|$. For example, if $\{\mathbf{M}_\sigma\}_{\sigma\in\mathcal{X}}$ comes from a fan in $\mathbb{R}^n$, then $|\mathcal{M}| = \bigcup_{\sigma\in\mathcal{X}} \mathbf{M}_\sigma \subset \mathbb{Z}^n$.

Let $a, b \in |\mathcal{M}|$. If there is some $\sigma \in \mathcal{X}$ with $a, b \in C_\sigma$, there is a unique minimal cell among these σ's. (In fact, if $C_{\sigma_1}, C_{\sigma_2} \in \mathcal{X}$ contain both a and b, there is a cell $\tau \in \mathcal{X}$ with $\bar{\tau} = \bar{\sigma}_1 \cap \bar{\sigma}_2$ by our assumption on $\mathcal{X}$, and C_τ contains both a and b.) If σ is the minimal one with this property, we have $a, b \in \mathbf{M}_\sigma$ and we can define $a+b \in \mathbf{M}_\sigma \subset |\mathcal{M}|$. If there is no $\sigma \in \mathcal{X}$ with $a, b \in C_\sigma$, then $a + b$ does not exist.

Definition 4.4. (See [4].) Let $\{\mathbf{M}_\sigma\}_{\sigma\in\mathcal{X}}$ be a monoidal complex with $|\mathcal{M}| := \varinjlim \mathbf{M}_\sigma$, and $\Bbbk$ a field. Then the $\Bbbk$-vector space

$$\Bbbk[\mathcal{M}] := \bigoplus_{a\in|\mathcal{M}|} \Bbbk x^a,$$

where x is a variable, equipped with the following multiplication

$$x^a \cdot x^b = \begin{cases} x^{a+b} & \text{if } a+b \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

has a $\Bbbk$-algebra structure. We call $\Bbbk[\mathcal{M}]$ the *toric face ring* of $\mathcal{M}$ over $\Bbbk$.

Clearly, $\dim \Bbbk[\mathcal{M}] = \dim \mathcal{X} + 1$. In the rest of this paper, we set $d := \dim \Bbbk[\mathcal{M}]$. Stanley–Reisner rings and affine semigroup rings (of positive semigroups) can be established as toric face rings. If $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, then $\Bbbk[\mathcal{M}]$ admits a $\mathbb{Z}^n$-grading with $\dim_\Bbbk \Bbbk[\mathcal{M}]_a \leq 1$ for all $a \in \mathbb{Z}^n$. But this is not true in general.

Example 4.5. (See [4, Example 4.6].) Consider the conical complex in Example 4.2. Assigning normal semigroup rings of the form $\Bbbk[a,b,c,d]/(ac - bd)$ to each rectangles, we have a toric face ring of the form

$$\Bbbk[x,y,z,u,v,w]/(xv - uy, vz - yw, xz - uw, uvw, uvz),$$

which does not admit a nice multi-grading. We can also get a similar example whose $\Bbbk[\mathbf{M}_\sigma]$ are not normal.

We say a toric face ring $R = \Bbbk[\mathcal{M}]$ is *cone-wise normal*, if $\Bbbk[\mathbf{M}_\sigma]$ is normal for all $\sigma \in \mathcal{X}$. The notion of cone-wise normal toric face rings coincides with that of the ring $\Bbbk[\mathcal{W}F]$ associated with a *weak fan* $\mathcal{W}F$ introduced by Bruns and Gubeladze [1]. They gave an example of a cone-wise normal toric face ring which does not admit a $\mathbb{Z}$-grading with $R_0 = \Bbbk$ [1, Example 2.7].

For $\sigma \in \mathcal{X}$, a monomial ideal $\mathfrak{p}_\sigma := (x^a \mid a \in |\mathcal{M}| \setminus \mathbf{M}_\sigma)$ of R is prime. In fact, the quotient ring $R/\mathfrak{p}_\sigma$ is isomorphic to the affine semigroup ring $\Bbbk[\mathbf{M}_\sigma]$. We regard $\Bbbk[\mathbf{M}_\sigma]$ as an R-module, through $R/\mathfrak{p}_\sigma \cong \Bbbk[\mathbf{M}_\sigma]$.

Set

$$I_R^{-i} := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} \Bbbk[\mathbf{M}_\sigma]$$

for $i = 0, \ldots, d$, and define $\partial : I_R^{-i} \to I_R^{-i+1}$ by

$$\partial(y) = \sum_{\substack{\dim \tau = i-2 \\ \tau \leq \sigma}} \varepsilon(\sigma, \tau) \cdot \pi_{\tau,\sigma}(y)$$

for $y \in \Bbbk[\mathbf{M}_\sigma] \subset I_R^{-i}$, where $\pi_{\tau,\sigma}$ is the natural surjection $\Bbbk[\mathbf{M}_\sigma] \to \Bbbk[\mathbf{M}_\tau]$ (note that if $\tau \leq \sigma$ then $\mathfrak{p}_\sigma \subset \mathfrak{p}_\tau$) and ε is an incidence function of $\mathcal{X}$. Then

$$I_R^\bullet : 0 \longrightarrow I_R^{-d} \longrightarrow I_R^{-d+1} \longrightarrow \cdots \longrightarrow I_R^0 \longrightarrow 0$$

is a cochain complex of finitely generated R-modules. The following is the main result of [13].

Theorem 4.6. *(See [13, Theorem 5.2].) If R is cone-wise normal, then $I_R^\bullet$ is quasi-isomorphic to a dualizing complex $D_R^\bullet$ of R.*

The proof of the main result in the next section largely depends on (the proof of) Theorem 4.6, but the proof in [13] is long and technical. So we summarize it here for the reader's convenience. See [13] for details.

An outline of the proof of Theorem 4.6. To prove the theorem, we realize $I_R^\bullet$ as a subcomplex of $D_R^\bullet$. Set $c(\sigma) := \dim \sigma + 1 = \dim \Bbbk[\mathbf{M}_\sigma]$ for a cell σ. The proof is divided into three steps.

Step 1. We have a canonical injection $i_\sigma : \Bbbk[\mathbf{M}_\sigma] \hookrightarrow D_R^{-c(\sigma)}$.

We fix a cell σ, and set $c := c(\sigma)$. Since $\Bbbk[\mathbf{M}_\sigma]$ is normal, it is Cohen–Macaulay and admits the canonical module simply denoted by ω_σ. Note that

$$H^{-c}\bigl(\operatorname{Hom}_R\bigl(\omega_\sigma, D_R^\bullet\bigr)\bigr) = \operatorname{Ext}_R^{-c}\bigl(\omega_\sigma, D_R^\bullet\bigr) \cong \Bbbk[\mathbf{M}_\sigma].$$

Since $\operatorname{Hom}_R(\omega_\sigma, D_R^{-c-1}) = 0$, the cohomology $H^{-c}(\operatorname{Hom}_R(\omega_\sigma, D_R^\bullet))$ is the kernel of the map

$$\operatorname{Hom}_R(\omega_\sigma, \partial_{D_R^\bullet}) : \operatorname{Hom}_R\bigl(\omega_\sigma, D_R^{-c}\bigr) \longrightarrow \operatorname{Hom}_R\bigl(\omega_\sigma, D_R^{-c+1}\bigr). \tag{4.1}$$

Through the identification,

$$\operatorname{Hom}_R\bigl(\omega_\sigma, D_R^{-c}\bigr) = \operatorname{Hom}_R\bigl(\Bbbk[\mathbf{M}_\sigma], D_R^{-c}\bigr) \cong \bigl\{y \in D_R^{-c} \mid \mathfrak{p}_\sigma y = 0\bigr\},$$

the kernel of the map (4.1) is

$$i_\sigma\bigl(\Bbbk[\mathbf{M}_\sigma]\bigr) := \bigl\{y \in D_R^{-c} \mid \mathfrak{p}_\sigma y = 0 \text{ and } \partial_{D_R^\bullet}(\mathfrak{q}_\sigma y) = 0\bigr\},$$

where $\mathfrak{q}_\sigma$ is the set $\{x^a \in R \mid a \in (\mathbf{M}_\sigma \cap \operatorname{int}(C_\sigma))\}$. (Note that ω_σ is the ideal of $\Bbbk[\mathbf{M}_\sigma]$ generated by $\mathfrak{q}_\sigma$.) Clearly, $i_\sigma(\Bbbk[\mathbf{M}_\sigma]) \cong \Bbbk[\mathbf{M}_\sigma]$.

Of course, we just chose the *subset* $i_\sigma(\Bbbk[\mathbf{M}_\sigma])$ of D_R^{-c}, not an injection $i_\sigma : \Bbbk[\mathbf{M}_\sigma] \hookrightarrow D_R^{-c}$. However, the R-module $\Bbbk[\mathbf{M}_\sigma]$ is generated by a single element, and the choice of a generator (i.e., the choice of i_σ) is unique up to constant multiplication. This small ambiguity does not affect the argument below.

Step 2. $\bigoplus_{\sigma \in \mathcal{X}} i_\sigma(\Bbbk[\mathbf{M}_\sigma])$ is a subcomplex of $D_R^\bullet$.

The dualizing complex $D_\sigma^\bullet := D_{\Bbbk[\mathbf{M}_\sigma]}^\bullet$ of $\Bbbk[\mathbf{M}_\sigma]$ coincides with $\operatorname{Hom}_R(\Bbbk[\mathbf{M}_\sigma], D_R^\bullet)$, which can be seen as a subcomplex of $D_R^\bullet$. Since $\Bbbk[\mathbf{M}_\sigma]$ is $\mathbb{Z}^{c(\sigma)}$-graded, we have the $\mathbb{Z}^{c(\sigma)}$-graded dualizing complex $J_\sigma^\bullet := J_{\Bbbk[\mathbf{M}_\sigma]}^\bullet$, and a quasi-isomorphism $J_\sigma^\bullet \to D_\sigma^\bullet$. Composing this morphism with $D_\sigma^\bullet \to D_R^\bullet$, we get a chain map $h_\sigma : J_\sigma^\bullet \to D_R^\bullet$ which induces

$$H^i\bigl(\operatorname{Hom}_R\bigl(\omega_\sigma, J_\sigma^\bullet\bigr)\bigr) \cong H^i\bigl(\operatorname{Hom}_R\bigl(\omega_\sigma, D_R^\bullet\bigr)\bigr). \tag{4.2}$$

Applying the same argument as Step 1, we have an injection ${}^*i_{\sigma,\tau} : \Bbbk[\mathbf{M}_\tau] \hookrightarrow J_\sigma^{-c(\tau)}$ for a cell τ with $\tau \le \sigma$. By (4.2), it is easy to see that

$$i_\tau\bigl(\Bbbk[\mathbf{M}_\tau]\bigr) = h_\sigma \circ {}^*i_{\sigma,\tau}\bigl(\Bbbk[\mathbf{M}_\tau]\bigr).$$

On the other hand, we have that

$$\bigl(J_\sigma^\bullet\bigr)_{C_\sigma} = \bigoplus_{\tau\le\sigma} {}^*i_{\sigma,\tau}\bigl(\Bbbk[\mathbf{M}_\tau]\bigr), \tag{4.3}$$

where C_σ is the polyhedral cone spanned by $\mathbf{M}_\sigma$. Since $J_\sigma^\bullet$ is a $\mathbb{Z}^{c(\sigma)}$-graded complex, the right side of (4.3) is a subcomplex of $J_\sigma^\bullet$. Since h_σ is a chain map, $\bigoplus_{\tau\le\sigma} i_\sigma(\Bbbk[\mathbf{M}_\tau])$ forms a subcomplex of $D_R^\bullet$. It implies that $\bigoplus_{\sigma\in\mathcal{X}} i_\sigma(\Bbbk[\mathbf{M}_\sigma])$ is also a subcomplex of $D_R^\bullet$.

Since $\bigoplus_{\sigma\in\mathcal{X}} i_\sigma(\Bbbk[\mathbf{M}_\sigma])$ is isomorphic to $I_R^\bullet$, it suffices to show the following.

Step 3. $D_R^\bullet$ is quasi-isomorphic to its subcomplex $\bigoplus_{\sigma\in\mathcal{X}} i_\sigma(\Bbbk[\mathbf{M}_\sigma])$.

The argument for this step will be used around the proof of Theorem 5.11 after a slight generalization. There, we explain this idea in detail, so we do not give a summary here. □

5. Dualizing complexes of seminormal toric face rings

We start from the following fact pointed out by Nguyen [12].

Proposition 5.1. *(See [12, Proposition 3.5].) For a toric face ring $\Bbbk[\mathcal{M}]$, the following are equivalent.*

(i) $\Bbbk[\mathcal{M}]$ *is seminormal.*
(ii) $\Bbbk[\mathbf{M}_\sigma]$ *is seminormal for all $\sigma \in \mathcal{X}$.*

Recall the precise definition of a monoidal complex $\mathcal{M}$ given in the previous section. For each $\sigma \in \mathcal{X}$, let ${}^+\mathbf{M}_\sigma \subset \mathbf{L}_\sigma$ be the monoid constructed from $\mathbf{M}_\sigma$ by the operation in (2.1), that is, $\Bbbk[{}^+\mathbf{M}_\sigma]$ is the seminormalization of $\Bbbk[\mathbf{M}_\sigma]$. Then ${}^+\mathcal{M} := \{{}^+\mathbf{M}_\sigma\}_{\sigma\in\mathcal{X}}$ forms a monoidal complex, and ${}^+R := \Bbbk[{}^+\mathcal{M}]$ is the seminormalization of $R := \Bbbk[\mathcal{M}]$. In particular, R is seminormal if and only if $\mathcal{M} = {}^+\mathcal{M}$.

On the other hand, $\Bbbk[\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma]$ is the normalization of $\Bbbk[\mathbf{M}_\sigma]$ (since we do not assume that $\mathbb{Z}\mathcal{M}_\sigma = \mathbf{L}_\sigma$, we have $\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma \ne \mathbf{L}_\sigma \cap C_\sigma$ in general), but $\{\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma\}_{\sigma\in\mathcal{X}}$ does *not* form a monoidal complex. The monoidal complex $\mathcal{M}$ of Example 5.3 below gives a counter example. In fact, the condition (3) of Definition 4.3 is violated.

We consider the following cochain complex

$${}^+I_R^\bullet : 0 \longrightarrow {}^+I_R^{-d} \longrightarrow {}^+I_R^{-d+1} \longrightarrow \cdots \longrightarrow {}^+I_R^0 \longrightarrow 0$$

with

$$^{+}I_R^{-i} := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} \Bbbk[\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma].$$

The differential map ∂ is given by

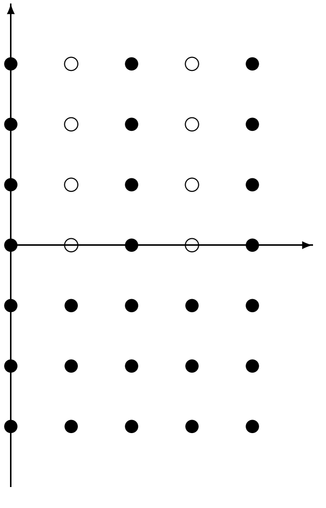
$$\partial(y) = \sum_{\substack{\dim \tau = i-2 \\ \tau \leq \sigma}} \varepsilon(\sigma, \tau) \cdot \pi_{\tau,\sigma}(y)$$

for $y \in \Bbbk[\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma] \subset I_R^{-i}$, where $\pi_{\tau,\sigma}$ is the natural surjection $\Bbbk[\mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma] \to \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]$. Clearly, $^{+}I_R^\bullet$ is a cochain complex of finitely generated R-modules.

Theorem 5.2. *If a toric face ring $R = \Bbbk[\mathcal{M}]$ is seminormal, then $^{+}I_R^\bullet$ is quasi-isomorphic to a dualizing complex $D_R^\bullet$.*

To prove the theorem, we need some preparation. For each $\sigma \in \mathcal{X}$, set $\widetilde{\mathbf{M}}_\sigma := \mathbf{L}_\sigma \cap C_\sigma$. Then $\{\widetilde{\mathbf{M}}_\sigma\}_{\sigma \in \mathcal{X}}$ is a monoidal complex again. We can regard that $|\widetilde{\mathcal{M}}| := \varinjlim \widetilde{\mathbf{M}}_\sigma$ contains $|\mathcal{M}|$ as a subset.

Example 5.3. While $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is always a normal semigroup ring, it is not the normalization of $\Bbbk[\mathbf{M}_\sigma]$. For example, consider the monoidal complex $\mathcal{M}$ illustrated below. Let $\mathbf{M}_\sigma$ be the monoid corresponding to the first quadrant, then $\Bbbk[\mathbf{M}_\sigma] = \Bbbk[x^2, y]$ is normal, but we have $\Bbbk[\widetilde{\mathbf{M}}_\sigma] = \Bbbk[x, y] \supsetneq \Bbbk[\mathbf{M}_\sigma]$.



Set $\widetilde{R} := \Bbbk[\widetilde{\mathcal{M}}]$. The next result holds, even if $\Bbbk[\mathcal{M}]$ is not seminormal.

Lemma 5.4. *For any $\mathcal{M}$, $\widetilde{R} = \Bbbk[\widetilde{\mathcal{M}}]$ is a finitely generated module over $R = \Bbbk[\mathcal{M}]$.*

Proof. It suffices to show that $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is finitely generated as a $\Bbbk[\mathbf{M}_\sigma]$-module for each $\sigma \in \mathcal{X}$. This must be a well-known result, but we give a proof here for the reader's convenience. If $\dim \sigma = 0$, then the assertion is clear (in fact, $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is a polynomial ring with one variable in this case). If $\dim \Bbbk[\mathbf{M}_\sigma] \geq 1$, set $A := \Bbbk[\mathbf{M}_\sigma]$, and let A' be the A-subalgebra of $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ generated by $\{x^a \mid a \in \widetilde{\mathbf{M}}_\tau, \tau < \sigma, \dim \tau = 0\}$. By the above remark, A' is a finitely generated A-module. Since $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is the normalization of A', it is a finitely generated as an A'-module, hence also as an A-module. □

We regard $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ as an R-module by the compositions of the ring homomorphisms $R \twoheadrightarrow R/\mathfrak{p}_\sigma (\cong \Bbbk[\mathbf{M}_\sigma]) \hookrightarrow \Bbbk[\widetilde{\mathbf{M}}_\sigma]$, which is the same thing as $R \hookrightarrow \widetilde{R} \twoheadrightarrow \Bbbk[\widetilde{\mathbf{M}}_\sigma]$.

As in the previous section, we set $c(\sigma) := \dim \sigma + 1 = \dim \Bbbk[\mathbf{M}_\sigma]$. For the simplicity, the dualizing complexes $D^\bullet_{\Bbbk[\mathbf{M}_\sigma]}$ (resp. $D^\bullet_{\Bbbk[\widetilde{\mathbf{M}}_\sigma]}$) of $\Bbbk[\mathbf{M}_\sigma]$ (resp. $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$) is denoted by $D^\bullet_\sigma$ (resp. $D^\bullet_{\widetilde{\sigma}}$). Since both $\Bbbk[\mathbf{M}_\sigma]$ and $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ are $\mathbb{Z}^{c(\sigma)}$-graded, they admit the $\mathbb{Z}^{c(\sigma)}$-graded dualizing complexes $J^\bullet_\sigma := J^\bullet_{\Bbbk[\mathbf{M}_\sigma]}$ and $J^\bullet_{\widetilde{\sigma}} := J^\bullet_{\Bbbk[\widetilde{\mathbf{M}}_\sigma]}$ respectively. Similarly, we also set ${}^+I^\bullet_\sigma := {}^+I^\bullet_{\Bbbk[\mathbf{M}_\sigma]}$ and $I^\bullet_{\widetilde{\sigma}} := I^\bullet_{\Bbbk[\widetilde{\mathbf{M}}_\sigma]} (= {}^+I^\bullet_{\Bbbk[\widetilde{\mathbf{M}}_\sigma]})$ for the simplicity.

Since $\widetilde{R}$ is cone-wise normal, $I^\bullet_{\widetilde{R}}$ is quasi-isomorphic to $D^\bullet_{\widetilde{R}}$ by Theorem 4.6. Moreover, we have the following.

Lemma 5.5. *There is a quasi-isomorphism $\psi : I^\bullet_{\widetilde{R}} \to D^\bullet_{\widetilde{R}}$ such that the induced map $\psi_\sigma := \operatorname{Hom}^\bullet_{\widetilde{R}}(\Bbbk[\widetilde{\mathbf{M}}_\sigma], \psi) : I^\bullet_{\widetilde{\sigma}} \to D^\bullet_{\widetilde{\sigma}}$ is a quasi-isomorphism for all $\sigma \in \mathcal{X}$.*

Proof. This fact has been shown in the proof of [13, Theorem 5.2] (Theorem 4.6 of the present paper). Recall the outline of the proof introduced in the previous section. □

Since $\widetilde{R}$ is finitely generated as an R-module by Lemma 5.4, we have $D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R)$. Via the canonical injection $R \hookrightarrow \widetilde{R}$, we have a chain map

$$\lambda : D^\bullet_{\widetilde{R}} = \operatorname{Hom}^\bullet_R(\widetilde{R}, D^\bullet_R) \longrightarrow \operatorname{Hom}^\bullet_R(R, D^\bullet_R) = D^\bullet_R.$$

Similarly, for each σ, the injection $\Bbbk[\mathbf{M}_\sigma] \hookrightarrow \Bbbk[\widetilde{\mathbf{M}}_\sigma]$ induces a chain map $\lambda_\sigma : D^\bullet_{\widetilde{\sigma}} \to D^\bullet_\sigma$. Since $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is a finitely generated $\mathbb{Z}^{c(\sigma)}$-graded module over $\Bbbk[\mathbf{M}_\sigma]$ and $J^\bullet_\sigma$ is the dualizing complex in the $\mathbb{Z}^{c(\sigma)}$-graded context, we have $\operatorname{Hom}^\bullet_{\Bbbk[\mathbf{M}_\sigma]}(\Bbbk[\widetilde{\mathbf{M}}_\sigma], J^\bullet_\sigma) = J^\bullet_{\widetilde{\sigma}}$. The injection $\Bbbk[\mathbf{M}_\sigma] \hookrightarrow \Bbbk[\widetilde{\mathbf{M}}_\sigma]$ induces the $\mathbb{Z}^{c(\sigma)}$-graded chain map $\mu'_\sigma : J^\bullet_{\widetilde{\sigma}} \longrightarrow J^\bullet_\sigma$.

Note that $\mathbf{M}_\sigma$ and $\widetilde{\mathbf{M}}_\sigma$ span the same polyhedral cone C_σ. Since $\Bbbk[\mathbf{M}_\sigma]$ is seminormal and $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$ is normal, we have $J^\bullet_\sigma \cong (J^\bullet_\sigma)_{C_\sigma} = {}^+I^\bullet_\sigma$ and $J^\bullet_{\widetilde{\sigma}} \cong (J^\bullet_{\widetilde{\sigma}})_{C_\sigma} = {}^+I^\bullet_{\widetilde{\sigma}} = I^\bullet_{\widetilde{\sigma}}$ as shown in the proof of Proposition 2.3. Taking the C_σ-graded part of μ'_σ, we have the chain map

$$\mu_\sigma : I^\bullet_{\widetilde{\sigma}} \longrightarrow {}^+I^\bullet_\sigma.$$

Lemma 5.6. *For the quasi-isomorphism $\psi_\sigma : I^\bullet_{\widetilde{\sigma}} \to D^\bullet_{\widetilde{\sigma}}$ of Lemma 5.5, we have a quasi-isomorphism $\phi_\sigma : {}^+I^\bullet_\sigma \to D^\bullet_\sigma$ which makes the following diagram commutative.*

$$\begin{tikzcd} I^\bullet_{\tilde\sigma} \arrow[r, "\psi_\sigma"] \arrow[d, "\mu_\sigma"'] & D^\bullet_{\tilde\sigma} \arrow[d, "\lambda_\sigma"] \\ {}^+I^\bullet_\sigma \arrow[r, "\phi_\sigma"'] & D^\bullet_\sigma \end{tikzcd}$$

Proof. It is easy to see that there exists a quasi-isomorphism $\psi'_\sigma : J^\bullet_{\tilde\sigma} \to D^\bullet_{\tilde\sigma}$ which is an extension of $\psi_\sigma : I^\bullet_{\tilde\sigma} \to D^\bullet_{\tilde\sigma}$. Since $\mu_\sigma : I^\bullet_{\tilde\sigma} \longrightarrow {}^+I^\bullet_\sigma$ is the restriction of $\mu'_\sigma : J^\bullet_{\tilde\sigma} \longrightarrow J^\bullet_\sigma$, it suffices to construct a quasi-isomorphism $\phi'_\sigma : J^\bullet_\sigma \to D^\bullet_\sigma$ with

$$\begin{tikzcd} J^\bullet_{\tilde\sigma} \arrow[r, "\psi'_\sigma"] \arrow[d, "\mu'_\sigma"'] & D^\bullet_{\tilde\sigma} \arrow[d, "\lambda_\sigma"] \\ J^\bullet_\sigma \arrow[r, "\phi'_\sigma"'] & D^\bullet_\sigma. \end{tikzcd}$$

In fact, the restriction of ϕ'_σ to ${}^+I^\bullet_\sigma$ gives ϕ_σ satisfying the expected condition.

Since $J^\bullet_\sigma \cong D^\bullet_\sigma$ in $\mathsf{D}^b(\operatorname{Mod} \Bbbk[\mathbf{M}_\sigma])$, we have a quasi-isomorphism $\xi : J^\bullet_\sigma \to D^\bullet_\sigma$. Taking $\operatorname{Hom}_{\Bbbk[\mathbf{M}_\sigma]}(\Bbbk[\widetilde{\mathbf{M}}_\sigma], -)$, we get a chain map

$$\xi_* : J^\bullet_{\tilde\sigma} = \operatorname{Hom}_{\Bbbk[\mathbf{M}_\sigma]}\big(\Bbbk[\widetilde{\mathbf{M}}_\sigma], J^\bullet_\sigma\big) \longrightarrow \operatorname{Hom}_{\Bbbk[\mathbf{M}_\sigma]}\big(\Bbbk[\widetilde{\mathbf{M}}_\sigma], D^\bullet_\sigma\big) = D^\bullet_{\tilde\sigma}.$$

Note that $J^\bullet_\sigma$ is a cochain complex of injective objects in the category ${}^*\operatorname{Mod}(\Bbbk[\mathbf{M}_\sigma])$ of $\mathbb{Z}^{c(\sigma)}$-graded $\Bbbk[\mathbf{M}_\sigma]$ modules, and $\Bbbk[\widetilde{\mathbf{M}}_\sigma] \in {}^*\operatorname{Mod}(\Bbbk[\mathbf{M}_\sigma])$. Hence ξ_* is a quasi-isomorphism.

Clearly, ξ_* is $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$-linear, and can be extended to a $\Bbbk[\widetilde{\mathbf{M}}_\sigma]$-linear automorphism $\bar\xi_*$ of $D^\bullet_{\tilde\sigma}$ uniquely (of course, the same is true for ψ'_σ). Since

$$\operatorname{Hom}_{\mathsf{D}^b(\operatorname{Mod} \Bbbk[\widetilde{\mathbf{M}}_\sigma])}\big(D^\bullet_{\tilde\sigma}, D^\bullet_{\tilde\sigma}\big) = \Bbbk[\widetilde{\mathbf{M}}_\sigma]$$

and $D^\bullet_{\tilde\sigma}$ is a cochain complex of injective modules, the automorphism $\bar\xi_*$ is homotopic to the multiplication by c for some $0 \neq c \in \Bbbk$. Moreover, since $D^\bullet_{\tilde\sigma}$ is of the form (1.1), $\bar\xi_*$ is *equal* to the multiplication by c. Since the same is true for ψ'_σ, we have $\psi'_\sigma = c'\xi_*$ for some $0 \neq c' \in \Bbbk$. Hence $\phi'_\sigma := c'\xi$ satisfies the desired condition. □

For each $i \in \mathbb{Z}$, ${}^+I^i_R$ is an R-submodule of $I^i_{\tilde R}$. However ${}^+I^\bullet_R$ is *not* a subcomplex of $I^\bullet_{\tilde R}$. This problem occurs even in the semigroup ring case. See Remark 3.2.

Let $\kappa : {}^+I^\bullet_R \dashrightarrow I^\bullet_{\tilde R}$ be the collection of the natural injections ${}^+I^i_R \hookrightarrow I^i_{\tilde R}$ (since this is not a chain map, we use the symbol "$\dashrightarrow$"). The similar map $\kappa_\sigma : {}^+I^\bullet_\sigma \dashrightarrow I^\bullet_{\tilde\sigma}$ is not a chain map in general again. For each i, ${}^+I^i_\sigma$ is a direct summand of $I^i_{\tilde\sigma}$ as a $\Bbbk[\mathbf{M}_\sigma]$-module, the i-th component $\mu^i_\sigma : I^i_{\tilde\sigma} \longrightarrow {}^+I^i_\sigma$ of the chain map $\mu_\sigma : I^\bullet_{\tilde\sigma} \longrightarrow {}^+I^{\bullet_\sigma}$ satisfies $\mu^i_\sigma \circ \kappa^i_\sigma = \mathrm{Id}$.

Lemma 5.7. *The composition* ${}^+I_R^\bullet \overset{\kappa}{\dashrightarrow} I_{\widetilde{R}}^\bullet \xrightarrow{\psi} D_{\widetilde{R}}^\bullet \xrightarrow{\lambda} D_R^\bullet$ *is a chain map.*

Proof. It suffice to check that

$$\partial_{D_R^\bullet}^{i+1} \circ \left(\lambda^i \circ \psi^i \circ \kappa^i\right)(y) = \left(\lambda^{i+1} \circ \psi^{i+1} \circ \kappa^{i+1}\right) \circ \partial^i_{{}^+I_R^\bullet}(y)$$

for all "homogeneous" element y (i.e., $y \in ({}^+I_R^i)_a$ for some $a \in |\mathcal{M}|$), since any element of ${}^+I_R^i$ is a sum of these elements. Then we can regard $y \in {}^+I_\sigma^i$ for some $\sigma \in \mathcal{X}$. We have the following commutative diagram.

$$\begin{tikzcd}
{}^+I_\sigma^i \arrow[r, "\kappa_\sigma^i"] \arrow[d, hook] & I_{\widetilde{\sigma}}^i \arrow[r, "\psi_\sigma^i"] \arrow[d, hook] & D_{\widetilde{\sigma}}^i \arrow[r, "\lambda_\sigma^i"] \arrow[d, hook] & D_\sigma^i \arrow[d, hook] \\
{}^+I_R^i \arrow[r, "\kappa^i"'] & I_{\widetilde{R}}^i \arrow[r, "\psi^i"'] & D_{\widetilde{R}}^i \arrow[r, "\lambda^i"'] & D_R^i
\end{tikzcd}$$

The commutativity of the left square is clear, that of the middle one is Lemma 5.5, and that of the right one follows from the fact that the composition $R \hookrightarrow \widetilde{R} \twoheadrightarrow \Bbbk[\widetilde{\mathbf{M}_\sigma}]$ coincides with the composition $R \twoheadrightarrow \Bbbk[\mathbf{M}_\sigma] \hookrightarrow \Bbbk[\widetilde{\mathbf{M}_\sigma}]$.

By Lemma 5.6, we have $\lambda_\sigma^i \circ \psi_\sigma^i \circ \kappa_\sigma^i = \phi_\sigma^i \circ \mu_\sigma^i \circ \kappa_\sigma^i = \phi_\sigma^i$. Since ϕ_σ is a chain map, we are done. □

Let ϕ denote the chain map $J_R^\bullet \to D_R^\bullet$ constructed in Lemma 5.7. To prove Theorem 5.2, we will show that ϕ is a quasi-isomorphism by a slightly indirect way.

Definition 5.8. Let $R = \Bbbk[\mathcal{M}]$ be a toric face ring. We say an R-module M is $|\widetilde{\mathcal{M}}|$-*graded* if the following are satisfied;

(i) $M = \bigoplus_{a \in |\widetilde{\mathcal{M}}|} M_a$ as $\Bbbk$-vector spaces;
(ii) Let $a \in |\mathcal{M}|$ and $b \in |\widetilde{\mathcal{M}}|$. If $a + b$ exists (equivalently, $a, b \in \widetilde{\mathbf{M}}_\sigma$ for some $\sigma \in \mathcal{X}$), then $x^a M_b \subset M_{a+b}$. Otherwise, $x^a M_b = 0$.

Let $\operatorname{Mod}_{\widetilde{\mathcal{M}}} R$ denote the subcategory of $\operatorname{Mod} R$ whose objects are $|\widetilde{\mathcal{M}}|$-graded and homomorphisms are $f : M \to N$ with $f(M_a) \subset N_a$ for all $a \in |\widetilde{\mathcal{M}}|$.

We say $M \in \operatorname{Mod}_{\widetilde{\mathcal{M}}} R$ is $|\mathcal{M}|$-*graded*, if $M = \bigoplus_{a \in |\mathcal{M}|} M_a$. Let $\operatorname{Mod}_{\mathcal{M}} R$ denote the subcategory of $\operatorname{Mod}_{\widetilde{\mathcal{M}}} R$ consisting of $|\mathcal{M}|$-graded modules.

Clearly, $\operatorname{Mod}_{\widetilde{\mathcal{M}}} R$ and $\operatorname{Mod}_{\mathcal{M}} R$ are abelian categories. It is easy to see that $R \in \operatorname{Mod}_{\mathcal{M}} R$ and $\widetilde{R} \in \operatorname{Mod}_{\widetilde{\mathcal{M}}} R$. Moreover, $I_R^\bullet$ (resp. ${}^+I_R^\bullet$) is a cochain complex in $\operatorname{Mod}_{\mathcal{M}} R$ (resp. $\operatorname{Mod}_{\widetilde{\mathcal{M}}} R$).

Definition 5.9. For each $a \in |\widetilde{\mathcal{M}}|$, there is a unique cell $\sigma \in \mathcal{X}$ with $a \in \operatorname{int}(C_\sigma)$ (equivalently, $a \in \widetilde{\mathbf{M}}_\sigma$ and σ is the minimal one with this property). This cell σ is denoted by $\operatorname{supp}(a)$.

An R-module $M \in \operatorname{Mod} R$ is said to be *squarefree* if it is $|\mathcal{M}|$-graded (*not* $|\widetilde{\mathcal{M}}|$-graded), finitely generated, and the multiplication map $M_a \ni y \longmapsto x^b y \in M_{a+b}$ is bijective for all $a, b \in |\mathcal{M}|$ with $\operatorname{supp}(a) \supset \operatorname{supp}(b)$.

For example, $\Bbbk[\mathbf{M}_\sigma]$ and R itself are squarefree R-modules. In [13], squarefree modules over a cone-wise normal toric face ring play a key role. Many properties are lost in the non-normal case. For example, ${}^+I_R^\bullet$ is no longer a complex of squarefree modules. In fact, ${}^+I_R^i$ is $|\widetilde{\mathcal{M}}|$-graded, not $|\mathcal{M}|$-graded. However, the next result still holds.

Lemma 5.10. *(Cf. [13, Lemma 4.2].) Let* $\operatorname{Sq} R$ *be the full subcategory of* $\operatorname{Mod}_{\mathcal{M}} R$ *consisting of squarefree modules. Then* $\operatorname{Sq} R$ *is an abelian category with enough injectives, and indecomposable injectives are objects isomorphic to* $\Bbbk[\mathbf{M}_\sigma]$ *for some* $\sigma \in \mathcal{X}$*. The injective dimension of any object is at most* d*.*

The proof is similar to the cone-wise normal case [13], and we omit it here. We just remark that $\operatorname{Sq} R$ is equivalent to the category of finitely generated left Λ-modules, where Λ is the incidence algebra of $\mathcal{X}$ (as a poset) over $\Bbbk$.

Let Inj-Sq be the full subcategory of $\operatorname{Sq} R$ consisting of all injective objects, that is, finite direct sums of copies of $\Bbbk[\mathbf{M}_\sigma]$ for various $\sigma \in \mathcal{X}$. Then the bounded homotopy category $\mathsf{K}^b(\text{Inj-Sq})$ is equivalent to $\mathsf{D}^b(\operatorname{Sq} R)$. We have an exact functor

$$\operatorname{Hom}_R^\bullet\big(-, {}^+I_R^\bullet\big) : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{D}^b(\operatorname{Mod} R)^{\mathsf{op}}.$$

Similarly, we have an exact functor

$$\operatorname{Hom}_R^\bullet\big(-, D_R^\bullet\big) : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{D}^b(\operatorname{Mod} R)^{\mathsf{op}}.$$

The chain map $\phi : {}^+I_R^\bullet \to D_R^\bullet$ gives a natural transformation

$$\Phi : \operatorname{Hom}_R^\bullet\big(-, {}^+I_R^\bullet\big) \to \operatorname{Hom}_R^\bullet\big(-, D_R^\bullet\big).$$

Theorem 5.11. *If* R *is seminormal,* Φ *is a natural isomorphism.*

Proof. By virtue of [8, Proposition 7.1], it suffices to show that

$$\Phi\big(\Bbbk[\mathbf{M}_\sigma]\big) : {}^+I_\sigma^\bullet = \operatorname{Hom}_R^\bullet\big(\Bbbk[\mathbf{M}_\sigma], {}^+I_R^\bullet\big) \to \operatorname{Hom}_R^\bullet\big(\Bbbk[\mathbf{M}_\sigma], D_R^\bullet\big) = D_\sigma^\bullet$$

is a quasi-isomorphism for all $\sigma \in \mathcal{X}$. Since $\Phi(\Bbbk[\mathbf{M}_\sigma]) = \operatorname{Hom}_R^\bullet(\Bbbk[\mathbf{M}_\sigma], \phi)$, it is factored as ${}^+I_\sigma^\bullet \overset{\kappa_\sigma}{\dashrightarrow} I_{\widetilde{\sigma}}^\bullet \xrightarrow{\psi_\sigma} D_{\widetilde{\sigma}}^\bullet \xrightarrow{\lambda_\sigma} D_\sigma^\bullet$. As shown in the proof of Lemma 5.7, this coincides with the quasi-isomorphism ϕ_σ of Lemma 5.6. □

The proof of Theorem 5.2. The assertion follows from Theorem 5.11. In fact, since $R \in \operatorname{Sq} R$, we have an isomorphism $\Phi(R) : \operatorname{Hom}^\bullet_R(E^\bullet, {}^+I^\bullet_R) \to \operatorname{Hom}^\bullet_R(E^\bullet, D^\bullet_R)$, where $E^\bullet$ is an injective resolution of R in $\operatorname{Sq} R$. It is clear that $\operatorname{Hom}^\bullet_R(E^\bullet, D^\bullet_R) \cong \operatorname{Hom}^\bullet_R(R, D^\bullet_R) \cong D^\bullet_R$, but we can also show that $\operatorname{Hom}^\bullet_R(E^\bullet, {}^+I^\bullet_R) \cong {}^+I^\bullet_R$ by the usual double complex argument. The key fact is that $\operatorname{Hom}^\bullet_R(E^\bullet, {}^+I^i_R)$ is an acyclic complex whose 0th cohomology is ${}^+I^i_R$ for each i. To see this, note that an indecomposable components of $E^\bullet$ and ${}^+I^i_R$ are $\Bbbk[\mathbf{M}_\sigma]$ and $\Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]$ respectively for some $\sigma, \tau \in \mathcal{X}$, moreover

$$\operatorname{Hom}_R\bigl(\Bbbk[\mathbf{M}_\sigma], \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]\bigr) \cong \begin{cases} \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau] & \text{if } \sigma \geq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Take $a \in |\mathcal{M}|$ with $\operatorname{supp}(a) = \tau$. Then, $\sigma \geq \tau$ if and only if $\Bbbk[\mathbf{M}_\sigma]_a \neq 0$. Hence we have $\operatorname{Hom}^\bullet_R(E^\bullet, \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]) \cong \operatorname{Hom}^\bullet_\Bbbk([E^\bullet]_a, \Bbbk) \otimes_\Bbbk \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]$. Since $[E^\bullet]_a$ is an acyclic complex whose 0th cohomology is $\Bbbk$, $\operatorname{Hom}^\bullet_R(E^\bullet, \Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau])$ is an acyclic complex whose 0th cohomology is $\Bbbk[\mathbb{Z}\mathbf{M}_\tau \cap C_\tau]$.

Anyway, we have ${}^+I^\bullet_R \cong \operatorname{Hom}^\bullet_R(E^\bullet, {}^+I^\bullet_R) \cong \operatorname{Hom}^\bullet_R(E^\bullet, D^\bullet_R) \cong D^\bullet_R$, where the middle isomorphism is given by $\Phi(R)$. □

The converse of Theorem 5.2 also holds.

Proposition 5.12. *Let $R = \Bbbk[\mathcal{M}]$ be a toric face ring. If ${}^+I^\bullet_R$ is quasi-isomorphic to the dualizing complex $D^\bullet_R$, then R is seminormal.*

Proof. Recall that ${}^+\mathcal{M} := \{{}^+\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$ forms a monoidal complex, and the toric face ring ${}^+R = \Bbbk[{}^+\mathcal{M}]$ is the seminormalization of R. Since ${}^+I^\bullet_{+R} = {}^+I^\bullet_R$, the proof of the latter half of Theorem 5.2 also works here. □

6. Local cohomologies

Recall that a monoidal complex $\mathcal{M} = \{\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$ is a collection of additive submonoids $\mathbf{M}_\sigma$ of lattices $\mathbf{L}_\sigma \cong \mathbb{Z}^{\dim \sigma + 1}$ for each $\sigma \in \mathcal{X}$, and we have an injective homomorphisms $\tilde{\iota}_{\sigma,\tau} : \mathbf{L}_\tau \to \mathbf{L}_\sigma$ for all $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$. Set

$$\mathcal{L} := \varinjlim_{\sigma \in \mathcal{X}} \mathbf{L}_\sigma.$$

Note that $\mathcal{L}$ is no longer a group in general. Since all $\tilde{\iota}_{\sigma,\tau}$ is injective, we can regard $\mathbf{L}_\sigma$ as a subset of $\mathcal{L}$. Let $a, b \in \mathcal{L}$. If there is some $\sigma \in \mathcal{X}$ with $a, b \in \mathbf{L}_\sigma$, we have $a+b \in \mathbf{L}_\sigma \subset \mathcal{L}$. If there is no $\sigma \in \mathcal{X}$ with $a, b \in \mathbf{L}_\sigma$, then $a + b$ does not exist. However, any $a \in \mathcal{L}$ has $-a \in \mathcal{L}$. We can regard that $|\widetilde{\mathcal{M}}| \subset \mathcal{L}$, and the structure of $\mathcal{L}$ defined above and that of $|\widetilde{\mathcal{M}}|$ are compatible with this injection.

Definition 6.1. Let $R := \Bbbk[\mathcal{M}]$ be a toric face ring. Then $M \in \operatorname{Mod} R$ is said to be $\mathcal{L}$-graded if the following conditions are satisfied;

(i) $M = \bigoplus_{a \in \mathcal{L}} M_a$ as $\Bbbk$-vector spaces;
(ii) $x^a M_b \subset M_{a+b}$ if $a \in \mathbf{M}_\sigma$ and $b \in \mathbf{L}_\sigma$ for some $\sigma \in \mathcal{X}$, and $x^a M_b = 0$ otherwise.

Let $\mathrm{Mod}_{\mathcal{L}} R$ be the category of $\mathcal{L}$-graded R-modules and R-homomorphisms $f : M \to N$ with $f(M_a) \subset N_a$ for all $a \in \mathcal{L}$.

Clearly, $\mathrm{Mod}_{\mathcal{M}} R$ and $\mathrm{Mod}_{\widetilde{\mathcal{M}}} R$ are full subcategories of $\mathrm{Mod}_{\mathcal{L}} R$. Note that $T_\sigma := \{x^a \mid a \in \mathbf{M}_\sigma\} \subset R$ is a multiplicatively closed subset. As shown in [13, Lemma 2.1], the localization $T_\sigma^{-1} R$ is $\mathcal{L}$-graded.

Well, set

$$\check{C}_R^i := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim \sigma = i-1}} T_\sigma^{-1} R$$

and define $\partial : \check{C}_R^i \to \check{C}_R^{i+1}$ by

$$\partial(x) = \sum_{\substack{\tau \geq \sigma \\ \dim \tau = i}} \varepsilon(\tau, \sigma) \cdot \iota_{\tau,\sigma}(x)$$

for $x \in T_\sigma^{-1} R \subset \check{C}_R^i$, where ε is an incidence function on $\mathcal{X}$ and $\iota_{\tau,\sigma}$ is a natural map $T_\sigma^{-1} R \to T_\tau^{-1} R$ for $\sigma \leq \tau$. Then $(\check{C}_R^\bullet, \partial)$ forms a cochain complex in $\mathrm{Mod}_{\mathcal{L}} R$:

$$0 \longrightarrow \check{C}_R^0 \longrightarrow \check{C}_R^1 \longrightarrow \cdots \longrightarrow \check{C}_R^d \longrightarrow 0.$$

We set $\mathfrak{m} := (x^a \mid 0 \neq a \in |\mathcal{M}|)$. This is a maximal ideal of R. The following result has been proved by Ichim and Römer [9] in the case $\mathcal{M}$ comes from a fan in $\mathbb{R}^d$, and Okazaki and the present author in the general case. (The proofs are essentially the same.)

Proposition 6.2. *(See [9, Theorem 4.2], [13, Proposition 3.2].) For any R-module M, we have*

$$H_{\mathfrak{m}}^i(M) \cong H^i\big(\check{C}_R^\bullet \otimes_R M\big),$$

for all i. In particular, $H_{\mathfrak{m}}^i(R)$ is $\mathcal{L}$-graded.

Corollary 6.3. *Let $\mathcal{X}$ be a CW complex supporting $R = \Bbbk[\mathcal{M}]$, and X the underlying topological space of $\mathcal{X}$. Then we have $[H_{\mathfrak{m}}^i(R)]_0 \cong \widetilde{H}^{i-1}(X; \Bbbk)$, where 0 is the zero element of $\mathcal{L}$ and $\widetilde{H}^{i-1}(X; \Bbbk)$ is the ith reduced cohomology of X with the coefficients in $\Bbbk$.*

Proof. Since $[T_\sigma^{-1} R]_0 = \Bbbk$ for all $\sigma \in \mathcal{X}$, the cochain complex $[\check{C}_R^\bullet]_0$ of $\Bbbk$-vector spaces is isomorphic to the reduced cochain complex of $\mathcal{X}$ with the coefficients in $\Bbbk$. Hence the assertion follows from Proposition 6.2. $\square$

For $M \in \operatorname{Mod}_{\mathcal{L}} R$, set $M_{-|\widetilde{\mathcal{M}}|} := \bigoplus_{a \in |\widetilde{\mathcal{M}}|} M_{-a}$. Since $M_{-|\widetilde{\mathcal{M}}|}$ is not an R-module in general, we just regard it as an $\mathcal{L}$-graded $\Bbbk$-vector space.

Lemma 6.4. *If a toric face ring $R = \Bbbk[\mathcal{M}]$ is seminormal, then we have*

$$H^i_{\mathfrak{m}}(R) = \left[H^i_{\mathfrak{m}}(R)\right]_{-|\widetilde{\mathcal{M}}|}$$

for all i.

Proof. We use the same idea as the proof of Theorem 5.11. Let $\operatorname{Sq} R$ be the category of squarefree R-modules. (See Definition 5.9.)

Let $\operatorname{Vect}_{\mathcal{L}} \Bbbk$ be the category of $\mathcal{L}$-graded $\Bbbk$-vector spaces, and $(-)_{-|\widetilde{\mathcal{M}}|} : \operatorname{Mod}_{\mathcal{L}} R \to \operatorname{Vect}_{\mathcal{L}} \Bbbk$ the functor which sends M to $M_{-|\widetilde{\mathcal{M}}|}$. We also have the forgetful functor $\mathbf{U} : \operatorname{Mod}_{\mathcal{L}} R \to \operatorname{Vect}_{\mathcal{L}} \Bbbk$.

Now, for each $i \in \mathbb{Z}$, we define the following two functors from $\mathsf{D}^b(\operatorname{Sq} R)$ to $\operatorname{Vect}_{\mathcal{L}} \Bbbk$:

$$\mathbf{F}_i : \mathbf{U} \circ H^i\left(- \otimes_R \check{C}^\bullet_R\right) \quad \text{and} \quad \mathbf{F}'_i : \left[H^i\left(- \otimes_R \check{C}^\bullet_R\right)\right]_{-|\widetilde{\mathcal{M}}|}.$$

Since $V_{-|\widetilde{\mathcal{M}}|}$ is a subspace of $V \in \operatorname{Mod}_{\mathcal{L}} \Bbbk$, we have the natural transformation $\Psi_i : \mathbf{F}'_i \to \mathbf{F}_i$. Since R is seminormal, $\Bbbk[\mathbf{M}_\sigma]$ is seminormal for all σ by Proposition 5.1. Hence $[H^i_{\mathfrak{m}}(\Bbbk[\mathbf{M}_\sigma])]_{-|\widetilde{\mathcal{M}}|} = H^i_{\mathfrak{m}}(\Bbbk[\mathbf{M}_\sigma])$, in fact, we have $[H^i_{\mathfrak{m}}(\Bbbk[\mathbf{M}_\sigma])]_{-C_\sigma} = H^i_{\mathfrak{m}}(\Bbbk[\mathbf{M}_\sigma])$ by Theorem 2.1. It means that $\Psi_i(\Bbbk[\mathbf{M}_\sigma])$ is an isomorphism, and hence Ψ_i is a natural isomorphism by the same reason as in the proof of Theorem 5.11. In particular, $\Psi_i(R) : \mathbf{F}'_i(R) \to \mathbf{F}_i(R)$ is an isomorphism. Hence $\mathbf{F}'_i(R) = [H^i_{\mathfrak{m}}(R)]_{-|\widetilde{\mathcal{M}}|}$ and $\mathbf{F}_i(R) = H^i_{\mathfrak{m}}(R)$ are isomorphic. $\square$

Proposition 6.5. *Let $R = \Bbbk[\mathcal{M}]$ be a toric face ring, and ${}^+R$ its seminormalization. Then we have*

$$H^i_{\mathfrak{m}}({}^+R) \cong \left[H^i_{\mathfrak{m}}(R)\right]_{-|\widetilde{\mathcal{M}}|}$$

as $\mathcal{L}$-graded $\Bbbk$-vector spaces for all i.

Proof. It is easy to see that

$$\left\{a \in |\widetilde{\mathcal{M}}| \mid \left[T_\sigma^{-1} R\right]_{-a} \neq 0\right\} = \mathbb{Z}\mathbf{M}_\sigma \cap C_\sigma = \left\{a \in |\widetilde{\mathcal{M}}| \mid \left[T_\sigma^{-1}({}^+R)\right]_{-a} \neq 0\right\}$$

for all $\sigma \in \mathcal{X}$. Hence we have $(\check{C}^\bullet_R)_{-a} = (\check{C}^\bullet_{{}^+R})_{-a}$ for all $a \in |\widetilde{\mathcal{M}}|$. Now the assertion follows from the following computation;

$$\left[H^i_{\mathfrak{m}}(R)\right]_{-|\widetilde{\mathcal{M}}|} \cong \left[H^i\left(\check{C}^\bullet_R\right)\right]_{-|\widetilde{\mathcal{M}}|} \cong \left[H^i\left(\check{C}^\bullet_{{}^+R}\right)\right]_{-|\widetilde{\mathcal{M}}|} \cong \left[H^i_{\mathfrak{m}}({}^+R)\right]_{-|\widetilde{\mathcal{M}}|} \cong H^i_{\mathfrak{m}}({}^+R).$$

Here the second "≅" follows from the fact stated above, and the last one is Lemma 6.4. $\square$

Remark 6.6. In some sense, Proposition 6.5 generalizes and refines the results and the problem in Section 4 of Nguyen [12] (especially, [12, Theorem 4.3]). However, the toric face rings in [12] are assumed to have nice multigradings, while the "$\mathcal{L}$-grading" of our $\Bbbk[\mathcal{M}]$ is not the grading in the usual sense.

Corollary 6.7. *Let $R = \Bbbk[\mathcal{M}]$ be a toric face ring, and ${}^+R$ its seminormalization. If R is Cohen–Macaulay, then so is ${}^+R$.*

Proof. We prove the contrapositive: if ${}^+R$ is not Cohen–Macaulay, then neither is R. Assume that ${}^+R$ is not Cohen–Macaulay. Then there is some $0 \le i < \dim R$ with $H^{-i}({}^+I^\bullet_{{}^+R}) \neq 0$. For $a \in |\widetilde{\mathcal{M}}|$, the cochain complex $[{}^+I^\bullet_{{}^+R}]_a$ of $\Bbbk$-vector spaces is isomorphic to the $\Bbbk$-dual of $[\check{C}^\bullet_{{}^+R}]_{-a}$. Hence it follows that $H^i_{\mathfrak{m}}({}^+R) \neq 0$. By Proposition 6.5, we have $H^i_{\mathfrak{m}}(R) \neq 0$, and hence the localization $R_{\mathfrak{m}}$ is not Cohen–Macaulay. □

Proposition 6.8. *For a monoidal complex $\mathcal{M} = \{\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$, set $\widetilde{\mathcal{M}} := \{\mathbf{L}_\sigma \cap C_\sigma\}_{\sigma \in \mathcal{X}}$ as before. Let $R := \Bbbk[\mathcal{M}]$ and $\widetilde{R} := \Bbbk[\widetilde{\mathcal{M}}]$ be their toric face rings. If R is Cohen–Macaulay, then so is $\widetilde{R}$. Moreover, $H^i_{\mathfrak{m}}(\widetilde{R}) \neq 0$ implies $H^i_{\mathfrak{m}}(R) \neq 0$.*

Lemma 6.9. *With the same notation as in Proposition 6.8, $H^i(D^\bullet_{\widetilde{R}}) \neq 0$ implies $H^i({}^+I^\bullet_R) \neq 0$.*

Proof. Recall that $D^\bullet_{\widetilde{R}} \cong I^\bullet_{\widetilde{R}}$. If $H^i(D^\bullet_{\widetilde{R}}) (\cong H^i(I^\bullet_{\widetilde{R}})) \neq 0$, then there is $a \in |\widetilde{\mathcal{M}}|$ with $[H^i(I^\bullet_{\widetilde{R}})]_a \neq 0$. Set $\sigma := \operatorname{supp}(a)$ (i.e., $a \in \widetilde{\mathbf{M}}_\sigma \cap \operatorname{int}(C_\sigma)$). Since $H^i(I^\bullet_{\widetilde{R}})$ is a squarefree $\widetilde{R}$-module, we have $[H^i(I^\bullet_{\widetilde{R}})]_a \cong [H^i(I^\bullet_{\widetilde{R}})]_b$ for all $b \in |\widetilde{\mathcal{M}}|$ with $\operatorname{supp}(b) = \sigma$.

For $b \in \mathbf{M}_\sigma$ with $\operatorname{supp}(b) = \sigma$, we have $b \in \mathbf{M}_\tau$ for all $\tau \in \mathcal{X}$ with $\tau \ge \sigma$. In this case, regarding $b \in |\mathcal{M}| \subset |\widetilde{\mathcal{M}}|$, we have $[{}^+I^\bullet_R]_b = [I^\bullet_{\widetilde{R}}]_b$ as cochain complexes of $\Bbbk$-vector spaces, and hence $[H^i({}^+I^\bullet_R)]_b \cong [H^i(I^\bullet_{\widetilde{R}})]_b \neq 0$. □

The proof of Proposition 6.8. By Corollary 6.7, we may assume that R is seminormal. Then ${}^+I^\bullet_R \cong D^\bullet_R$ by Theorem 5.2, and the assertion easily follows from Lemma 6.9. □

Let $R = \Bbbk[\mathcal{M}]$ be a general toric face ring, ${}^+R = \Bbbk[{}^+\mathcal{M}]$ its seminormalization, and $\widetilde{R} = \Bbbk[\widetilde{\mathcal{M}}]$. Proposition 6.8 and Corollary 6.7 state that

$$R \text{ is Cohen–Macaulay} \quad \Longrightarrow \quad {}^+R \text{ is Cohen–Macaulay} \quad \Longrightarrow \quad \widetilde{R} \text{ is Cohen–Macaulay.}$$

By a result of Caijun [6] (see also [13]), the Cohen–Macaulay property of $\widetilde{R}$ is a topological property of the underlying space X of $\mathcal{X}$, while it may depend on $\operatorname{char}(\Bbbk)$.

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