On group gradings on PI-algebras

Eli Aljadeff [1], Ofir David [*]

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

A R T I C L E   I N F O

A B S T R A C T

We show that there exists a constant K such that for any PI-algebra W and any nondegenerate G-grading on W where G is any group (possibly infinite), there exists an abelian subgroup U of G with $[G : U] \leq \exp(W)^K$. A G-grading $W = \bigoplus_{g \in G} W_g$ is said to be nondegenerate if $W_{g_1} W_{g_2} \cdots W_{g_r} \neq 0$ for any $r \geq 1$ and any r tuple $(g_1, g_2, \ldots, g_r)$ in G^r.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

In the last two decades there were significant efforts to extend important results in the theory of polynomial identities for (ordinary) associative algebras to G-graded algebras, where G is a *finite group*, and more generally to H-comodule algebras where H is a finite dimensional Hopf algebra. For instance Kemer's representability theorem and the solution of the Specht problem were established for G-graded associative algebras over a field of characteristic zero (see [5,23]). Recall that Kemer's representability theorem says that any associative PI-algebra over a field F of characteristic zero is PI-equivalent to the Grassmann envelope of a finite dimensional $\mathbb{Z}_2$-graded algebra A over some field extension L of F (see below the precise statement and Proposition 3.1). Another instance

---

* Corresponding author.
  E-mail addresses: aljadeff@tx.technion.ac.il (E. Aljadeff), ofirdav@tx.technion.ac.il (O. David).

of these efforts is the proof of Amitsur's conjecture which was originally proved for ungraded associative algebras over F by Giambruno and Zaicev [14], and was extended to the context of G-graded algebras by Giambruno, La Mattina and the first named author of this article (see [4,12]) and considerable more generally for H-comodule algebras by Gordienko [17]. Amitsur's conjecture states that the sequence $c_n^{1/n}$, where $c_n = c_n(W)$ is the nth term of the codimension sequence of W, has an integer limit (denoted by $\exp(W)$).

In [2] a different point of view was considered (in combining PI-theory and G-gradings, still under the condition that G is finite), namely asymptotic PI-theory was applied in order to prove invariance of the order of the grading group on an associative algebra whenever the grading is minimal regular (as conjectured by Bahturin and Regev [6]). In fact, it is shown there that the order of the grading group coincides with $\exp(W)$.

Suppose now G is arbitrary (i.e. not necessarily finite). Our goal in this paper, roughly speaking, is to exploit the invariant $\exp(W)$ of the algebra W in order to put an effective bound on the minimal index of an abelian subgroup of G whenever the algebra W admits a G-gradings satisfying a natural condition which we call nondegenerate (see Definition 1.1). Our results extend considerable known results for PI group algebras (which are obviously nondegenerately G-graded). Let us remark here that a big part of our analysis is devoted to the case where the group G is finite (a case where Kemer and asymptotic PI theory can be applied) and then we pass to infinite groups.

In this paper we only consider fields of characteristic zero. Let W be an associative PI-algebra over a field F. Suppose $W \cong \bigoplus_{g \in G} W_g$ is G-graded where G is arbitrary.

Definition 1.1. We say the G-grading on W is *nondegenerate* if for any positive integer r and any tuple $(g_1, \ldots, g_r) \in G^{(r)}$, we have $W_{g_1} W_{g_2} \cdots W_{g_r} \neq 0$.

Theorem 1.2 *(Main theorem). There exists an integer K such that for any PI-algebra W and for any nondegenerate G-grading on W by any group G, there exists an abelian subgroup U of G with $[G : U] \leq \exp(W)^K$.*

It is known (and not difficult to prove; see Lemma 4.5) that if a group G has an abelian subgroup of index n, then it contains a characteristic abelian subgroup whose index is bounded by a function of n. We therefore have the following corollary.

Corollary 1.3. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for any PI-algebra W and for any nondegenerate G-grading on W by any group G, there exists a characteristic abelian subgroup U of G with $[G : U] \leq f(\exp(W))$.*

In order to put our main result in an "appropriate" context, we recall (i) different type of G-gradings on associative algebras, (ii) three conditions on groups which are closely related to the content of the main theorem, namely n-permutability, n-rewritability and PI_n (the group algebra FG satisfies a polynomial identity of degree n, $char(F) = 0$).

A G-grading on $W \neq 0$ is called *strong* if $W_g W_h = W_{gh}$ for every $g, h \in G$. Note that this condition is considerable stronger than a nondegenerate grading. For instance, the well known $\mathbb{Z}_2$-grading on the infinite dimensional Grassmann algebra is nondegenerate but not strong. The fact that the $\mathbb{Z}_2$-grading on the Grassmann algebra is nondegenerate will play an important role in the proof of the main theorem. Strong grading is considerably weaker than *crossed product grading* which requires that every homogeneous component has an invertible element (e.g. group algebras). In the other direction we may consider conditions on G-gradings which are weaker than nondegenerate G-gradings as G-gradings where $W_g \neq 0$ for every $g \in G$ (call it *connected grading*). A somewhat stronger condition to the latter but yet weaker than nondegenerate grading is a condition which we call bounded nondegenerate: by definition a G-grading on an algebra W is *bounded nondegenerate* if any product of homogeneous components $W_{g_1} W_{g_2} \cdots W_{g_r}$ does not vanish unless $r > r_0$ for some (large) fixed integer r_0. We thus have *crossed product grading $\Rightarrow$ strong grading $\Rightarrow$ nondegenerate grading $\Rightarrow$ bounded nondegenerate grading $\Rightarrow$ connected grading*.

In Section 5 we show that if a PI algebra W is "bounded nondegenerately" G-graded than the main theorem is false in general.

Definition 1.4. (See [10,11,21].) Let $n > 1$ be an integer.

(1) We say that a group G is n-permutable (resp. n-rewritable), denoted by P_n (resp. Q_n), if for any n-tuple $(g_1, \ldots, g_n) \in G^{(n)}$ there exists a nontrivial permutation $\sigma \in Sym(n)$ (resp. distinct permutations $\sigma, \tau \in Sym(n)$) such that

$$g_1 g_2 \cdots g_n = g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)} \in G$$

(resp.

$$g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)} = g_{\tau(1)} g_{\tau(2)} \cdots g_{\tau(n)} \in G).$$

(2) We say that a group G satisfies PI_n if the group algebra FG satisfies a (multilinear) identity of degree n (it is well known that since F is a field of characteristic zero, the T-ideal of identities is generated by multilinear polynomials).

Clearly, $P_n \Rightarrow P_{n+1}$, $Q_n \Rightarrow Q_{n+1}$ and $P_n \Rightarrow Q_n$. We say that group is permutable (resp. rewritable, PI), if it is n-permutable (resp. n-rewritable, PI_n) for some n. We denote (with a slight abuse of notation) by P, Q, PI the families of all permutable, rewritable or PI groups. It was proved in [11] that if a group is n-rewritable then it is m-permutable where m is bounded by a function of n.

As for the condition PI_n, it is easy to show that if FG satisfies a (multilinear) polynomial identity of degree n then the group G is n-permutable and in particular n-rewritable (indeed, if $f(x_1, \ldots, x_n) = x_1 \cdots x_n + \sum_{e \neq \sigma \in Sym(n)} \alpha_\sigma x_\sigma(1) \cdots x_\sigma(n)$, $\alpha_\sigma \in F$, is a multilinear identity of FG and $(g_1, \ldots, g_n) \in G^{(n)}$ is any nth tuple, the evaluation $x_i = g_i$,

$i = 1, \ldots, n$, yields $g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)}$ for some $e \neq \sigma \in Sym(n)$). Thus we have that $PI_n \Rightarrow P_n \Rightarrow Q_n$. As for the reverse direction of arrows the following is known (see [11]).

(1) Q_n is strictly weaker than P_n (although, as mentioned above, there exists a function f such that $Q_n \Rightarrow P_{f(n)}$).
(2) $P_n \nRightarrow PI_m$ for any n and m. In particular it is known that if G satisfies PI_m, then G has a finite index abelian subgroup whose index is bounded by a function of m whereas for any $n > 2$ there exists an infinite family of finite groups $\{G_i\}_i$ which satisfy P_n, whose PI degree is d_i and $\lim d_i = \infty$ (the PI degree of G is the minimal degree of a nontrivial polynomial identity of FG).

Remark 1.5. As for the existence of a finite index abelian subgroup in G and the permutability or rewritability conditions there is an interesting distinction between finitely/nonfinitely generated groups. If G is finitely generated and satisfies P_n (or Q_n) then it has an abelian subgroup of finite index (note however, as mentioned above, the index is not bounded by a function of n; see example in Section 5). If G is not finitely generated, it may not have a finite index abelian subgroup. However it does have a characteristic subgroup H whose index $[G : H]$ is bounded by a function of n and whose commutator subgroup H' is finite, and its order is bounded by a function of n.

In view of the above considerations it is natural to introduce the following condition on a group G.

Definition 1.6. Let G be any group. We say that G satisfies T_n if there exists a PI algebra W of PI degree n which admits a nondegenerate G-grading. We say that G has T if it has T_n for some n.

It is easy to see that the argument which shows $PI_n \Rightarrow P_n$ shows also that $T_n \Rightarrow P_n$. This simple fact will play an important role while extending the proof of the main theorem from finitely generated residually finite groups to arbitrary finitely generated groups.

Note that since the group algebra FG is nondegenerately G-graded we have $PI_n \Rightarrow T_n$. In the other direction it follows from our main theorem that if G satisfies T_n, then G has PI_m for some m (indeed, G is abelian by finite and hence, by [20], the group algebra FG is PI). As for the relation between m and n we have the following result.

Theorem 1.7. *Let G be any group and suppose it grades nondegenerately a PI algebra W of PI degree n. Then the group algebra FG is PI and its PI degree is bounded by n^2. Similarly, $\exp(FG) \leq \exp(W)^2$.*

It is somewhat surprising that $T_n \nRightarrow PI_n$ (intuitively, the group algebra FG seems to be the "smallest or simplest" G-graded algebra whose grading is nondegenerate). The following example shows that twisted group algebras may have lower PI degree.

Example 1.8. Let A_4 be the alternating group of order 12. It is known that the largest irreducible complex representation is of degree 3 and hence by Amitsur–Levitzki theorem the PI degree is 6. On the other hand, the group A_4 admits a nontrivial cohomology class $\alpha \in H^2(A_4, \mathbb{C}^*)$ (corresponding to the binary tetrahedral group of order 24). Since twisted group algebras with nontrivial cohomology class cannot admit the trivial representation, we have $\mathbb{C}^\alpha A_4 \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and hence the PI degree is 4.

Conjecture 1.9. *Let W be an algebra over an algebraically closed field F of characteristic zero satisfying a PI of degree n. Suppose W is nondegenerately graded by a group G. Then there exists a class $\alpha \in H^2(G, F^*)$ such that the twisted group algebra $F^\alpha G$ has PI degree bounded by the same integer n.*

Theorem 1.10. *Notation as above. The conjecture holds whenever the group G is finite.*

The main tools used in the proof of the main theorem are the representability theorem for G-graded algebras where G is a *finite group* [5] and Giambruno and Zaicev's result on the exponent of W [14]. The representability theorem allows us to replace the G-graded algebra W by a finite dimensional G-graded algebra A (or the Grassmann envelope of a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra A) whereas Giambruno and Zaicev's result provides an interpretation of $\exp(W)$ in terms of the dimension of a certain subalgebra of A. The proof of Theorem 1.2 in case the group G is finite is presented in Section 3. In Section 4 we show how to pass from finite groups to arbitrary groups and by this we complete the proof of Theorem 1.2.

In Section 2 we recall some background on group gradings and PI theory needed for the proofs of the main results of the paper. In the last section of the paper, Section 5, we present (1) a family of n-permutable with no uniform bound on the index of abelian subgroups and (2) an example which shows that we cannot replace in the main theorem nondegenerate G-gradings with bounded nondegenerate G-grading.

We close the introduction by explaining why one would prefer bounding the index of an abelian subgroup by a function of the $\exp(W)$ (as in the main theorem) rather than by the PI degree of W. It is known that $\exp(W)$ is bounded by a function of the PI degree (e.g. $\exp(W) \le (PI\deg(W) - 1)^2$, see Theorem 4.2.4 [15]) but such function does not exist in the reverse direction. Indeed, since $\exp(W)$ is an *asymptotic* invariant it remains invariant if we consider the G-graded T-ideal generated by all polynomials in $\mathrm{Id}_G(W)$ of degree at least m (any m) whereas the G-graded PI degree and hence the ordinary PI degree is at least m.

2. Background and some preliminary reductions

We start by recalling some facts on G-graded algebras W over a field F of characteristic zero and their corresponding G-graded identities. We refer the reader to [5] for a detailed account on this topic.

Remark 2.1. In this section we consider only finite groups. Although some of the basic results in G-graded PI theory hold for arbitrary groups, one of our main tools, namely the "representability theorem" for G-graded PI algebras, is false for infinite groups.

2.1. G-graded identities

Let W be a PI-algebra over F. Suppose W is G-graded where G is a finite group. Denote by $I = \mathrm{Id}_G(W)$ the ideal of G-graded polynomial identities of W. It consists of all elements in the free G-graded algebra $F\langle X_G \rangle$ over F, that vanish upon any admissible evaluation on W. Here, $X_G = \bigcup_{g \in G} X_g$ and X_g is a set of countably many variables of degree g. An evaluation on W is admissible if the variables from X_g are replaced only by elements of W_g. The ideal I is a G-graded T-ideal, i.e. it is invariant under all G-graded endomorphisms of $F\langle X_G \rangle$.

We recall from [5] that the G-graded T-ideal I is generated by multilinear polynomials. Consequently, it remains invariant when passing to any field extension L of F, that is $\mathrm{Id}_G(W \otimes_F L) = \mathrm{Id}_G(W) \otimes_F L$.

The following observations play an important role in the proofs.

Observation 2.2. The condition nondegenerate G-grading on W can be easily translated into the language of G-graded polynomial identities. Indeed a G-grading on W is nondegenerate if and only if for any integer r and any tuple $(g_1, \ldots, g_r) \in G^{(r)}$, the G-graded multilinear monomial $x_{g_1,1} \cdots x_{g_r,r}$ is a G-graded *nonidentity* of W (in short we say that $\mathrm{Id}_G(W)$ contains no multilinear G-graded monomials). Consequently, if G-graded algebras W_1 and W_2 are G-graded PI-equivalent (i.e. have the same T-ideal of G-graded identities), then the grading on W_1 is nondegenerate if and only if the grading on W_2 is nondegenerate.

Observation 2.3. If W_1, W_2 are two G-graded algebras with $\mathrm{Id}_G(W_1) = \mathrm{Id}_G(W_2)$, then $\mathrm{Id}(W_1) = \mathrm{Id}(W_2)$ (the ungraded identities). In particular we have $\exp(W_1) = \exp(W_2)$. Indeed, this follows easily from the fact that a polynomial $p(x_1, \ldots, x_n)$ is an ungraded identity of an algebra W with a G-grading if and only if the polynomial $p(\sum_{g \in G} x_{g,1}, \ldots, \sum_{g \in G} x_{g,n})$ is a graded identity of W as a G-graded algebra.

As noted above, the nondegeneracy condition satisfied by a G-grading on W depends only on the T-ideal of G-graded identities, hence if the grading on a G-graded algebra W over a field F is nondegenerate, the same holds for the G-graded algebra $W_L = W \otimes_F L$. Similarly, the numerical invariant $\exp(W)$ of the algebra W remains unchanged if we extend scalars.

Remark 2.4. In the main steps of the proof (in case G is a finite group), roughly speaking, we "pass" to simpler algebras without increasing too much the exponent or the PI degree. More precisely, given an arbitrary G-grading on a PI algebra W we first pass to a finite

dimensional G-graded algebra A, then to a G-simple algebra and finally to a group algebra, a case which was solved by Gluck using the classification of finite simple groups (see [16]).

Let us recall some terminology and some facts from Kemer's theory extended to the context of G-graded algebras as they appear in [5].

Let W be a G-graded algebra over F. Suppose that W is PI (as an ungraded algebra). Kemer's representability theorem for G-graded algebras assures that there exists a field extension L/F and a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra A over L such that the Grassmann envelope $E(A)$ (with respect to the $\mathbb{Z}_2$-grading) yields a G-graded algebra which is G-graded PI-equivalent to W_L (see Proposition 3.1). In case the algebra W is affine, or more generally in case it satisfies a Capelli identity (it is known that any affine PI algebra satisfies a Capelli identity), there exists a field extension L/F such that the algebra W_L is G-graded PI-equivalent to a finite dimensional G-graded algebra A over L. This result will be used to reduce our discussion from infinite dimensional algebras to finite dimensional ones in case the group G is finite. As extensions of scalars do not change the exponent (nor the PI-degree) we assume that the field L is algebraically closed.

2.2. G-simple algebras

The next ingredient we need is a result of Bahturin, Sehgal and Zaicev, which determines the G-graded structure of finite dimensional G-simple algebra over an algebraically closed field of characteristic zero.

Let A be the algebra of $r \times r$-matrices over F and let G be any group (here, G may be infinite). Fix an r-tuple $\alpha = (g_1, \ldots, g_r) \in G^{(r)}$. Consider the G-grading on A given by

$$A_g = \operatorname{span}_F\{e_{i,j} : g = g_i^{-1}g_j\}.$$

One checks easily that this indeed determines a G-grading on A. Clearly, since the algebra A is simple, it is G-simple as a G-graded algebra.

Next we present a different type of G-gradings on semisimple algebras which turn out to be G-simple. Let H be any finite subgroup of G and consider the group algebra FH. By Maschke's theorem FH is semisimple and of course H-simple (any nonzero homogeneous element is invertible). More generally we consider twisted group algebras $F^\alpha H$ as H-graded algebras, where α is a 2-cocycle in $Z^2(H, F^*)$ (H acts trivially on F). Recall that $F^\alpha H = \operatorname{span}_F\{U_h : h \in H\}$, $U_{h_1}U_{h_2} = \alpha(h_1, h_2)U_{h_1 h_2}$, for all $h_1, h_2 \in H$. We say that the basis $\{U_h : h \in H\}$ *corresponds to the 2-cocycle* α. Finally, we may view the twisted group algebra $F^\alpha H$ as a G-graded algebra by setting $A_g = 0$ for $g \in G \setminus H$ and as such it is G-simple. We refer to the G-grading on $F^\alpha H$ as a *fine grading* (i.e. every homogeneous component is of dimension ≤ 1).

Remark 2.5. In the sequel, whenever we say that $\{U_h : h \in H\}$ is a basis of $F^\alpha H$, we mean that the basis corresponds to the cocycle α. One knows that in general a homogeneous basis of that kind corresponds to a cocycle α' cohomologous to α.

In case the field F is algebraically closed of characteristic zero, we have that these two gradings (*elementary* and *fine*) are the building blocks of any G-grading on a finite dimensional algebra so that it is G-simple. This is a theorem of Bahturin, Sehgal and Zaicev.

Theorem 2.6. *(See [7].) Let A be a finite dimensional G-graded simple algebra. Then there exists a finite subgroup H of G, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer r and an r-tuple $(g_1, g_2, \ldots, g_r) \in G^{(r)}$ such that A is G-graded isomorphic to $\Lambda = F^\alpha H \otimes M_r(F)$ where $\Lambda_g = \mathrm{span}_F\{U_h \otimes e_{i,j} \mid g = g_i^{-1} h g_j\}$. Here $U_h \in F^\alpha H$ is a representative of $h \in H$ and $e_{i,j} \in M_r(F)$ is the (i,j) elementary matrix.*

In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity element of A are homogeneous of degree $e \in G$.

2.3. Asymptotic PI-theory

The last ingredient we need is Regev, Giambruno and Zaicev's PI-asymptotic theory. Let W be an ordinary PI-algebra over an algebraically closed field F of characteristic zero and let $\mathrm{Id}(W)$ be its T-ideal of identities. Consider the $n!$-dimensional vector space

$$P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} : \sigma \in Sym(n)\}$$

and let $c_n(W) = \dim_F(P_n/P_n \cap \mathrm{Id}(W))$ be the nth term of the codimension sequence of the algebra W. It was proved by Regev in 1972 [22] that the sequence $\{c_n(W)\}$ is exponentially bounded and conjectured by Amitsur that the limit $\lim_{n\to\infty} c_n^{1/n}$ exists (the exponent of W) and is a nonnegative integer. The conjecture was established by Giambruno and Zaicev in the late 1990s by showing that the limit coincides, roughly speaking, with the dimension of a certain subspace "attached" to W. In particular, for a matrix algebra $M_d(F)$ we have $\exp(M_d(F)) = d^2$ and by the Amitsur–Levitzki theorem it has PI-degree $2d$. Any finite dimensional G-simple algebra is a direct product of matrix algebra (as an ungraded algebra), hence its T-ideal of identities coincides with the ideal of identities (and therefore the exponent and PI-degree) of the largest matrix algebra appearing in its decomposition.

Remark 2.7. It follows from the Amitsur–Levitzki theorem that if A is a finite dimensional G-simple algebra A we have $\exp(A) = \frac{1}{4}(PI\,deg(A))^2$. For an arbitrary PI-algebra we only have the bound $\exp(A) \leq (PI\,deg(A) - 1)^2$. Recall (from the last paragraph of the introduction) that the PI-degree cannot be bounded from above by any function of $\exp(A)$.

3. Proof of main theorem – finite groups

All groups considered in this section are finite.

For a PI-algebra W over a field F of characteristic zero we denote by $\exp(W)$ its exponent.

Proposition 3.1. *Let W be a PI-algebra over a field F. Suppose W is graded nondegenerately by a group G. Then there exists a field extension L of F and a finite dimensional L-algebra W_0 which is nondegenerately G-graded, such that $\exp(W_0) \leq \exp(W)$.*

Proof. Let us consider first the case where W is affine. Applying [5] there exists a finite dimensional G-graded algebra B over a field extension L of F such that $\mathrm{Id}_G(W \otimes_F L) = \mathrm{Id}_G(B)$. Clearly, we may assume that L is algebraically closed by further extending the scalars if needed. Next, by Observations 2.2 and 2.3 we know that the G-grading on B is nondegenerate and $\exp(W_L) = \exp(B)$, thus proving the proposition for this case.

Suppose now that W is arbitrary (i.e., not necessarily affine). By [5] there exists a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra $C \cong \bigoplus_{(\epsilon,g) \in \mathbb{Z}_2 \times G} C_{(\epsilon,g)}$ over an (algebraically closed) field extension L of F such that W_L is G-PI-equivalent to $E(C) = (E_0 \otimes C_0) \oplus (E_1 \otimes C_1)$ (the Grassmann envelope of C) where $C_0 = \bigoplus_{g \in G} C_{(0,g)}$ and $C_1 = \bigoplus_{g \in G} C_{(1,g)}$. The G-grading on $E(C)$ is given by

$$E(C)_g = (E_0 \otimes C_{(0,g)}) \oplus (E_1 \otimes C_{(1,g)}).$$

We claim that the G-grading on C is nondegenerate (where $C_g = C_{(0,g)} \oplus C_{(1,g)}$). To this end fix an nth tuple $(g_1, \ldots, g_n) \in G^{(n)}$. By linearity we need to show that at least one of the 2^n monomials of the form

$$x_{(\epsilon_1,g_1),1} x_{(\epsilon_2,g_2),2} \cdots x_{(\epsilon_n,g_n),n}$$

is not in $\mathrm{Id}_{\mathbb{Z}_2 \times G}(C)$. Let us show that if this is not the case, then the monomial $x_{g_1,1} \cdots x_{g_n,n}$ is a G-graded identity of $E(C)$, contradicting the fact that the G-grading on $E(C)$ and hence on W is nondegenerate. To see this consider the evaluation $x_{g_i,i} = z_{0,i} \otimes a_{0,i} + z_{1,i} \otimes a_{1,i}$ for $i = 1, \ldots, n$ where $z_{\epsilon,i} \in E_\epsilon$ and $a_{\epsilon,i} \in C_{(\epsilon,g_i)}$. This evaluation yields an expression with 2^n summands of the form

$$z_{\epsilon_1,i_1} z_{\epsilon_2,i_2} \cdots z_{\epsilon_n,i_n} \otimes a_{(\epsilon_1,i_1)} a_{(\epsilon_2,i_2)} \cdots a_{(\epsilon_n,i_n)}$$

which are all zero and the claim follows.

Finally, by a theorem of Giambruno and Zaicev (see [14, proof of main theorem] or [1, Theorem 2.3]) we have that $\exp(C) \leq \exp_{\mathbb{Z}_2}(C) = \exp(E(C)) = \exp(W)$, which is precisely what we need. $\quad\square$

Our next step is to reduce the main theorem from finite dimensional algebras to G-simple algebras.

Proposition 3.2. *Let W be a finite dimensional PI F-algebra graded nondegenerately by a group G. Then there exists a G-simple algebra W_0 such that $\mathrm{Id}_G(W) \subseteq \mathrm{Id}_G(W_0)$ and the grading on W_0 is nondegenerate (in fact W_0 is a homomorphic image of W). In particular $\exp(W_0) \leq \exp(W)$.*

Proof. Denote by $J = J(W)$ the Jacobson radical of W. Since the characteristic of the field is zero, it is known that J is G-graded and so W/J is a semi-simple G-graded algebra (see [8]).

We claim that the G-grading on W/J is still nondegenerate. Indeed, if W/J satisfies a monomial identity f, then any evaluation of this monomial on W yields an element in J. Since J is nilpotent (say of nilpotency degree is k) we have that the product of k copies of f (with distinct variables) is a monomial identity of W. This contradicts the assumption that the G-grading on W is nondegenerate and the claim is proved.

The algebra W/J is G-semisimple and therefore a direct product of G-simple algebra $\prod_1^n A_i$. If for each i there is a multilinear monomial identity f_i of A_i, then the product $\prod f_i$ is a multilinear monomial identity of W/J, contradicting our assumption on the G-grading on W/J. Consequently, there is an i such that A_i is nondegenerately G-graded. Letting $W_0 = A_i$ we have $\mathrm{Id}_G(W) \subseteq \mathrm{Id}_G(W/J) \subseteq \mathrm{Id}_G(W_0)$ as desired. $\square$

In the next lemma we characterize (in terms of Bahturin, Sehgal and Zaicev's theorem) when the grading on a G-simple algebra is nondegenerate. Recall that a G-grading on A is *strong* if for any $g, h \in G$ we have $A_g A_h = A_{gh}$.

Lemma 3.3. *Let $A \neq 0$ be a finite dimensional G-simple algebra. Then the following conditions are equivalent.*

(1) *The G-grading on A is nondegenerate.*
(2) *The G-grading on A is strong. In particular $A_g \neq 0$, for every $g \in G$.*
(3) *Let $F^\alpha H \otimes M_r(F)$ be a presentation of the G-grading on A (as given by Theorem 2.6) where H is a finite subgroup of G and $(g_1, \ldots, g_r) \in G^{(r)}$ is the r-tuple which determines the elementary grading on $M_r(F)$. Then every right coset of H in G is represented in the r-tuple.*

Remark 3.4. Note that in general (i.e. in case the algebra A is not necessarily G-simple) the first two conditions are not equivalent. For instance (as mentioned in the introduction), the $\mathbb{Z}_2$-grading on the infinite dimensional Grassmann algebra is nondegenerate but not strong. Indeed, $E_1 E_1 \subsetneq E_0$ (or $E_0 E_0 \subsetneq E_0$ in case the algebra E is assumed to have no identity element).

Proof of Lemma 3.3. Note that since A is assumed to be finite dimensional G-simple, each one of the conditions (1)–(3) implies that G is finite. As for the 3rd condition of the lemma we replace (as we may by [3, Lemma 1.3]) the given presentation with another so that the r-tuple has the following form

$$(g_{(1,1)}, \ldots, g_{(1,d_1)}, g_{(2,1)}, \ldots, g_{(2,d_2)}, \ldots, g_{(s,1)}, \ldots, g_{(s,d_s)})$$

where:

- $r = d_1 + \cdots + d_s$.
- $g_{i,1} = g_{i,2} = \ldots = g_{i,d_i}$ (denoted by z_i), and for $i \neq k$ the elements $g_{i,j}$, $g_{k,l}$ represent different right H-cosets in G.
- $g_{1,j} = e$ for $j = 1, \ldots, d_1$.

(2) → (1): This is clear.

(1) → (3): Suppose (3) does not hold. We claim there exists a multilinear monomial of degree at most r which is a G-graded identity of $F^\alpha H \otimes M_r(F) = \text{span}_F\{U_h \otimes e_{i,j} : h \in H, 1 \leq i, j \leq r\}$.

It is convenient to view the matrices in $M_r(F)$ as $s \times s$ block matrices corresponding to the decomposition $d_1 + \cdots + d_s = r$. More precisely, let $D_k = d_1 + \cdots + d_k$ and decompose $M_r(F) = \bigoplus_{i,j=1}^s M_{[i,j]}$ into the direct sum of vector spaces $M_{[i,j]} = \text{span}\{e_{k,l} \mid D_{i-1} < k \leq D_i, \ D_{j-1} < l \leq D_j\}$. Note that $M_{[i,j]}$ are submatrices supported on a single block of size $d_i \times d_j$. This decomposition is natural in the sense that $(F^\alpha H \otimes M_r(F))_g$ is the direct sum of the vector spaces $U_h \otimes M_{[i,j]}$ such that $z_i^{-1} h z_j = g$.

For a fixed index $i \in \{1, \ldots, s\}$ and an element $g \in G$, consider the equation $h z_j = z_i g$. It has a solution if and only if $H z_i g$ has a representative in $(z_1, \ldots, z_s)$. It follows that if $U_h \otimes B$ is homogeneous of degree g and $H z_i g$ has no representative in $(z_1, \ldots, z_s)$, then the ith row of blocks in B must be zero.

Consider the multilinear monomial

$$x_{w_1,1} x_{w_2,2} \cdots x_{w_n,n}$$

where $x_{w_i,i}$ is homogeneous of degree $w_i \in G$. We will show there exist $w_i \in G$, $i = 1, \ldots, n$, so that the monomial above is a G-graded identity.

To this end, note that such a monomial (being multilinear) is a G-graded identity if and only if it is zero on graded assignments of the form $x_{w_i,i} = U_{h_i} \otimes A_i$ which span the algebra. In particular the value of $x_{w_1,1} x_{w_2,2} \cdots x_{w_n,n}$ under this assignment is $U_{h_1} \cdots U_{h_n} \otimes A_1 \cdots A_n$ which is zero if and only if $A = A_1 \cdots A_n = 0$. It follows that if for any such homogeneous assignment, the ith row of blocks in the matrix $B_i = A_1 A_2 \cdots A_i$ is zero, then A must be zero (since each of its blocks rows is zero).

Following the argument above we choose $w_i \in G$ such that for each i the right coset $H z_i w_1 \cdots w_i$ (i.e. the right coset of H represented by z_i times the homogeneous degree

of $U_{h_1} \cdots U_{h_i} \otimes B_i$) has no representative in $(z_1, \ldots, z_s)$. Now, by assumption, there is some $z \in G$ such that Hz has no representative in $(z_1, \ldots, z_s)$. Thus, choosing $w_i = (z_i w_1 \cdots w_{i-1})^{-1} z$ we obtain the required result.

$(3) \rightarrow (2)$: Suppose that all right H-cosets are represented in the tuple $(g_1, \ldots, g_r)$. To show that the grading is strong, it is enough to show that any basis element $U_h \otimes e_{i,j}$ can be written as a product in $A_{w_1} A_{w_2}$ where $w_1 \cdot w_2 = g_i^{-1} h g_j$. Indeed, since each right coset has a representative in the tuple $(g_1, \ldots, g_r) \in G^{(r)}$, we can find k such that $g_k \in H g_i w_1 = H g_j w_2^{-1}$. Letting $h_1 = g_i w_1 g_k^{-1}$ and $h_2 = g_k w_2 g_j^{-1}$, we get that $a = U_{h_1} \otimes e_{i,k}, b = U_{h_2} \otimes e_{k,j}$ are in A_{w_1}, A_{w_2} respectively and $a \cdot b = \alpha(h_1, h_2) U_h \otimes e_{i,j}$. The lemma is now proved. $\quad\square$

Our next step is to pass from G-simple algebras to the group algebra FG.

Let $V = \bigoplus V_g$ be a G-graded F-vector space. Then the algebra of endomorphisms $End_F(V)$ has a natural G-grading where an endomorphism $\psi \in End(V)$ has homogeneous degree g if $\psi(V_h) \subseteq V_{gh}$ for every $h \in G$. In particular, this grading on $End(FG)$ is isomorphic to the elementary grading by a tuple $(g_1, \ldots, g_n)$ where each element of G appears exactly once. It is clear that the left regular action of G on FG induces a natural G-graded embedding of FG in $End(FG) \cong M_{|G|}(F)$.

This statement can be generalized as follows.

Lemma 3.5. *Let G be a finite group, H a subgroup and $\{w_1, \ldots, w_k\}$ a complete set of representatives for the right cosets of H in G. Then the group algebra FG can be embedded in $FH \otimes M_k(F)$ where the tuple of the elementary grading is $(w_1, \ldots, w_k)$.*

Proof. For any $g \in G$ and any H-right coset representative $w \in \{w_1, \ldots, w_k\}$, there are $h \in H$ and $w' \in \{w_1, \ldots, w_k\}$ such that $wg = hw'$. We denote these elements by $h := h_{w,g}$ and $w' := w^g$. From associativity of G, we get that

$$h_{w,g_1 g_2} = h_{w,g_1} h_{w^{g_1}, g_2}, \qquad w^{g_1 g_2} = \left(w^{g_1}\right)^{g_2}.$$

Define a map $\psi : FG \rightarrow FH \otimes M_k(F)$ by

$$\psi(U_g) = \sum_{i=1}^{k} V_{h_{w_i,g}} \otimes E_{i,j(i)}$$

where $\{U_g\}_g$ and $\{V_h\}_h$ are the corresponding bases of FG and FH, $E_{i,j(i)}$ is the (i,j) elementary matrix and $j(i)$ is determined by the equation $w_{j(i)} = w_i^g$. It is easy to show (left to the reader) that ψ is a homomorphism. Furthermore, by definition of the G-grading on $FH \otimes M_k(F)$ (see Theorem 2.6), we have that the homogeneous degree of $V_{h_{w_i,g}} \otimes E_{i,j(i)}$ is $w_i^{-1} h_{w_i,g} w_i^g = w_i^{-1} w_i g = g$, and hence ψ is a G-graded map. Finally, since FG is G-simple and $\psi \neq 0$, it follows that ψ is an embedding. $\quad\square$

Returning to our proof, we have a G-simple algebra $F^{\alpha}H \otimes M_k(F)$, nondegenerately G-graded. Recall that this means that any right coset of H in G appears at least once in the tuple corresponding to the elementary grading $\{w_1, \ldots, w_k\}$. If $\alpha = 1$, then by the previous lemma the group algebra FG embeds in $FH \otimes M_k(F)$ and hence $\mathrm{Id}(FG) \supseteq \mathrm{Id}(FH \otimes M_k(F))$. This proves the reduction to FG in that case.

In general (i.e. α not necessarily trivial), FG may not be G-graded embedded in $F^{\alpha}H \otimes M_k(F)$. We might hope however, that even if such an embedding is not possible, still $\exp(FG) \leq \exp(F^{\alpha}H \otimes M_k(F))$. It turns out that this is also false as Example 1.8 shows.

The next lemma shows how to get rid of the 2-cocycle α.

Lemma 3.6. *Let $A = F^{\alpha}H \otimes M_k(F)$ be a nondegenerate G-simple graded algebra. Let $\rho : F^{\alpha}H \to M_d(F)$ be a nonzero (ungraded) representation and denote by $B = M_d(F)$ the trivially G-graded algebra (and therefore trivially H-graded). Then FH can be embedded in $F^{\alpha}H \otimes B$ and FG can be embedded in $A \otimes B$ as H and G-graded algebras respectively.*

Proof. Define the map $\psi : FH \to F^{\alpha}H \otimes M_d(F)$ by $\psi(U_h) = V_h \otimes \rho(V_h^{-1})^t$, where $\{U_h\}_h$ and $\{V_h\}_h$ are the corresponding bases of FH and $F^{\alpha}H$. This is easily checked to be an H-graded homomorphism, and it is an embedding since FH is H-simple. This proves the first claim of the lemma.

The second claim follows from the last lemma using the graded embeddings

$$FG \hookrightarrow FH \otimes M_k(F) \hookrightarrow F^{\alpha}H \otimes M_d(F) \otimes M_k(F) \cong A \otimes B. \qquad \square$$

Corollary 3.7. *Let $A = F^{\alpha}H \otimes M_k(F)$ be a nondegenerate G-simple graded algebra. Then $\exp(FG) \leq \exp(A)^2$.*

Proof. Recall that the exponent of $F^{\alpha}H$ is d^2, where d is the dimension of its largest irreducible representation. It follows that FG can be embedded in $A \otimes M_d(F)$ where $d^2 = \exp(F^{\alpha}H) \leq \exp(A)$ and therefore

$$\exp(FG) \leq \exp\big(A \otimes M_d(F)\big) = \exp(A)\exp\big(M_d(F)\big) \leq \exp(A)^2. \qquad \square$$

Corollary 3.8. *Let W be an associative PI F-algebra nondegenerately G-graded. Then the following hold.*

$$\exp(FG) \leq \exp(W)^2$$

and

$$PI\,deg(FG) \leq 2\big(PI\,deg(W) - 1\big)^2.$$

Proof. The first inequality follows from Propositions 3.1, 3.2 and Corollary 3.7. For the proof of the second inequality recall that in general $\exp(W) \leq (PI\,deg(W) - 1)^2$ (see Theorem 4.2.4 in [15]). Now, since FG is semisimple, it follows by Amitsur–Levitzki theorem that $\frac{1}{2} PI\,deg(FG) = \sqrt{\exp(FG)}$ and so, we conclude that

$$PI\,deg(FG) = 2\sqrt{\exp(FG)} \leq 2\exp(W) \leq 2\big(PI\,deg(W) - 1\big)^2. \qquad \square$$

The last step in our analysis concerns with group algebras. Here we refer to the following result of D. Gluck (see [16]) in which he bounds the minimal index of an abelian subgroup U in G in terms of the maximal character degree of G. We emphasize that the proof uses the classification of finite simple groups.

Theorem 3.9 *(D. Gluck). There exists a constant m with the following property. For any finite group G there exists an abelian subgroup U of G such that $[G : U] \leq b(G)^m$, where $b(G)$ is the largest irreducible character degree of G.*

We can now complete the proof of the main theorem for finite groups.

We note that by Giambruno and Zaicev's result $\exp(FG) = b(G)^2$ and hence, any finite group has an abelian subgroup U with $[G : U] \leq b(G)^m = \exp(FG)^{m/2}$. Combining with our results above, we see that if a PI-algebra W admits a nondegenerate G-grading where G is a finite group, then $\exp(FG) \leq \exp(W)^2$, hence then there is an abelian subgroup U with $[G : U] \leq \exp(W)^m$. In particular, taking $K = m$ where m is determined by the theorem above, will do.

4. Proof of main theorem – infinite groups

In this section we prove the main theorem for arbitrary groups. Let us sketch briefly the structure of our proof. In the preceding section we proved the main theorem for arbitrary finite groups. Our first step in this section is to prove the main theorem for groups which are finitely generated and residually finite. Next, we pass to finitely generated groups (not necessarily residually finite) by the following argument. Any group G which grades nondegenerately a PI algebra is permutable and hence being finitely generated, it is abelian by finite (see [9,10] or Remark 1.5) and hence residually finite. Finally we show how to pass from finitely generated groups to arbitrary groups. We emphasize that the constant K (which appears in the main theorem) remains unchanged when passing from finite groups to arbitrary group.

Proposition 4.1. *Suppose the main theorem holds for arbitrary finite groups with the constant K, that is, for any finite group G and any PI algebra W which is nondegenerately G-graded, there exists an abelian subgroup $U \subseteq G$ with $[G : U] \leq \exp(W)^K$. Then the main theorem holds for finitely generated residually finite groups with the same constant K.*

Proof. Since G is finitely generated, by Hall's theorem [18] there are finitely many subgroups of index $\leq \exp(A)^K$. Denoting these groups by $U_1, \ldots, U_n$, we wish to show that one of them is abelian. Suppose the contrary holds. Hence we can find $g_i, h_i \in U_i$ such that $e \neq [g_i, h_i]$ for any $i = 1, \ldots, n$ and we let N be a normal subgroup of finite index which doesn't contain any of the $[g_i, h_i]$.

Define an induced G/N-grading on A by setting $A_{gN} = \bigoplus_{h \in N} A_{gh}$. Clearly, the induced G/N-grading on A is nondegenerate, thus by the main theorem there is some $U \leq G$ (containing N) such that $[G : U] \leq \exp(A)^K$ and U/N is abelian. By the construction, $U = U_i$ for some i, and we get that $[g_i, h_i] \in N$ – a contradiction. $\square$

The next step is to remove the condition of residually finiteness.

Proposition 4.2. *Suppose the main theorem holds for finitely generated residually finite groups with the constant K. Then the main theorem holds for arbitrary finitely generated groups with the same constant K.*

Proof. As mentioned above this is obtained using permutability.

Let G be a finitely generated group and suppose it grades nondegenerately a PI algebra A. Let us show that G must be permutable. To this end let $f = \sum c_\sigma x_{\sigma(1),1} \cdots x_{\sigma(n),n}$ be a nonzero ordinary identity of A and assume that $c_{id} = 1$. Fix a tuple $g_1, \ldots, g_n \in G$ and consider the graded identity

$$\tilde{f} = f(x_{g_1,1}, \ldots, x_{g_n,n}) = \sum_{h \in G} f_h(x_{g_1,1}, \ldots, x_{g_n,n})$$

where f_h is the h homogenous part of $\tilde{f}$. Since $\tilde{f}$ is a graded identity, its homogenous parts are also graded identities. Letting $g = g_1 \cdots g_n$, the polynomial f_g contains the monomial $x_{g_1,1} \cdots x_{g_n,n}$ (with coefficient 1). Since the grading is nondegenerate, f_g is not a monomial and therefore has another monomial with nonzero coefficient corresponding to some permutation $\sigma \neq id$, hence $g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)}$. This can be done for any tuple of length n, so it follows that G is n-permutable. $\square$

The main theorem now follows from the following proposition.

Proposition 4.3. *Let G be any group and d be a positive integer. Suppose that any finitely generated subgroup H of G contains an abelian subgroup U_H with $[H : U_H] \leq d$. Then there exists an abelian subgroup U of G with $[G : U] \leq d$.*

Remark 4.4. The proposition above generalizes a statement which appears in [19] but the proof is basically the same (see Lemma 3.5 and the proof of Theorem II). We believe the result of the proposition is well known but we were unable to find an appropriate reference in the literature.

Proof of Proposition 4.3. Let $A \leq F \leq G$. We say that (F, A) is a *pair* if F is f.g., A is abelian and $[F : A] \leq d$. We write $(F, A) \leq (F_1, A_1)$ if $F \leq F_1$ and $F \cap A_1 = A$. Note in particular that $[F : A] \leq [F_1 : A_1]$.

A pair (F, A) is called *good* if whenever $F \leq F_1 \leq G$ with F_1 finitely generated, there is a pair (F_1, A_1) with $(F, A) \leq (F_1, A_1)$. Note that the assumption of the proposition says that (e, e) is a good pair.

(1) We claim that if (F, A) is good pair and $F \leq H \leq G$ with H finitely generated, we can find $B \leq H$ such that (H, B) is a good pair and $(F, A) \leq (H, B)$.
    Indeed, since (F, A) is a good pair, there are pairs (H, B_i) with $(F, A) \leq (H, B_i)$, and by Hall's theorem there exist only finitely many such pairs. Suppose by negation that none of them are good pairs. Thus we can find $H \leq F_i \leq G$ (F_i-f.g.) such that there are no abelian subgroups A_i with $(H, B_i) \leq (F_i, A_i)$. The group $K = \langle F_1, \dots, F_n \rangle$ is finitely generated, so there is some abelian subgroup $A_K \leq K$ of index $\leq d$ such that $(F, A) \leq (K, A_K)$. Clearly, there is some i such that $A_K \cap H = B_i$, but then $(H, B_i) \leq (F_i, F_i \cap A_K)$ – contradiction.

(2) Let (F, A) be a good pair with $s = [F : A]$ maximal. Note that if $(F, A) \leq (H, B)$ are good pairs, then we must have $[F : A] = [H : B]$. Claim: for any such B we have $[G : C_G(B)] \leq d$. Let us show that if $g_1, \dots, g_s$ represent the left cosets of A in F then they also represent the left cosets of $C_G(B)$ in G. Fix an element $g \in G$. Then, by (1) above, $\langle H, g \rangle$ has an abelian subgroup C such that $(H, B) \leq (\langle H, g \rangle, C)$ are good pairs. It follows that $g_1, \dots, g_s$ represent also the left cosets of C in $\langle H, g \rangle$ and hence $g \in g_i C$ for some i. Since $B \leq C$ are abelian groups we get that $g_i C \subseteq g_i C_G(B)$ and the claim follows.

(3) Assume now that (F, A) is a good pair with $[F : A] = s$ and $[G : C_G(A)]$ maximal. Define

$$J = \left\langle B \mid (H, B) \geq (F, A) \text{ is a good pair} \right\rangle.$$

We claim that J is abelian and $[G : J] \leq d$.
Let $(H_i, B_i) \geq (F, A)$, $i = 1, 2$, be good pairs, and let $b_i \in B_i$. Since $A \leq B_1$, we have that $C_G(B_1) \leq C_G(A)$, but from the maximality of $[G : C_G(A)]$, it follows that there is an equality. Similarly, we have that $C_G(B_2) = C_G(A)$ and since B_2 is abelian we get that $b_2 \in B_2 \subseteq C_G(B_2) = C_G(B_1)$, so that b_1, b_2 commute. This proves J is abelian.
Suppose now that $[G : J] > d$, and let $g_0, \dots, g_d$ different coset representatives of J in G. The group $F_1 = \langle F, g_1, \dots, g_d \rangle$ is finitely generated and so we can find $A_1 \leq F_1$ such that (F_1, A_1) is a good pair larger than (F, A), and in particular $[F_1 : A_1] \leq d$. But this means that there are some $0 \leq i < j \leq d$ with $g_i^{-1} g_j \in A_1 \subseteq J$ which is a contradiction. Thus, $[G : J] \leq d$ and we are done. $\square$

As mentioned in the introduction, once a group has an abelian subgroup of finite index (say d), then it also has a characteristic abelian subgroup of (finite) index bounded by a function of d. For completeness of the article we provide a simple proof here (shown to us by Uri Bader).

Lemma 4.5. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if a group G contains an abelian subgroup A of index at most n, then G contains a characteristic abelian subgroup of index $\leq f(n)$.*

Proof. Let N be the characteristic subgroup of G generated by A (the group generated by all images of A under all automorphisms of G). Let $Z = Z(N)$ (the center of N). We claim $[N : Z]$ (and hence $[G : Z]$) is bounded by a function of n. Indeed, there are n images of A which already generate N and Z contains their intersection. This proves the lemma. $\square$

In the preceding section, we proved that for a finite group G and a nondegenerate G-graded algebra A, we have the inequalities $\exp(FG) \leq \exp(A)^2$ and $PI\,deg(FG) \leq 2(PI\,deg(A) - 1)^2$. The rest of this section is dedicated to generalize these results for infinite groups.

Lemma 4.6. *Let G be a finitely generated group such that FG is PI. Then there exists a finite index normal subgroup N of G such that $\mathrm{Id}(FG) = \mathrm{Id}(FG/N)$. In particular $\mathrm{Id}(FG) = \mathrm{Id}(M_k(F))$ for some integer k.*

Proof. We know that G is abelian by finite. Furthermore, since it is finitely generated it is residually finite. If N is any finite index normal subgroup of G, then $\mathrm{Id}(FG) \subseteq \mathrm{Id}(FG/N)$ and hence $I := \bigcap_{[G:N]<\infty} \mathrm{Id}(FG/N) \supseteq \mathrm{Id}(FG)$. On the other hand, the algebra FG/N is semisimple, so that $\mathrm{Id}(FG/N) = \mathrm{Id}(M_k(F))$ where $k^2 = \exp(FG/N) \leq \exp(FG)$ and in particular the set $\{\exp(FG/N)\}_N$ is bounded. It follows that $I = \mathrm{Id}(M_k(F))$ for some k and there is some finite index normal subgroup N with $I = \mathrm{Id}(FG/N)$. The lemma will follow if we can show that $\mathrm{Id}(FG) = I$.

Let $f(x_1, \ldots, x_m) \in I$ be any multilinear polynomial. If f is not an identity of FG, we can find some $g_1, \ldots, g_m \in G$ such that $f(U_{g_1}, \ldots, U_{g_m}) \neq 0$ (U_g represents g in FG). Let $h_1, \ldots, h_k \in G$ and $a_1, \ldots, a_k \in F^\times$ such that $f(U_{g_1}, \ldots, U_{g_m}) = \sum_1^k a_i U_{h_i}$. Since G is residually finite, there is some finite index normal subgroup N not containing $h_i^{-1} h_j$ for any $i \neq j$. Reducing the equation above modulo N, the elements $U_{h_i}, i = 1, \ldots, k$, remain linearly independent by the choice of N, so in particular f is not an identity of FG/N. We obtain that $I \subseteq \mathrm{Id}(FG)$ and the result follows. $\square$

Lemma 4.7. *Let G be any group such that FG is PI. Then there exists some finitely generated subgroup H of G such that $\mathrm{Id}(FG) = \mathrm{Id}(FH)$. Consequently, $\mathrm{Id}(FG) = \mathrm{Id}(M_k(F))$ for some integer k.*

Proof. By the preceding lemma, for each finitely generated subgroup H of G we have $\mathrm{Id}(FH) = \mathrm{Id}(M_k(F))$ for some integer k which is uniformly bounded over the finitely generated subgroups (by $\exp(FG)^{1/2}$). Since any multilinear nonidentity of FG is already a nonidentity of FH for some finitely generated subgroup H of G, we have that $\mathrm{Id}(FG) = \bigcap_{H \leq G} \mathrm{Id}(FH)$, H is f.g., and the lemma follows. $\square$

We can now generalize to arbitrary groups the result of the previous section.

Theorem 4.8. *Let G be any group and A be a nondegenerate G-graded algebra. Then $\exp(FG) \leq \exp(A)^2$ and $PI\deg(FG) \leq 2(PI\deg(A) - 1)^2$.*

Proof. Since G grades nondegenerately the algebra A, it is abelian by finite and therefore the group algebra FG is PI. It follows from the previous two lemmas that there is some finitely generated subgroup H and a finite index normal subgroup N in H such that $\mathrm{Id}(FG) = \mathrm{Id}(FH/N)$, hence it is enough to bound the exponent and PI-degree of FH/N.

If A_H is the subalgebra of A supported on the H homogeneous components of A, we have $\mathrm{Id}(A_H) \supseteq \mathrm{Id}(A)$, $\exp(A_H) \leq \exp(A)$ and $PI\deg(A_H) \leq PI\deg(A)$. Moreover, since A is nondegenerately G-graded, A_H is nondegenerately H-graded and hence A_H is also H/N-nondegenerately graded where N is any normal subgroup of H (by the induced grading).

By [Corollary 3.8](#) we have

$$\exp(FH/N) \leq \exp(A_H)^2 \leq \exp(A)^2$$

and

$$PI\deg(FH/N) \leq 2\big(PI\deg(A_H) - 1\big)^2 \leq 2\big(PI\deg(A) - 1\big)^2$$

and the result follows. $\square$

5. Some examples

Let G be a finitely generated group and suppose it grades nondegenerately a PI algebra A. We know that G is n permutable for some $n \in \mathbb{N}$. While G must be abelian by finite, the minimal index of an abelian subgroup is not bounded by a function of the permutability index. Indeed, if there was such a function $f(n)$, then given an arbitrary n-permutable group H, its finitely generated subgroup would be n-permutable as well. By the assumption, each such subgroup has an abelian subgroup of index $\leq f(n)$, and hence, by [Proposition 4.3](#), the group G would contain an abelian subgroup of index bounded by $f(n)$. This is known to be false. In fact G need not have an abelian subgroup of finite index (see [10]).

Let us give a concrete example, i.e. a family of (finite) n-permutable groups $\{G_k\}_{k \in \mathbb{N}}$, with $d_k = min\{[G_k : U_k] \mid U_k \text{ abelian subgroup}\}$ and $\lim d_k = \infty$.

Example 5.1. Let $G = C_p^{2n}$ for some n and let $\alpha \in Z^2(G, \mathbb{C}^*)$ be a *nontrivial* two cocycle. It is well known that up to a coboundary α takes values which are roots of unity, and for G above, the values must be p-roots of unity. Thus, we may consider α as a cocycle in $Z^2(G, C_p)$ which corresponds to a central extension

$$1 \to C_p \to H \to G \to 1.$$

Since the group G is abelian and the cocycle α is nontrivial we have that $[H, H] = Z(H) \cong C_p$ and hence the group H is $p + 1$ permutable (see [10, (3.3)]).

Let $B = \mathbb{C}^\alpha G$ be the corresponding twisted group algebra with basis $\{U_g\}_{g \in G}$. If $A \leq H$ is an abelian group of minimal index, we have that $[H, H] = Z(H) \leq A$, and thus we have $\tilde{A} = A/[H, H] \leq G$. Clearly, the group A is abelian if and only if $[U_{g_1}, U_{g_2}] = 1$ (the multiplicative commutator) for any $g_1, g_2 \in \tilde{A}$.

For $g, h \in G$, set $\mu(g, h) = \frac{\alpha(g,h)}{\alpha(h,g)}$, namely the scalar satisfying $U_g U_h = \mu(g, h) U_h U_g$. It is easily seen that $\mu : G \times G \to \mathbb{C}^*$ is a bicharacter, i.e. $\mu(g_1 g_2, h) = \mu(g_1, h) \mu(g_2, h)$ and $\mu(g, h_1 h_2) = \mu(g, h_1) \mu(g, h_2)$. With this notation we have that $\mu(g_1, g_2) = 1$ for any $g_1, g_2 \in \tilde{A}$.

Identifying C_p with the additive group of the field F_p with p elements, we see that μ is a bilinear map. In particular, if $g \in G$, then $\dim\{h \in G \mid \mu(h, g) = 1\} \geq n - 1$. If $\dim_{F_p}(\tilde{A}) > \frac{1}{2} \dim_{F_p}(G)$, or equivalently $[G : \tilde{A}] < p^n$, then there is some $e \neq u \in \tilde{A}$ such that $\mu(u, g) = 1$ for all $g \in G$ (by dimension counting). Thus, if μ is nondegenerate, i.e. for any $e \neq h \in G$ there is some $g \in G$ such that $\mu(h, g) \neq 1$, then $[H : A] = [G : \tilde{A}] \geq p^n$.

Note that to say that μ is nondegenerate is equivalent to saying that U_g is in the center of the twisted group algebra if and only if $g = e$, which in turn is equivalent to the twisted group algebras $\mathbb{C}^\alpha C_p^{2n}$ being isomorphic to a matrix algebra $M_{p^n}(\mathbb{C})$.

Fix a prime p and let σ, τ be generators for $C_p \times C_p$. Let B be the twisted group algebra $B = \bigoplus_{0 \leq i, j \leq p-1} \mathbb{C} U_{\sigma^i \tau^j}$ where the multiplication is defined by

$$U_{\sigma^i \tau^j} = U_\sigma{}^i U_\tau{}^j, \qquad U_\sigma U_\tau = \zeta U_\tau U_\sigma$$

and ζ is a primitive p-root of unity. It is well known that $B \cong M_p(\mathbb{C})$, and hence $\bigotimes_1^n B$ is on one hand isomorphic to a twisted group algebra with the group C_p^{2n} and on the other hand isomorphic to $M_{p^n}(\mathbb{C})$. This completes the construction of the required family of groups. We remark here that the function μ defined above plays a central role in the theory of twisted group algebras and their polynomial identities (see [2]).

Remark 5.2. Let $\alpha_n \in Z^2(C_p^{2n}, \langle \zeta \rangle)$ be the nondegenerate 2-cocycle as constructed in the previous example, and let H_n be the central extensions (of C_p^{2n} by $\langle \zeta \rangle$) defined by such cocycle. The last example shows that the group algebra $\mathbb{C} H_n$ has an irreducible representation of degree p^n. On the other hand, Kaplansky's theorem [20] states that if a group has an abelian subgroup of index m, then all of its irreducible representations are finite with degree at most m. This provides another proof that the minimal index of an abelian subgroup of H_n tends to infinity.

Next we provide some examples/counter examples to statements that are related to the main theorem.

Example 5.3. Let F be an algebraically closed field of characteristic zero. For any finite abelian group G, the group algebra FG is isomorphic to a product of $|G|$ copies of F. In particular, we get that $\exp(FG) = 1$. Hence, we cannot hope to get an inequality of the form $|G| \leq \exp(A)^K$ for any constant K.

More generally, given an H-graded algebra A with a nondegenerate grading, the algebra $B = FG \otimes A$ has a natural $G \times H$-grading which is also nondegenerate. In addition we have that $\exp(B) = \exp(A)$. While the grading group is of course larger, the index of the largest abelian group remains the same.

Example 5.4. Suppose we omit the requirement that $\mathrm{Id}_G(A)$ has no G-graded monomials and only assume that $\mathrm{Id}_G(A)$ has G-graded monomials of high degrees (as a function of $\dim(A)$ or the cardinality of G). In other words we drop the assumption that A is nondegenerately G-graded and we only assume that the G-grading on A is nondegenerately bounded. We show that the consequence of the main theorem does not hold in general.

Consider the algebras A_m of upper triangular matrices $m \times m$ where the diagonal matrices consist only of scalar matrices. Note that by Giambruno and Zaicev's theorem (see [13]) we have $\exp(A) = 1$. Let G be a group of order n and assume that $m = n^2 + 1$. Let $s' = (g_1, \ldots, g_n) \in G^n$ be a tuple such that each element of G appears in s' exactly once and let $s \in G^{(n^2+1)}$ be n copies of s' with additional g_1 at the end. Consider the algebra A_m with the elementary grading corresponding to the tuple s. We claim that A_m has no graded multilinear monomial identities of degree $\leq n$.

Fix $1 \leq i \leq n^2 + 1 - n$ and $h \in G$. We first note that by the definition of the grading we have that $e_{i,j}$ is homogeneous of degree $s_i^{-1} s_j$ for each $1 \leq i < j \leq n^2 + 1$. By the choice of the tuple s, the elements $\{s_i^{-1} s_{i+1}, s_i^{-1} s_{i+2}, \ldots, s_i^{-1} s_{i+n}\}$ are all distinct, and therefore, for any $h \in G$ and $i \leq n^2 + 1 - n$ we can choose $j = j(i, h)$ such that $i < j \leq i + n$ and $e_{i,j} \in A_h$.

Let $x_{h_1,1} \cdots x_{h_n,n}$ be any multilinear monomial, $h_1, \ldots, h_n \in G$.

Set $i_1 = 1$. Given i_k, define i_{k+1} to be $j(i_k, h_k)$ so that $e_{i_k,i_{k+1}}$ is homogeneous of degree h_k and $i_k < i_{k+1} \leq i_k + n$. It is now easy to see by induction that $i_k \leq 1 + (k-1)n \leq 1 + n^2$ for all $1 \leq k \leq n$ so that $e_{i_1,i_2} \cdots e_{i_n,i_{n+1}}$ is well defined as an element of A and it is a nonzero evaluation of $x_{h_1,1} \cdots x_{h_n,n}$.

For a finite group G, denote by $\gamma(G)$ the smallest index of an abelian subgroup in G. Let G_n be any sequence of groups where $\gamma(G_n)$ goes to infinity with n. By the above construction, the algebras $B_n = A_{|G_n|^2+1}$ have G_n-gradings such that:

- $\dim(B_n)$ and $\gamma(G_n)$ tend to infinity with n.
- B_n has no multilinear monomial identities of degrees smaller then $|G_n|$.
- $\exp(B_n) = 1$.

Example 5.5. Suppose we have a sequence of algebras A_n with $d_n = \exp(A_n)$ monotonically increasing (i.e. to infinity). Can we necessarily find groups G_n and nondegenerate G_n-gradings such that the index of any abelian subgroups U_n of G_n tends to infinity?

The answer is negative as the algebras of upper triangular matrices show. More precisely, let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices, which have exponent $\exp(UT_n(F)) = n$. By a theorem of Valenti and Zaicev [24], every G-grading on $UT_n(F)$ is isomorphic to an elementary grading. Unless the grading is trivial, the grading cannot be nondegenerate since $UT_n(F)_g$ contains only upper triangular matrices with zero on the diagonal for every $e \neq g \in G$, so $x_{g,1} \cdots x_{g,n}$ is an identity. We conclude that the only nondegenerate grading is obtained with the trivial group, so in particular there is no nondegenerate grading such that the index of the largest abelian subgroup tends to infinity.

References

[1] E. Aljadeff, Group graded PI algebras and their codimension growth, Israel J. Math. 189 (2012) 189–205.
[2] E. Aljadeff, O. David, On regular G-gradings, Trans. Amer. Math. Soc. (2015), http://dx.doi.org/10.1090/S0002-9947-2014-06200-4, in press, electronically published on Dec. 5, 2014.
[3] E. Aljadeff, D. Haile, Simple G-graded algebras and their polynomial identities, Trans. Amer. Math. Soc. 366 (4) (2014) 1749–1771.
[4] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011) 83–100.
[5] E. Aljadeff, A. Kanel-Belov, Representability and Specht problem for G-graded algebras, Adv. Math. 225 (5) (2010) 2391–2428.
[6] Y. Bahturin, A. Regev, Graded tensor products, J. Pure Appl. Algebra 213 (9) (2009) 1643–1650.
[7] Y. Bahturin, M. Zaicev, S.K. Sehgal, Finite-dimensional simple graded algebras, Mat. Sb. 199 (7) (2008) 21–40 (in Russian); translation Sb. Math. 199 (7) (2008) 965–983.
[8] M. Cohen, S. Montgomery, Group-graded rings, smash products, and group actions, Trans. Amer. Math. Soc. 282 (1) (1984) 237–258.
[9] M. Curzio, P. Longobardi, M. Maj, Su di un problema combinatorio in teoria dei gruppi, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 74 (1983) 136–142.
[10] M. Curzio, P. Longobardi, M. Maj, D.J.S. Robinson, A permutational property of groups, Arch. Math. 44 (5) (1985) 385–389.
[11] M.I. Elashiry, D.S. Passman, Rewritable groups, J. Algebra 345 (1) (2011) 190–201.
[12] A. Giambruno, D. La Mattina, Graded polynomial identities and codimensions: computing the exponential growth, Adv. Math. 225 (2010) 859–881.
[13] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (2) (1998) 145–155.
[14] A. Giambruno, M. Zaicev, Exponential codimension growth of PI-algebras: an exact estimate, Adv. Math. 142 (2) (1999) 221–243.
[15] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Math. Surveys Monogr., vol. 122, American Mathematical Society, Providence, RI, 2005.
[16] D. Gluck, The largest irreducible character degree of a finite group, Canad. J. Math. 37 (3) (1985) 442–451.
[17] A.S. Gordienko, Amitsur conjecture for associative algebras with a generalized Hopf action, J. Pure Appl. Algebra 217 (8) (2013) 1395–1411.
[18] M. Hall, Subgroups of finite index in free groups, Canad. J. Math. 1 (1949) 187–190.
[19] I.M. Isaacs, D.S. Passman, Groups with representations of bounded degree, Canad. J. Math. 16 (1964) 299–309.
[20] I. Kaplansky, Groups with representations of bounded degree, Canad. J. Math. 1 (1949) 105–112.
[21] D.S. Passman, Group rings satisfying a polynomial identity, J. Algebra 20 (1) (1972) 103–117.
[22] A. Regev, Existence of identities in $A \otimes B$, Israel J. Math. 11 (1972) 131–152.

[23] I. Sviridova, Identities of pi-algebras graded by a finite abelian group, Comm. Algebra 39 (2011) 3462–3490.
[24] A. Valenti, M.V. Zaicev, Group gradings on upper triangular matrices, Arch. Math. 89 (1) (2007) 33–40.