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On the index of a free abelian subgroup in the group of central units of an integral group ring

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Let $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ denote the group of central units in the integral group ring $\mathbb{Z}[G]$ of a finite group G. A bound on the index of the subgroup generated by a virtual basis in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is computed for a class of strongly monomial groups. The result is illustrated with application to the groups of order p^n, p prime, $n \leq 4$. The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ and the Wedderburn decomposition of the rational group algebra of these p-groups have also been obtained.

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1. Introduction

Let $\mathcal{U}(\mathbb{Z}[G])$ denote the unit group of the integral group ring $\mathbb{Z}[G]$ of a finite group G. The center of $\mathcal{U}(\mathbb{Z}[G])$ is denoted by $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$. It is well known that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G) \times A$, where A is a free abelian subgroup of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of finite rank. In order to study $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, a multiplicatively independent subset of such a subgroup A, i.e., a $\mathbb{Z}$-basis for such a free $\mathbb{Z}$-module A, is of importance, and is known only for a few groups ([1,2,7,18], see also [20], Examples 8.3.11 and 8.3.12). However, other papers deal with determining a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, i.e., a multiplicatively independent subset of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ which generates a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ (see e.g. [6,9–17]).

Analogously to well known cyclotomic units in cyclotomic fields, Bass [4] constructed units, so called *Bass cyclic units*, which generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$, when G is cyclic. A virtual basis consisting of certain Bass cyclic units was also given by Bass. Generalizing the notion of Bass cyclic units, Jespers et al. [13] defined *generalized Bass units* and have shown that the group generated by these units contains a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for an arbitrary strongly monomial group G. Recently, for a class of groups properly contained in finite strongly monomial groups, Jespers et al. [15] provided a subset, denoted by $\mathcal{B}(G)$ (say), of the group generated by generalized Bass units, which forms a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

In this paper, we determine a bound on the index of the subgroup generated by $\mathcal{B}(G)$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for the same class of groups as considered in [15] (Theorem 2). Our result is based on the ideas contained in [15] and Kummer's work (see [23], Theorem 8.2) on the index of cyclotomic units. Further in [15], Jespers et al. have provided the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ in terms of strong Shoda pairs of G, when G is a strongly monomial group. In Section 4, we compute a complete and irredundant set of strong Shoda pairs of the non abelian groups of order p^n, p prime, $n \leq 4$, and provide, in terms of p, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of these p-groups along with the Wedderburn decomposition of their rational group algebras. We also illustrate Theorem 2 for the non abelian groups of order 16 and those of order p^3, $p \leq 5$. It may be mentioned that for a given group G, the calculation of the bound on the index given by Theorem 2 requires the values $n_{H,K}$ corresponding to the strong Shoda pairs (H, K) of G, the computation of which is not always obvious.

2. Notation and preliminaries

We begin by fixing some notation.

G	a finite group
$\lvert g\rvert$	the order of the element g in G
g^t	$t^{-1}gt,\ g, t \in G$
$\langle X\rangle$	the subgroup generated by the subset X of G
$\lvert X\rvert$	the cardinality of the set X
$K \leq G$	K is a subgroup of G

$K \trianglelefteq G$	K is a normal subgroup of G
$[G : K]$	the index of the subgroup K in G
$N_G(K)$	the normalizer of K in G
$\mathrm{core}(K)$	$\bigcap_{x\in G} xKx^{-1},$ the largest normal subgroup of G contained in K
$\hat{K}$	$\frac{1}{\lvert K\rvert}\sum_{k\in K} k$
$\mathcal{M}(G/K)$	the set of minimal normal subgroups of G containing K properly
$\varepsilon(H,K)$	$\begin{cases} \hat{H}, & \text{if } H = K; \\ \prod_{M/K\in\mathcal{M}(H/K)}(\hat{K}-\hat{M}), & \text{otherwise, where } K \trianglelefteq H \leq G \end{cases}$
$e(G,H,K)$	the sum of all the distinct G-conjugates of $\varepsilon(H,K)$
φ	Euler's phi function
$\mathbb{Z}/n\mathbb{Z}$	the ring of integers modulo $n,\ n \geq 1$
ζ_n	a primitive nth root of unity in the field of complex numbers
$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$	the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$
h_n^+	the class number of the maximal real subfield of $\mathbb{Q}(\zeta_n)$
$l.c.m.(k,n)$	the least common multiple of the integers k and n
(k,n)	the greatest common divisor of the integers k and n
$o_n(k)$	the multiplicative order of k modulo $n,$ where $(k,n) = 1$
$\eta_k(\zeta_n)$	$\begin{cases} 1, & \text{if } n = 1; \\ 1+\zeta_n+\zeta_n^2+\ldots+\zeta_n^{k-1}, & \text{if } n > 1, \text{ where } k \geq 1 \end{cases}$
$\mathcal{U}(R)$	the unit group of the ring R
$M_n(R)$	the ring of $n \times n$ matrices over the ring $R,\ n \geq 1$
$M_n(R)^{(s)}$	$M_n(R) \oplus M_n(R) \oplus \ldots \oplus M_n(R),$ direct sum of s copies, $s \geq 1$
I_n	the $n \times n$ identity matrix

A *strong Shoda pair* ([19], Definition 3.1) of G is a pair (H,K) of subgroups of G with the properties that

(i) $K \trianglelefteq H \trianglelefteq N_G(K)$;
(ii) H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$;
(iii) the distinct G-conjugates of $\varepsilon(H,K)$ are mutually orthogonal.

Note that (G,G) is always a strong Shoda pair of G.

If (H,K) is a strong Shoda pair of G, then $e(G,H,K)$ is a primitive central idempotent of the rational group algebra $\mathbb{Q}[G]$ ([19], Proposition 3.3). A group G is called *strongly monomial* if every primitive central idempotent of $\mathbb{Q}[G]$ is of the form $e(G,H,K)$ for some strong Shoda pair (H,K) of G.

Two strong Shoda pairs (H_1,K_1) and (H_2,K_2) of G are said to be *equivalent* if $e(G,H_1,K_1) = e(G,H_2,K_2)$. A complete set of representatives of distinct equivalence classes of strong Shoda pairs of G is called *a complete irredundant set of strong Shoda* *pairs* of G. In case G is strongly monomial, one can calculate the primitive central idempotents of $\mathbb{Q}[G]$ from a complete irredundant set of strong Shoda pairs of G.

Recall that a group G is called *normally monomial* if every complex irreducible character of G is induced from a linear character of a normal subgroup of G. Theorem 1, as stated below, provides an algorithm to determine a complete irredundant set of strong Shoda pairs of a normally monomial group G and also, in particular, yields that a normally monomial group is strongly monomial.

Let $\mathcal{N}$ be the set of all the distinct normal subgroups of a finite group G. For $N \in \mathcal{N}$, set

A_N: a normal subgroup of G containing N such that A_N/N is an abelian normal subgroup of maximal order in G/N.

$\mathcal{D}_N$: the set of all subgroups D of A_N containing N such that $\mathrm{core}(D) = N$, A_N/D is cyclic and is a maximal abelian subgroup of $N_G(D)/D$.

$\mathcal{T}_N$: a set of representatives of $\mathcal{D}_N$ under the equivalence relation defined by conjugacy of subgroups in G.

$\mathcal{S}_N$: $\{(A_N, D) \mid D \in \mathcal{T}_N\}$.

Note that if $N \in \mathcal{N}$ is such that G/N is abelian, then, by ([3], Eq. (1)),

$$\mathcal{S}_N = \begin{cases} \{(G, N)\}, & \text{if } G/N \text{ cyclic;} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 1. *(See [3], Theorem 1, Corollaries 1 and 2.) The following statements are equivalent:*

(i) *G is normally monomial;*

(ii) *$\mathcal{S}(G) := \bigcup_{N \in \mathcal{N}} \mathcal{S}_N$ is a complete irredundant set of strong Shoda pairs of G;*

(iii) *$\{e(G, A_N, D) \mid (A_N, D) \in S_N,\ N \in \mathcal{N}\}$ is a complete set of primitive central idempotents of $\mathbb{Q}[G]$;*

(iv) $|G| = \sum_{N \in \mathcal{N}} \sum_{D \in \mathcal{D}_N} [G : A_N]\varphi([A_N : D])$.

Let $n \geq 1$ and let k be an integer coprime to n. Then, $\eta_k(\zeta_n)$ is a unit of $\mathbb{Z}[\zeta_n]$. The units of the form $\eta_k(\zeta_n^j)$ with integers j, k and n such that $(k, n) = 1$ are called *cyclotomic units* of $\mathbb{Q}(\zeta_n)$.

Let $g \in G$ and let k, m be positive integers such that $k^m \equiv 1 (\mathrm{mod}\ n)$, where $n = |g|$. Then,

$$u_{k,m}(g) = (1 + g + \ldots + g^{k-1})^m + \frac{1 - k^m}{n}(1 + g + \ldots + g^{n-1})$$

is a unit in the integral group ring $\mathbb{Z}[G]$. The units in $\mathbb{Z}[G]$ of this form are called *Bass cyclic units* (see [21], (10.3)).

Next, we recall the definition of *generalized Bass units* of $\mathbb{Z}[G]$ introduced by Jespers et al. [13]. For $M \trianglelefteq G$, $g \in G$, and positive integers k, m such that $k^m \equiv 1 (\text{mod } |g|)$, let

$$u_{k,m}(1 - \hat{M} + g\hat{M}) = 1 - \hat{M} + u_{k,m}(g)\hat{M}.$$

This element is a unit in $\mathbb{Z}[G](1-\hat{M}) + \mathbb{Z}[G]\hat{M}$. As both $\mathbb{Z}[G](1-\hat{M}) + \mathbb{Z}[G]\hat{M}$ and $\mathbb{Z}[G]$ are orders in $\mathbb{Q}[G]$, there is a positive integer $n_{g,M}$ such that

$$(u_{k,m}(1 - \hat{M} + g\hat{M}))^{n_{g,M}} \in \mathcal{U}(\mathbb{Z}[G]). \tag{2}$$

Suppose $n_{G,M}$ is the minimal positive integer satisfying Eq. (2) for all $g \in G$. Then, the element

$$(u_{k,m}(1 - \hat{M} + g\hat{M}))^{n_{G,M}} = 1 - \hat{M} + u_{k,mn_{G,M}}(g)\hat{M}$$

is called the *generalized Bass unit* of $\mathbb{Z}[G]$ based on g and M with parameters k and m. Observe that $n_{G,M} = 1$, if M is trivial i.e., $M = \langle 1 \rangle$ or G.

Remark 1. For a non trivial normal subgroup M of G, using Lemma 3.1 of [8], it may be noted that $n_{G,M} = 1$, if every $g \in G \setminus M$ is of order 2; otherwise, $n_{G,M}$ is the minimal positive integer satisfying Eq. (2) for all $g \in G \setminus M$ with $|g| > 2$ and integers k, m such that $1 < k < |g|$, $(k, |g|) = 1$ and $m = o_{|g|}(k)$.

Let G be a strongly monomial group such that there is a complete irredundant set $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ of strong Shoda pairs of G with the property that each $[H_i : K_i]$ is a prime power, say $p_i^{n_i}$. Assume that $(H_1, K_1) = (G, G)$. For such a group G, we recall the virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ provided by Jespers et al. [15].

For $1 \leq i \leq m$, we adopt the following notation:

$$\begin{array}{ll}
\varepsilon_i & := \varepsilon(H_i, K_i) \\
e_i & := e(G, H_i, K_i) \\
[H_i : K_i] & := p_i^{n_i}, p_i \text{ prime}, \ n_i \geq 0 \ (n_i = 0 \text{ only if } i = 1) \\
g_i K_i & := \text{a generator of the cyclic group } H_i/K_i \\
L_j^{(i)} & := \langle g_i^{p_i^{n_i - j}}, K_i \rangle, \ 0 \leq j \leq n_i \\
N_i & := N_G(K_i) \\
m_i & := [G : N_i] \\
T_i & := \text{a right transversal of } N_i \text{ in } G.
\end{array}$$

For $2 \leq i \leq m$, consider the action of N_i/H_i on $\mathbb{Q}(\zeta_{p_i^{n_i}})$ given by the map

$$\begin{aligned}
N_i/H_i &\longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q}) \\
n_i H_i &\longmapsto \alpha_{n_i H_i},
\end{aligned} \tag{3}$$

where $\alpha_{n_i H_i}(\zeta_{p_i^{n_i}}) = \zeta_{p_i^{n_i}}^j$, if $n_i^{-1} g_i n_i K_i = g_i^j K_i$. As H_i/K_i is a maximal abelian subgroup of N_i/K_i, it turns out that the above action is faithful. Hence, N_i/H_i is isomorphic to a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q}) \cong \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. For the convenience of notation, we regard N_i/H_i as a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q})$ and that of $\mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. (Notice that N_i/H_i can be regarded as a subgroup of $\mathcal{U}(\mathbb{Z}/[H_i : K_i]\mathbb{Z})$, even if $[H_i : K_i]$ is not a prime power.) With this identification, N_i/H_i is equal to either $\langle \phi_{r_i}\rangle$ or $\langle \phi_{r_i}\rangle \times \langle \phi_{-1}\rangle$ (resp. $\langle r_i\rangle$ or $\langle r_i\rangle \times \langle -1\rangle$) for some $r_i \in \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$, where ϕ_{r_i} denotes the automorphism of $\mathbb{Q}(\zeta_{p_i^{n_i}})$ which maps $\zeta_{p_i^{n_i}}$ to $\zeta_{p_i^{n_i}}^{r_i}$. The later case arises only if $p_i = 2$ and $n_i \geq 3$. Set

$$d_i := \begin{cases} 1, & \text{if } -1 \in \langle r_i\rangle; \\ 2, & \text{otherwise,} \end{cases} \tag{4}$$

and

$$o_i := \begin{cases} 4, & \text{if } p_i = 2,\ N_i/H_i = \langle r_i\rangle,\ r_i \equiv 1 (\mathrm{mod}\ 4),\ n_i \geq 2; \\ 6, & \text{if } p_i = 3,\ N_i/H_i = \langle r_i\rangle,\ r_i \equiv 1 (\mathrm{mod}\ 3); \\ 2, & \text{otherwise.} \end{cases} \tag{5}$$

Further, choose a subset I_i of $\{k \mid 1 \leq k \leq \frac{{p_i}^{n_i}}{2}, (k, p_i) = 1\}$ containing 1, which forms a set of representatives of $\mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$ modulo $\langle N_i/H_i, -1\rangle$. We extend the notation by setting $I_1 = \{1\}$, in view of the trivial action of the identity group N_1/H_1 on $\mathbb{Q}(\zeta_1) = \mathbb{Q}$.

Let k be a positive integer coprime to p_i and let r be an arbitrary integer. For $0 \leq j \leq s \leq n_i$, consider the following products of generalized Bass units of $\mathbb{Z}[H_i]$, defined recursively:

$$c_s^s(H_i, K_i, k, r) = 1$$

and, for $0 \leq j \leq s - 1$,

$$\begin{aligned} c_j^s(H_i, K_i, k, r) = \Big(\prod\nolimits_{h \in L_j^{(i)}} u_{k, o_{p_i^{n_i}}(k) n_{H_i,K_i}}\big(g_i^{r p_i^{n_i - s}} h\hat{K}_i + 1 - \hat{K}_i\big)\Big)^{p_i^{s-j-1}} \times \\ \Big(\prod\nolimits_{l=j+1}^{s-1} c_l^s(H_i, K_i, k, r)^{-1}\Big)\Big(\prod\nolimits_{l=0}^{j-1} c_l^{s+l-j}(H_i, K_i, k, r)^{-1}\Big), \end{aligned}$$

where the empty products equal 1.

Define

$$\begin{aligned} B(H_i, K_i) &:= \{\prod\nolimits_{x \in N_i/H_i} c_0^{n_i}(H_i, K_i, k, x) \mid k \in I_i \setminus \{1\}\}, \\ \mathcal{B}(H_i, K_i) &:= \{\prod\nolimits_{t \in T_i} u^t \mid u \in B(H_i, K_i)\}, \end{aligned}$$

and

$$\mathcal{B}(G) := \bigcup_{i=1}^{m} B(H_i, K_i). \tag{6}$$

Jespers et al. ([15], Theorem 3.5) proved that $\mathcal{B}(G)$ is a *virtual basis of* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

3. A bound on the index of $\langle \mathcal{B}(G) \rangle$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$

In this section, we continue with the notation developed in Section 2.

Theorem 2. *Let G be a strongly monomial group and let $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ be a complete irredundant set of strong Shoda pairs of G with $(H_1, K_1) = (G, G)$. For $2 \leq i \leq m$, let I_i be a subset of $\{k \mid 1 \leq k \leq \frac{[H_i:K_i]}{2}, (k, [H_i : K_i]) = 1\}$ containing 1, which forms a set of representatives of $\mathcal{U}(\mathbb{Z}/[H_i : K_i]\mathbb{Z})$ modulo $\langle N_i/H_i, -1\rangle$, where $N_i = N_G(K_i)$. Set $I_1 = \{1\}$.*

(i) *The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = 0$ (equivalently, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is finite) if and only if $|I_i| = 1$ for all i, $1 \leq i \leq m$, and in this case, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$.*
(ii) *In addition, if $[H_i : K_i]$ is a prime power, say $p_i^{n_i}$, for all i, $1 \leq i \leq m$, and $\mathcal{B}(G)$ is the virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ as defined in Eq. (6), then,*

$$[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle \mathcal{B}(G)\rangle] \leq 2 \prod_{\substack{i=2\\|I_i|=1}}^{m} o_i \prod_{\substack{i=2\\|I_i|\neq 1}}^{m} h^{+}_{p_i^{n_i}} l_i p_i^{n_i-1} \mathfrak{o}_i (l_i^{d_i-1}[N_i : H_i])^{|I_i|-1},$$

where $l_i = l.c.m.(2, p_i)$; $\mathfrak{o}_i = \prod_{\substack{1<k<\frac{p_i^{n_i}}{2}\\(k,p_i)=1}} o_{p_i^{n_i}}(k) p_i^{n_i-1} n_{H_i,K_i}$; d_i and o_i are as defined in Eq. (4) and Eq. (5) respectively.

We first prove the following:

Lemma 1. *Let G be as in Theorem 2. Let $\mathcal{A}(H_i, K_i) = \mathcal{Z}(1 - e_i + \mathcal{U}(\mathbb{Z}[G]e_i))$ and $A(H_i, K_i) = \mathcal{Z}(1 - \varepsilon_i + \mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, where $1 \leq i \leq m$. Then,*

$$[\mathcal{A}(H_i, K_i) : \langle \mathcal{B}(H_i, K_i)\rangle] = [A(H_i, K_i) : \langle B(H_i, K_i)\rangle].$$

Proof. Let $\{t_j \mid 1 \leq j \leq m_i\}$ be a right transversal of N_i in G with $t_1 = 1$. For $\alpha \in \mathbb{Q}[G]e_i$ and integers r and s such that $1 \leq r, s \leq m_i$, let $\alpha_{rs} = \varepsilon_i t_r \alpha t_s^{-1} \varepsilon_i$. We notice that $\alpha_{rs} \in \mathbb{Q}[N_i]\varepsilon_i$. To see this, write $\alpha = (\sum_{g\in G} \alpha_g g)e_i$ with $\alpha_g \in \mathbb{Q}$. Then $\alpha_{rs} = \sum_{g\in G} \alpha_g \varepsilon_i t_r g t_s^{-1} \varepsilon_i$. By ([19], Proposition 3.3), the centralizer of ε_i in G equals N_i. Therefore, if $t_r g t_s^{-1} \notin N_i$, then $\varepsilon_i t_r g t_s^{-1} \varepsilon_i = t_r g t_s^{-1} \varepsilon_i^{t_r g t_s^{-1}} \varepsilon_i = 0$. Also, if $t_r g t_s^{-1} \in N_i$, then $\varepsilon_i t_r g t_s^{-1} \varepsilon_i = t_r g t_s^{-1} \varepsilon_i$. Consequently, $\alpha_{rs} = \sum_{t_r g t_s^{-1} \in N_i} \alpha_g t_r g t_s^{-1} \varepsilon_i \in \mathbb{Q}[N_i]\varepsilon_i$.

Now consider the map

$$\theta_i : \mathbb{Q}[G]e_i \longrightarrow M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)$$

given by

$$\alpha \xrightarrow{\theta_i} (\alpha_{rs})_{m_i \times m_i}.$$

As ε_i^t, $t \in T_i$, are mutually orthogonal idempotents and $\sum_{t\in T_i} \varepsilon_i^t = e_i$, it can be checked that θ_i is an isomorphism of $\mathbb{Q}$-algebras. This isomorphism in turn yields the group isomorphism

$$\mathcal{Z}(\mathcal{U}(\mathbb{Q}[G]e_i)) \overset{\theta_i}{\cong} \mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)))$$

given by

$$\alpha \xrightarrow{\theta_i} \varepsilon_i \alpha \varepsilon_i I_{m_i}.$$

Let ψ_i denote the canonical isomorphism from $\mathcal{Z}(\mathcal{U}(\mathbb{Q}[N_i]\varepsilon_i))$ to $\mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)))$ given by $u \overset{\psi_i}{\mapsto} uI_{m_i}$. Denote by ϕ_i, the restriction of $\psi_i^{-1} o\ \theta_i$ to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$. We assert that ϕ_i is an isomorphism from $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$ to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$. For this, we need to show that

$$\phi_i(\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i)). \tag{7}$$

Consider $\alpha = (\sum_{g\in G} \alpha_g g)e_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$ with $\alpha_g \in \mathbb{Z}$. We have $\varepsilon_i \alpha \varepsilon_i = \sum_{g\in N_i} \alpha_g g \varepsilon_i \in \mathbb{Z}[N_i]\varepsilon_i$. Consequently, $\phi_i(\alpha) = \varepsilon_i \alpha \varepsilon_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, as we already have $\phi_i(\alpha) \in \mathcal{Z}(\mathcal{U}(\mathbb{Q}[N_i]\varepsilon_i))$. On the other hand, to see that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$ is contained in $\phi_i(\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i)))$, let $u \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$. Following the argument as in the proof of Theorem 3.5 of [15], it can be seen that $\sum_{t\in T_i} u^t$ belongs to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$, as ε^t, $t \in T_i$, are mutually orthogonal idempotents. One checks that $\sum_{t\in T_i} u^t$ maps to u under ϕ_i and hence Eq. (7) follows. The isomorphism ϕ_i now provides the group isomorphism

$$\Theta_i : \mathcal{A}(H_i, K_i) \longrightarrow A(H_i, K_i)$$

by setting

$$1 - e_i + \alpha \overset{\Theta_i}{\mapsto} 1 - \varepsilon_i + \varepsilon_i \alpha \varepsilon_i,$$

where $\alpha \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$. We further see that if $u = 1 - \varepsilon_i + \gamma \in B(H_i, K_i)$, with $\gamma \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, then $\Theta_i(\prod_{t\in T_i} u^t) = \Theta_i(1 - e_i + \sum_{t\in T_i} \gamma^t) = 1 - \varepsilon_i + \varepsilon_i(\sum_{t\in T_i} \gamma^t)\varepsilon_i = 1 - \varepsilon_i + \gamma = u$. This yields $\Theta_i(\mathcal{B}(H_i, K_i)) = B(H_i, K_i)$ and consequently, Lemma 1 follows. $\square$

Lemma 2. *Let p be a prime and let $n \geq 1$ be an integer. For a subgroup A of $\mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})(\cong Aut(\langle \zeta_{p^n} \rangle))$, let $\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A)$ denote the unit group of the fixed ring $\mathbb{Z}[\zeta_{p^n}]^A$. If $\langle A, -1 \rangle = \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$, then*

$$\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A) = \begin{cases} \langle \zeta_4 \rangle, & \text{if } p = 2,\ A = \langle r \rangle,\ r \equiv 1 (\text{mod } 4),\ n \geq 2; \\ \langle \zeta_6 \rangle, & \text{if } p = 3,\ A = \langle r \rangle,\ r \equiv 1 (\text{mod } 3); \\ \langle \zeta_2 \rangle, & \text{otherwise.} \end{cases}$$

Proof. Let $F = \mathbb{Q}(\zeta_{p^n})^A$, the subfield of $\mathbb{Q}(\zeta_{p^n})$ fixed by A and let $R = \mathbb{Z}[\zeta_{p^n}]^A$. The assumption $\langle A, -1\rangle = \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ implies that F is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. Thus the group of units of R is finite and hence it is formed by roots of unity of order dividing 4 or 6. If $\zeta_4 \in F$, then $F = \mathbb{Q}(\zeta_4)$ and hence $\mathcal{U}(R) = \langle\zeta_4\rangle$. If $\zeta_3 \in F$, then $F = \mathbb{Q}(\zeta_6)$ and hence $\mathcal{U}(R) = \langle\zeta_6\rangle$. Otherwise, $\mathcal{U}(R) = \{1, -1\} = \langle\zeta_2\rangle$. Furthermore, we see that $\zeta_4 \in F$ if and only if $p = 2$, $A = \langle r\rangle$ with $r \equiv 1 (\mathrm{mod}\ 4)$ and $n \geq 2$. Also, $\zeta_3 \in F$ if and only if $p = 3$, $A = \langle r\rangle$ with $r \equiv 1(\mathrm{mod}\ 3)$. This yields the desired result. $\square$

Proof of Theorem 2. (i) From ([15], Theorem 3.1), it follows immediately that the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = 0$ if and only if $|I_i| = 1$, $\forall i$, $1 \leq i \leq m$. Further, ([22], Corollary 7.3.3) implies that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$ in this case.

(ii) Since $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is a subgroup of $\prod_{i=1}^m \mathcal{A}(H_i, K_i)$, we have

$$\begin{aligned}[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle\mathcal{B}(G)\rangle] &\leq [\prod_{i=1}^m \mathcal{A}(H_i, K_i) : \langle\mathcal{B}(G)\rangle] \\ &= [\prod_{i=1}^m \mathcal{A}(H_i, K_i) : \langle\cup_{i=1}^m \mathcal{B}(H_i, K_i)\rangle] \\ &= \prod_{i=1}^m [\mathcal{A}(H_i, K_i) : \langle\mathcal{B}(H_i, K_i)\rangle] \\ &= \prod_{i=1}^m [A(H_i, K_i) : \langle B(H_i, K_i)\rangle]. \qquad (8)\end{aligned}$$

The last equality follows from Lemma 1. We now show that

$$[A(H_1, K_1) : \langle B(H_1, K_1)\rangle] \leq 2 \qquad (9)$$

and for $2 \leq i \leq m$,

$$[A(H_i, K_i) : \langle B(H_i, K_i)\rangle] \leq \begin{cases} o_i, & \text{if } |I_i| = 1; \\ 2h^+_{p_i^{n_i}} l_i p_i^{n_i - 1}\mathfrak{o}_i(l_i^{d_i-1}[N_i : H_i])^{|I_i|-1}, & \text{if } |I_i| \neq 1. \end{cases} \qquad (10)$$

Let $1 \leq i \leq m$. We have that the center of $\mathbb{Q}[N_i]\varepsilon_i$ is equal to $(\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$, where $(\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$ denotes the fixed field under the action of N_i/H_i on $\mathbb{Q}(\zeta_{p_i^{n_i}}) \cong \mathbb{Q}[H_i]\varepsilon_i$. Now, the center of $\mathbb{Q}(1-\varepsilon_i) + \mathbb{Q}[N_i]\varepsilon_i$, which is equal to $\mathbb{Q}(1-\varepsilon_i) + (\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$, is embedded inside the algebra $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1-\hat{K}_i)$, via the embedding

$$r(1-\varepsilon_i) + u \overset{\iota}{\mapsto} (r(1-\varepsilon_i) + u)\hat{K}_i + r(1-\hat{K}_i), \qquad (11)$$

where $r \in \mathbb{Q}$ and $u \in (\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$.

As $H_i/K_i = \langle g_iK_i\rangle$, any element $x\hat{K}_i \in \mathbb{Q}[H_i]\hat{K}_i$ can be expressed as $x\hat{K}_i = \sum_{j=0}^{p_i^{n_i}-1} x_j g_i^j \hat{K}_i$ with $x_j \in \mathbb{Q}$. Let π denote the projection of $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1-\hat{K}_i)$ onto $\mathbb{Q}(\zeta_{p_i^{n_i}})$ under the isomorphism $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1-\hat{K}_i) \overset{\tau}{\cong} \oplus_{k=0}^{n_i}\mathbb{Q}(\zeta_{p_i^k}) \oplus \mathbb{Q}(1-\hat{K}_i)$ given by

$$x\hat{K}_i + a(1-\hat{K}_i) \overset{\tau}{\mapsto} (\sum_{j=0}^{p_i^{n_i}-1} x_j, \sum_{j=0}^{p_i^{n_i}-1} x_j\zeta_{p_i}^j, \ldots, \sum_{j=0}^{p_i^{n_i}-1} x_j\zeta_{p_i^{n_i}}^j, a(1-\hat{K}_i)), \tag{12}$$

where $x\hat{K}_i = \sum_{j=0}^{p_i^{n_i}-1} x_j g_i^j \hat{K}_i \in \mathbb{Q}[H_i]\hat{K}_i$ and $a \in \mathbb{Q}$.

Now observe that $\pi\circ\tau\circ\iota$ is injective on $\mathcal{Z}(\mathbb{Q}(1-\varepsilon_i)+\mathbb{Q}[N_i]\varepsilon_i)$ and $\pi\circ\tau\circ\iota(A(H_i,K_i))$ is contained in $\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i})$. Hence,

$$\begin{aligned}[A(H_i,K_i) : \langle B(H_i,K_i)\rangle] &= [\pi\circ\tau\circ\iota(A(H_i,K_i)) : \pi\circ\tau\circ\iota(\langle B(H_i,K_i)\rangle)] \\ &\le [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi\circ\tau\circ\iota(\langle B(H_i,K_i)\rangle)].\end{aligned} \tag{13}$$

If $i = 1$, i.e., $(H_i,K_i) = (G,G)$, then $[A(H_i,K_i) : \langle B(H_i,K_i)\rangle] \le |\mathcal{U}(\mathbb{Z})| = 2$. Thus Eq. (9) holds.

If $2 \le i \le m$ is such that $|I_i| = 1$, then $B(H_i,K_i)$ is an empty set and therefore, $[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi\circ\tau\circ\iota(\langle B(H_i,K_i)\rangle)] = |\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i})|$. We have using Lemma 2,

$$[A(H_i,K_i) : \langle B(H_i,K_i)\rangle] \le o_i, \tag{14}$$

as $\langle N_i/H_i, -1\rangle = \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$ in this case.

We next assume that $|I_i| \ne 1$.

Set

$$\begin{aligned}
N_{(H_i,K_i)} &= \langle \pi_{N_i/H_i}(\eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i,K_i}}) \mid k \in I_i \setminus \{1\}\rangle, \\
F_{(H_i,K_i)} &= \langle \eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i,K_i}} \mid 1 < k < \frac{p_i^{n_i}}{2}, (k,p_i)=1\rangle, \\
O_{(H_i,K_i)} &= F_{(H_i,K_i)} \times \langle \zeta_{p_i^{n_i}}^{p_i^{n_i-1}}, -1\rangle, \\
P_{(H_i,K_i)} &= \langle \eta_k(\zeta_{p_i^{n_i}}) \mid k \in \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})\rangle \\
&= \langle \eta_k(\zeta_{p_i^{n_i}}) \mid 1 < k < \frac{p_i^{n_i}}{2}, (k,p_i)=1\rangle \times \langle \zeta_{p_i^{n_i}}, -1\rangle, \\
Q_{(H_i,K_i)} &= \mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) \cap O_{(H_i,K_i)},
\end{aligned}$$

where $\pi_{N_i/H_i}(u) = \prod_{\sigma\in N_i/H_i}\sigma(u)$, for $u \in \mathbb{Q}(\zeta_{p_i{}^{n_i}})$.

By ([15], Proposition 3.4),

$$\pi\circ\tau\circ\iota(\langle B(H_i,K_i)\rangle) = N_{(H_i,K_i)}. \tag{15}$$

Therefore,

$$\begin{aligned}&[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi\circ\tau\circ\iota(\langle B(H_i,K_i)\rangle)]\\&\quad= [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : N_{(H_i,K_i)}]\\&\quad= [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : Q_{(H_i,K_i)}][Q_{(H_i,K_i)} : N_{(H_i,K_i)}]\\&\quad\leq [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : O_{(H_i,K_i)}][Q_{(H_i,K_i)} : N_{(H_i,K_i)}]. \end{aligned}\tag{16}$$

Further,

$$[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : O_{(H_i,K_i)}] = [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : P_{(H_i,K_i)}][P_{(H_i,K_i)} : O_{(H_i,K_i)}].\tag{17}$$

Clearly,

$$[P_{(H_i,K_i)} : O_{(H_i,K_i)}] = p_i^{n_i-1}\prod_{\substack{1<k<\frac{p_i^{n_i}}{2}\\(k,p_i)=1}} o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i,K_i} = p_i^{n_i-1}\mathfrak{o}_i.\tag{18}$$

Also, by ([23], Theorem 8.2),

$$[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : P_{(H_i,K_i)}] = h^+_{p_i^{n_i}}.\tag{19}$$

Next, observe that $Q_{(H_i,K_i)}\cap F_{(H_i,K_i)}$ is a free abelian group, and by ([15], Lemma 3.2), it has rank at most $|I_i|-1$. Furthermore, any element of $Q_{(H_i,K_i)}\cap F_{(H_i,K_i)}$ is of order at most $l_i^{d_i-1}|N_i/H_i|$ modulo $N_{(H_i,K_i)}\cap F_{(H_i,K_i)}$. To see this, let $u\in Q_{(H_i,K_i)}\cap F_{(H_i,K_i)}$ and write $u=\prod_{\substack{1<k<\frac{p_i^{n_i}}{2}\\(k,p_i)=1}}(\eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i,K_i}})^{\alpha_k}$, $\alpha_k\geq 0$. Since $\pi_{N_i/H_i}(\eta_{r_i^t}(\zeta^j_{p_i^{n_i}}))=1$ and $\pi_{N_i/H_i}(\eta_{-j}(\zeta_{p_i^{n_i}}))=\pi_{N_i/H_i}(-\zeta^{-j}_{p_i^{n_i}})\pi_{N_i/H_i}(\eta_j(\zeta_{p_i^{n_i}}))$, for $i,j\geq 0$, it turns out that $u^{|N_i/H_i|l_i^{d_i-1}}=(\pi_{N_i/H_i}(u))^{l_i^{d_i-1}}\in N_{(H_i,K_i)}\cap F_{(H_i,K_i)}$.

Consequently,

$$[Q_{(H_i,K_i)}\cap F_{(H_i,K_i)} : N_{(H_i,K_i)}\cap F_{(H_i,K_i)}]\leq (l_i^{d_i-1}|N_i/H_i|)^{|I_i|-1}\tag{20}$$

and therefore,

$$\begin{aligned}&[Q_{(H_i,K_i)} : N_{(H_i,K_i)}]\\&\quad\leq [Q_{(H_i,K_i)} : N_{(H_i,K_i)}\cap F_{(H_i,K_i)}]\\&\quad= [Q_{(H_i,K_i)} : Q_{(H_i,K_i)}\cap F_{(H_i,K_i)}][Q_{(H_i,K_i)}\cap F_{(H_i,K_i)} : N_{(H_i,K_i)}\cap F_{(H_i,K_i)}]\\&\quad\leq l_i(l_i^{d_i-1}|N_i/H_i|)^{|I_i|-1}.\end{aligned}\tag{21}$$

Finally, Eqs. (13)–(21) yield Eq. (10), which in view of Eq. (8) and Eq. (9) complete the proof. □

It is known that if G is an abelian group, then $\mathcal{U}(\mathbb{Z}[G]) = \pm G$ if and only if G is of exponent 1, 2, 3, 4 or 6 (see [21], Theorem 2.7). We have the following:

Corollary 1. *Let G be a strongly monomial group with a complete irredundant set $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ of strong Shoda pairs such that $[H_i : K_i] = 1, 2, 3, 4$ or 6 for all i, $1 \leq i \leq m$. Then, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$.*

In particular, if G is a strongly monomial (e.g. abelian by supersolvable) group of exponent 1, 2, 3, 4 or 6, then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$. However, the converse need not be true.

Proof. Here, $|I_i| = 1$ for all i, $1 \leq i \leq m$. Therefore, Theorem 2(i) is applicable. The group $\mathcal{G}_1$ defined in Section 4.1 for $p = 3$ is an example of a strongly monomial group of exponent 9 satisfying $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$. □

For abelian p-groups, Theorem 2 gives the following:

Corollary 2. *Let G be an abelian p-group, p prime, and let K_i, $1 \leq i \leq m$, be all the subgroups of G with cyclic quotient groups. Suppose $[G : K_i] = p^{n_i}$, for $1 \leq i \leq m$. Then, the rank of $\mathcal{U}(\mathbb{Z}[G])$ is non zero if and only if $p^{n_i} \geq 5$ for some i. In this case, the index of $\langle \mathcal{B}(G) \rangle$ in $\mathcal{U}(\mathbb{Z}[G])$ is at most*

$$2\prod_{i=1}^{m} 2^{n_i} h^{+}_{2^{n_i}} \Big(\prod_{\substack{1<k<2^{n_i-1} \\ (k,2)=1}} 2^{n_i} o_{2^{n_i}}(k) n_{G,K_i} \Big), \ \text{if } p = 2.$$

and

$$\prod_{i=1}^{m} 2p^{n_i} h^{+}_{p^{n_i}} \Big(\prod_{\substack{1<k<\frac{p^{n_i}}{2} \\ (k,p)=1}} 2p^{n_i} o_{p^{n_i}}(k) n_{G,K_i} \Big), \ \text{if } p \neq 2;$$

where an empty product equals 1.

4. Non Abelian groups of order p^n, $n \leq 4$

Let G be a non abelian group of order p^n, $n \leq 4$. Observe that any such group, being metabelian, is normally monomial.

4.1. Non Abelian groups of order p^3

If $p = 2$, then G is either isomorphic to D_4, the dihedral group of order 8 or is isomorphic to Q_8, the group of quaternions. Both groups satisfy the hypothesis of

(H,K)	$n_{H,K}$
$(\mathcal{G}_1,\mathcal{G}_1)$	1
$(\langle a^5,b\rangle,\langle b\rangle)$	5
$(\mathcal{G}_1,\langle a\rangle)$	5^2
$(\mathcal{G}_1,\langle a^5,b\rangle)$	5^2
$(\mathcal{G}_1,\langle ab\rangle)$	5^2
$(\mathcal{G}_1,\langle a^2b\rangle)$	5^2
$(\mathcal{G}_1,\langle a^3b\rangle)$	5^2
$(\mathcal{G}_1,\langle a^4b\rangle)$	5^2

Fig. 1. Strong Shoda pairs of $\mathcal{G}_1$.

(H,K)	$n_{H,K}$
$(\mathcal{G}_2,\mathcal{G}_2)$	1
$(\langle a,c\rangle,\langle a\rangle)$	5
$(\mathcal{G}_2,\langle b,c\rangle)$	5^2
$(\mathcal{G}_2,\langle a,c\rangle)$	5^2
$(\mathcal{G}_2,\langle ab,c\rangle)$	5^2
$(\mathcal{G}_2,\langle a^2b,c\rangle)$	5^2
$(\mathcal{G}_2,\langle a^3b,c\rangle)$	5^2
$(\mathcal{G}_2,\langle a^4b,c\rangle)$	5^2

Fig. 2. Strong Shoda pairs of $\mathcal{G}_2$.

([21], Theorem 6.1). Therefore, we already know that the group of central units in the integral group ring of these groups consists of only the trivial units.

If p is an odd prime, then G is isomorphic to one of the following groups:

- $\mathcal{G}_1 = \langle a,b \mid a^{p^2} = b^p = 1,\ ab = ba^{p+1}\rangle$;
- $\mathcal{G}_2 = \langle a,b,c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb\rangle$.

In ([3], Theorems 3 and 4), a complete and irredundant set of strong Shoda pairs of these groups has been found. Applying Theorem 3.1 of [15], we obtain that

$$\text{Rank of } \mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = \frac{(p-3)(p+2)}{2},\ i = 1,2.$$

We now illustrate Theorem 2 in the particular cases, when $p = 3$ or 5.

$\underline{\mathbf{p=3}}$: In this case, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = 0$ and therefore, by Theorem 2(i), $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = \pm\mathcal{Z}(\mathcal{G}_i)$, $i = 1,2$.

$\underline{\mathbf{p=5}}$: In this case, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = 7$, $i = 1,2$. Using Remark 1, we have computed the value of $n_{H,K}$ corresponding to each strong Shoda pair (H,K) of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$, which are tabulated in Figs. 1 and 2.

Theorem 2 and ([23], §11.5) yield that $[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) : \langle\mathcal{B}(\mathcal{G}_i)\rangle] \leq 2^{29}5^{27}$, $i = 1,2$.

4.2. Non Abelian groups of order p^4

We first take the case, when $p = 2$. Up to isomorphism, there are 9 non isomorphic groups of order 2^4 as listed in ([5], §118). Except the following two groups:

- $\mathcal{H}_1 = \langle a,b : a^8 = b^2 = 1, ba = a^7b\rangle$;
- $\mathcal{H}_2 = \langle a,b : a^8 = b^4 = 1, ba = a^7b, a^4 = b^2\rangle$,

the other non abelian groups of order 2^4 again satisfy the hypothesis of ([21], Theorem 6.1). Hence, if G is a non abelian group of order 2^4 other than the dihedral group $\mathcal{H}_1$ and the quaternion group $\mathcal{H}_2$, then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G)$.

For the groups $\mathcal{H}_1$ and $\mathcal{H}_2$, we obtain using Theorem 1, that $\{(\langle a\rangle, \langle 1\rangle), (\langle a\rangle, \langle a^4\rangle), (\mathcal{H}_1, \langle a\rangle), (\mathcal{H}_1, \langle a^2, b\rangle), (\mathcal{H}_1, \langle a^2, ab\rangle), (\mathcal{H}_1, \mathcal{H}_1)\}$ and $\{(\langle a\rangle, \langle 1\rangle), (\langle a\rangle, \langle a^4\rangle), (\mathcal{H}_2, \langle a\rangle), (\mathcal{H}_2, \langle a^2, b\rangle), (\mathcal{H}_2, \langle a^2, ab\rangle), (\mathcal{H}_2, \mathcal{H}_2)\}$ are complete irredundant sets of strong Shoda pairs of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Theorem 3.1 of [15] now yields that

$$\text{Rank of } \mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{H}_i])) = 1,\ i = 1, 2.$$

Also, by Theorem 2 and ([23], §11.5), it follows that

$$[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{H}_i])) : \langle \mathcal{B}(\mathcal{H}_i)\rangle] \leq 2^{12},\ i = 1, 2.$$

We next assume that p is an odd prime.

Up to isomorphism, the following are all the non abelian groups of order p^4 (see [5], §117):

1. $G_1 = \langle a, b : a^{p^3} = b^p = 1, ba = a^{1+p^2}b\rangle$;
2. $G_2 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, cb = a^p bc, ab = ba, ac = ca\rangle$;
3. $G_3 = \langle a, b : a^{p^2} = b^{p^2} = 1, ba = a^{1+p}b\rangle$;
4. $G_4 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = a^{1+p}c, ba = ab, cb = bc\rangle$;
5. $G_5 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = abc, ab = ba, bc = cb\rangle$;
6. $G_6 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = abc, cb = bc\rangle$;
7. $G_7 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^p, ab = ba^{1+p}, ac = cab^{-1}, cb = bc\rangle, & \text{if } p = 3, \\ \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^p bc\rangle, & \text{if } p > 3; \end{cases}$
8. $G_8 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^{-p}, ab = ba^{1+p}, ac = cab^{-1}, cb = bc\rangle, & \text{if } p = 3, \\ \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+dp}bc, cb = a^{dp}bc\rangle, & \text{if } p > 3 \end{cases}$
   $d \not\equiv 0, 1 (\bmod\ p)$;
9. $G_9 = \langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba\rangle$;
10. $G_{10} = \begin{cases} \langle a, b, c : a^{p^2} = b^p = c^p = 1, ab = ba, ac = cab, bc = ca^{-p}b\rangle, & \text{if } p = 3, \\ \langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, \\ \qquad bc = cb, ac = ca, ab = ba\rangle, & \text{if } p > 3. \end{cases}$

Theorem 3. *For $1 \leq i \leq 10$, the set $\mathcal{S}(G_i)$, given below, is a complete irredundant set of strong Shoda pairs of G_i:*

$$(i)\ \mathcal{S}(G_1) = \{(\langle a\rangle,\ \langle 1\rangle),\ \ (G_1,\ \langle a\rangle),\ \ (G_1, G_1)\}\ \cup \\ \{(G_1,\ \langle a^{p^2}, a^{pi}b\rangle),\ \ (G_1,\ \langle a^p, a^i b\rangle) \mid 0 \leq i \leq p-1\};$$

$$\begin{aligned}
(ii)\ \mathcal{S}(G_2) = {} & \{(\langle a,b\rangle,\ \langle b\rangle),\ \ (G_2,\ \langle a,b\rangle),\ \ (G_2,\ G_2)\}\ \cup \\
& \{(G_2,\ \langle a,b^ic\rangle),\ \ (G_2,\ \langle a^ib,a^jc\rangle) \mid 0 \le i,j \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(iii)\ \mathcal{S}(G_3) = {} & \{(G_3,\ \langle a,b^p\rangle),\ \ (G_3,\ \langle a\rangle),\ \ (G_3,\ G_3)\}\ \cup \\
& \{(\langle a,b^p\rangle,\ \langle a^{pi}b^p\rangle),\ \ (G_3,\ \langle a^p,a^ib\rangle) \mid 0 \le i \le p-1\}\ \cup \\
& \{(G_3,\ \langle a^p,a^kb^p\rangle) \mid 1 \le k \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(iv)\ \mathcal{S}(G_4) = {} & \{(G_4,\ \langle a,b\rangle),\ \ (G_4,\ G_4)\}\ \cup \\
& \{(\langle a,b\rangle,\ \langle a^{pi}b\rangle),\ \ (G_4,\ \langle a,b^ic\rangle),\ \ (G_4,\langle a^p,a^ib,a^jc\rangle) \mid 0 \le i,j \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(v)\ \mathcal{S}(G_5) = {} & \{(\langle a,b\rangle,\ \langle a\rangle),\ \ (G_5,\ \langle a^p,b,c\rangle),\ \ (G_5,\ \langle a,b\rangle),\ \ (G_5,\ G_5)\}\ \cup \\
& \{(G_5,\ \langle b,a^{pi}c\rangle) \mid 0 \le i \le p-1\}\ \cup \\
& \{(\langle a,b\rangle,\ \langle a^pb^k\rangle),\ \ (G_5,\langle b,a^kc\rangle) \mid 1 \le k \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(vi)\ \mathcal{S}(G_6) = {} & \{(\langle a^p,b,c\rangle,\ \langle a^p,c\rangle),\ (G_6,\ \langle a,b\rangle),\ (G_6,\ \langle a^p,b,c\rangle,),\ (G_6,\ G_6)\}\ \cup \\
& \{(\langle a^p,b,c\rangle,\ \langle b,a^{pi}c\rangle) \mid 0 \le i \le p-1\}\ \cup \\
& \{(G_6,\ \langle b,a^kc\rangle) \mid 1 \le k \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(vii)\ \mathcal{S}(G_7) = {} & \{(\langle b,c\rangle,\ \langle b\rangle),\ \ (\langle b,c\rangle,\ \langle c\rangle),\ \ (G_7,\ \langle a,b\rangle),\ \ (G_7,\ G_7)\}\ \cup \\
(p=3)\quad & \{(G_7,\ \langle b,a^ic\rangle) \mid 0 \le i \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(viii)\ \mathcal{S}(G_7) = {} & \{(\langle b,ac\rangle,\ \langle b\rangle),\ \ (\langle b,ac\rangle,\ \langle ac\rangle),\ \ (G_7,\ \langle a,b\rangle),\ \ (G_7,\ G_7)\}\ \cup \\
(p>3)\quad & \{(G_7,\ \langle b,a^ic\rangle) \mid 0 \le i \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(ix)\ \mathcal{S}(G_8) = {} & \{(\langle b,c\rangle,\ \langle b\rangle),\ \ (\langle b,c\rangle,\ \langle c\rangle),\ \ (G_8,\ \langle a,b\rangle),\ \ (G_8,\ G_8)\}\ \cup \\
(p=3)\quad & \{(G_8,\ \langle b,a^ic\rangle) \mid 0 \le i \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(x)\ \mathcal{S}(G_8) = {} & \{(\langle b,a^dc\rangle,\ \langle b\rangle),\ \ (\langle b,a^dc\rangle,\ \langle a^dc\rangle),\ \ (G_8,\ \langle a,b\rangle),\ (G_8,\ G_8)\}\ \cup \\
(p>3)\quad & \{(G_8,\ \langle b,a^ic\rangle) \mid 0 \le i \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(xi)\ \mathcal{S}(G_9) = {} & \{(G_9,\ \langle a,b,d\rangle),\ \ (G_9,\ G_9)\}\ \cup \\
& \{(\langle a,b,d\rangle,\ \langle d,a^ib\rangle),\ \ (G_9,\ \langle a,b,cd^i\rangle), \\
& \ (G_9,\ \langle a,b^ic,b^jd\rangle) \mid 0 \le i,j \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(xii)\ \mathcal{S}(G_{10}) = {} & \{(\langle a,b\rangle,\ \langle a\rangle),\ \ (\langle a,b\rangle,\ \langle b\rangle),\ \ (G_{10},\ \langle a,b\rangle),\ \ (G_{10},\ G_{10})\}\ \cup \\
(p=3)\quad & \{(G_{10},\ \langle b,a^ic\rangle) \mid 0 \le i \le p-1\};
\end{aligned}$$

$$\begin{aligned}
(xiii)\ \mathcal{S}(G_{10}) = {} & \{(\langle a,b,c\rangle,\langle a,c\rangle),\ \ (G_{10},\langle a,b,d\rangle)\}\ \cup \\
(p>3)\quad & \{(\langle a,b,c\rangle,\ \langle a^ic,b\rangle),\ \ (G_{10},\langle a,b,cd^i\rangle),\ \ (G_{10},G_{10}) \mid 0 \le i \le p-1\}.
\end{aligned}$$

Proof. (i) Define $N_0 := \langle 1\rangle$, $N_1 := \langle a^{p^2}\rangle$, $N_2 := \langle a^p\rangle$, $N_3 := \langle a\rangle$, $H_i := \langle a^{p^2}, a^{pi}b\rangle$, $K_j := \langle a^p, a^j b\rangle$ where $0 \le i, j \le p-1$. Observe that these subgroups are normal in G_1. Using Eq. (1), we have $\mathcal{S}_{N_1} = \mathcal{S}_{N_2} = \phi$, $\mathcal{S}_{N_3} = \{(G_1, N_3)\}$, $\mathcal{S}_{H_i} = \{(G_1, H_i)\}$, $\mathcal{S}_{K_j} = \{(G_1, K_j)\}$, $0 \le i, j \le p-1$. In order to find $\mathcal{S}_{N_0}$, we see that $\langle a\rangle$ is a maximal abelian subgroup of G_1. Further, the only subgroup D of $\langle a\rangle$ satisfying $\mathrm{core}(D) = \langle 1\rangle$ is $D = \langle 1\rangle$. This gives $\mathcal{S}_{N_0} = \{(\langle a\rangle,\ \langle 1\rangle)\}$. Define

$$\mathcal{N}_1 = \{\langle 1\rangle,\ \langle a^{p^2}\rangle,\ \langle a^p\rangle,\ \langle a\rangle,\ \langle a, b\rangle\} \ \cup\ \{\langle a^{p^2}, a^{pi}b\rangle,\ \langle a^p, a^j b\rangle \mid 0 \le i, j \le p-1\}.$$

Observe that $\sum\limits_{N\in\mathcal{N}_1} \sum\limits_{D\in\mathcal{D}_N} [G : A_N]\varphi([A_N : D]) = p^4$. Now, if $\mathcal{N}$ is the set of all normal subgroups of G_1, then

$$\begin{aligned} p^4 = |G_1| &= \sum_{N\in\mathcal{N}} \sum_{D\in\mathcal{D}_N} [G : A_N]\varphi([A_N : D]) \quad \text{(by Theorem 1)} \\ &\ge \sum_{N\in\mathcal{N}_1} \sum_{D\in\mathcal{D}_N} [G : A_N]\varphi([A_N : D]) \quad (\text{as } \mathcal{N}_1 \subseteq \mathcal{N}) \\ &= p^4. \end{aligned}$$

This yields $\mathcal{S}_N = \phi$, if $N \notin \mathcal{N}_1$ and consequently, by Theorem 1, $\bigcup_{N\in\mathcal{N}_1} \mathcal{S}_N$ is a complete irredundant set of strong Shoda pairs of G_1.

(ii)–(xiii) For $2 \le i \le 10$, consider the following set $\mathcal{N}_i$ of normal subgroups of G_i:

$$\begin{aligned} \mathcal{N}_2 &= \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a^p, b\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\} \ \cup \\ &\quad \{\langle a^p, b^i c\rangle,\ \langle a, b^i c\rangle,\ \langle ab^i c^j\rangle,\ \langle a^i b, a^j c\rangle \mid 0 \le i, j \le p-1\}; \end{aligned}$$

$$\begin{aligned} \mathcal{N}_3 &= \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a\rangle,\ \langle a, b^p\rangle,\ \langle a^p, b^p\rangle,\ \langle a, b\rangle\} \ \cup \\ &\quad \{\ \langle a^{pi} b^p\rangle,\ \langle a^p, a^i b\rangle \mid 0 \le i \le p-1\rangle\} \ \cup\ \{\ \langle a^p, a^k b^p\rangle \mid 1 \le k \le p-1\}; \end{aligned}$$

$$\begin{aligned} \mathcal{N}_4 &= \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a^p, b\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\} \ \cup \\ &\quad \{\langle a^{pi} b\rangle,\ \langle a^p, b^i c\rangle,\ \langle a, b^i c\rangle,\ \langle ab^i c^j\rangle,\ \langle a^p, a^i b, a^j c\rangle \mid 0 \le i, j \le p-1\rangle\}; \end{aligned}$$

$$\begin{aligned} \mathcal{N}_5 &= \{\langle 1\rangle,\ \langle b\rangle,\ \langle a^p, b\rangle,\ \langle a^p\rangle,\ \langle a^p, b, c\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\} \ \cup \\ &\quad \{\ \langle b, a^{pi} c\rangle \mid 0 \le i \le p-1\rangle\} \ \cup\ \{\ \langle a^p b^k\rangle,\ \langle b, a^k c\rangle \mid 1 \le k \le p-1\}; \end{aligned}$$

$$\mathcal{N}_6 = \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a^p, b\rangle,\ \langle a, b\rangle,\ \langle a^p, b, c\rangle,\ \langle a, b, c\rangle\} \ \cup\ \{\langle b, a^k c\rangle \mid 1 \le k \le p-1\};$$

$$\mathcal{N}_7 = \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a^p, b\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\} \ \cup\ \{\langle b, a^i c\rangle,\ \mid 0 \le i \le p-1\};$$

$$\mathcal{N}_8 = \{\langle 1\rangle,\ \langle a^p\rangle,\ \langle a^p, b\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\} \ \cup\ \{\langle b, a^i c\rangle,\ \mid 0 \le i \le p-1\};$$

$$\begin{aligned} \mathcal{N}_9 &= \{\langle 1\rangle,\ \langle a\rangle,\ \langle a, d\rangle,\ \langle a, b, d\rangle,\ \langle a, b, c, d\rangle\} \ \cup \\ &\quad \{\langle a^i b\rangle,\ \langle a, bc^i d^j\rangle,\ \langle a, cd^i\rangle,\ \langle a, b, cd^i\rangle\ \langle a, b^i c, b^j d\rangle \mid 0 \le i, j \le p-1\rangle\}; \end{aligned}$$

$$\mathcal{N}_{10} = \begin{cases} \{\langle 1\rangle,\ \langle a^3\rangle,\ \langle a^3, b\rangle,\ \langle b, c\rangle,\ \langle b, ac\rangle,\ \langle b, a^2c\rangle,\ \langle a, b\rangle,\ \langle a, b, c\rangle\}, \text{ if } p = 3; \\ \{\langle 1\rangle,\ \langle a\rangle,\ \langle a, b\rangle,\ \langle a, b, d\rangle,\ \langle a, b, c, d\rangle\} \ \cup\ \{\langle a, b, cd^i\rangle \mid 0 \le i \le p-1\}, \text{ if } p > 3. \end{cases}$$

Now proceeding as in (i), we get the required complete and irredundant set of strong Shoda pairs of G_i, $2 \le i \le 10$. □

For a particular odd prime p, the computation of $n_{H,K}$ corresponding to a strong Shoda pair $(H, K) \in \mathcal{S}(G_i)$, $1 \le i \le 10$, may be done using Remark 1. An explicit bound on the index of $\langle \mathcal{B}(G_i)\rangle$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \le i \le 10$, may thus be computed using Theorems 2 and 3.

Remark 2. It would be of interest to compute the integer $n_{H,K}$ corresponding to each strong Shoda pairs (H, K) of the groups discussed in this section, explicitly in terms of p.

Finally, Theorem 3 along with ([19], Proposition 3.4) and ([15], Theorem 3.1) also yield the following:

Corollary 3. *The Wedderburn decomposition of* $\mathbb{Q}[G_i]$ *and the rank of* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \le i \le 10$, *are as follows:*

G	$\mathbb{Q}[G]$	Rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$
G_1	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$\frac{(p+1)(p^2-5)}{2}$
G_2	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$\frac{p^3-p^2-3p-5}{2}$
G_3	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(Q(\zeta_p))^{(p)}$	$\frac{p^3+p^2-7p-3}{2}$
G_4	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{(p-3)(p+1)^2}{2}$
G_5	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{p^3+p^2-7p-3}{2}$
G_6	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$	$(p-3)(p+1)$
G_7	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$p^2 - p - 4$
G_8	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$p^2 - p - 4$
G_9	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{(p-3)(p+1)^2}{2}$
G_{10} $(p = 3)$	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_3)^{(4)} \oplus M_3(\mathbb{Q}(\zeta_3)) \oplus M_3(\mathbb{Q}(\zeta_9))$	2
G_{10} $(p > 3)$	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$	$(p-3)(p+1)$

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