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Generalized Hilbert coefficients and Northcott's inequality

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ABSTRACT

Let R be a Cohen–Macaulay local ring of dimension d with infinite residue field. Let I be an R-ideal that has analytic spread $\ell(I) = d$, satisfies the G_d condition and the weak Artin–Nagata property AN^-_{d-2}. We provide a formula relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where J is a general minimal reduction of I, $P_I(n)$ and $H_I(n)$ are respectively the generalized Hilbert–Samuel polynomial and the generalized Hilbert–Samuel function. We then use it to establish formulas to compute the generalized Hilbert coefficients of I. As an application, we extend Northcott's inequality to non-$\mathfrak{m}$-primary ideals. Furthermore, when equality holds, we prove that the ideal I enjoys nice properties. Indeed, if this is the case, then the reduction number of I is at most one and the associated graded ring of I is Cohen–Macaulay. We also recover results of G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie on the positivity of the generalized first Hilbert coefficient $j_1(I)$. Our work extends that of S. Huckaba, C. Huneke and A. Ooishi to ideals that are not necessarily $\mathfrak{m}$-primary.

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1. Introduction

Hilbert functions play an important role in commutative algebra and algebraic geometry for the reason that they capture many useful numerical invariants which reflect various algebraic and geometric properties of an ideal of a projective variety or a Noetherian local ring. Besides the application in intersection theory and singularity theory, these invariants are also used in the study of the arithmetical properties, like the depth, of the blowup algebras such as the associated graded rings (see for instance, [27,25,21]).

The classical Hilbert functions are only defined for ideals that are primary to the maximal ideal. It is well-known that the Hilbert multiplicity (i.e., the normalized leading coefficient of the Hilbert polynomial obtained from the Hilbert function) is an important invariant that is used to describe proper intersections and isolated singularities. In order to study improper intersections and non-isolated singularities, in 1993, R. Achilles and M. Manaresi introduced the concept of the j-multiplicity as a generalization of the Hilbert multiplicity [1] and, in 1997, they also defined a generalized (bivariate) Hilbert function using the bigraded ring of the associated graded ring with respect to the maximal ideal [2]. In 1999, H. Flenner, L. O' Carroll and W. Vogel defined the generalized Hilbert function using the 0-th local cohomology functor [9, Definition 6.1.5]. Later in 2003, C. Ciupercă [5] introduced the generalized (bivariate) Hilbert coefficients via the approach of R. Achilles and M. Manaresi in [2]. In 2012, C. Polini and Y. Xie defined the concepts of the generalized Hilbert polynomial and the generalized Hilbert coefficients following the approach of H. Flenner, L. O' Carroll and W. Vogel in [9], and they proved that the generalized Hilbert coefficients as defined using the 0-th local cohomology functor can also be obtained from the generalized (bivariate) Hilbert function of the bigraded ring of the associated graded ring with respect to a suitable ideal [26]. One of the fundamental properties proved by C. Polini and Y. Xie illustrates the behavior of the generalized Hilbert function under a hyperplane section [26]. Indeed, they proved that the first $d-1$ generalized Hilbert coefficients $j_0(I), \dots, j_{d-2}(I)$, where I is an ideal in a Noetherian local ring of dimension d, are preserved after modding a general element. This nice property allows us to study the generalized Hilbert coefficients by reduction to the lower dimensional case.

The generalized Hilbert coefficients are important invariants of an ideal I in a Noetherian local ring $(R, \mathfrak{m})$. It is well-known that the normalized leading coefficient $j_0(I)$ (i.e., the *j-multiplicity* of I) was used to prove the refined Bezout's theorem [9], to detect integral dependence of non-$\mathfrak{m}$-primary ideals (extension of the fundamental theorem of Rees) [8], and to study the depth of the associated graded rings of arbitrary ideals (see [25] and [21]). The next normalized coefficient $j_1(I)$ is called the *generalized first Hilbert coefficient* of I. If I is $\mathfrak{m}$-primary, $j_1(I) = e_1(I)$ is called the first Hilbert coefficient. It is also called the *Chern number* by W. V. Vasconcelos for its tracking position in distinguishing Noetherian filtrations with the same Hilbert multiplicity [31]. The first Hilbert coefficient $e_1(Q)$, where Q is a parameter ideal, was used to characterize the Cohen–Macaulay property for large classes of rings [10]. Moreover, G. Colomé-Nin, C. Polini,

B. Ulrich and Y. Xie used the generalized first Hilbert coefficient $j_1(I)$ to bound the number of steps in a process of normalization of ideals of maximal analytic spread [6]. Therefore it is very important to establish properties such as the positivity for the generalized Hilbert coefficients.

In the case of $\mathfrak{m}$-primary ideals, there are a number of formulas to compute the Hilbert coefficients (see for instance, [17] and [15]). In 1987, C. Huneke provided a formula relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where I is an $\mathfrak{m}$-primary ideal in a 2-dimensional Cohen–Macaulay local ring, J is a minimal reduction of I, $P_I(n)$ and $H_I(n)$ are respectively the usual Hilbert–Samuel polynomial and the usual Hilbert–Samuel function of I [17]. This formula was extended later by S. Huckaba to $\mathfrak{m}$-primary ideals in Cohen–Macaulay local rings of arbitrary dimension [15]. S. Huckaba also established some formulas to compute the Hilbert coefficients of $\mathfrak{m}$-primary ideals I, and provided conditions in terms of the first Hilbert coefficient $e_1(I)$ for the associated graded ring of I to be almost Cohen–Macaulay [15].

If I is an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay local ring R, the positivity of $e_1(I)$ can be observed from the well-known Northcott's inequality

$$e_1(I) \geq e_0(I) - \lambda(R/I) = \lambda(R/J) - \lambda(R/I) = \lambda(I/J),$$

where J is a minimal reduction of I. By this inequality, one has that $e_1(I) = 0$ if and only if I is a complete intersection. Furthermore, when equality holds, the ideal I enjoys nice properties. Indeed, it was shown that $e_1(I) = \lambda(I/J)$ if and only if the reduction number of I is at most 1, and when this is the case, the associated graded ring of I is Cohen–Macaulay (see [17] and [24]).

This paper generalizes the above classical results to ideals of maximal analytic spread. In Section 2, we fix the notation and recall some basic concepts and facts that will be used throughout the paper. For an ideal I in a d-dimensional Noetherian local ring that has maximal analytic spread $\ell(I) = d$ and satisfies the G_d condition, we establish a formula to compute $e_1(\overline{I})$, where $\overline{I}$ is a 1-dimensional reduction of I (see Section 2 for the definition of $\overline{I}$). We then give a condition in terms of $e_1(\overline{I})$ for the associated graded ring of I to be almost Cohen–Macaulay. This result generalizes [15, Theorem 3.1]. In Section 3, we provide a generalized version of [15, Theorem 2.4] relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where I is an ideal in a d-dimensional Cohen–Macaulay local ring that satisfies $\ell(I) = d$, the G_d condition and the AN^-_{d-2}, J is a general minimal reduction of I, $P_I(n)$ and $H_I(n)$ are respectively the generalized Hilbert–Samuel polynomial and the generalized Hilbert–Samuel function of I. As an application, we establish some formulas to compute the generalized Hilbert coefficients. In the last section, we apply our formula to prove a generalized version of Northcott's inequality, and recover the work of G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie on the positivity of the generalized first Hilbert coefficient $j_1(I)$. At the same time, we prove that, if equality holds in the generalized Northcott's inequality, the reduction number of I is at most one and the associated graded ring of I is Cohen–Macaulay. This result generalizes the classical results of [17] and [24].

2. Formula for $e_1(\overline{I})$

In this paper, we always assume that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$ and infinite residue field k (one can always enlarge the residue field to be infinite by replacing R by $R(z) = R[z]_{\mathfrak{m}R[z]}$, where z is a variable over R). Let I be an R-ideal. We recall the concept of the generalized Hilbert–Samuel function of I. Let $G = \mathrm{gr}_I(R) = \oplus_{n=0}^{\infty} I^n/I^{n+1}$ be the *associated graded ring* of I. As the homogeneous components of G may not have finite length, one considers the G-submodule of elements supported on $\mathfrak{m}$: $W = \{\xi \in G \mid \exists\, t > 0 \text{ such that } \xi \cdot \mathfrak{m}^t = 0\} = H^0_{\mathfrak{m}}(G) = \oplus_{n=0}^{\infty} H^0_{\mathfrak{m}}(I^n/I^{n+1})$. Since W is a finite graded module over $\mathrm{gr}_I(R) \otimes_R R/\mathfrak{m}^{\alpha}$ for some $\alpha \geq 0$, its Hilbert–Samuel function $H_W(n) = \sum_{j=0}^{n} \lambda(H^0_{\mathfrak{m}}(I^j/I^{j+1}))$ is well defined. The *generalized Hilbert–Samuel function* of I is defined to be: $H_I(n) = H_W(n)$ for every $n \geq 0$.

The definition of generalized Hilbert–Samuel function was introduced by H. Flenner, L. O' Carroll and W. Vogel in 1999 [9, Definition 6.1.5], and studied later by C. Polini and Y. Xie [26] as well as G. Colomé Nin, C. Polini, B. Ulrich and Y. Xie [6]. Since $\dim_G W \leq \dim R = d$, $H_I(n)$ is eventually a polynomial of degree at most d

$$P_I(n) = \sum_{i=0}^{d} (-1)^i j_i(I) \binom{n+d-i}{d-i}.$$

C. Polini and Y. Xie defined $P_I(n)$ to be the *generalized Hilbert–Samuel polynomial* of I and $j_i(I)$, $0 \leq i \leq d$, the *generalized Hilbert coefficients* of I [26]. The normalized leading coefficient $j_0(I)$ is called the *j-multiplicity* of I (see [1,22], or [26]). The next normalized coefficient $j_1(I)$ is called the *generalized first Hilbert coefficient*.

Recall the Krull dimension of the special fiber ring $G/\mathfrak{m}G$ is called the *analytic spread* of I and is denoted by $\ell(I)$. In general, $\dim_G W \leq \ell(I) \leq d$ and equalities hold if and only if $\ell(I) = d$. Therefore $j_0(I) \neq 0$ if and only if $\ell(I) = d$ (see [2] or [22]).

If I is $\mathfrak{m}$-primary, each homogeneous component of G has finite length, thus $W = G$ and the generalized Hilbert–Samuel function coincides with the usual Hilbert–Samuel function; in particular, the generalized Hilbert coefficients $j_i(I)$, $0 \leq i \leq d$, coincide with the usual Hilbert coefficients $e_i(I)$.

The definition of generalized Hilbert coefficients is different from the one given by C. Ciupercă [5] where he used the bigraded ring $\mathrm{gr}_{\mathfrak{m}}(G)$. Polini and Xie proved that the generalized Hilbert coefficients as defined above can also be obtained from the generalized (bivariate) Hilbert–Samuel function of the bigraded ring $\mathrm{gr}_q(G)$, where q is a suitable $\mathfrak{m}$-primary ideal, and that the generalized Hilbert coefficients $j_0(I), \ldots, j_{d-2}(I)$ are preserved under a general hyperplane section [26].

We are going to use the tool of general elements to study the generalized Hilbert–Samuel function. Let $I = (a_1, \dots, a_t)$ and write $x_i = \sum_{j=1}^{t} \lambda_{ij} a_j$ for $1 \le i \le s$ and $(\lambda_{ij}) \in R^{st}$. The elements $x_1, \dots, x_s$ form a *sequence of general elements* in I (equivalently $x_1, \dots, x_s$ are *general* in I) if there exists a Zariski dense open subset U of k^{st} such that the image $(\overline{\lambda_{ij}}) \in U$. When $s = 1$, $x = x_1$ is said to be *general* in I.

Recall an ideal $J \subseteq I$ is called a *reduction* of I if $JI^r = I^{r+1}$ for some non-negative integer r. The least such r is denoted by $r_J(I)$. A reduction is *minimal* if it is minimal with respect to inclusion. The *reduction number* $r(I)$ of I is defined as $\min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}$. Since R has infinite residue field, the minimal number of generators $\mu(J)$ of any minimal reduction J of I equals the analytic spread $\ell(I)$. Furthermore, general $\ell(I)$ elements in I form a minimal reduction J whose $r_J(I)$ coincides with the reduction number $r(I)$ (see [28, 2.2] or [18, 8.6.6]). One says that J is a *general minimal reduction* of I if it is generated by $\ell(I)$ general elements in I.

The ideal I is said to satisfy the G_{s+1} *condition* if for every $\mathfrak{p} \in V(I)$ with $\operatorname{ht}\mathfrak{p} = i \le s$, the ideal $I_{\mathfrak{p}}$ is generated by i elements, i.e., $I_{\mathfrak{p}} = (x_1, \dots, x_i)_{\mathfrak{p}}$ for some $x_1, \dots, x_i$ in I.

From now on, we will assume I has $\ell(I) = d$ and satisfies the G_d condition. Let $J = (x_1, \dots, x_d)$, where $x_1, \dots, x_d$ are general elements in I, i.e., J is a general minimal reduction of I. For $i \le d-1$, set $J_i = (x_1, \dots, x_i)$ (with the convention $J_i = (0)$ if $i \le 0$), $\overline{R} = R/J_{d-1} : I^\infty$, where $J_{d-1} : I^\infty = \{a \in R \mid \exists\, t > 0 \text{ such that } a \cdot I^t \subseteq J_{d-1}\}$, and use $^-$ to denote images in the quotient ring $\overline{R}$. Then $\overline{R}$ is a 1-dimensional Cohen–Macaulay local ring and $\overline{I}$ is $\overline{\mathfrak{m}}$-primary. Hence the generalized Hilbert–Samuel function $H_{\overline{I}}(n)$ and the generalized Hilbert–Samuel polynomial $P_{\overline{I}}(n)$ are respectively the usual Hilbert–Samuel function and the usual Hilbert–Samuel polynomial of $\overline{I}$. Note $H_{\overline{R}}(I)$ and hence $P_{\overline{I}}(n)$ do not depend on choices of general elements $x_1, \dots, x_{d-1}$ in I, and $P_{\overline{I}}(n) = e_0(\overline{I})(n+1) - e_1(\overline{I})$, where $e_0(\overline{I}) = \lambda(\overline{R}/(\overline{x_d})) = j_0(I)$ (see [26]). If R is Cohen–Macaulay and I is $\mathfrak{m}$-primary, then $e_1(\overline{I}) = e_1(I)$ (see for instance [27, Proposition 1.2]). But they are in general not the same.

We will show in Theorem 2.3 that $e_1(\overline{I})$ (like $e_1(I)$, see [15, Theorem 3.1]) characterizes the depth of the associated graded ring G. For $\operatorname{depth}(G)$, we mean the depth of the local ring G_M, where $M = \mathfrak{m}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$ denotes the maximal homogeneous ideal of G. Since $\operatorname{depth}(G) \le \dim G = \dim R = d$, G is said to be *Cohen–Macaulay* if $\operatorname{depth}(G) = d$ and *almost Cohen–Macaulay* if $\operatorname{depth}(G) = d - 1$. The condition $\operatorname{depth}(G) \ge d - 1$ is a useful one, especially when one considers questions about the behavior of I^n. It reduces greatly the computation of the generalized Hilbert coefficients (see Corollary 3.4 in Section 3).

Theorem 2.3 is achieved from a formula computing $e_1(\overline{I})$ (see Lemma 2.2 in the following). Since we do not have the finite length on R/I, to compare the length $\lambda(I^{n+1}/JI^n)$ (this length is finite by the G_d condition) with $\lambda(\overline{I}^{n+1}/\overline{JI}^n)$, where J is a general minimal reduction of I, we need the following lemma.

Lemma 2.1. *Let $D \subseteq B \subseteq A$ and $D \subseteq C \subseteq A$ be finite modules over R such that A/B and C/D have finite lengths (while the lengths of B/D and A/C are not necessarily finite). Then*

$$\lambda(A/B) + \lambda(B \cap C/D) = \lambda(C/D) + \lambda(A/(B+C)).$$

Proof. By the exact sequences

$$\begin{gathered} 0 \to B \cap C/D \to B/D \xrightarrow{\pi_1} (B+C)/C \to 0, \\ 0 \to (B+C)/C \xrightarrow{i_1} A/C \to A/(B+C) \to 0, \\ 0 \to C/D \to A/D \xrightarrow{\pi_2} A/C \to 0, \\ 0 \to B/D \xrightarrow{i_2} A/D \to A/B \to 0, \end{gathered}$$

we have the following long exact sequences

$$\begin{gathered} 0 \to B \cap C/D \to H^0_{\mathfrak{m}}(B/D) \to H^0_{\mathfrak{m}}((B+C)/C) \to 0 \to H^1_{\mathfrak{m}}(B/D) \xrightarrow{\widetilde{\pi_1}} H^1_{\mathfrak{m}}((B+C)/C) \to 0, \\ 0 \to H^0_{\mathfrak{m}}((B+C)/C) \to H^0_{\mathfrak{m}}(A/C) \to A/(B+C) \xrightarrow{\Delta_1} H^1_{\mathfrak{m}}((B+C)/C) \xrightarrow{\widetilde{i_1}} H^1_{\mathfrak{m}}(A/C) \to 0, \\ 0 \to C/D \to H^0_{\mathfrak{m}}(A/D) \to H^0_{\mathfrak{m}}(A/C) \to 0 \to H^1_{\mathfrak{m}}(A/D) \xrightarrow{\widetilde{\pi_2}} H^1_{\mathfrak{m}}(A/C) \to 0, \\ 0 \to H^0_{\mathfrak{m}}(B/D) \to H^0_{\mathfrak{m}}(A/D) \to A/B \xrightarrow{\Delta_2} H^1_{\mathfrak{m}}(B/D) \xrightarrow{\widetilde{i_2}} H^1_{\mathfrak{m}}(A/D) \to 0, \end{gathered}$$

and the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Im}(\Delta_2) & \to & H^1_{\mathfrak{m}}(B/D) & \xrightarrow{\widetilde{i_2}} & H^1_{\mathfrak{m}}(A/D) & \to & 0 \\ & & & & \downarrow id & & \downarrow \widetilde{\pi_2} & & \\ 0 & \to & \mathrm{Ker}(\widetilde{i_1} \circ \widetilde{\pi_1}) & \to & H^1_{\mathfrak{m}}(B/D) & \xrightarrow{\widetilde{i_1} \circ \widetilde{\pi_1}} & H^1_{\mathfrak{m}}(A/C) & \to & 0 \end{array}$$

with exact rows and isomorphic vertical maps id and $\widetilde{\pi_2}$, hence $\mathrm{Im}(\Delta_2) \cong \mathrm{Ker}(\widetilde{i_1} \circ \widetilde{\pi_1})$. Since $\mathrm{Ker}(\widetilde{i_1} \circ \widetilde{\pi_1}) \cong \mathrm{Ker}(\widetilde{i_1}) = \mathrm{Im}(\Delta_1)$, we have $\mathrm{Im}(\Delta_2) \cong \mathrm{Im}(\Delta_1)$. Now by the above exact sequences

$$\begin{aligned} \lambda(A/B) + \lambda(B \cap C/D) &= \lambda(\mathrm{Im}(\Delta_2)) + \lambda(H^0_{\mathfrak{m}}(A/D)) - \lambda(H^0_{\mathfrak{m}}(B/D)) + \lambda(B \cap C/D) \\ &= \lambda(\mathrm{Im}(\Delta_1)) + \lambda(H^0_{\mathfrak{m}}(A/D)) - \lambda(H^0_{\mathfrak{m}}(B/D)) + \lambda(B \cap C/D) \\ &= \lambda(A/(B+C)) + \lambda(H^0_{\mathfrak{m}}((B+C)/C)) - \lambda(H^0_{\mathfrak{m}}(A/C)) \\ &\quad + \lambda(H^0_{\mathfrak{m}}(A/D)) - \lambda(H^0_{\mathfrak{m}}(B/D)) + \lambda(B \cap C/D) \\ &= \lambda(A/(B+C)) + \lambda(H^0_{\mathfrak{m}}(B/D)) - \lambda(B \cap C/D) + \lambda(C/D) \\ &\quad - \lambda(H^0_{\mathfrak{m}}(B/D)) + \lambda(B \cap C/D) \\ &= \lambda(C/D) + \lambda(A/(B+C)). \qquad \square \end{aligned}$$

Applying Lemma 2.1, we obtain the following proposition. Recall if $f : \mathbb{Z} \to \mathbb{Z}$ is a function then Δ is the first difference function defined by $\Delta[f(n)] = f(n) - f(n-1)$, and Δ^i is defined as $\Delta^i[f(n)] = \Delta^{i-1}[\Delta f(n)]$. By convention, $\Delta^0[f(n)] = f(n)$.

Proposition 2.2. *Let I be an R-ideal with $\ell(I) = d$ and that satisfies the G_d condition. For general elements $x_1, \dots, x_d$ in I, set $J = (x_1, \dots x_d)$, $J_{d-1} = (x_1, \dots, x_{d-1})$, and $\overline{R} = R/J_{d-1} : I^\infty$ as above. Then for every $n \geq 0$, one has*

(a) $\lambda(I^{n+1}/JI^n) - \lambda\big[(J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n\big] = \Delta[P_{\overline{I}}(n) - H_{\overline{I}}(n)]$.
(b) $\sum_{n=0}^{\infty} \big[\lambda(I^{n+1}/JI^n) - \lambda[(J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n]\big] = e_1(\overline{I})$.

Proof. (a) For every $n \geq 0$, we have

$$\begin{array}{ccc} I^{n+1} + J_{d-1} : I^\infty & \to & I^{n+1} \\ \downarrow & & \downarrow \\ JI^n + J_{d-1} : I^\infty & \to & JI^n \end{array}$$

with $\left(I^{n+1} + J_{d-1} : I^\infty\right) / (JI^n + J_{d-1} : I^\infty)$ and I^{n+1}/JI^n all having finite lengths by the G_d condition. By Lemma 2.1,

$$\begin{aligned} \lambda(I^{n+1}/JI^n) = {} & \lambda\left(\left(I^{n+1} + J_{d-1} : I^\infty\right) / (JI^n + J_{d-1} : I^\infty)\right) \\ & + \lambda\big[(JI^n + J_{d-1} : I^\infty) \cap I^{n+1}/JI^n\big]. \end{aligned}$$

Since $(JI^n + J_{d-1} : I^\infty) \cap I^{n+1}/JI^n \cong (J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n$, we have

$$\begin{aligned} \lambda(I^{n+1}/JI^n) - \lambda\big[(J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n\big] &= \lambda(\overline{I}^{n+1}/\overline{JI}^n) \\ &= \left(\Delta(P_{\overline{I}} - H_{\overline{I}})\right)(n), \end{aligned}$$

where the latter equality follows from [15, Theorem 2.4]. Now (b) follows by (a) and [15, Corollary 2.10]. $\square$

We recall some residual intersection properties. Let $J_i = (x_1, \dots, x_i)$ (by convention, $J_i = (0)$ if $i \leq 0$), where $x_1, \dots, x_i$ are elements in I. Define $J_i : I = \{a \in R \,|\, a \cdot I \subseteq J_i\}$. One says that $J_i : I$ is an *i-residual intersection* of I if $I_\mathfrak{p} = (x_1, \dots, x_i)_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_\mathfrak{p} \leq i-1$. An i-residual intersection $J_i : I$ is called a *geometric i-residual intersection* of I if, in addition, $I_\mathfrak{p} = (x_1, \dots, x_i)_\mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leq i$. It was shown that if I satisfies the G_{s+1} condition, then for general elements $x_1, \dots, x_{s+1}$ in I and each $0 \leq i \leq s$, the ideal $J_i : I$ is a geometric i-residual intersection of I, and $J_{s+1} : I$ is an $(s+1)$-residual intersection of I (see [29] or [25, Lemma 3.1]).

Assume R is Cohen–Macaulay. The ideal I is *s-weakly residually* (S_2) (respectively, has the *weak Artin–Nagata property* AN_s^-) if for every geometric i-residual intersection

$J_i : I$, $0 \leq i \leq s$, the quotient ring $R/J_i : I$ satisfies Serre's condition (S_2) (respectively, is Cohen–Macaulay).

The notion of residual intersections was introduced by Artin and Nagata as a generalization of the concept of linkage to the case where the two "linked" ideals do not necessarily have the same height [3]. The issue on the Cohen–Macaulayness of residual intersections has been addressed in a series of results (for instance, [16,14,19,29]), which require either depth conditions on all of the Koszul homology modules of I such as the "strong Cohen–Macaulayness", or weaker "sliding depth condition", or depth conditions on sufficiently many powers of I.

The following theorem generalizes [15, Theorem 3.1] to ideals that are not necessarily $\mathfrak{m}$-primary. Notice if R is Cohen–Macaulay and I is $\mathfrak{m}$-primary, then $e_1(I) = e_1(\overline{I})$ [27, Proposition 1.2], and I automatically satisfies $\ell(I) = d$, the G_d condition, the $(d-2)$-weakly residually (S_2) as well as the weak Artin–Nagata property AN^-_{d-2}.

Theorem 2.3. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and the AN^-_{d-2}. Then for a general minimal reduction $J = (x_1, \dots, x_d)$ of I, the following statements are equivalent:*

(a) $\sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) = e_1(\overline{I})$.
(b) *For every $n \geq 0$, $J_{d-1} \cap I^{n+1} = J_{d-1}I^n$, where $J_{d-1} = (x_1, \dots, x_{d-1})$ is defined as before.*
(c) $\operatorname{depth}(G) \geq d - 1$.

Proof. Since I satisfies $\ell(I) = d$ and the G_d condition, one has that $J_i : I$ is a geometric i-residual intersection of I, where $J_i = (x_1, \dots, x_i)$, $0 \leq i \leq d-1$ [25]. By the weak Artin–Nagata property AN^-_{d-2}, one has that $J_i : I^\infty = J_i : I = J_i : x_{i+1}$ and $(J_i : I^\infty) \cap I = (J_i : I) \cap I = J_i$ for $0 \leq i \leq d-1$ [29]. In particular, $(J_{d-1} : I^\infty) \cap I = (J_{d-1} : I) \cap I = J_{d-1}$. Therefore for $n \geq 0$, $(J_{d-1} : I^\infty) \cap I^{n+1} = J_{d-1} \cap I^{n+1}$ and $(J_{d-1} : I^\infty) \cap JI^n = J_{d-1} \cap JI^n$.

Assume (b) is true. Then for $n \geq 0$,

$$\begin{aligned}\lambda\left[(J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n\right] &= \lambda\left[J_{d-1} \cap I^{n+1}/J_{d-1} \cap JI^n\right] \\ &= \lambda\left[J_{d-1}I^n/J_{d-1} \cap JI^n\right] = 0.\end{aligned}$$

And (a) follows by Proposition 2.2 (b).

Now assume (a). By Proposition 2.2 (b), one has that for every $n \geq 0$,

$$\lambda\left[(J_{d-1} : I^\infty) \cap I^{n+1}/(J_{d-1} : I^\infty) \cap JI^n\right] = 0.$$

Hence

$$J_{d-1} \cap I^{n+1} = (J_{d-1} : I^\infty) \cap I^{n+1} = (J_{d-1} : I^\infty) \cap JI^n = J_{d-1} \cap JI^n.$$

We use induction on n to prove that for every $n \geq 0$, $J_{d-1} \cap I^{n+1} = J_{d-1}I^n$. This is clear if $n = 0$. Assume $n \geq 1$ and $J_{d-1} \cap I^n = J_{d-1}I^{n-1}$. Then (b) follows by the following equalities:

$$\begin{aligned}
J_{d-1} \cap I^{n+1} &= J_{d-1} \cap JI^n \\
&= J_{d-1} \cap (J_{d-1}I^n + x_d I^n) \\
&= J_{d-1}I^n + J_{d-1} \cap x_d I^n \\
&= J_{d-1}I^n + x_d[(J_{d-1} : x_d) \cap I^n] \\
&= J_{d-1}I^n + x_d[J_{d-1} \cap I^n] \\
&= J_{d-1}I^n + x_d J_{d-1}I^{n-1} \\
&= J_{d-1}I^n.
\end{aligned}$$

Finally we show (b) is equivalent to (c). Set $\delta(I) = d - g$, where height $I = g$. We use the induction on δ. If $\delta = 0$, the assertion follows because (b) is equivalent to that $x_1^*, \dots, x_{d-1}^*$ form a regular sequence on G (see [30, Proposition 2.6]), and the latter is equivalent to that $\operatorname{depth}(G) \geq d-1$. Thus we may assume $\delta(I) \geq 1$ and the theorem holds for smaller values of $\delta(I)$. In particular, $d \geq g+1$. Since $x_1^*, \dots, x_g^*$ form a regular sequence on G, we may factor out $x_1, \dots, x_g$ to assume $g = 0$. Now $d = \delta(I) \geq 1$. Set $S = R/0 : I$. Then S is Cohen–Macaulay since I satisfies the AN_{d-2}^-. Note $\dim S = \dim R = d$, grade $(IS) \geq 1$, IS still satisfies $\ell(IS) = \ell(I) = d$, the G_d condition and the AN_{d-2}^- (see for instance [29]). We claim that (b) is equivalent to $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$. Indeed, if (b) holds, then clearly $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$. On the other hand, if $J_{d-1}S \cap I^{n+1}S = J_{d-1}I^nS$ for every $n \geq 0$, then

$$J_{d-1} \cap I^{n+1} \subseteq J_{d-1}I^n + (0 : I) \cap I^{n+1} = J_{d-1}I^n,$$

by the fact that $(0 : I) \cap I = 0$ (see [29]).

By the exact sequence $0 \to 0 : I \to R \to R/0 : I \to 0$, one has that $\operatorname{depth}(0 : I) \geq d$. Since $(0 : I) \cap I = 0$, there is a graded exact sequence

$$0 \to 0 : I \to G \to \operatorname{gr}_{IS}(S) \to 0.$$

Hence one has that $\operatorname{depth}(G) \geq d - 1 \Leftrightarrow \operatorname{depth}(\operatorname{gr}_{IS}(S)) \geq d - 1$. We are done by induction hypothesis since $\delta(IS) = d - \operatorname{grade}(IS) < d = \delta(I)$. $\square$

3. Formulas for $j_i(I)$, $1 \leq i \leq d$

In this section we will provide a formula relating the length $\lambda(I^{n+1}/JI^n)$ to the difference $P_I(n) - H_I(n)$, where I is an ideal with $\ell(I) = d$, and satisfies the G_d condition and the AN_{d-2}^-, J is a general minimal reduction of I, $P_I(n)$ and $H_I(n)$ are respectively the generalized Hilbert–Samuel polynomial and the generalized Hilbert–Samuel function of I (see Theorem 3.2). This formula generalizes [15, Theorem 2.4]. Before we state this result, we first prove the following lemma. Recall that for any two integers n and t, $t \geq 1$, the binomial coefficients are defined by $\binom{n}{t} = \frac{n\cdot(n-1)\cdots(n-t+1)}{t!}$, and $\binom{n}{0} = 1$.

Lemma 3.1. *Let $H : \mathbb{Z} \to \mathbb{Z}$ be a polynomial function with $H(n) = 0$ for all $n < 0$. Let $P(n) = \sum_{i=0}^{d}(-1)^i e_i \binom{n+d-i}{d-i}$ be the polynomial of H. Then $e_0 = \Delta^d[P(n)]$ and*

$$e_i = \sum_{n=i-1}^{\infty} \binom{n}{i-1} \Delta^d[P(n) - H(n)] \quad \textit{for} \quad 1 \leq i \leq d.$$

Proof. Let $h(t) = \sum_{n=0}^{\infty} H(n)t^n$ be the generating function of H. Then $h(t) = \frac{Q(t)}{(1-t)^{d+1}}$, where $Q(t) = a_0 + a_1 t + \ldots + a_N t^N \in \mathbb{Z}[t]$, $Q(1) \neq 0$. By [4, Proposition 4.1.9], one has

$$e_i = \frac{Q^{(i)}(1)}{i!} = \sum_{m=i}^{N} \binom{m}{i} a_m \quad \text{for} \quad 0 \leq i \leq d.$$

Set $a_m = 0$ if $m < 0$ or $m > N$. By computation, for $n \in \mathbb{Z}$,

$$\Delta^d[P(n)] = e_0 = \sum_{m=0}^{\infty} a_m \quad \text{and} \quad \Delta^d[H(n)] = \sum_{m=0}^{n} a_m.$$

Hence

$$\Delta^d[P(n) - H(n)] = \sum_{m=n+1}^{\infty} a_m$$

and

$$\sum_{n=i-1}^{\infty} \binom{n}{i-1} \Delta^d[P(n) - H(n)] = \sum_{n=i-1}^{\infty} \binom{n}{i-1} \sum_{m=n+1}^{\infty} a_m$$
$$= \sum_{m=i}^{\infty} \sum_{s=i-1}^{m-1} \binom{s}{i-1} a_m = \sum_{m=i}^{\infty} \binom{m}{i} a_m = e_i \quad \text{for} \quad 1 \leq i \leq d. \qquad \square$$

Theorem 3.2. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and the AN^-_{d-2}. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has that for all $n \geq 0$,*

$$\lambda(I^{n+1}/JI^n) + \omega_n(J, I) = \Delta^d[P_I(n) - H_I(n)],$$

where $\omega_0(J, I) = \lambda(R/(J_{d-1} : I + I)) - \lambda[H^0_{\mathfrak{m}}(R/I)]$, and for $n \geq 1$,

$$\omega_n(J,I)=\sum_{i=0}^{d-2}\Delta^{d-1-i}[\lambda(\widetilde{K}^i_{n-1})]+\sum_{i=0}^{d-2}\Delta^{d-2-i}\big[\lambda(\widetilde{L}^i_n)-\lambda(L^i_n)+\lambda(N^i_n)\big]$$
$$-\sum_{i=1}^{d-1}\lambda\left[J_i\cap I^{n+1}/(J_i\cap JI^n+J_{i-1}\cap I^{n+1})\right]-(-1)^n\binom{d-1}{n}\beta,$$

and for $0\le i\le d-2$ and any integer n (here by convention $I^n=R$ if $n\le 0$),

$$\widetilde{K}^i_{n-1}=I^{n+1}:x_{i+1}/\left((I^{n+1}\cap J_i):I+I^n\right),$$
$$\widetilde{L}^i_n=J_{i+1}\cap I^n/[J_i\cap I^n+J_{i+1}\cap I^{n+1}+x_{i+1}I^{n-1}],$$
$$L^i_n=((J_i:I)\cap I^n+I^{n+1}):_{(J_{i+1}:I)\cap I^n}\mathfrak{m}^\infty/\big[(J_i:I)\cap I^n$$
$$+(J_{i+1}:I)\cap I^{n+1}+x_{i+1}[((J_i:I)\cap I^{n-1}+I^n):_{I^{n-1}}\mathfrak{m}^\infty]\big],$$
$$N^i_n=((J_{i+1}:I)\cap I^n+I^{n+1}):_{I^n}\mathfrak{m}^\infty/\big[(J_{i+1}:I)\cap I^n+((J_i:I)\cap I^n+I^{n+1}):_{I^n}\mathfrak{m}^\infty\big],$$
$$\beta=\lambda[H^0_\mathfrak{m}(R/I)]-\lambda\left[H^0_\mathfrak{m}\left(R/(0:I+I)\right)\right].$$

Proof. Recall for each $0\le i\le d-1$, $J_i:I$ is a geometric i-residual intersection of I, where $J_i=(x_1,\dots,x_i)$ (by convention, $J_i=(0)$ if $i\le 0$). Moreover, for $0\le i\le d-1$, $J_i:I^\infty=J_i:I=J_i:x_{i+1}$ and $(J_i:I^\infty)\cap I=(J_i:I)\cap I=J_i$ (see [29]). Set $R^i=R/J_i:I$ and $G^i=\mathrm{gr}_{IR^i}(R^i)$. Then one has $[G^0]_0=R/(0:I+I)$ and $[G^0]_n=[G]_n=I^n/I^{n+1}$ for every $n\ge 1$. (Here $[G]_i$ denotes the i-th homogeneous component of the graded ring G.) Hence

$$\begin{aligned}&\Delta[H_I(0)]=\lambda[H^0_\mathfrak{m}(R/I)]\\&=\lambda\left[H^0_\mathfrak{m}\left(R/(0:I+I)\right)\right]+\left[\lambda[H^0_\mathfrak{m}(R/I)]-\lambda\left[H^0_\mathfrak{m}\left(R/(0:I+I)\right)\right]\right]\\&=\Delta[H_{IR^0}(0)]+\beta,\end{aligned}$$

with β defined above, and $\Delta[H_I(n)]=\Delta[H_{IR^0}(n)]=\lambda[H^0_\mathfrak{m}(I^n/I^{n+1})]$ for $n\ge 1$. Therefore we have that for $n\ge 0$,

$$\Delta^d[H_I(n)]=\Delta^d[H_{IR^0}(n)]+(-1)^n\binom{d-1}{n}\beta,\tag{1}$$

with the binomial coefficient $\binom{d-1}{n}=0$ if $n\ge d$.

We use induction on d to prove the theorem. First assume $d=1$. If $n=0$, one has

$$\begin{aligned}&\lambda(I/J)+\omega_0(J,I)\\&=\lambda(IR^0/JR^0)+\lambda\left(R/(0:I+I)\right)-\lambda[H^0_\mathfrak{m}(R/I)]\\&=\Delta[P_{IR^0}(0)-H_{IR^0}(0)]+\lambda\left(R/(0:I+I)\right)-\lambda[H^0_\mathfrak{m}(R/I)]\\&=\Delta[P_{IR^0}(0)]-\lambda\left(R/(0:I+I)\right)+\lambda\left(R/(0:I+I)\right)-\lambda[H^0_\mathfrak{m}(R/I)]\\&=\Delta[P_I(0)]-\lambda[H^0_\mathfrak{m}(R/I)]=\Delta[P_I(0)-H_I(0)],\end{aligned}$$

where the second equality follows from [15, Theorem 2.4] since R^0 is a 1-dimensional Cohen–Macaulay local ring and IR^0 is $\mathfrak{m}R^0$-primary, and the fourth equality follows by $\Delta[P_{IR^0}(0)] = \Delta[P_I(0)] = j_0(I)$ by Lemma 3.1. If $n \geq 1$, then $\omega_n(J, I) = 0$, and one has

$$\begin{aligned}\lambda(I^{n+1}/JI^n) + \omega_n(J, I) &= \lambda(I^{n+1}R^0/JI^nR^0)\\ &= \Delta[P_{IR^0}(n) - H_{IR^0}(n)] = \Delta[P_I(n) - H_I(n)],\end{aligned}$$

where the second equality again follows from [15, Theorem 2.4] and the third equality follows by Lemma 3.1 and the fact that $\Delta[H_{IR^0}(n)] = \Delta[H_I(n)]$ for $n \geq 1$.

Now assume $d \geq 2$ and the assertion holds for $d - 1$. Notice $\Delta[P_{IR^{d-1}}(n)] = \Delta^d[P_I(n)] = j_0(I)$ by Lemma 3.1. Furthermore, by Proposition 2.2 (a) and $(J_{d-1} : I^\infty) \cap I = J_{d-1}$, one has

$$\begin{aligned}&\lambda(I^{n+1}/JI^n) - \lambda[J_{d-1} \cap I^{n+1}/J_{d-1} \cap JI^n]\\ &= \Delta[P_{IR^{d-1}}(n) - H_{IR^{d-1}}(n)] = \Delta^d[P_I(n)] - \Delta[H_{IR^{d-1}}(n)].\end{aligned} \tag{2}$$

If $n = 0$, one has

$$\lambda[J_{d-1} \cap I/J_{d-1} \cap J] = \lambda(J_{d-1}/J_{d-1}) = 0,$$

and therefore by equation (2),

$$\begin{aligned}&\lambda(I/J) + \omega_0(J, I)\\ &= \Delta^d[P_I(0)] - \Delta[H_{IR^{d-1}}(0)] + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda[H^0_{\mathfrak{m}}(R/I)]\\ &= \Delta^d[P_I(0)] - \lambda\left(R/(J_{d-1} : I + I)\right) + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda[H^0_{\mathfrak{m}}(R/I)]\\ &= \Delta^d[P_I(0)] - \lambda[H^0_{\mathfrak{m}}(R/I)]\\ &= \Delta^d[P_I(0) - H_I(0)].\end{aligned}$$

Assume $n \geq 1$. Then we have the following exact sequences for any integer n

$$\begin{gathered}0 \to K^0_{n-1} \to H^0_{\mathfrak{m}}([G^0]_{n-1}) \xrightarrow{x_1^*} H^0_{\mathfrak{m}}([G^0]_n) \to H^0_{\mathfrak{m}}([G^0]_n)/x_1^* H^0_{\mathfrak{m}}([G^0]_{n-1}) \to 0,\\ 0 \to L^0_n \to H^0_{\mathfrak{m}}([G^0]_n)/x_1^* H^0_{\mathfrak{m}}([G^0]_{n-1}) \to H^0_{\mathfrak{m}}([G^1]_n) \to N^0_n \to 0,\end{gathered}$$

where

$$\begin{aligned}K^0_{n-1} &= \left[((0 : I) \cap I^n + I^{n+1}) :_{I^{n-1}} x_1\right]\\ &\qquad \cap \left[((0 : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty\right]/((0 : I) \cap I^{n-1} + I^n),\\ L^0_n &= ((0 : I) \cap I^n + I^{n+1}) :_{(J_1:I)\cap I^n} \mathfrak{m}^\infty/\left[(0 : I) \cap I^n + (J_1 : I) \cap I^{n+1}\right.\\ &\qquad \left.+ x_1[((0 : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty]\right],\\ N^0_n &= ((J_1 : I) \cap I^n + I^{n+1}) :_{I^n} \mathfrak{m}^\infty/\left[(J_1 : I) \cap I^n + ((0 : I) \cap I^n + I^{n+1}) :_{I^n} \mathfrak{m}^\infty\right].\end{aligned}$$

Note $((0:I)\cap I^n+I^{n+1}):_{I^{n-1}} x_1/((0:I)\cap I^{n-1}+I^n)$ has finite length because G is Cohen–Macaulay on the punctured spectrum by [20, Theorem 3.1]. Hence

$$K^0_{n-1}=((0:I)\cap I^n+I^{n+1}):_{I^{n-1}} x_1/((0:I)\cap I^{n-1}+I^n).$$

Therefore

$$\begin{aligned}\Delta^d[H_{IR^0}(n)]&=\Delta^{d-2}\left[\lambda[H^0_{\mathfrak m}([G^0]_n)]-\lambda[H^0_{\mathfrak m}([G^0]_{n-1})]\right]\\&=\Delta^{d-2}\left[\lambda[H^0_{\mathfrak m}([G^0]_n)/x_1^*H^0_{\mathfrak m}([G^0]_{n-1})]-\lambda(K^0_{n-1})\right]\\&=\Delta^{d-2}\left[\lambda[H^0_{\mathfrak m}([G^1]_n)]\right]+\Delta^{d-2}[\lambda(L^0_n)]-\Delta^{d-2}[\lambda(N^0_n)]-\Delta^{d-2}[\lambda(K^0_{n-1})]\\&=\Delta^{d-1}[H_{IR^1}(n)]+\Delta^{d-2}[\lambda(L^0_n)-\lambda(N^0_n)-\lambda(K^0_{n-1})].\end{aligned}\tag{3}$$

Observe for every $n\geq 0$, one has the following diagram

$$\begin{array}{ccc}I^{n+1}+J_1:I^\infty & \to & I^{n+1}\\ \downarrow & & \downarrow\\ JI^n+J_1:I^\infty & \to & JI^n.\end{array}$$

By Lemma 2.1, the induction hypothesis, and equation (3), one has

$$\begin{aligned}&\lambda(I^{n+1}/JI^n)\\&=\lambda(I^{n+1}R^1/JI^nR^1)+\lambda[J_1\cap I^{n+1}/J_1\cap JI^n]\\&=\Delta^{d-1}[P_{IR^1}(n)-H_{IR^1}(n)]-\omega_n(JR^1,IR^1)+\lambda[J_1\cap I^{n+1}/J_1\cap JI^n]\\&=\Delta^d[P_{IR^0}(n)]-\Delta^d[H_{IR^0}(n)]+\Delta^{d-2}[\lambda(L^0_n)-\lambda(N^0_n)-\lambda(K^0_{n-1})]\\&\quad-\omega_n(JR^1,IR^1)+\lambda[J_1\cap I^{n+1}/J_1\cap JI^n]\\&=\Delta^d[P_{IR^0}(n)-H_{IR^0}(n)]\\&\quad-\left[\omega_n(JR^1,IR^1)+\Delta^{d-2}[\lambda(K^0_{n-1})]+\Delta^{d-2}[-\lambda(L^0_n)+\lambda(N^0_n)]\right.\\&\quad\left.-\lambda[J_1\cap I^{n+1}/J_1\cap JI^n]\right].\end{aligned}\tag{4}$$

Again by $(0:I)\cap I^n=0$ for $n\geq 1$, one has

$$\begin{aligned}&\lambda(K^0_{n-1})=\lambda\left[((0:I)\cap I^n+I^{n+1}):_{I^{n-1}} x_1/\left((0:I)\cap I^{n-1}+I^n\right)\right]\\&=\lambda\left[\left(((0:I)\cap I^n+I^{n+1}):_{I^{n-1}} x_1+0:I\right)/(I^n+0:I)\right]\\&=\lambda\left[((0:I)\cap I^n+I^{n+1}):x_1/(I^n+0:I)\right]\\&\quad-\lambda\left[((0:I)\cap I^n+I^{n+1}):x_1/\left(((0:I)\cap I^n+I^{n+1}):_{I^{n-1}} x_1+0:I\right)\right]\\&=\lambda\left[I^{n+1}:x_1/(I^n+0:I)\right]-\lambda\left[I^{n+1}:x_1/\left(I^{n+1}:_{I^{n-1}} x_1+0:I\right)\right]\\&=\Delta[\lambda(\widetilde{K}^0_{n-1})]+\lambda\left[I^n:x_1/(I^{n-1}+0:I)\right]-\lambda[(I^{n+1}:x_1+I^{n-1})/(0:I+I^{n-1})]\end{aligned}$$

$$
\begin{aligned}
&= \Delta[\lambda(\widetilde{K}^0_{n-1})] + \lambda\big[I^n : x_1/(I^{n+1} : x_1 + I^{n-1})\big] \\
&= \Delta[\lambda(\widetilde{K}^0_{n-1})] + \lambda\big[(x_1) \cap I^n/(x_1) \cap I^{n+1} + x_1 I^{n-1}\big] \\
&= \Delta[\lambda(\widetilde{K}^0_{n-1})] + \lambda(\widetilde{L}^0_n).
\end{aligned}
$$

Now by the definition of ω_n,

$$
\begin{aligned}
&\omega_n(JR^1, IR^1) + \Delta^{d-2}[\lambda(K^0_{n-1})] + \Delta^{d-2}[-\lambda(L^0_n) + \lambda(N^0_n)] - \lambda[J_1 \cap I^{n+1}/J_1 \cap JI^n] \\
&= \omega_n(JR^1, IR^1) + \Delta^{d-1}[\lambda(\widetilde{K}^0_{n-1})] + \Delta^{d-2}[\lambda(\widetilde{L}^0_n) - \lambda(L^0_n) + \lambda(N^0_n)] \\
&\quad - \lambda[J_1 \cap I^{n+1}/J_1 \cap JI^n] \\
&= \omega_n(JR^0, IR^0).
\end{aligned} \tag{5}
$$

Therefore by equations (4), (5) and (1), we have for $n \geq 1$,

$$
\begin{aligned}
\lambda(I^{n+1}/JI^n) &= \Delta^d[P_{IR^0}(n) - H_{IR^0}(n)] - \omega_n(JR^0, IR^0) \\
&= \Delta^d[P_I(n) - H_I(n)] - \Big[\omega_n(JR^0, IR^0) - (-1)^n \binom{d-1}{n}\beta\Big] \\
&= \Delta^d[P_I(n) - H_I(n)] - \omega_n(J, I). \qquad \square
\end{aligned}
$$

By Theorem 3.2 and Lemma 3.1, we obtain formulas to compute the generalized Hilbert coefficients.

Corollary 3.3. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and the AN^-_{d-2}. Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has*

$$
\sum_{n=i-1}^{\infty} \binom{n}{i-1} \big[\lambda(I^{n+1}/JI^n) + \omega_n(J, I)\big] = j_i(I) \text{ for } 1 \leq i \leq d,
$$

where $\omega_n(J, I)$ is defined as in Theorem 3.2. In particular, if $d = 1$,

$$
j_1(I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) + \lambda\left(R/(0 : I + I)\right) - \lambda[H^0_{\mathfrak{m}}(R/I)],
$$

and if $d \geq 2$,

$$
\begin{aligned}
j_1(I) = &\sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda[H^0_{\mathfrak{m}}\left(R/(0 : I + I)\right)] \\
&+ \sum_{i=0}^{d-3} \left[\lambda(L^i_0) - \lambda(N^i_0)\right]
\end{aligned}
$$

$$+\sum_{n=1}^{\infty}\left[\lambda(\widetilde{L}_n^{d-2})-\lambda(L_n^{d-2})+\lambda(N_n^{d-2})\right]$$
$$-\sum_{n=1}^{\infty}\sum_{i=1}^{d-1}\lambda\left[J_i\cap I^{n+1}/\left(J_i\cap JI^n+J_{i-1}\cap I^{n+1}\right)\right].$$

Proof. If $d=1$, by Theorem 3.2, one has $\omega_0(J,I)=\lambda\left(R/(0:I+I)\right)-\lambda[H^0_{\mathfrak{m}}(R/I)]$ and $\omega_n(J,I)=0$ for $n\geq 1$. Hence

$$\begin{aligned}j_1(I)&=\sum_{n=0}^{\infty}\left[\lambda(I^{n+1}/JI^n)+\omega_n(J,I)\right]\\&=\sum_{n=0}^{\infty}\lambda(I^{n+1}/JI^n)+\lambda\left(R/(0:I+I)\right)-\lambda[H^0_{\mathfrak{m}}(R/I)].\end{aligned}$$

Assume $d\geq 2$. Observe for $0\leq i\leq d-2$, since $\widetilde{K}^i_{-1}=0$ and $\widetilde{L}^i_0=0$, one has $\sum_{n=1}^{\infty}\Delta^j[\lambda(\widetilde{K}^i_{n-1})]=0$ and $\sum_{n=1}^{\infty}\Delta^j[\lambda(\widetilde{L}^i_n)]=0$ for $j\geq 1$. Since for $0\leq i\leq d-3$, $L^i_{-1}=0$ and $N^i_{-1}=0$, one has $\sum_{n=1}^{\infty}\Delta^j[\lambda(L^i_n)]=-L^i_0$ and $\sum_{n=1}^{\infty}\Delta^j[\lambda(N^i_n)]=-N^i_0$ for $j\geq 1$. Hence by Theorem 3.2,

$$\begin{aligned}j_1(I)&=\sum_{n=0}^{\infty}\left[\lambda(I^{n+1}/JI^n)+\omega_n(J,I)\right]\\&=\sum_{n=0}^{\infty}\lambda(I^{n+1}/JI^n)+\lambda\left(R/(J_{d-1}:I+I)\right)-\lambda[H^0_{\mathfrak{m}}(R/I)]\\&+\sum_{i=0}^{d-3}\left[\lambda(L^i_0)-\lambda(N^i_0)\right]\\&+\sum_{n=1}^{\infty}\left[\lambda(\widetilde{L}_n^{d-2})-\lambda(L_n^{d-2})+\lambda(N_n^{d-2})\right]\\&-\sum_{n=1}^{\infty}\sum_{i=1}^{d-1}\lambda\big[J_i\cap I^{n+1}/J_i\cap JI^n+J_{i-1}\cap I^{n+1}\big]\\&-\beta\left[\sum_{n=0}^{d-1}(-1)^n\binom{d-1}{n}\right]+\beta,\end{aligned}$$

which is equal to the desired result since $\sum_{n=0}^{d-1}(-1)^n\binom{d-1}{n}=0$ and

$$\beta=\lambda\big[H^0_{\mathfrak{m}}(R/I)\big]-\lambda\big[H^0_{\mathfrak{m}}\left(R/(0:I+I)\right)\big].\qquad\square$$

With some depth conditions, we can greatly reduce the computation of the generalized Hilbert coefficients.

Corollary 3.4. *Assume R is Cohen–Macaulay of dimension $d \geq 2$. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and the AN_{d-2}^-. If* $\operatorname{depth}(G) \geq d-1$ *and* $\operatorname{depth}(R/I) \geq \min\{\dim R/I, 1\}$, *then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has*

$$j_1(I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda\left[H^0_{\mathfrak{m}}\left(R/(0 : I + I)\right)\right]$$
$$+ \sum_{n=1}^{\infty} \lambda\left[I^n/\left(J_{d-1}I^{n-1} + (J_{d-2}I^{n-1} + I^{n+1}) :_{I^n} \mathfrak{m}^{\infty}\right)\right].$$

Proof. First notice if I is m-primary, then $j_1(I) = e_1(I) = \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n)$ by [15, Theorem 3.1]. Since $\operatorname{grade} I = d$ and $\lambda(I^n/I^{n+1}) < \infty$ for every $n \geq 0$,

$$\lambda\left(R/(J_{d-1} : I + I)\right) - \lambda\left[H^0_{\mathfrak{m}}\left(R/(0 : I + I)\right)\right]$$
$$+ \sum_{n=1}^{\infty} \lambda\left[I^n/\left(J_{d-1}I^{n-1} + (J_{d-2}I^{n-1} + I^{n+1}) :_{I^n} \mathfrak{m}^{\infty}\right)\right]$$
$$= \lambda(R/I) - \lambda(R/I) + \sum_{n=1}^{\infty} \lambda\left[I^n/(J_{d-1}I^{n-1} + I^n)\right] = 0$$

and the result holds.

Now we assume I is not $\mathfrak{m}$-primary. Hence $\operatorname{depth}(R/I) \geq 1$. We use induction on i to prove $\operatorname{depth}\left(R/(J_i : I + I)\right) \geq 1$ for $0 \leq i \leq d-2$, where $J_i = (x_1, \ldots, x_i)$ is defined as before. If $i = 0$, by the exact sequence

$$0 \to 0 : I \to R/I \to R/(0 : I + I) \to 0,$$

and $\operatorname{depth}(0 : I) = d \geq 2$ (see [29]), we have $\operatorname{depth}\left(R/(0 : I + I)\right) \geq \min\{\operatorname{depth}(0 : I) - 1, \operatorname{depth}(R/I)\} \geq 1$. Let $1 \leq i \leq d-2$ and assume $\operatorname{depth}\left(R/(J_{i-1} : I + I)\right) \geq 1$. By the exact sequence

$$0 \to J_i : I/\left(J_{i-1} : I + (x_i)\right) \to R/\left(J_{i-1} : I + (x_i)\right) \to R/J_i : I \to 0,$$

one has $\operatorname{depth}\left(J_i : I/\left(J_{i-1} : I + (x_i)\right)\right) \geq \min\{\operatorname{depth}\left(R/\left(J_{i-1} : I + (x_i)\right)\right), \operatorname{depth}(R/J_i : I) + 1\} = \min\{d-i,\, d-i+1\} = d-i \geq 2$. Since

$$(J_i : I + I)/(J_{i-1} : I + I) \cong J_i : I/\left(J_{i-1} : I + (J_i : I) \cap I\right) = J_i : I/\left(J_{i-1} : I + (x_i)\right),$$

we have the following exact sequence

$$0 \to J_i : I/\left(J_{i-1} : I + (x_i)\right) \to R/(J_{i-1} : I + I) \to R/(J_i : I + I) \to 0,$$

and therefore depth $(R/(J_i : I + I)) \geq \min\{\text{depth}\,(J_i : I/\,(J_{i-1} : I + (x_i)))-1, \text{depth}(R/(J_{i-1} : I + I))\} \geq \min\{2-1,\, 1\} = 1$. Hence we have $(J_i : I + I) :_R \mathfrak{m}^\infty = J_i : I + I$ for $0 \leq i \leq d-2$.

For $1 \leq i \leq d-3$, one has

$$\begin{aligned}
L_0^i &= (J_i : I + I) :_{J_{i+1}:I} \mathfrak{m}^\infty / \big[J_i : I + J_{i+1} + x_{i+1}[(J_i : I + R) :_R \mathfrak{m}^\infty]\big] \\
&= [((J_i : I + I) :_R \mathfrak{m}^\infty) \cap (J_{i+1} : I)] \,/ (J_i : I + J_{i+1}) \\
&= (J_i : I + I) \cap (J_{i+1} : I)/(J_i : I + J_{i+1}) \\
&= [J_i : I + I \cap (J_{i+1} : I)] \,/ (J_i : I + J_{i+1}) = (J_i : I + J_{i+1})/(J_i : I + J_{i+1}) = 0,
\end{aligned}$$

$$\begin{aligned}
N_0^i &= (J_{i+1} : I + I) :_R \mathfrak{m}^\infty / \big[J_{i+1} : I + (J_i : I + I) :_R \mathfrak{m}^\infty\big] \\
&= (J_{i+1} : I + I)/(J_{i+1} : I + J_i : I + I) = 0.
\end{aligned}$$

Since $\text{depth}(G) \geq d-1$, for $n \geq 1$ and $1 \leq i \leq d-1$, by a similar proof as in Theorem 2.3,

$$J_i \cap I^{n+1} / \left(J_i \cap JI^n + J_{i-1} \cap I^{n+1}\right) = J_i I^n / \left(J_i \cap JI^n + J_{i-1} \cap I^{n+1}\right) = 0,$$

$$\begin{aligned}
\widetilde{L}_n^{d-2} &= J_{d-1} \cap I^n / [J_{d-2} \cap I^n + J_{d-1} \cap I^{n+1} + x_{d-1} I^{n-1}] \\
&= J_{d-1} I^{n-1} / \left(J_{d-2} I^{n-1} + J_{d-1} I^n + x_{d-1} I^{n-1}\right) = J_{d-1} I^{n-1} / J_{d-1} I^{n-1} = 0,
\end{aligned}$$

$$\begin{aligned}
L_n^{d-2} &= (J_{d-2} \cap I^n + I^{n+1}) :_{J_{d-1} \cap I^n} \mathfrak{m}^\infty / \big[J_{d-2} \cap I^n + J_{d-1} \cap I^{n+1} \\
&\quad + x_{d-1}[((J_{d-2} : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty]\big] \\
&= (J_{d-2} I^{n-1} + I^{n+1}) :_{J_{d-1} I^{n-1}} \mathfrak{m}^\infty / \big[J_{d-2} I^{n-1} + J_{d-1} I^n \\
&\quad + x_{d-1}[((J_{d-2} : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty]\big] \\
&= (J_{d-2} I^{n-1} + I^{n+1}) :_{J_{d-1} I^{n-1}} \mathfrak{m}^\infty / \big[J_{d-2} I^{n-1} \\
&\quad + x_{d-1}[((J_{d-2} : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty]\big] = 0,
\end{aligned}$$

since

$$\begin{aligned}
&(J_{d-2} I^{n-1} + I^{n+1}) :_{J_{d-1} I^{n-1}} \mathfrak{m}^\infty \\
&= \left[(J_{d-2} I^{n-1} + I^{n+1}) :_R \mathfrak{m}^\infty\right] \cap (J_{d-2} I^{n-1} + x_{d-1} I^{n-1}) \\
&= J_{d-2} I^{n-1} + \left[(J_{d-2} I^{n-1} + I^{n+1}) :_R \mathfrak{m}^\infty\right] \cap x_{d-1} I^{n-1} \\
&= J_{d-2} I^{n-1} + x_{d-1} \left[\left[(J_{d-2} I^{n-1} + I^{n+1}) :_R \mathfrak{m}^\infty\right] :_{I^{n-1}} x_{d-1}\right] \\
&= J_{d-2} I^{n-1} + x_{d-1} \left[(J_{d-2} I^{n-1} + I^{n+1}) :_{I^{n-1}} x_{d-1} \mathfrak{m}^\infty\right] \\
&= J_{d-2} I^{n-1} + x_{d-1} \left[\left[(J_{d-2} I^{n-1} + I^{n+1}) :_R x_{d-1}\right] :_{I^{n-1}} \mathfrak{m}^\infty\right] \\
&= J_{d-2} I^{n-1} + x_{d-1} \left[((J_{d-2} : I) \cap I^{n-1} + I^n) :_{I^{n-1}} \mathfrak{m}^\infty\right].
\end{aligned}$$

Finally since for $n \geq 1$, $\lambda(I^n/J_{d-1} \cap I^n + I^{n+1}) < \infty$, one has

$$
\begin{aligned}
N_n^{d-2} &= (J_{d-1} \cap I^n + I^{n+1}) :_{I^n} \mathfrak{m}^\infty / \left[J_{d-1} \cap I^n + (J_{d-2} \cap I^n + I^{n+1}) :_{I^n} \mathfrak{m}^\infty\right] \\
&= I^n / \left[J_{d-1} I^{n-1} + (J_{d-2} I^{n-1} + I^{n+1}) :_{I^n} \mathfrak{m}^\infty\right],
\end{aligned}
$$

and by Corollary 3.3,

$$
\begin{aligned}
j_1(I) &= \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda\left[H^0_{\mathfrak{m}}\left(R/(0 : I + I)\right)\right] + \sum_{n=1}^{\infty} \lambda(N_n^{d-2}) \\
&= \sum_{n=0}^{\infty} \lambda(I^{n+1}/JI^n) + \lambda\left(R/(J_{d-1} : I + I)\right) - \lambda[H^0_{\mathfrak{m}}\left(R/(0 : I + I)\right)] \\
&+ \sum_{n=1}^{\infty} \lambda\left[I^n/\left(J_{d-1} I^{n-1} + (J_{d-2} I^{n-1} + I^{n+1}) :_{I^n} \mathfrak{m}^\infty\right)\right]. \qquad \square
\end{aligned}
$$

4. Generalized Northcott's inequality

In 1959, D. G. Northcott proved a basic lower bound for the first Hilbert coefficient $e_1(I)$, which can be restated as $e_1(I) \geq \lambda(I/J)$, where I is an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay local ring R, J is a minimal reduction of I [23]. By this inequality, he obtained that, in order that I is generalized by a system of parameters, it is necessary and sufficient that the first Hilbert coefficient $e_1(I)$ vanishes [23]. J. P. Fillmore extended Northcott's result to Cohen–Macaulay modules (see [7]). Later, C. Huneke (see [17]) and A. Ooishi (see [24]) proved that equality holds (i.e., $e_1(I) = \lambda(I/J)$) if and only if the reduction number of I is at most 1 (i.e., $I^2 = JI$). When this is the case, by Valabrega–Valla criterion, the associated graded ring of I is Cohen–Macaulay and the Hilbert function $H_I(n)$ is easily described (see for instance [27]). This result has been extended to the ideal filtrations of Cohen–Macaulay rings by Guerrieri and Rossi in [13]. Goto and Nishida in [11] generalized the inequality, with suitable correction terms, to any local ring not necessarily Cohen–Macaulay and they studied the equality in the Buchsbaum case. All of their results are based on the condition that the ideal filtrations must have finite colength (like the $\mathfrak{m}$-primary case), a condition that is required to define the classical Hilbert function.

As an application of Corollary 3.3, we generalize Northcott's inequality to ideals that are not necessarily m-primary.

Theorem 4.1. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition and the weakly $(d-2)$-residually (S_2). Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has the following generalized Northcott's inequality:*

$$
j_1(I) \geq \lambda(I/J) + \lambda\left[R/\left(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty\right)\right].
$$

In particular, if I is $\mathfrak{m}$-primary, then the above residual conditions are automatically satisfied, the length $\lambda\left[R/\left(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty\right)\right]=0$, and the above inequality becomes the original Northcott's inequality $j_1(I)\geq \lambda(I/J)$.

Proof. Set $S=R/J_{d-2}:I$, where $J_{d-2}=(x_1,\ldots,x_{d-2})$. Then S is Cohen–Macaulay of dimension 2 since R is weakly $(d-2)$-residually (S_2). Furthermore, $j_1(I)=j_1(IS)$, IS satisfies $\ell(IS)=2$, the G_2 condition and the AN_0^- (see [26] and [29]). By Corollary 3.3, we have

$$
\begin{aligned}
&j_1(I)=j_1(IS)\\
&=\sum_{n=0}^{\infty}\lambda(I^{n+1}S/JI^nS)+\lambda\left(S/(x_{d-1}S:_S IS+IS)\right)-\lambda\big[H^0_{\mathfrak{m}}(S/IS)\big]\\
&+\sum_{n=1}^{\infty}\Big\{\lambda\left[(x_{d-1}S)\cap I^nS/\left((x_{d-1}S)\cap I^{n+1}S+x_{d-1}I^{n-1}S\right)\right]\\
&-\lambda\left[I^{n+1}S:_{(x_{d-1}S:_S IS)\cap I^nS}\mathfrak{m}^\infty/\left((x_{d-1}S:_S IS)\cap I^{n+1}S+x_{d-1}(I^nS:_{I^{n-1}S}\mathfrak{m}^\infty)\right)\right]\Big\}\\
&+\sum_{n=1}^{\infty}\lambda\big[((x_{d-1}S:_S IS)\cap I^nS+I^{n+1}S):_{I^nS}\mathfrak{m}^\infty/\big((x_{d-1}S:_S IS)\cap I^nS\\
&+I^{n+1}S:_{I^nS}\mathfrak{m}^\infty\big)\big]\\
&-\sum_{n=1}^{\infty}\lambda\left[x_{d-1}S\cap I^{n+1}S/x_{d-1}S\cap JI^nS\right]\\
&\geq\lambda(I/J)+\lambda\left[R/\left(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R\mathfrak{m}^\infty\right)\right].
\end{aligned}
$$

The reason is in the following. First by Lemma 2.1 and the diagram

$$
\begin{array}{ccc}
I^{n+1}S & \to & x_{d-1}S\cap I^{n+1}S\\
\downarrow & & \downarrow\\
JI^nS & \to & x_{d-1}S\cap JI^nS,
\end{array}
$$

one has $\lambda(I^{n+1}S/JI^nS)=\lambda(x_{d-1}S\cap I^{n+1}S/x_{d-1}S\cap JI^nS)+\lambda\big(I^{n+1}S/\big(JI^nS+x_{d-1}S\cap I^{n+1}S\big)\big)$. Therefore

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\lambda(I^{n+1}S/JI^nS)-\sum_{n=1}^{\infty}\lambda\big[x_{d-1}S\cap I^{n+1}S/x_{d-1}S\cap JI^nS\big]\\
&=\lambda(IS/JS)+\sum_{n=1}^{\infty}\left[\lambda(I^{n+1}S/JI^nS)-\lambda\big[x_{d-1}S\cap I^{n+1}S/x_{d-1}S\cap JI^nS\big]\right]\\
&=\lambda(I/J)+\sum_{n=1}^{\infty}\lambda\left[I^{n+1}S/\left(JI^nS+x_{d-1}S\cap I^{n+1}S\right)\right]\\
&\geq\lambda(I/J).
\end{aligned}
\tag{6}
$$

Next, since $(x_{d-1}S :_S IS) \cap IS = x_{d-1}S$, one has

$$\begin{aligned}
&\lambda\left(S/(x_{d-1}S :_S IS + IS)\right) \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS)\right) - \lambda\left((x_{d-1}S :_S IS + IS)/(x_{d-1}S :_S IS + JS)\right) \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS)\right) - \lambda\left(IS/\left((x_{d-1}S :_S IS) \cap IS + JS\right)\right) \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS)\right) - \lambda(IS/JS). \qquad (7)
\end{aligned}$$

By the fact that $\lambda(IS/JS) < \infty$, one also has

$$\lambda\big[H^0_{\mathfrak{m}}(S/IS)\big] = \lambda\big[IS :_S \mathfrak{m}^\infty/IS\big] = \lambda\big[JS :_S \mathfrak{m}^\infty/IS\big] = \lambda\big[JS :_S \mathfrak{m}^\infty/JS\big] - \lambda(IS/JS). \qquad (8)$$

Moreover, since $\operatorname{depth}(S/x_{d-1}S) \geq 1$ (see [29]), for every $\mathfrak{p} \in \operatorname{Ass}(S/x_{d-1}S)$, one has that $\mathfrak{p}$ is not maximal and $IS_\mathfrak{p} = (x_{d-1})S_\mathfrak{p} = J_\mathfrak{p}$ if $\mathfrak{p} \in V(I)$, or otherwise $I_\mathfrak{p} = R_\mathfrak{p}$. Hence

$$[(x_{d-1}S :_S IS) \cap (JS :_S \mathfrak{m}^\infty)]_\mathfrak{p} = x_{d-1}S_\mathfrak{p}$$

for every $\mathfrak{p} \in \operatorname{Ass}(S/x_{d-1}S)$, which yields that $(x_{d-1}S :_S IS) \cap (JS :_S \mathfrak{m}^\infty) = x_{d-1}S$. Therefore

$$(x_{d-1}S :_S IS + JS) \cap (JS :_S \mathfrak{m}^\infty) = JS + (x_{d-1}S :_S IS) \cap (JS :_S \mathfrak{m}^\infty) = JS. \qquad (9)$$

Now by equations (7), (8), (9), Lemma 2.1 and the diagram

$$\begin{array}{ccc}
S & \to & JS :_S \mathfrak{m}^\infty \\
\downarrow & & \downarrow \\
x_{d-1}S :_S IS + JS & \to & JS,
\end{array}$$

one has

$$\begin{aligned}
&\lambda\left(S/(x_{d-1}S :_S IS + IS)\right) - \lambda\left[H^0_{\mathfrak{m}}(S/IS)\right] \\
&= [\lambda\left(S/(x_{d-1}S :_S IS + JS)\right) - \lambda(IS/JS)] - [\lambda\left(JS :_S \mathfrak{m}^\infty/JS\right) - \lambda(IS/JS)] \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS)\right) - \lambda\left(JS :_S \mathfrak{m}^\infty/JS\right) \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS :_S \mathfrak{m}^\infty)\right) - \lambda\left((x_{d-1}S :_S IS + JS) \cap (JS :_S \mathfrak{m}^\infty)/JS\right) \\
&= \lambda\left(S/(x_{d-1}S :_S IS + JS :_S \mathfrak{m}^\infty)\right) - \lambda(JS/JS) \\
&= \lambda\left[R/\left(J_{d-1} :_R I + (J_{d-2} :_R I + I) : \mathfrak{m}^\infty\right)\right].
\end{aligned}$$

Finally for $n \geq 1$,

$$
\begin{aligned}
&\lambda\left[x_{d-1}S \cap I^n S/(x_{d-1}S \cap I^{n+1}S + x_{d-1}I^{n-1}S)\right] \\
&- \lambda\left[I^{n+1}S :_{(x_{d-1}S :_S IS)\cap I^n S} \mathfrak{m}^\infty / \left((x_{d-1}S :_S IS) \cap I^{n+1}S + x_{d-1}(I^n S :_{I^{n-1}S} \mathfrak{m}^\infty)\right)\right] \\
&= \lambda\left[x_{d-1}S \cap I^n S/(x_{d-1}S \cap I^{n+1}S + x_{d-1}I^{n-1}S)\right] \\
&\quad - \lambda\left[I^{n+1}S :_{x_{d-1}S \cap I^n S} \mathfrak{m}^\infty / \left(x_{d-1}S \cap I^{n+1}S + x_{d-1}(I^n S :_{I^{n-1}S} \mathfrak{m}^\infty)\right)\right] \\
&\geq 0,
\end{aligned}
$$

since there is a map

$$
I^{n+1}S :_{x_{d-1}S \cap I^n S} \mathfrak{m}^\infty \to x_{d-1}S \cap I^n S/(x_{d-1}S \cap I^{n+1}S + x_{d-1}I^{n-1}S)
$$

with kernel

$$
\begin{aligned}
&\left[I^{n+1}S :_{x_{d-1}S \cap I^n S} \mathfrak{m}^\infty\right] \cap \left[x_{d-1}S \cap I^{n+1}S + x_{d-1}I^{n-1}S\right] \\
&= x_{d-1}S \cap I^{n+1}S + \left[I^{n+1}S :_{x_{d-1}S \cap I^n S} \mathfrak{m}^\infty\right] \cap x_{d-1}I^{n-1}S \\
&= x_{d-1}S \cap I^{n+1}S + \left[x_{d-1}I^n S :_{x_{d-1}S \cap I^n S} \mathfrak{m}^\infty\right] \cap x_{d-1}I^{n-1}S \\
&= x_{d-1}S \cap I^{n+1}S + x_{d-1}(I^n S :_{I^{n-1}S} \mathfrak{m}^\infty),
\end{aligned}
$$

where the second equality holds because $\lambda(I^{n+1}S/x_{d-1}I^n S) < \infty$. □

The following theorem shows that the ideal I enjoys nice properties when equality holds. It generalizes the classical result of [17] and [24].

Theorem 4.2. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I) = d$, the G_d condition, the AN_{d-2}^- and* $\operatorname{depth}(R/I) \geq \min\{1, \dim R/I\}$. *Then for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has that $j_1(I) = \lambda(I/J) + \lambda\left[R/\left(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty\right)\right]$ if and only if $r(I) \leq 1$. When this is the case, the associated graded ring of I is Cohen–Macaulay.*

Proof. By Eq. (6) (see p. 195), if $j_1(I) = \lambda(I/J) + \lambda[R/(J_{d-1} :_R I + (J_{d-2} :_R I + I) :_R \mathfrak{m}^\infty)]$ then for every $n \geq 1$, the length $\lambda\left[I^{n+1}S/\left(JI^n S + (x_{d-1}S :_S IS) \cap I^{n+1}S\right)\right] = 0$. Hence

$$
I^2 \subseteq JI + (J_{d-1} :_R I) \cap I^2 = JI
$$

since $(J_{d-1} :_R I) \cap I^2 = J_{d-1}I$ by [25, Lemma 3.2]. Now the desired result follows from [20, Theorem 3.1]. □

In the following example, we provide an ideal with equality holds in the generalized Northcott's inequality. Therefore by Theorem 4.2, the reduction number of the ideal is 1 and the associated graded ring is Cohen–Macaulay. This example is taken from [21].

Example 4.3. Let $R = \mathbb{C}[\![x,y,z]\!]/(x,y)\cap(x^2,z) = \mathbb{C}[\![x,y,z]\!]/(x^2,xz,yz)$ and $I=(x,y)$. Then R is a 1-dimensional Cohen–Macaulay local ring and I is a Cohen–Macaulay ideal of height 0 which satisfies $\ell(I)=1$, the G_1 condition, and the AN_{-1}^{-}. By computations, the generalized Hilbert–Samuel polynomial is $P_I(n) = 2(n+1)-2$. Hence $j_0(I)=j_1(I)=2$. For a general minimal reduction $J=(\xi)$ of I, one has

$$\begin{aligned}&\lambda(I/J)+\lambda\left[R/\left(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty\right)\right]\\ &=\lambda\big[(x,y)/(\xi)\big]+\lambda\big[\mathbb{C}[\![x,y,z]\!]/(x,y,z)\big]=1+1=j_1(I).\end{aligned}$$

Therefore by Theorem 4.2, the reduction number $r(I)=1$ and the associated graded ring $\mathrm{gr}_I(R)$ is Cohen–Macaulay (indeed, by computations, $JI = I^2$ and $\mathrm{gr}_I(R) \cong \mathbb{C}[x,y,z,t,u]/(x,y,zu,t^2,zt)$).

As an application of Theorem 4.2, we obtain the following corollary.

Corollary 4.4. *Assume R is Cohen–Macaulay. Let I be an R-ideal which satisfies $\ell(I)=d$, the G_d condition and the weakly $(d-2)$ residually (S_2). Then for a general minimal reduction $J=(x_1,\dots,x_d)$ of I, one has*

(a) $j_1(I)\geq 0$.
(b) $j_1(I)=\lambda\left[R/\left(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty\right)\right]$ *if and only if $I=J$ is a minimal reduction.*
(c) *Assume R is excellent. Then $j_1(I)=\lambda(I/J)$ if and only if I is $\mathfrak{m}$-primary.*
(d) *Assume R is excellent. Then $j_1(I)=0$ if and only if I is a complete intersection.*

Proof. (a) and (b) are clear. To prove (c), assume R is excellent. Then

$$\lambda\left[R/\left(J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty\right)\right]=0$$

implies $J_{d-1}:_R I+(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty = R$. Since $\ell(I)=d$, one has $J_{d-1}:_R I\neq R$. Hence $(J_{d-2}:_R I+I):_R \mathfrak{m}^\infty=R$, i.e., height$\,(J_{d-2}:_R I+I)=d$. Since R is excellent, by [6], height$(J_{d-2}:_R I+I)=\max\{\text{height } I, d-1\}=d$, which yields height $I=d$, i.e., I is $\mathfrak{m}$-primary. The assertion (d) follows by (b) and (c). $\square$

We remark that (a) and (d) recover the work on the positivity of $j_1(I)$ by G. Colomé-Nin, C. Polini, B. Ulrich and Y. Xie [6].

We will finish the paper by an example from [6] that shows if residual properties do not satisfy then the generalized Northcott's inequality fails to hold. The Macaulay2 code for computing this example can be found in [6] which will appear later.

Example 4.5. Let $R=k[\![x,y]\!]/(x^3-x^2y)$ and $J=(xy^t)$ for any $t\geq 0$. Notice that R is a one-dimensional Cohen–Macaulay local ring and $\ell(J)=1$. However, J does not satisfy the G_1. By Macaulay2 [12], one sees that $j_0(J) = t + 1$, $j_1(J) = 2 - t$, which is strictly less than 0 if $t > 2$.

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References

[1] R. Achilles, M. Manaresi, Multiplicity for ideals of maximal analytic spread and intersection theory, J. Math. Kyoto Univ. 33 (4) (1993) 1029–1046.
[2] R. Achilles, M. Manaresi, Multiplicities of a bigraded ring and intersection theory, Math. Ann. 309 (1997) 573–591.
[3] M. Artin, M. Nagata, Residual intersections in Cohen–Macaulay rings, J. Math. Kyoto Univ. 12 (1972) 307–323.
[4] W. Bruns, J. Herzog, Cohen–Macaulay Local Rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.
[5] C. Ciupercă, A numerical characterization of the S_2-ification of a Rees algebra, J. Pure Appl. Algebra 178 (2003) 25–48.
[6] G. Colome Nin, C. Polini, B. Ulrich, Y. Xie, Generalized Hilbert coefficients and normalization of ideals, in progress.
[7] J.P. Fillmore, On the coefficients of the Hilbert–Samuel polynomial, Math. Z. 97 (1967) 212–228.
[8] H. Flenner, M. Manaresi, A numerical characterization of reduction ideals, Math. Z. 238 (2001) 205–214.
[9] H. Flenner, L. O'Carroll, W. Vogel, Joins and Intersections, Monographs in Mathematics, Springer-Verlag, Berlin, 1999.
[10] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T.T. Phuong, W.V. Vasconcelos, Cohen–Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals, J. Lond. Math. Soc. 81 (2010) 679–695.
[11] S. Goto, K. Noshida, Hilbert coefficients and Buchsbaumness of associated graded rings, J. Pure Appl. Algebra 181 (2003) 61–74.
[12] D.R. Grayson, M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2.
[13] A. Guerrieri, M.E. Rossi, Hilbert coefficients of Hilbert filtrations, J. Algebra 199 (1998) 40–61.
[14] J. Herzog, W.V. Vasconcelos, R.H. Villarreal, Ideals with sliding depth, Nagoya Math. J. 99 (1985) 159–172.
[15] S. Huckaba, A d-dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients, Proc. Amer. Math. Soc. 124 (1996) 1393–1401.
[16] C. Huneke, Strongly Cohen–Macaulay schemes and residual intersections, Trans. Amer. Math. Soc. 277 (1983) 739–763.
[17] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. 34 (1987) 293–318.
[18] C. Huneke, I. Swanson, Integral Closure of Ideals, Rings, and Modules, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.
[19] C. Huneke, B. Ulrich, Residue intersections, J. Reine Angew. Math. 390 (1988) 1–20.
[20] M. Johnson, B. Ulrich, Artin–Nagata properties and Cohen–Macaulay associated graded rings, Compos. Math. 103 (1996) 7–29.
[21] P. Mantero, Y. Xie, Generalized stretched ideals and Sally's conjecture, J. Pure Appl. Algebra 220 (2016) 1157–1177.
[22] K. Nishida, B. Ulrich, Computing j-multiplicities, J. Pure Appl. Algebra 214 (2010) 2101–2110.
[23] D.G. Northcott, A note on the coefficients of the abstract Hilbert function, J. Lond. Math. Soc. 35 (1960) 209–214.
[24] A. Ooishi, Δ-genera and sectional genera of commutative rings, Hiroshima Math. J. 17 (1987) 361–372.

[25] C. Polini, Y. Xie, j-multiplicity and depth of associated graded modules, J. Algebra 372 (2012) 35–55.
[26] C. Polini, Y. Xie, Generalized Hilbert functions, Comm. Algebra 42 (2014) 2411–2427.
[27] M.E. Rossi, G. Valla, Hilbert Functions of Filtered Modules, Lecture Notes of the Unione Matematica Italiana, vol. 9, Springer-Verlag, Berlin, UMI, Bologna, 2010.
[28] N.V. Trung, Constructive characterization of the reduction numbers, Compos. Math. 137 (2003) 99–113.
[29] B. Ulrich, Artin–Nagata properties and reductions of ideals, Contemp. Math. 159 (1994) 373–400.
[30] P. Valabrega, G. Valla, Form rings and regular sequences, Nagoya Math. J. 72 (1978) 91–101.
[31] W.V. Vasconcelos, The Chern coefficients of local rings, Michigan Math. J. 57 (2008) 725–743.