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Fraïssé structures with universal automorphism groups

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ABSTRACT

We prove that the automorphism group of a Fraïssé structure M equipped with a notion of stationary independence is universal for the class of automorphism groups of substructures of M. Furthermore, we show that this applies to certain homogeneous n-gons.

1. Introduction

Certain homogeneous structures are universal with respect to the class of their substructures: The Rado graph is universal for the class of all countable graphs, the rationals as a dense linear order for the class of all countable linear orders and Urysohn's universal Polish space for the class of all Polish spaces. Jaligot asked whether a universal structure M transfers its universality onto its automorphism group, i.e. whether $\mathrm{Aut}(M)$ is universal for the class of automorphism groups of substructures of M (cf. [6]). Recently, Doucha showed that, for an uncountable structure M, the answer to Jaligot's question is

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rarely positive [3]. Countable homogeneous structures on the contrary, most often have universal automorphism groups. In fact, the only known counterexample was pointed out by Piotr Kowalski and is given by the Fraïssé limit of finite fields in fixed characteristic p, which coincides with the algebraic closure of $\mathbb{F}_p$. Its automorphism group is $\hat{\mathbb{Z}}$, which is torsion free and hence does not embed any automorphism group of a finite field. It is still unknown if there is a relational countable counterexample.

We will prove that in the case where M is a Fraïssé structure admitting a certain stationary independence relation, the automorphism group $\mathrm{Aut}(M)$ will be universal for the class of automorphism groups of substructures of M.

Uspenskij [11], using a careful construction of Urysohn's universal Polish space given by Katětov [7], proved that its isometry group is universal for the class of all Polish groups, which corresponds to the class of isometry groups of Polish spaces [4]. The idea of Katětov thereby can be described as follows: Given a Polish space X, he constructed a new metric space $E_1(X)$ consisting of X together with all possible 1-point metric extensions, while assigning the smallest possible distance between new points. Under minor restrictions, the space obtained is again Polish, denoted by the first Katětov space of X. Iterating this, i.e. building one Katětov space over the other, he constructed a copy of Urysohn's space itself. Furthermore, all isometries of X extend in a unique way at every step of the construction, which yields the desired embedding of $\mathrm{Isom}(X)$ into $\mathrm{Isom}(\mathbb{U})$.

In [2] Bilge adapted this construction to Fraïssé limits of rational structures with free amalgamation by gluing extensions freely over the given space. Both Urysohn's spaces and Fraïssé classes with free amalgamation carry an independence relation as introduced by Tent and Ziegler [10]. In this paper, we will show that the mere presence of a stationary independence relation within a Fraïssé structure M allows us to mimic Katětov's construction of Urysohn's universal metric space, starting with any structure X embeddable in M. With the help of the given independence relation, we will glue "small" extensions of X independently and construct an analog of Katětov spaces in the non-metric setting, thereby ensuring that the automorphisms of X extend canonically to its Katětov spaces and that the extensions behave well under composition. In particular, we will give a positive answer to the question of Jaligot for the class of Fraïssé limits with stationary independence relation by proving the following result (Theorem 4.9):

Theorem. *Let M be a Fraïssé structure with stationary independence relation and $\mathcal{K}_\omega$ the class of all countable structures embeddable into M. Then for any $X \in \mathcal{K}_\omega$, there is an embedding $f : X \to M$ such that every automorphism of $f(X)$ extends to an automorphism of M in such a way that this extension yields a continuous embedding of* $\mathrm{Aut}(X)$ *into* $\mathrm{Aut}(M)$*. In particular, the automorphism group* $\mathrm{Aut}(M)$ *is universal for the class* $\mathrm{Aut}(\mathcal{K}_\omega) := \{\mathrm{Aut}(X) \mid X \in \mathcal{K}_\omega\}$*, i.e. every group in* $\mathrm{Aut}(\mathcal{K}_\omega)$ *can be continuously embedded as a subgroup into* $\mathrm{Aut}(M)$*.*

Note, that every automorphism group of a countable first order structure M can be considered as a Polish group if we equip it with the topology of pointwise convergence.

The basic open sets for that topology

$$\mathcal{O}_u := \{f \in \mathrm{Aut}(M) \mid f|_A = u\}$$

are determined by finite partial isomorphisms $u : A \to M$, where $A \subseteq M$ is a finite subset of M.

2. Preliminaries

Let us briefly recall the central concepts of Fraïssé theory used in the article. For further reading and proofs in this topic, there is a plethora of sources, see for example [5, p. 158ff.] or [9, p. 69ff.].

Let L be a countable language and $\mathcal{K}$ a class of finitely generated L-structures which is countable up to isomorphism types. We call $\mathcal{K}$ a *Fraïssé class* if the following three conditions are satisfied:

- HP For any finitely generated L-structure A which is embeddable into some $B \in \mathcal{K}$, there is a structure A' in $\mathcal{K}$ isomorphic to A.
- JEP For every B and C in $\mathcal{K}$, there is some $D \in \mathcal{K}$ such that both B and C are embeddable into D.
- AP For every A, B and C in $\mathcal{K}$ together with embeddings $f_1 : A \to B$ and $f_2 : A \to C$, there is some D in $\mathcal{K}$ together with embeddings $g_1 : B \to D$ and $g_2 : C \to D$ such that the following diagram commutes:

$$\begin{tikzcd}
 & B \arrow[dr, dashed, "g_1"] & \\
A \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & D. \\
 & C \arrow[ur, dashed, "g_2"'] &
\end{tikzcd}$$

We call the class of all finitely generated substructures of an L-structure M the *skeleton* of M. An L-structure M is called *rich* with respect to a class $\mathcal{K}$ of finitely generated L-structures if, for all A and B in $\mathcal{K}$ together with embeddings $f : A \to B$ and $g : A \to M$, there is an embedding $h : B \to M$ such that $h \circ f = g$. Finally, we say that an L-structure is $\mathcal{K}$*-saturated* if its skeleton is exactly $\mathcal{K}$ and it is furthermore rich with respect to $\mathcal{K}$.

The following fact is the main theorem of Fraïssé theory:

Fact 2.1. *Assume $\mathcal{K}$ to be a class of finitely generated L-structures, countable up to isomorphism types. Then there is a countable $\mathcal{K}$-saturated structure M if and only if $\mathcal{K}$ is a Fraïssé class. Furthermore, any two countable $\mathcal{K}$-saturated structures are isomorphic.*

Given a Fraïssé-class $\mathcal{K}$, the corresponding $\mathcal{K}$-saturated structure as above is called the *Fraïssé limit* of $\mathcal{K}$. Note that a countable structure M is the Fraïssé limit of its skeleton if and only if M is *homogeneous*, i.e. every partial isomorphism between finitely generated substructures can be extended to an automorphism of M. We call such structures *Fraïssé structures.*

3. Stationary independence

The main ingredient to generalize Katětov's construction to arbitrary Fraïssé structures is the presence of a stationary independence relation. For the following $a, b, \dots$ denote finite tuples, by $A, B, C, \dots$ we denote small, i.e. finitely generated structures, whereas $X, Y, \dots$ stand for arbitrary countable ones. Given substructures A and B, the substructure generated by their union is denoted by $\langle AB \rangle$. By a type over X, we mean a set of $L(X)$ formulas $p(x)$ with free variables x which is maximal satisfiable in M. The type $\mathrm{tp}(a/X)$ of a tuple a over X is the set of all $L(X)$ formulas which are satisfied by a.

Definition 3.1 *((Local) stationary independence relation).* Assume M to be a homogeneous L-structure. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on the finitely generated substructures of M is called a *stationary independence relation* (*SIR*) if the following conditions are satisfied:

SIR1 (Invariance). The independence of finitely generated substructures in M only depends on their type. In particular, for any automorphism f of M, we have $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$.
SIR2 (Symmetry). If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.
SIR3 (Monotonicity). If $A \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{\langle BC \rangle} D$.
SIR4 (Existence). For any A, B and C in M, there is some $A' \models \mathrm{tp}(A/C)$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
SIR5 (Stationarity). If A and A' have the same type over C and are both independent over C from some set B, then they also have the same type over $\langle BC \rangle$.

If the relation $A \mathop{\smile\!\!\!\!\!\!|}_C B$ is only defined for nonempty C, we call $\mathop{\smile\!\!\!\!\!\!|}$ a *local* stationary independence relation. $\dashv$

Remark 3.2. Any SIR also fulfills the following property, which was part of the original definition in [10]:

SIR6 (Transitivity). If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{\langle BC \rangle} D$, then $A \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$.

To see that, consider A, B, C and D with $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{\langle BC \rangle} D$. We have to show that this implies the independence of A and $\langle BD \rangle$ over C. By Existence there is some $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$. By Monotonicity and Stationarity we get $A' \equiv_{BC} A$. Again by Monotonicity and Stationarity, we obtain $A \equiv_{BCD} A'$ and hence $A \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$, as desired. $\dashv$

For a (local) SIR $\mathop{\smile\!\!\!\!\!\!\;|}$ defined on some homogeneous structure M, we call the pair $(M, \mathop{\smile\!\!\!\!\!\!\;|})$ a *(local) SI-structure*. If the interpretation of $\mathop{\smile\!\!\!\!\!\!\;|}$ in M is clear or irrelevant, we will refer to M alone as an SI-structure.

Remark 3.3. If A and A' in M have the same quantifier free (qf-)type over some $B \subseteq M$, then the map $AB \mapsto A'B$ is a partial isomorphism. In homogeneous structures such a map extends to an automorphism of the whole structure M, fixing B and sending A to A'. As we will exclusively work inside homogeneous structures for the rest of the article, note that A and A' have the same qf-type over B (denoted by $\mathrm{tp}^{qf}(A/B) = \mathrm{tp}^{qf}(A'/B)$) if and only if there is an automorphism of M that fixes B pointwise and maps A to A' (write $A \equiv_B A'$).

A necessary condition for a given structure to carry a stationary independence relation is given by the following fact (cf. [10], Proof of Lemma 5.1).

Fact 3.4. *Algebraic and definable closures coincide in an SI-structure M, i.e.* $\mathrm{acl}(X) = \mathrm{dcl}(X)$ *for all* $X \subset M$.

Note that Fact 3.4 also holds in local SI-structures for nonempty X. Furthermore, the equality $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ is true if and only if either $\mathrm{acl}(\emptyset) = \emptyset$ or every automorphism has a fixed point.

This characterization of algebraic closures in SI-structures already implies that we cannot define a stationary independence relation on every Fraïssé structure: The class of finite fields in a fixed characteristic p forms a Fraïssé class. In its limit, the algebraically closed field $\bar{\mathbb{F}}_p$ of characteristic p, algebraic and definable closure differ. Thus Fact 3.4 states that no stationary independence relation can be defined on $\bar{\mathbb{F}}_p$. As mentioned before, its automorphism group is the torsionfree group $\hat{\mathbb{Z}}$, which does not embed any finite group. Thus, the group $\mathrm{Aut}(\bar{\mathbb{F}}_p)$ is not universal for the class of automorphism groups of substructures of $\bar{\mathbb{F}}_p$.

On the other hand, the rationals as a dense linear order form another example of a Fraïssé structure which does not allow a notion of stationary independence, but still has a universal automorphism group. These two examples show that the absence of a notion of stationary independence within a Fraïssé structure does not decide about the universality of its automorphism group.

Nevertheless, several examples of Fraïssé structures admitting a stationary independence relation are known. Amongst them are the rational Urysohn space and the Urysohn sphere as well as Fraïssé limits of rational free amalgamation classes [2]. More examples, also including a non-relational SI-structure, will be discussed in detail in section 5.

Unlike forking in simple theories, which is uniquely determined by its properties, a Fraïssé structure can carry different notions of stationary independence: For an example, consider the random graph, and define finite subgraphs A and B to be independent over some finite subgraph C if and only if $A \cap B \subseteq C$ and every vertex in $A \setminus C$ is connected to every vertex in $B \setminus C$. It is not hard to verify that this defines a stationary independence relation. On the other hand, the class of finite graphs is a free amalgamation class, whence another stationary independence relation is given by the free amalgam of A and B over C, i.e. A and B are defined to be independent over C if and only if $A \cap B \subseteq C$ and no vertex in $A \setminus C$ is connected to a vertex in $B \setminus C$.

We have to develop some tools to mimic Katětov's construction of Urysohn's space. In order to merge certain small extensions over any embeddable infinite substructure in an independent way, we will need to extend the independence notion to arbitrary base sets:

Definition 3.5. Let M be an SI-structure. Two substructures A and B are *independent* over $X \subseteq M$ (write $A \perp_X B$), if and only if there is some finitely generated $C \subset X$ such that $A \perp_{C'} B$ for every finitely generated $C' \subset X$ containing C.

Notice that in the examples of SI-structures mentioned above, the independence relation is naturally defined between arbitrary sets and coincides with the one given in the previous definition.

Lemma 3.6. *The independence relation $\perp$ extended to arbitrary base sets satisfies all the properties of an SIR except possibly Existence.*

Proof. Invariance and Monotonicity easily follow from the definition. To see Transitivity, assume $X \subseteq M$ and finitely generated A, B and $D \subset M$ given with

$$A \underset{X}{\perp} B \text{ and } A \underset{\langle XB\rangle}{\perp} D.$$

By definition, there are C_1 and $C_2 \subset X$ such that C_1 and $\langle C_2B\rangle$ are finite supports for the first and the second independence respectively, i.e. $A \perp_{C'} B$ (resp. $A \perp_{C'} D$) for every finitely generated $C' \subset X$ (resp. $C' \subset \langle XB\rangle$) containing C_1 (resp. $\langle C_2B\rangle$). If we set $C := \langle C_1C_2\rangle$, then every finitely generated $C' \subset X$ containing C satisfies

$$A \underset{C'}{\perp} B \text{ and } A \underset{\langle C'B\rangle}{\perp} D, \text{ hence } A \underset{C'}{\perp} \langle BD\rangle,$$

which yields Transitivity.

To prove Stationarity, note that two different realizations of the same type over X, both independent from some finite set B, have different types over $\langle XB\rangle$ if and only if there is some finite subset C' of X such that their types differ already over $\langle C'B\rangle$. □

The homogeneity of M allows us furthermore to speak of independence between subsets of embeddable structures. We denote by $\mathcal{K}_\omega$ the class of all structures embeddable into M, i.e. the class of all structures whose skeleton is contained in the skeleton of M. For some $Y \in \mathcal{K}_\omega$ with substructures A, B and X, we say that A is independent from B over X if the same is true for one, and hence for every embedding of Y into M.

4. A general Katětov construction

Since the independence relation is defined only in one model and need not be part of the theory, we cannot ensure the existence of independent extensions for types over base sets which are not necessarily finitely generated. Nevertheless, a variant of Existence for certain types, called finitely supported, can be deduced.

Definition 4.1. Let $(M, \mathop{\smile\!\!\!\!\!\mid}\,)$ be an SI-structure, and $\langle AX\rangle$ arbitrary in $\mathcal{K}_\omega$. We say that $\langle AX\rangle$ is *finitely supported* (over X), if there is a finitely generated subset $C \subseteq X$ such that $A \mathop{\smile\!\!\!\!\!\mid}_C D$ for all finitely generated $D \subseteq X$. In that case we also write $A \mathop{\smile\!\!\!\!\!\mid}_C X$ and refer to C as a *support* of $\langle AX\rangle$ over X. Furthermore, we call a quantifier free type $\pi(x)$ over X *finitely supported*, if it defines a finitely supported $\mathcal{K}_\omega$-structure.

Loosely speaking, a type $\pi(x)$ over X is finitely supported if its realizations are independent from the base set over some finitely generated substructure C of X. It is not hard to see that every finitely generated $D \subseteq X$ that contains C is again a support for $\pi(x)$, so that for any finite family of finitely supported types over X we can choose a common support.

Let us denote by $S^{qf}(X)$ the set of all quantifier-free types over X. In the following, we show some useful properties of finitely supported types and structures.

Lemma 4.2. *Assume X to be a $\mathcal{K}_\omega$-structure.*

i) Suppose $C \subseteq X$ is finitely generated and $\pi := \pi(x)$ a qf-type over C realized in M. Then π has a unique extension $\tilde{\pi} \in S^{qf}(X)$ which is finitely supported over X with support C.
ii) Let $\langle AX\rangle$ and $\langle BX\rangle \in \mathcal{K}_\omega$ be finitely supported over X. Then there is some $\langle A'B'X'\rangle \in \mathcal{K}_\omega$ with

$$\langle A'X'\rangle \cong \langle AX\rangle,\ \langle B'X'\rangle \cong \langle BX\rangle \text{ and } A' \mathop{\smile\!\!\!\!\!\mid}_{X'} B'.$$

Furthermore, the structure $\langle A'B'X'\rangle$ is again finitely supported over $X' \cong X$.

Proof. By choosing an arbitrary embedding, we may assume X to be a substructure of M.

i) Write X as the limit of a chain $X = \bigcup_{n\in\omega} C_n$ with $C_0 := C$. Inductively we can construct a chain of types $\pi_0 \subseteq \pi_1 \subseteq \dots$ by setting $\pi_0 := \pi$ and for each $n > 0$, we set $\pi_n := \text{tp}^{qf}(A_n/C_n)$, where $A_n \subseteq M$ with $A_n \models \pi_{n-1}$ and $A_n \downarrow_{C_{n-1}} C_n$. By compactness and Transitivity, the set $\tilde{\pi} := \bigcup_{n\in\omega} \pi_n$ is a finitely supported type over X with support C. Note that $\tilde{\pi}$ defines again a $\mathcal{K}_\omega$-structure $\langle AX\rangle$, as every finite subset of $\langle AX\rangle$ is embeddable in some $\langle A_nC_n\rangle \subseteq M$. The uniqueness of $\tilde{\pi}$ now follows from Stationarity.

ii) Let $C \subset X$ be some common support of $\langle AX\rangle$ and $\langle BX\rangle$ over X. By Existence we find realizations A_1 (resp. B_1) of the qf-type of A (resp. B) over C in M such that $A_1 \downarrow_C B_1$. Part i) allows us to extend the type $\text{tp}^{qf}(A_1B_1/C)$ to some finitely supported type π over X with support C, which defines a $\mathcal{K}_\omega$-structure $\langle A_2B_2X\rangle$. As A_2 (resp. B_2) and A (resp. B) have the same qf-type over C and are both independent from X over C, Stationarity implies that $\langle AX\rangle \cong \langle A_2X\rangle$ (resp. $\langle BX\rangle \cong \langle B_2X\rangle$). Furthermore, for any finitely generated $C' \subset X$ containing C we have

$$\langle A_2B_2\rangle \underset{C}{\downarrow} C' \text{ and } A_2 \underset{C}{\downarrow} B_2.$$

So $A_2 \downarrow_{C'} B_2$ by Monotonicity and Transitivity, and thus $A_2 \downarrow_X B_2$. □

The second part of the above lemma shows how to independently glue certain structures over arbitrary $\mathcal{K}_\omega$-base sets. As we will see below, the $\mathcal{K}_\omega$-structure described in Lemma 4.2.ii) is unique up to isomorphism. This justifies the following definition.

Definition 4.3. Assume $\langle AX\rangle$ and $\langle BX\rangle$ to be finitely supported $\mathcal{K}_\omega$-structures. The structure $\langle A'B'X'\rangle \in \mathcal{K}_\omega$ obtained in Lemma 4.2.ii) is called the *SI-amalgam* of $\langle AX\rangle$ and $\langle BX\rangle$ over X and denoted by $A *_X B$.

Since an SI-amalgam is again a finitely supported $\mathcal{K}_\omega$-structure, we can amalgamate finite families of finitely supported $\mathcal{K}_\omega$-structures. This process behaves well under permutations of the given finite family.

Lemma 4.4. *The SI-amalgam of two structures is unique up to isomorphism. Moreover, SI-amalgamation is commutative and associative, meaning that for given finitely supported $\langle AX\rangle$, $\langle BX\rangle$ and $\langle CX\rangle$, the SI-amalgams $A *_X B$ and $B *_X A$ (resp. $(A *_X B) *_X C$ and $A *_X (B *_X C)$) are isomorphic.*

Proof. Let two $\mathcal{K}_\omega$-structures $\langle A_1B_1X_1\rangle$ and $\langle A_2B_2X_2\rangle$ be given with $A_iX_i \cong AX$ and $B_iX_i \cong BX$ as well as $A_i \downarrow_{X_i} B_i$. By Lemma 4.2.ii), the structures $\langle A_iB_iX_i\rangle$ are again finitely supported with support $C_i \subseteq X_i$. Because $A_iX_i \cong AX$, we may assume that $A_iC_i \cong AC$ for some $C \subset X$. Note that we can pick the C_i's such that they also witness the independence of A_i and B_i over X_i, i.e. $A_i \downarrow_{C'} B_i$ for all $C' \subset X_i$ with $C_i \subseteq C'$. By Stationarity and Invariance, it suffices to show that $A_1B_1C_1 \cong A_2B_2C_2$. Let us assume that the structures are embedded into M. By homogeneity, the partial isomorphism $f : A_1C_1 \to A_2C_2$ extends to $\langle A_1B_1C_1\rangle$, yielding a copy $f(B_1) := B_2'$ of B_1.

The structures B_2 and B_2' have the same type over C_2 and are both independent from A_2 over it. Hence, there is an automorphism $g \in \mathrm{Aut}(M)$ that fixes A_2C_2 and sends B_2' to B_2. Finally, the map $g \circ f : \langle A_1B_1C_1\rangle \to \langle A_2B_2C_2\rangle$ provides the desired isomorphism.

Commutativity follows directly from Symmetry of our independence relation. It remains to show that the amalgamation process is associative. To see that, assume $\langle A_1B_1C_1X_1\rangle = A *_X (B *_X C)$. By definition of the SI-amalgam, we have $A_1 \mathop{\smile\!\!\!\!\!|}_{X_1} B_1C_1$ and $B_1 \mathop{\smile\!\!\!\!\!|}_{X_1} C_1$, whence

$$(1)\ A_1 \mathop{\smile\!\!\!\!\!|}_{X_1} B_1 \text{ and } (2)\ A_1B_1 \mathop{\smile\!\!\!\!\!|}_{X_1} C_1,$$

by Monotonicity and Transitivity. Now (1) yields $\langle A_1B_1X_1\rangle = A *_X B$, whereas (2) concludes that

$$\langle A_1B_1C_1X_1\rangle = (A *_X B) *_X C.$$

This implies $A *_X (B *_X C) \cong (A *_X B) *_X C$, as desired. $\square$

Lemma 4.4 guarantees that the order in which we amalgamate a finite family of structures is irrelevant. Hence, for a given finite family of finitely supported $\mathcal{K}_\omega$-structures $\{\langle A_iX\rangle, i \in n\}$, it makes sense to write:

$$\mathop{*_X}_{i\in n} A_i := ((\ldots((A_0 *_X A_1) *_X A_2)\ldots) *_X A_{n-1}).$$

This amalgamation will be the main tool for developing a general analog of the so-called Katětov spaces in the non-metric setting. When we now move on to countable families $\mathcal{F} := \{\langle A_iX\rangle, i \in \omega\}$ of finitely supported structures over X, note that every finite amalgam $\mathop{*_X}_{i\leq n} A_i$ can naturally be embedded into $\mathop{*_X}_{i\leq n+1} A_i$, so that the family $(\mathop{*_X}_{i\leq n} A_i)_{i\in\omega}$ is a directed system. The structure $\mathop{*_X}_{i\in\omega} A_i$ generated by the limit of this system is still a $\mathcal{K}_\omega$-structure, called the *SI-amalgam* of $\{\langle A_iX\rangle, i \in \omega\}$. As we are mainly interested in extensions of automorphisms, the following lemma will be useful further on.

Lemma 4.5. *Let $\{\langle A_iX\rangle, i \in \omega\}$ be a countable family of $\mathcal{K}_\omega$-structures, finitely supported over X. Let furthermore $\{f_i : A_iX \to A_{\sigma(i)}X, i \in \omega\}$ be a family of isomorphisms, where σ is a permutation of ω and $f_i|_X = f_j|_X$ for all i, j. Then the union $\bigcup_{i\in\omega} f_i$ induces an automorphism of the SI-amalgam $\mathop{*_X}_{i\in\omega} A_i$.*

Proof. We will establish the statement for the SI-amalgam of two structures. The claim follows via induction. Hereby, surjectivity in the limit process is given by the surjectivity of σ. As all of the structures are in $\mathcal{K}_\omega$, we may take $A_0 *_X A_1$ and $A_{\sigma(0)} *_X A_{\sigma(1)}$ to be substructures of M and hence the f_i's to be partial isomorphisms. It suffices to show that $g := f_0 \cup f_1$ restricted to any finitely generated substructure of $A_0 *_X A_1$ is again a partial isomorphism.

Choose $D \subset X$ arbitrary. As $g_{|A_0D} = f_{0|A_0D}$, the restriction of g to A_0D defines a partial isomorphism between finitely generated substructures of M and hence it extends to an automorphism $\tilde{g} \in \mathrm{Aut}(M)$. Denote by $B_{\sigma(1)}$ the image of A_1 under $\tilde{g}$. Then $A_{\sigma(1)}$ and $B_{\sigma(1)}$ have the same type over D, as $\tilde{g}_{|D} = f_{0|D} = f_{1|D}$, and are both independent from $A_{\sigma(0)}$ over D by Invariance. Stationarity implies now that they even have the same type over $DA_{\sigma(0)}$, whence there is some $h \in \mathrm{Aut}(M)$ fixing $DA_{\sigma(0)}$ and sending $B_{\sigma(1)}$ to $A_{\sigma(1)}$. Thus $h\tilde{g}_{|A_0A_1D} = g_{|A_0A_1D}$ is a partial isomorphism and we are done. □

The above lemma yields a crucial property of SI-amalgams needed to imitate Katětov's construction. We can now define a chain of Katětov spaces in this setting.

Given an arbitrary $\mathcal{K}_\omega$-structure X, denote by $S^{\mathrm{fin}}(X)$ the space of all finitely supported qf-types over X. The set $S^{\mathrm{fin}}(X)$ gives rise to a countable family $\mathcal{F}$ of finitely supported structures and, after fixing an arbitrary enumeration, we can build the SI-amalgam of that family.

Definition 4.6. After choosing an arbitrary enumeration $S^{\mathrm{fin}}(X) = \{\langle A_i X\rangle \mid i \in \omega\}$ of the space of qf-types over some $\mathcal{K}_\omega$-structure X, let $E_1(X) := \underset{i\in\omega}{*_X} A_i$ be the SI-amalgam of that family and call it the *first Katětov space* of X.

One can show that $E_1(X)$ does not depend on the chosen enumeration of $S^{\mathrm{fin}}(X)$. Moreover, the space $E_1(X)$ is again a $\mathcal{K}_\omega$-structure, whence we may iterate the procedure and thereby construct inductively the n-th Katětov spaces $E_n(X)$ of X as follows:

$$E_0(X) := X,$$
$$E_{n+1}(X) := E_1(E_n(X)).$$

This family of Katětov spaces comes equipped with natural embeddings between its members and therefore it forms an inductive system. The limit $E(X)$ of that system will be called the *Katětov limit* of X.

In order to prove Theorem 4.9, it remains to show that the Katětov limit of an arbitrary $\mathcal{K}_\omega(M)$-structure X is isomorphic to M and $\mathrm{Aut}(X)$ embeds continuously into $\mathrm{Aut}(E_1(X))$.

Lemma 4.7. *Assume M to be an SI-structure and consider an arbitrary $X \in \mathcal{K}_\omega(M)$. The Katětov limit $E(X)$ is isomorphic to M.*

Proof. Let $\mathcal{K}$ be the skeleton of M. As M is homogeneous, the class $\mathcal{K}$ is a Fraïssé class and $M = \mathrm{Fr}(\mathcal{K})$ is $\mathcal{K}$-saturated. Any two countable $\mathcal{K}$-saturated structures are isomorphic, whence it suffices to prove $\mathcal{K}$-saturation for $E(X)$ to establish the lemma.

We will first show that $\mathcal{K}$ is exactly the skeleton of $E(X)$. It is easy to see that $\mathcal{K}$ is contained in the skeleton, as for any finitely generated structure $A = \langle a \rangle$ of M, the type $\mathrm{tp}^{qf}(a/\emptyset)$ determines completely A and it can be extended to a finitely supported type over X by Lemma 4.2.i). Hence, there is a copy of each structure from $\mathcal{K}$ inside the first Katětov space $E_1(X)$, and thus also in the limit $E(X)$.

For the other direction, let $A \subset E(X)$ be an arbitrary finitely generated substructure. Then there is some $n \in \omega$ such that $A \subset E_n(X)$. Since all the Katětov-spaces $E_n(X)$ are $\mathcal{K}_\omega$-structures, it follows that $A \in \mathcal{K}$ by definition, so $\mathcal{K} = \mathcal{K}(E(X))$ as desired.

It remains to show that $E(X)$ is rich with respect to $\mathcal{K}$. Consider some finitely generated substructure $A \subset E(X)$ and a $\mathcal{K}$-structure B with $f : A \hookrightarrow B = \langle Ab \rangle$. Again, the structure A is contained in some $E_n(X)$ and we can extend $\mathrm{tp}^{qf}(b/A)$ to a finitely supported type over $E_n(X)$. Since a realization of this type occurs in $E_{n+1}(X)$, we can embed B in $E(X)$ over A.

Consequently, the countable Katětov-limit of an arbitrary $\mathcal{K}_\omega$-structure X is $\mathcal{K}$-saturated and hence isomorphic to M. $\square$

Lemma 4.8. *Let M be an SI-structure and $X \in \mathcal{K}_\omega$ arbitrary. Then every automorphism in* $\mathrm{Aut}(X)$ *can be canonically extended to an automorphism of $E_1(X)$ such that the extension yields a continuous embedding of* $\mathrm{Aut}(X)$ *into* $\mathrm{Aut}(E_1(X))$*. Moreover, the group* $\mathrm{Aut}(X)$ *embeds continuously into* $\mathrm{Aut}(E(X))$*.*

Proof. Consider an automorphism $f \in \mathrm{Aut}(X)$ and observe that f induces a permutation σ_f of $S^{\mathrm{fin}}(X) = (\langle A_i X \rangle \mid i \in \omega)$, the space of all finitely supported qf-types over X. By Lemma 4.5, this gives rise to an automorphism of the amalgam $*_{i\in\omega} A_i = E_1(X)$. Hence, for every $f \in \mathrm{Aut}(X)$, there is an automorphism $\hat{f} \in \mathrm{Aut}(E_1(X))$ which extends f. As for every finitely supported type $\pi(x)$ over X there exists a unique $k \in \omega$ with $A_k \models \pi(x)$, these extensions behave well under multiplication, i.e. $\sigma_g \circ (\sigma_f)^{-1} = \sigma_{g\circ(f^{-1})}$. Therefore, the set $\{\hat{f} \mid f \in \mathrm{Aut}(X)\}$ forms a subgroup of $\mathrm{Aut}(E_1(X))$. Denote by ι the map that sends $f \in \mathrm{Aut}(X)$ to $\hat{f} \in \mathrm{Aut}(E_1(X))$. If we identify the structures coming from $S^{\mathrm{fin}}(X)$ in $E_1(X)$ again with $\{\langle A_i X \rangle \mid i \in \omega\}$, the isomorphic copy of $\mathrm{Aut}(X)$ inside $\mathrm{Aut}(E_1(X))$ consists of the subgroup of all $f \in \mathrm{Aut}(E_1(X))$ such that $f|_X$ is in $\mathrm{Aut}(X)$ and f induces a permutation on $\{A_i \mid i \in \omega\}$.

It remains to show that $\iota : \mathrm{Aut}(X) \to \mathrm{Aut}(E_1(X))$ is a continuous embedding: Let $\hat{f} \in \mathrm{im}(\iota)$ be an automorphism of $E_1(X)$. For an arbitrary finite subset $a \subseteq E_1(X)$, let $u := \hat{f}|_a : a \to E_1(X)$ be the restriction of $\hat{f}$ to a and $\mathcal{O}_u := \{g \in \mathrm{Aut}(E_1(X)) \mid g|_a = u\}$ the basic open set defined by u containing $\hat{f}$. We have to show that the preimage of $\mathcal{O}_u$ under ι contains again an open subset. As a is finite, we can choose $A_{i_1}, \ldots, A_{i_n}$ from above and $C_0 \subseteq X$ such that a is definable over $C_0 \cup \bigcup_{j=1,\ldots,n} A_{i_j}$. Since the A_{i_j} correspond to finitely supported extensions of X, for each $j = 1, \ldots, n$ there exists a $C_j \subseteq X$ with $A_{i_j} \mathop{\smile\!\!\!\!\!\!|}_{C_j} X$. Set by $C := \bigcup_{i\le n} C_i$ and $v := \hat{f}|_C$ the restriction of $\hat{f}$ to C. We claim that $\mathcal{O}_v := \{g \in \mathrm{Aut}(X) \mid g|_C = v\} \subseteq \mathrm{Aut}(X)$ is contained in the preimage of $\mathcal{O}_u$ under ι. Let $g \in \mathcal{O}_v$ be an arbitrary automorphism of X that extends v and $\hat{g}$ its extension to $E_1(X)$. As by assumption $A_{i_j} \mathop{\smile\!\!\!\!\!\!\,|}_C X$ and $\hat{g}(C) = \hat{f}(C)$, Invariance implies

$$\hat{g}(A_{i_j}) \underset{\hat{f}(C)}{\downarrow} X \text{ and } \hat{f}(A_{i_j}) \underset{\hat{f}(C)}{\downarrow} X. \tag{1}$$

Thus, Stationarity yields $\hat{g}(A_{i_j}) \equiv_X \hat{f}(A_{i_j})$. As both $\hat{f}$ and $\hat{g}$ are in the image of ι, the image of A_{i_j} under each of the two maps is again one of the A_k. On the other hand, every finitely supported extension of X has only been realized once within the A_k's, whence $\hat{g}(A_{i_j}) = \hat{f}(A_{i_j})$ for all $j = 1, \ldots, n$. In particular, we get $\hat{g}(a) = \hat{f}(a)$ and $\hat{g} \in \mathcal{O}_u$. This proves that the embedding $\iota : \mathrm{Aut}(X) \to \mathrm{Aut}(E_1(X))$ is continuous. □

With Lemmas 4.7 and 4.8 at hand, the main theorem now follows easily.

Theorem 4.9 *(Main theorem). Let M be a Fraïssé structure with stationary independence relation and $\mathcal{K}_\omega$ the class of all countable structures embeddable into M. Then for any $X \in \mathcal{K}_\omega$, there is an embedding $f : X \to M$ such that every automorphism of $f(X)$ extends to an automorphism of M in such a way that this extension yields a continuous embedding of $\mathrm{Aut}(X)$ into $\mathrm{Aut}(M)$. In particular, the automorphism group $\mathrm{Aut}(M)$ is universal for the class $\mathrm{Aut}(\mathcal{K}_\omega) := \{\mathrm{Aut}(X) \mid X \in \mathcal{K}_\omega\}$, i.e. every group in $\mathrm{Aut}(\mathcal{K}_\omega)$ can be continuously embedded as a subgroup into $\mathrm{Aut}(M)$.*

Proof. Assume X to be a $\mathcal{K}_\omega$-structure. Lemma 4.8 yields that the automorphism group of X can be continuously embedded as a subgroup into the automorphism group of its Katětov limit $E(X)$. As this limit is isomorphic to M by Lemma 4.7, we conclude that $\mathrm{Aut}(X)$ can also be continuously embedded as a subgroup into $\mathrm{Aut}(M)$. □

5. Generalized n-gons

Several examples of Fraïssé structures with (local) stationary independence were already mentioned in the introduction, amongst them pure structures, relational Fraïssé limits with free amalgamation and the rational Urysohn space and sphere. In all these cases the construction given above yields that their automorphism groups are universal with respect to the class of automorphism groups of their countable substructures.

Another SI-structure in a relational language, yet without free amalgamation, is the countable universal partial order $\mathcal{P}$. Given finite partial orders A and B with a common substructure C, we define the amalgam $A \otimes_C B$ of A and B over C as the structure consisting of the disjoint union of A and B over C such that $a \leq b$ (resp. $b \leq a$) if and only if there is some $c \in C$ with $a \leq c \leq b$ (resp. $b \leq c \leq a$). It is easy to check that the relation $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ defined by $\langle ABC \rangle \cong \langle AC \rangle \otimes_C \langle BC \rangle$ provides a stationary independence relation on $\mathcal{P}$.

Further, non-relational examples have recently been provided by Baudisch [1], who shows that graded Lie algebras over finite fields and c-nilpotent groups of exponent p with an extra predicate for a central Lazard series are SI-structures, whence their automorphism group is universal.

In the following, we will exhibit yet another homogeneous non-relational structure that admits a stationary independence relation: The countable universal generalized n-gon Γ_n, which arises as the Fraïssé limit of certain bipartite graphs [8].

A graph $G = (V_G, E_G)$ is *bipartite* if we can partition its vertex set $V_G = V_1 \dot{\cup} V_2$ in such a way that every vertex in V_1 is only connected to vertices in V_2 and vice versa. We equip graphs with the *graph metric* d^G, where $d^G(x, y)$ is the length of the shortest path from x to y in G. Furthermore, the *diameter* of G is the smallest number $n \in \mathbb{N}$ such that the distance between every two vertices x and y is at most n. If no such number exists, we say that G is of infinite diameter. Finally, the *girth* of G is the length of a shortest cycle in G. By a *subgraph* $H \subseteq G$ we mean an induced subgraph.

Definition 5.1. A *generalized n-gon* Γ is a bipartite graph of diameter n and girth $2n$.

Generalized n-gons were introduced by Jaques Tits, who developed the theory of buildings. For any n, a finite example is given by the *ordinary n-gon*, i.e. a cycle of length $2n$. A projective plane provides an example of a generalized 3-gon. The class of generalized n-gons coincides with the class of spherical buildings of rank 2.

We consider generalized n-gons Γ in the language $L_n = \langle P, f_k \mid k = 0, \dots, n\rangle$, where P is a predicate for the sort of the partition of the vertex set and the f_k's are binary functions with $f_k(x, y) := x_k$ if $d^\Gamma(x, y) = l \geq k$ and there is a unique shortest path $p = (x = x_0, \dots, x_k, \dots, x_l = y)$ from x to y. For $l < n$, such a unique path always exists, as otherwise there would be a non-trivial cycle of length $2l < 2n$, contradicting the assumption on the girth of Γ. If there is no such unique path or $d^\Gamma(x, y) < k$, we set $f_k(x, y) := x$. Note that the edge relation is definable within this language as two vertices x and y are incident if and only if $x \neq y$ and $f_1(x, y) = y$. Furthermore, if $\Delta \subseteq \Gamma$ is a generalized n-gon contained in Γ, then Δ is generated by a subgraph $A \subseteq \Delta$ as an L_n-structure, if and only if Δ is the smallest generalized n-gon in Γ containing A.

Given any connected bipartite graph G without cycles of length less than $2n$, we can build an L_n-structure inductively as follows: Set $\mathcal{F}_0(G) := G$. Assume the graph $\mathcal{F}_i(G)$ has already been constructed. For any pair (x, y) in $\mathcal{F}_i(G)$ with distance $n+1$ in $\mathcal{F}_i(G)$, we add a *new path* $p_{x,y} = (x = x_0, x_1, \dots, x_{n-2}, x_{n-1} = y)$ from x to y of length $n-1$, i.e. a path from x to y of length $n-1$ such that all the x_i for $i = 1, \dots, n-2$ are new vertices. Clearly, the graph $\mathcal{F}_i(G) \cup p_{x,y}$ still does not have cycles of length less than $2n$, as every such cycle would have to contain the whole path $p_{x,y}$, but if we could complete $p_{x,y}$ to a cycle of length less than $2n$, there existed a path of length at most n between x and y in $\mathcal{F}_i(G)$, contradicting $d^{\mathcal{F}_i(G)}(x, y) = n+1$. Now we define

$$\mathcal{F}_{i+1}(G) := \mathcal{F}_i(G) \cup \{p_{x,y} \mid x, y \in \mathcal{F}_i(G) \text{ and } d^{\mathcal{F}_i(G)}(x,y) = n+1\}$$

and call the graph $\mathcal{F}(G) := \bigcup_{i\in\omega} \mathcal{F}_i(G)$ the *free n-completion* of G. Observe that $\mathcal{F}(G)$ is a generalized n-gon.

We now consider the class $\mathcal{C}_n$ of all free n-completions of finite connected bipartite graphs without cycles of length less than $2n$ and their L_n-substructures. Note that an arbitrary L_n-substructure of a generalized n-gon does not need to be a generalized n-gon itself: If we consider for example two points $\{x, y\}$ of distance exactly n in an ordinary n-gon Δ, then there is no unique path of shortest length between them in Δ, whence the substructure generated by them consists merely of $\{x, y\}$ without any edges. In particular, substructures of generalized n-gons need not be connected. Note though, that two vertices x and y in some generalized n-gon Δ are in different connected components in some L_n-substructure of Δ, only if their graph distance in Δ equals n. In particular, a non-connected L_n-substructure of Δ consists of vertices $\{x_i \mid i \in I\}$ such that the x_i have mutual distance n in Δ.

Tent shows in [8] that $\mathcal{C}_n$ is a Fraïssé class and hence it admits a countable homogeneous limit Γ_n. She also provides a characterization to recognize free n-completions of finite graphs via some corresponding weighted Euler characteristic. We will lift that characterization to infinite graphs and use it to define an independence relation on Γ_n.

Definition 5.2. Consider the function χ_n on finite graphs H with vertex set V_H and edge set E_H defined by $\chi_n(H) := (n-1)|V_H| - (n-2)|E_H|$. For arbitrary, possibly infinite graphs $X \subseteq Y$, we say that X is *n-strong* in Y (write $X \leq_n Y$) if and only if for all finite $H \subseteq Y$ we have $\chi_n(H/H \cap X) := \chi_n(H) - \chi_n(H \cap X) \geq 0$.

Lemma 5.3. *Assume $X \subseteq Y$ to be graphs. If Y arises from X by successively patching new paths of length $n-1$, then the following hold:*

i) $X \leq_n Y$ and
ii) if $X \subseteq Y \subseteq \Delta$ for some graph Δ with $X \leq_n \Delta$, then $Y \leq_n \Delta$.

Proof. It suffices to consider the case where Y arises from X by adding one new path $p_{x,y} = (x = x_0, x_1, \ldots, x_{n-2}, x_{n-1} = y)$ of length $n-1$ to two vertices x and y of X, i.e. $Y := X \cup p_{x,y}$ and $x_i \notin X$ for $i = 1, \ldots, n-2$.

i) We have to show that

$$\chi_n(H/H \cap X) = (n-1)(|V_H| - |V_{H\cap X}|) - (n-2)(|E_H| - |E_{H\cap X}|) \geq 0 \tag{2}$$

for any finite subgraph $H \subseteq Y$. Clearly, the inequality (2) holds, if $|V_H| - |V_{H\cap X}| \geq |E_H| - |E_{H\cap X}|$, i.e. if there are more vertices in H outside of X than there are edges in H outside of X. Thus, as $Y = X \cup p_{x,y}$, the inequality (2) could only fail, if H contained the entire path $p_{x,y}$. But then,

$$\begin{aligned}\chi_n(H/H\cap X) &= (n-1)(|V_H|-|V_{H\cap X}|)-(n-2)(|E_H|-|E_{H\cap X}|)\\ &= (n-1)(|V_{p_{x,y}}|-|\{x,y\}|)-(n-2)|E_{p_{x,y}}|\\ &= (n-1)(n-2)-(n-2)(n-1)\\ &= 0.\end{aligned}$$

Hence, we get $X \leq_n Y$, as desired.

ii) Consider $H \subseteq \Delta$ finite. We have to show that $\chi_n(H/H\cap Y)\geq 0$. Let H' be the smallest subgraph of Δ that contains H and the path $p_{x,y}$. As $Y = X\cup p_{x,y}$, all vertices in $H'\setminus H$ are also in $H'\cap Y$. This yields

$$\begin{aligned}\chi_n(H/H\cap Y) &= (n-1)(|V_H|-|V_{H\cap Y}|)-(n-2)(|E_H|-|E_{H\cap Y}|)\\ &= (n-1)(|V_{H'}|-|V_{H'\cap Y}|)-(n-2)(|E_H|-|E_{H\cap Y}|)\\ &\geq (n-1)(|V_{H'}|-|V_{H'\cap Y}|)-(n-2)(|E_{H'}|-|E_{H'\cap Y}|)\\ &= \chi_n(H'/H'\cap Y).\end{aligned}$$

One calculates as above that $\chi_n(H'\cap Y)=\chi_n(H'\cap X)$, because $H'\cap Y=(H'\cap X)\cup p_{x,y}$. This yields

$$\chi_n(H/H\cap Y)\geq \chi_n(H'/H'\cap Y)=\chi_n(H'/H'\cap X)\geq 0. \qquad \square$$

With Lemma 5.3 at hand, we can now prove the following characterization of free n-completions. The proof follows the proof of Proposition 2.5 from [8], which we included for completeness.

Lemma 5.4. *For any L-structure $\Delta\in\mathcal{C}_n$ generated by a subset $X\subset\Delta$, the following are equivalent:*

i) X is n-strong in Δ;
ii) Δ is the free n-completion of X.

Proof. $\underline{\Rightarrow:}$ Assume X is n-strong in Δ. As above, we will denote by $\mathcal{F}_k(X)$ the k-th step of the free n-completion of X. We show that for all k we can embed $\mathcal{F}_k(X)$ as X_k over X_{k-1} in Δ such that $X_k\leq_n\Delta$. As X generates Δ and $\mathcal{F}(X)$ is a generalized n-gon, this implies $\mathcal{F}(X)\cong\Delta$.

For $k=0$, the statement is given by the assumption, as $\mathcal{F}_0(X)=X$. Now assume that $\mathcal{F}_k(X)\cong X_k\leq_n\Delta$ is isomorphic to X_k over X and consider $x,y\in X_k$ arbitrary with $d^{X_k}(x,y)=n+1$. As Δ is in $\mathcal{C}_n$, the conditions on diameter and girth imply that there is a unique path $p_{x,y}=(x,x_1,\dots,x_{n-2},y)$ between x and y of length $n-1$ in Δ given by $x_i=f_i(x,y)$ for $i\leq n-2$. We claim that $x_i\notin X_k$ for $i=1,\dots,n-2$, i.e. that $p_{x,y}$ is a new path. Note that necessarily one of the x_i is outside of X_k, as otherwise $d^{X_k}(x,y)=n-1$, contradicting the assumptions. Choose $1 \leq j_1 + 1 < j_2 \leq n - 1$ such that $x_{j_1}, x_{j_2} \in X_k$ and $x_l \notin X_k$ for all l with $j_1 < l < j_2$. Let $H := \{x_{j_1}, x_{j_1+1}, \ldots, x_{j_2}\}$ be the path between x_{j_1} and x_{j_2}. Because X_k is n-strong in Δ, we know that $\chi_n(H/H \cap X_k) \geq 0$. An easy calculation shows that $\chi_n(H/H \cap X_k) \geq 0$ if and only if the path H has at least length $n - 1$, i.e. $j_1 = 0$ and $j_2 = n - 1$. This proves that $p_{x,y}$ indeed is a new path and thus we can embed $\mathcal{F}_{k+1}(X)$ as $X_{k+1} \subseteq \Delta$ over X_k. By Lemma 5.3.ii) the structure X_{k+1} is even n-strong in Δ, which concludes the induction.

$\Leftarrow$: If Δ is the free n-completion of X, it arises from X by successively patching new paths of length $n - 1$. Thus, Lemma 5.3.i) immediately implies that $X \leq_n \Delta$. $\square$

We can now define a stationary independence relation on Γ_n which corresponds to free amalgamation. For given graphs A, B and C with $C \subseteq A, B$, we denote the free amalgam of A and B over C by $A \otimes_C B$.

Definition 5.5 *(Independence relation on Γ_n).* Let Γ_n be the countable universal homogeneous generalized n-gon as introduced above. For finitely generated substructures A, B and $C \subset \Gamma_n$, we define A and B to be independent over C (write $A \mathop{\smile\!\!\!\!\!\!|}_C B$) if and only if the free amalgam $\langle AC \rangle \otimes_C \langle BC \rangle$ is n-strong in the substructure $\langle ABC \rangle \subseteq \Gamma_n$ generated by A, B and C in Γ_n.

Lemma 5.4 yields that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if the substructure generated by ABC is exactly the free n-completion of $\langle AC \rangle \otimes_C \langle BC \rangle$. We will use both characterizations to prove the following main lemma.

Lemma 5.6. *The relation $\mathop{\smile\!\!\!\!\!\!|}$ as stated in Definition 5.5 defines a stationary independence relation on Γ_n.*

Proof. *Invariance* and *Symmetry* are immediate and *Existence* follows since Γ_n is rich with respect to $\mathcal{C}_n$-structures.

Monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_C \langle BD \rangle$, we know that

$$\langle AC \rangle \otimes_C \langle BCD \rangle \leq_n \langle ABCD \rangle. \tag{3}$$

We will first prove that this yields the equality

$$\langle ABC \rangle \cap \langle BCD \rangle = \langle BC \rangle. \tag{4}$$

Recall that for any subgraph $\Gamma' \subseteq \langle ABCD \rangle$ and points x and y in Γ' with distance $d^{\Gamma'}(x, y) = n + 1$, there is a unique path $p_{x,y} = (x = x_0, x_1, \ldots, x_{n-1} = y) \subseteq \langle ABCD \rangle$ from x to y of length $n - 1$, by the conditions on diameter and girth in generalized n-gons. As before, we will denote by $\mathcal{F}_k := \mathcal{F}_k(\langle AC \rangle \otimes_C \langle BCD \rangle)$ the k-th step of the free n-completion of $\langle AC \rangle \otimes_C \langle BCD \rangle$. As in the proof of Lemma 5.4 we know that we can embed $\mathcal{F}_k$ as X_k over X_{k-1} into $\langle ABCD\rangle$ with $X_k \le_n \langle ABCD\rangle$. Inductively we define

$$\Gamma_0 := \langle AC\rangle \otimes_C \langle BC\rangle \text{ and}$$
$$\Gamma_{k+1} := \Gamma_k \cup \{p_{x,y} \mid x, y \in \Gamma_k, d^{X_k}(x, y) = n + 1\}.$$

Note that $\Gamma_k \subseteq \langle ABC\rangle$ for each k and $\bigcup_{k\in\omega} \Gamma_k$ is a generalized n-gon, whence $\bigcup_{k\in\omega} \Gamma_k = \langle ABC\rangle$. Thus, in order to prove (4), it suffices to show that $\Gamma_k \cap \langle BCD\rangle = \langle BC\rangle$ for all k. We furthermore need to prove $\Gamma_k \subseteq X_k$, so that the calculation of the distance $d^{X_k}(x, y)$ is well defined for arbitrary x and y in Γ_k.

Both statements are clear for $k = 0$ by the inequality in (3). Now, assume that $\Gamma_k \subseteq X_k$ and that $\Gamma_k \cap \langle BCD\rangle = \langle BC\rangle$. Consider x and y in Γ_k with $d^{X_k}(x, y) = n + 1$. By the definition of a free n-completion, the unique path $p_{x,y}$ of length $n-1$ between x and y will be contained in X_{k+1}, whence $\Gamma_{k+1} \subseteq X_{k+1}$. As $\langle BCD\rangle \subset X_k$ and $p_{x,y} \cap X_k = \{x, y\}$, we have

$$\begin{aligned} p_{x,y} \cap \langle BCD\rangle &= \{x, y\} \cap \langle BCD\rangle \\ &\subseteq \Gamma_k \cap \langle BCD\rangle \\ &= \langle BC\rangle, \end{aligned}$$

and thus $\Gamma_{k+1} \cap \langle BCD\rangle = \langle BC\rangle$, as desired.

Now, note that for arbitrary graphs X and Y, whenever $X \le_n Y$, then $X\cap U \le_n Y\cap U$ for any $U \subseteq \Gamma_k$. Thus, for $U = \langle ABC\rangle$, the equations (3) and (4) from above imply

$$\langle AC\rangle \otimes_C \langle BC\rangle \overset{(4)}{=} (\langle AC\rangle \otimes_C \langle BCD\rangle) \cap \langle ABC\rangle \overset{(3)}{\le_n} \langle ABCD\rangle \cap \langle ABC\rangle = \langle ABC\rangle,$$

so we got $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$, as desired.

It remains to show that $A \mathop{\smile\!\!\!\!\!\!\;|}_{\langle BC\rangle} D$. Again by the equality in (4), we know that $\langle ABC\rangle \otimes_{\langle BC\rangle} \langle BCD\rangle$ is contained in $\langle ABCD\rangle$. Furthermore, the independence $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ implies that $\langle ABC\rangle \otimes_{\langle BC\rangle} \langle BCD\rangle$ arises from $\langle AC\rangle \otimes_C \langle BCD\rangle$ by successively patching new paths of length $n - 1$. As $\langle AC\rangle \otimes_C \langle BCD\rangle \le_n \langle ABCD\rangle$, Lemma 5.3.ii) yields $\langle ABC\rangle \otimes_{\langle BC\rangle} \langle BCD\rangle \le_n \langle ABCD\rangle$ and thus $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$ as desired.

Stationarity: Assume A_1, A_2, B and C are given such that A_1 and A_2 have the same type over C and are both independent from B over C. Then in particular, the graphs $\langle A_iC\rangle$ and $\langle BC\rangle$ form a free amalgam over C, whence there is partial isomorphism $\varphi : \langle A_1C\rangle \otimes_C \langle BC\rangle \to \langle A_2C\rangle \otimes_C \langle BC\rangle$ sending A_1 to A_2 while fixing $\langle BC\rangle$. Clearly, the map φ extends to the free completions $\tilde{\varphi} : \langle A_1BC\rangle \to \langle A_2BC\rangle$. As Γ_n is homogeneous, the partial isomorphism $\tilde{\varphi}$ extends to an automorphism of Γ_n, which still fixes $\langle BC\rangle$ and maps A_1 to A_2. Thus, the substructures A_1 and A_2 have the same type over $\langle BC\rangle$. □

Corollary 5.7. *The automorphism group of the countable homogeneous universal generalized n-gon Γ_n is universal for the class of all automorphism groups of generalized n-gons that are free over a finite subset.*

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