Semi-direct products of Lie algebras and covariants ☆

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A R T I C L E   I N F O

A B S T R A C T

Let Q be a connected algebraic group with Lie algebra $\mathfrak{q}$. Symmetric invariants of $\mathfrak{q}$, i.e., the Q-invariants in the symmetric algebra $S(\mathfrak{q})$ of $\mathfrak{q}$, is a first approximation to the understanding of the coadjoint action $(Q : \mathfrak{q}^*)$ and coadjoint Q-orbits. In this article, we study a class of non-reductive Lie algebras, where the description of the symmetric invariants is possible and the coadjoint representation has a number of nice invariant-theoretic properties. If G is a semisimple group with Lie algebra $\mathfrak{g}$ and V is G-module, then we define $\mathfrak{q}$ to be the semi-direct product of $\mathfrak{g}$ and V. Then we are interested in the case, where the generic isotropy group for the G-action on V is reductive and commutative. It turns out that in this case symmetric invariants of $\mathfrak{q}$ can be constructed via certain G-equivariant maps from $\mathfrak{g}$ to V ("covariants").

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Introduction

The coadjoint representation of an algebraic group Q is a thrilling and fascinating object. It encodes information about many other representations of Q and $\mathfrak{q} = \operatorname{Lie} Q$. Yet, it is a very difficult object to study. Symmetric invariants of $\mathfrak{q}$ can be considered as a first approximation to the understanding of the coadjoint action $(Q : \mathfrak{q}^*)$ and coadjoint orbits. The goal of this article is to describe and study a class of non-reductive Lie algebras, where the description of the symmetric invariants is possible and the coadjoint representation has a number of nice invariant-theoretic properties. The ground field $\Bbbk$ is algebraically closed and of characteristic 0.

Let $G \to GL(V)$ be a (finite-dimensional rational) representation of a connected algebraic group G with $\operatorname{Lie} G = \mathfrak{g}$. We form a new Lie algebra $\mathfrak{q}$ as the semi-direct product $\mathfrak{q} = \mathfrak{g} \ltimes V^*$, where V^* is an abelian ideal. Then $Q = G \times V^*$ can be regarded as a connected algebraic group with $\operatorname{Lie} Q = \mathfrak{q}$, where $1 \ltimes V^*$ is a commutative unipotent normal subgroup. Here $\mathfrak{q}^* = \mathfrak{g}^* \oplus V$ and the algebra of symmetric invariants $\mathcal{S}(\mathfrak{q})^Q = \Bbbk[\mathfrak{q}^*]^Q$ contains $\Bbbk[V]^G$ as a subalgebra. But finding the other invariants is a difficult and non-trivial problem. Nevertheless, one can use certain G-equivariant morphisms $F : V \to \mathfrak{g}$ for constructing Q-invariants in $\Bbbk[\mathfrak{q}^*]$. Our observation is that if a generic stabiliser for $(G : V)$ is toral, then this is usually sufficient for obtaining a generating set for $\Bbbk[\mathfrak{q}^*]^Q$.

For G-modules V and N, let $\operatorname{Mor}(V, N)$ denote the graded $\Bbbk[V]$-module of polynomial morphisms $F : V \to N$. There is the natural map $\phi : \operatorname{Mor}(V, \mathfrak{g}) \to \operatorname{Mor}(V, V)$ such that $(\phi(F))(v) := F(v) \cdot v$ for $v \in V$. If $F \in \operatorname{Ker}(\phi)$, then one obtains a $(1 \ltimes V^*)$-invariant polynomial $\hat{F} \in \Bbbk[\mathfrak{q}^*]$ by letting $\hat{F}(\xi, v) = \langle F(v), \xi \rangle$ (Lemma 3.1). Furthermore, if F is also G-equivariant, then $\hat{F} \in \Bbbk[\mathfrak{q}^*]^Q$. Likewise, if $\operatorname{Mor}_G(V, N)$ denotes the $\Bbbk[V]^G$-module of G-equivariant morphisms (*covariants*), then there is the map $\operatorname{Mor}_G(V, \mathfrak{g}) \xrightarrow{\phi_G} \operatorname{Mor}_G(V, V)$, which is the restriction of ϕ. Suppose that G is reductive and $H \subset G$ is a generic isotropy group for $(G : V)$, with $\mathfrak{h} = \operatorname{Lie} H$. It is known that $\operatorname{rk}_{\Bbbk[V]} \operatorname{Ker}(\phi) = \dim \mathfrak{h}$ [9], and we prove that $\operatorname{rk}_{\Bbbk[V]^G} \operatorname{Ker}(\phi_G) = \dim \mathfrak{h}^H$ whenever the action $(G : V)$ is stable (Theorem 2.1). Hence $\operatorname{rk}_{\Bbbk[V]} \operatorname{Ker}(\phi) = \operatorname{rk}_{\Bbbk[V]^G} \operatorname{Ker}(\phi_G)$ if and only if the adjoint representation of H is trivial; in particular, $\mathfrak{h}$ must be toral. The main hope behind our considerations is that if $\operatorname{Ker}(\phi)$ is generated by G-equivariant morphisms, then $\Bbbk[V]^G$ and the polynomials $\hat{F}$ with $F \in \operatorname{Ker}(\phi_G)$ together generate the whole ring $\Bbbk[\mathfrak{q}^*]^Q$. Actually, we prove this under certain additional constraints, see below. For our general theorems, we also need the *codimension-2 condition* $(= \mathbf{C}\text{·}2\text{·}\mathbf{C})$ on the set V_{reg} of G-regular elements in V. This means that $V \setminus V_{\mathsf{reg}} := \{v \in V \mid \dim G \cdot v \text{ is not maximal}\}$ does not contain divisors.

Our results concern the case in which G is semisimple and $\mathbf{C}\text{·}2\text{·}\mathbf{C}$ holds for $(G : V)$. Suppose that there are linearly independent homogeneous morphisms $F_1, \ldots, F_l \in \operatorname{Ker}(\phi)$ such that $l = \dim \mathfrak{h}$ and $\sum_i \deg F_i = \dim V - q(V /\!/ G)$, where $q(V /\!/ G)$ is the minus degree of the Poincaré series of $\Bbbk[V]^G$. Then we prove that $\operatorname{Ker}(\phi)$ is a free $\Bbbk[V]$-module with basis $F_1, \ldots, F_l$ and $\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*} \simeq \Bbbk[V][\hat{F}_1, \ldots, \hat{F}_l]$ is a polynomial ring (Theorem 3.3). Un-

der certain additional assumptions (namely, $\mathfrak{h} = \mathfrak{h}^H$ and H is not contained in a proper normal subgroup of G), we then prove that such $F_1, \ldots, F_l$ are necessarily G-equivariant and hence $\mathrm{Ker}(\phi_G)$ is a free $\Bbbk[V]^G$-module and $\Bbbk[\mathfrak{q}^*]^Q \simeq \Bbbk[V]^G[\hat{F}_1, \ldots, \hat{F}_l]$. Furthermore, if $\Bbbk[V]^G$ is a polynomial ring, then the Kostant (regularity) criterion holds for $\mathfrak{q}$ (Theorem 3.6). In case $\dim \mathfrak{h} = 1$, our results are stronger and more precise, see Theorem 3.11.

Using Elashvili's classification [2,3], one can write down the arbitrary representations of simple groups and irreducible representations of arbitrary semisimple groups with toral generic stabilisers. We then demonstrate that for most of these representations, the assumptions of our general theorems are satisfied. In each example, an emphasise is made on an explicit construction of morphisms $F_1, \ldots, F_l$ and verification that they belong to $\mathrm{Ker}(\phi)$. In some cases, the construction is rather intricate and involved, cf. Examples 5.1 and 6.2.

The structure of the paper is as follows. In Section 1, we gather some standard well-known facts on semi-direct products, regular elements, and generic stabilisers. In Section 2, we consider the $\Bbbk[V]$-module of polynomial morphisms $\mathrm{Mor}(V, \mathfrak{g})$ and the associated exact sequence $0 \to \mathrm{Ker}(\phi) \to \mathrm{Mor}(V, \mathfrak{g}) \xrightarrow{\phi} \mathrm{Mor}(V, V)$. We also compute the rank of the $\Bbbk[V]^G$-module $\mathrm{Ker}(\phi_G)$. Section 3 is the heart of the article. Here we present our main results on semi-direct products related to the case in which the **C·2·C** holds for $(G : V)$, a generic stabiliser $\mathfrak{h}$ for $(G : V)$ is toral, and there are linearly independent morphisms $F_1, \ldots, F_l \in \mathrm{Ker}(\phi)$ such that $l = \dim \mathfrak{h}$ and $\sum_{i=1}^{l} \deg F_i = \dim V - q(V /\!\!/ G)$. In Section 4, we explain how to verify that the **C·2·C** holds for a G-module V. Examples of representations with toral generic stabilisers are presented in Sections 5 and 6. For each example, we explicitly construct the morphisms $F_1, \ldots, F_l$ such that the assumptions of our theorems from Section 3 are satisfied. Our results are summarised in Appendix A, where we provide tables of the representations with toral generic stabilisers.

This is a part of a general project initiated by the second author [25]: to classify all semi-direct products $\mathfrak{q} = \mathfrak{g} \ltimes V^*$ with semisimple $\mathfrak{g}$ such that the ring $\Bbbk[\mathfrak{q}^*]^Q$ is polynomial.

Notation. If an algebraic group G acts on an irreducible affine variety X, then $\Bbbk[X]^G$ is the algebra of G-invariant regular functions on X and $\Bbbk(X)^G$ is the field of G-invariant rational functions. If $\Bbbk[X]^G$ is finitely generated, then $X /\!\!/ G := \mathsf{Spec}\, \Bbbk[X]^G$, and the *quotient morphism* $\pi_{X,G} : X \to X /\!\!/ G$ is induced by the inclusion $\Bbbk[X]^G \hookrightarrow \Bbbk[X]$. If $X = V$ is a G-module, then $\mathcal{N}_G(V) := \pi_{V,G}^{-1}(\pi_{V,G}(0))$ is the null-cone in V. Whenever the ring $\Bbbk[X]^G$ is graded polynomial, the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*. For a G-module V and $v \in V$, $\mathfrak{g}_v = \{s \in \mathfrak{g} \mid s{\cdot}v = 0\}$ is the *stabiliser* of v in $\mathfrak{g}$ and $G_v = \{g \in G \mid g{\cdot}v = v\}$ is the *isotropy group* of v in G.

An explanation of the multiplicative (highest weight) notation for representations of semisimple groups is given in 4.5.

1. Preliminaries

Let G be a connected affine algebraic group with Lie algebra $\mathfrak{g}$. The symmetric algebra $\mathcal{S}(\mathfrak{g})$ is identified with the algebra of polynomial functions on $\mathfrak{g}^*$ and we also write $\Bbbk[\mathfrak{g}^*]$ for it. The algebra $\mathcal{S}(\mathfrak{g})$ has the natural Poisson structure $\{\,,\,\}$ such that $\{x, y\} = [x, y]$ for $x, y \in \mathfrak{g}$. A subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{g})$ is said to be *Poisson-commutative*, if it is a subalgebra in the usual (associative–commutative) sense and also $\{f, g\} = 0$ for all $f, g \in \mathcal{A}$. The algebra of invariants $\mathcal{S}(\mathfrak{g})^G = \Bbbk[\mathfrak{g}^*]^G$ is the centraliser of $\mathfrak{g}$ w.r.t $\{\,,\,\}$, therefore it is the Poisson-centre of $\mathcal{S}(\mathfrak{g})$.

Definition 1. The index of $\mathfrak{g}$, denoted $\mathsf{ind}\,\mathfrak{g}$, is $\min_{\xi \in \mathfrak{g}^*} \dim \mathfrak{g}_\xi$, where $\mathfrak{g}_\xi$ is the stabiliser of ξ with respect to the coadjoint representation of $\mathfrak{g}$.

Set $b(\mathfrak{g}) = (\dim \mathfrak{g} + \mathsf{ind}\,\mathfrak{g})/2$. If $\mathfrak{g}$ is reductive, then $\mathsf{ind}\,\mathfrak{g} = \mathsf{rk}\,\mathfrak{g}$ and $b(\mathfrak{g})$ equals the dimension of a Borel subalgebra. If $\mathcal{A} \subset \mathcal{S}(\mathfrak{g})$ is Poisson-commutative, then

$$\mathsf{tr.deg}\,\mathcal{A} \leqslant b(\mathfrak{g}). \tag{1.1}$$

It is also known that this upper bound is attained for some $\mathcal{A}$.

Let V be a (finite-dimensional rational) G-module. The set of G-*regular* elements of V is defined to be

$$V_{\mathsf{reg}} = \{v \in V \mid \dim G{\cdot}v \geqslant \dim G{\cdot}v' \text{ for all } v' \in V\}\,.$$

As is well-known, V_{reg} is a dense open subset of V [22]. In particular, $\mathfrak{g}^*_{\mathsf{reg}}$ is the set of G-regular elements w.r.t. the coadjoint representation of G.

Definition 2. We say that the *codimension-n condition* ($= \mathbf{C}{\cdot}n{\cdot}\mathbf{C}$) holds for the action $(G : V)$, if $\mathsf{codim}_V(V \setminus V_{\mathsf{reg}}) \geqslant n$.

Suppose that $\mathsf{tr.deg}\,\mathcal{S}(\mathfrak{g})^G = \mathsf{ind}\,\mathfrak{g}(=: l)$. Then $\max_{\xi \in \mathfrak{g}^*} \dim G\xi = \dim \mathfrak{g} - l$. For any $f \in \mathcal{S}(\mathfrak{g})$, let $(df)_\xi \in \mathfrak{g}$ denote the differential of f at ξ. We say that $\mathfrak{g}$ satisfies *the Kostant (regularity) criterion* if the following properties hold for $\mathcal{S}(\mathfrak{g})^G$ and $\xi \in \mathfrak{g}^*$:

- $\mathcal{S}(\mathfrak{g})^G = \Bbbk[f_1, \ldots, f_l]$ is a graded polynomial ring (with basic invariants $f_1, \ldots, f_l$);
- $\xi \in \mathfrak{g}^*_{\mathsf{reg}}$ if and only if $(df_1)_\xi, \ldots, (df_l)_\xi$ are linearly independent.

A very useful fact is that if $\mathbf{C}{\cdot}2{\cdot}\mathbf{C}$ holds for $(G : \mathfrak{g}^*)$, $\mathsf{tr.deg}\,\mathcal{S}(\mathfrak{g})^G = \mathsf{ind}\,\mathfrak{g} = l$, and there are algebraically independent $f_1, \ldots, f_l \in \mathcal{S}(\mathfrak{g})^G$ such that $\sum_{i=1}^{l} \deg f_i = b(\mathfrak{g})$, then $f_1, \ldots, f_l$ freely generate $\mathcal{S}(\mathfrak{g})^G$ and the Kostant criterion holds for $\mathfrak{g}$, see [12, Theorem 1.2].

Example. If $\mathfrak{g}$ is reductive and nonabelian, then $\mathsf{codim}\,(\mathfrak{g}\backslash\mathfrak{g}_{\mathsf{reg}}) = 3$. Hence the (co)adjoint representation of a reductive Lie algebra satisfies the $\mathbf{C}{\cdot}3{\cdot}\mathbf{C}$.

For a G-module V, the vector space $\mathfrak{g} \oplus V^*$ has a natural structure of Lie algebra, the *semi-direct product of $\mathfrak{g}$ and V^**. Explicitly, if $x, x' \in \mathfrak{g}$ and $\zeta, \zeta' \in V^*$, then

$$[(x, \zeta), (x', \zeta')] = ([x, x'], x\cdot\zeta' - x'\cdot\zeta) \ .$$

This Lie algebra is denoted by $\mathfrak{q} = \mathfrak{g} \ltimes V^*$, and $V^* \simeq \{(0, \zeta) \mid \zeta \in V^*\}$ is an abelian ideal of $\mathfrak{q}$. The corresponding connected algebraic group Q is the semi-direct product of G and the commutative unipotent group $\exp(V^*) \simeq V^*$. The group Q can be identified with $G \times V^*$, the product being given by

$$(s, \zeta)(s', \zeta') = (ss', (s')^{-1}\cdot\zeta + \zeta'), \quad \text{where} \ \ s, s' \in G.$$

In particular, $(s, \zeta)^{-1} = (s^{-1}, -s\cdot\zeta)$. Then $\exp(V^*)$ can be identified with $1 \ltimes V^* := \{(1, \zeta) \mid \zeta \in V^*\} \subset G \ltimes V^*$. If G is reductive, then the subgroup $1 \ltimes V^*$ is the unipotent radical of Q, also denoted by $R_u(Q)$.

Let $\mu : V \times V^* \to \mathfrak{g}^*$ be the *moment map*, i.e., $\mu(v, \zeta)(g) := \langle \zeta, g\cdot v \rangle$, where $g \in \mathfrak{g}$ and $\langle \ , \ \rangle$ is the pairing of V and V^*. The restriction of the coadjoint representation of Q to $1 \ltimes V^*$ is explicitly described as follows. If $\zeta \in V^*$ and $\eta = (\xi, v) \in \mathfrak{q}^* = \mathfrak{g}^* \times V$, then

$$(1 \ltimes \zeta)\cdot\eta = (\xi + \mu(v, \zeta), v). \tag{1.2}$$

Since $\mu(v, \zeta) = 0$ if and only if $\zeta \in (\mathfrak{g}\cdot v)^{\perp}$, the maximal dimension of the $(1 \ltimes V^*)$-orbits in $\mathfrak{q}^*$ equals $\max_{v \in V} \dim(\mathfrak{g}\cdot v) = \dim \mathfrak{g} - \min_{v \in V} \dim \mathfrak{g}_v$.

Lemma 1.1. *For $\mathfrak{q} = \mathfrak{g} \ltimes V^*$. There is a dense open subset $\tilde{\Omega} \in V_{\text{reg}}$ such that for any $x \in \tilde{\Omega}$*

(i) $b(\mathfrak{q}) = \dim V + b(\mathfrak{g}_x)$;
(ii) $\mathrm{tr.deg}\,(\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}) = \dim V + \dim \mathfrak{g}_x$.

Proof. (i) By [15], there is a dense open subset $\tilde{\Omega} \in V_{\text{reg}}$ such that $\mathrm{ind}\,\mathfrak{q} = \dim V - \max_{v \in V} \dim \mathfrak{g}\cdot v + \mathrm{ind}\,\mathfrak{g}_x = \dim V - \dim \mathfrak{g} + \dim \mathfrak{g}_x + \mathrm{ind}\,\mathfrak{g}_x$ for any $x \in \tilde{\Omega}$. This yields the desired formula for $b(\mathfrak{q})$.

(ii) By Rosenlicht's theorem [22, 2.3],

$$\mathrm{tr.deg}\,(\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}) = \dim \mathfrak{q} - \max_{\eta \in \mathfrak{q}^*} \dim\big((1 \ltimes V^*)\cdot\eta\big) = \dim \mathfrak{q} - \dim \mathfrak{g} + \dim \mathfrak{g}_x. \quad \square$$

It follows from this lemma that $\mathrm{tr.deg}\,(\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}) \geqslant b(\mathfrak{q})$ and the equality holds if and only if $\mathrm{ind}\,\mathfrak{g}_x = \dim \mathfrak{g}_x$, i.e., $\mathfrak{g}_x$ is abelian for generic elements of V. By [24], if there is a dense open subset $\tilde{\Omega}$ of V such that $\mathfrak{g}_x$ is abelian for all $x \in \tilde{\Omega}$, then $\mathcal{P} := \Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$ is Poisson-commutative. Having in mind the general upper bound (1.1), we conclude that in such a case $\mathcal{P}$ is a Poisson-commutative subalgebra of $\Bbbk[\mathfrak{q}^*]$ of maximal

dimension. Moreover, since $\mathcal{P}$ is the centraliser of V^* in $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \})$, it is also a maximal Poisson-commutative subalgebra, cf. [13, Theorem 3.3].

We say that the action $(G{:}V)$ has a generic stabiliser, if there exists a dense open subset $\Omega \subset V$ such that all stabilisers $\mathfrak{g}_v$, $v \in \Omega$, are G-conjugate. Then any subalgebra $\mathfrak{g}_v$, $v \in \Omega$, is called a *generic stabiliser* ($=$ g.s.). Likewise, one defines a *generic isotropy group* ($=$ g.i.g.), which is a subgroup of G. By [16, §4], the linear action $(G : V)$ has a generic stabiliser if and only if it has a generic isotropy group. It is also known that g.i.g. always exists if G is reductive. A systematic treatment of generic stabilisers in the context of reductive group actions can be found in [22, §7].

2. On the rank of certain modules of covariants

For finite-dimensional $\Bbbk$-vector spaces V and N, let $\mathrm{Mor}(V, N)$ denote the set of polynomial morphisms $F : V \to N$. Clearly, $\mathrm{Mor}(V, N) \simeq \Bbbk[V] \otimes N$ and it is a free graded $\Bbbk[V]$-module of rank $\dim N$. Here $\deg F = d$, if $F(tv) = t^d F(v)$ for any $t \in \Bbbk^\times$ and $v \in V$.

If both V and N are G-modules, then G acts on $\mathrm{Mor}(V, \mathfrak{g})$ by $(g*F)(v) = g(F(g^{-1}v))$. Therefore, $g * F = F$ for all $g \in G$ if and only if F is G-equivariant. Write $\mathrm{Mor}_G(V, N)$ for the set of G-*equivariant* polynomial morphisms $V \to N$. It is also called the *module of covariants of type N*. We have $\mathrm{Mor}_G(V, N) \simeq (\Bbbk[V] \otimes N)^G$. In the rest of the section, we assume that G is reductive. Then $\mathrm{Mor}_G(V, N)$ is a finitely generated $\Bbbk[V]^G$-module, see e.g. [22, 3.12].

Given a G-module V, consider the exact sequence of $\Bbbk[V]$-modules

$$0 \to \mathrm{Ker}(\phi) \to \mathrm{Mor}(V, \mathfrak{g}) \overset{\phi}{\to} \mathrm{Mor}(V, V) \ ,$$

where $\phi(F)(v) := F(v){\cdot}v$ for $F \in \mathrm{Mor}(V, \mathfrak{g})$ and $v \in V$. Therefore,

$$\mathrm{Ker}(\phi) = \{F \in \mathrm{Mor}(V, \mathfrak{g}) \mid F(v) \in \mathfrak{g}_v \quad \forall v \in V\} \ .$$

Here $\mathrm{rk}\,\phi = \max_{v \in V} \dim \mathfrak{g}{\cdot}v$ [9, Prop. 1.7] and hence $\mathrm{rk}\,\mathrm{Ker}(\phi) = \min_{v \in V} \dim \mathfrak{g}_v$. Recall that if R is a domain and M is a finitely generated R-module, then the rank of M is $\mathrm{rk}\,M = \mathrm{rk}_R(M) = \dim_{\mathsf{Quot}(R)} M \otimes \mathsf{Quot}(R)$.

We also consider the "equivariant sequence" that comprises $\Bbbk[V]^G$-modules:

$$0 \to \mathrm{Ker}(\phi_G) \to \mathrm{Mor}_G(V, \mathfrak{g}) \overset{\phi_G}{\longrightarrow} \mathrm{Mor}_G(V, V) \ .$$

Here ϕ_G is the restriction of ϕ to $\mathrm{Mor}_G(V, \mathfrak{g})$. We are interested in conditions under which the $\Bbbk[V]$-module $\mathrm{Ker}(\phi)$ is generated by G-equivariant morphisms. In other words, when is it true that the $\Bbbk[V]$-modules $\mathrm{Ker}(\phi)$ and $\Bbbk[V] \otimes_{\Bbbk[V]^G} \mathrm{Ker}(\phi_G)$ are isomorphic?

If H is a generic isotropy group for $(G : V)$ and $\mathfrak{h} = \mathsf{Lie}\,H$, then we write $\mathfrak{h} = \mathsf{g.s.}(\mathfrak{g} : V)$ and $H = \mathsf{g.i.g.}(G : V)$ for this. Then $\min_{v \in V} \dim \mathfrak{g}_v = \dim \mathfrak{h}$ and hence

$$\mathrm{rk}\,\mathrm{Ker}(\phi) = \dim\mathfrak{h}. \tag{2.1}$$

Recall that the G-action on V is said to be *stable*, if the union of closed G-orbits is dense in V, see [22, §7]. Then H is a reductive (not necessarily connected) group. By a general result of Vust [23, Chap. III], if the action $(G:V)$ is stable, then

$$\text{the rank of the } \Bbbk[V]^G\text{-module } \mathrm{Mor}_G(V, N) \text{ equals } \dim N^H. \tag{2.2}$$

For the reader's convenience, we outline a proof:

• If F is G-equivariant, then $F(v) \in N^{G_v}$ for any $v \in V$. Applying this to the open set of G-generic elements in V, we obtain that $\mathrm{rk}\,\mathrm{Mor}_G(V, N) \leqslant \dim N^H$.

• On the other hand, the "evaluation" map $\epsilon_v : \mathrm{Mor}_G(V, N) \to N^{G_v}$, $F \mapsto F(v)$, is onto whenever $G{\cdot}v = \overline{G{\cdot}v}$, see [10, Theorem 1]. Hence if generic G-orbits in V are closed (and isomorphic to G/H), then the upper bound $\dim N^H$ is attained.

Our goal is to compute the rank of the $\Bbbk[V]^G$-module $\mathrm{Ker}(\phi_G)$.

Theorem 2.1. *If the action $(G{:}V)$ is stable and $H = \mathsf{g.i.g.}(G{:}V)$, then $\mathrm{rk}\,\mathrm{Ker}(\phi_G) = \dim\mathfrak{h}^H$.*

Proof. The reductive group $W = N_G(H)/H$ acts on V^H. By the Luna–Richardson theorem [8], the restriction homomorphism $\Bbbk[V] \to \Bbbk[V^H]$ induces an isomorphism of rings of invariants $\Bbbk[V]^G \simeq \Bbbk[V^H]^W$. This common ring will be denoted by $\mathfrak{J}$. Consider the commutative diagram of $\mathfrak{J}$-modules

$$
\begin{array}{ccccc}
0 \to \mathrm{Ker}(\phi_G) \to & \mathrm{Mor}_G(V, \mathfrak{g}) & \xrightarrow{\phi_G} & \mathrm{Mor}_G(V, V) \\
& \downarrow & & \downarrow \\
0 \to \mathrm{Ker}(\psi_W) \to \mathrm{Mor}_W(V^H, \mathfrak{g}^H) & \xrightarrow{\psi_W} & \mathrm{Mor}_W(V^H, V^H),
\end{array}
$$

where the vertical arrows denote the restriction of G-equivariant morphisms to $V^H \subset V$. Note that the although W-module $\mathfrak{g}^H$ is not the Lie algebra of W, the $\mathfrak{J}$-module homomorphism ψ_W is being defined similarly to ϕ_G. By construction, the action $(W{:}V^H)$ is again stable and has trivial generic isotropy groups. Therefore, using Eq. (2.2), we conclude that

$$\mathrm{rk}\,\mathrm{Mor}_W(V^H, \mathfrak{g}^H) = \dim\mathfrak{g}^H = \mathrm{rk}\,\mathrm{Mor}_G(V, \mathfrak{g})$$

$$\mathrm{rk}\,\mathrm{Mor}_W(V^H, V^H) = \dim V^H = \mathrm{rk}\,\mathrm{Mor}_G(V, V).$$

Since H is a generic isotropy group, $\overline{G{\cdot}V^H} = V$. It follows that both vertical arrows are *injective* homomorphisms of $\mathfrak{J}$-modules of equal ranks. Therefore, they give rise to isomorphisms over the field of fractions of $\mathfrak{J}$ and hence $\mathrm{rk}\,\mathrm{Ker}(\psi_W) = \mathrm{rk}\,\mathrm{Ker}(\phi_G)$. Here

$$\mathrm{Ker}(\psi_W) = \{F : V^H \to \mathfrak{g}^H \mid F \text{ is } W\text{-equivariant and } F(v) \in (\mathfrak{g}^H)_v \ \forall v \in V^H\} =$$

$$= \{F \mid F \text{ is } W\text{-equivariant and } F(v) \in \mathfrak{h}^H \;\; \forall v \in V^H \} \simeq \mathrm{Mor}_W(V^H, \mathfrak{h}^H).$$

The second equality follows from the fact that $\mathfrak{g}_v = \mathfrak{h}$ for a generic $v \in V^H$ and hence $F(v) \in \mathfrak{h}$ for any $v \in V^H$. Since $\mathbf{g.i.g.}(W{:}V^H) = \{1\}$, Eq. (2.2) implies that $\mathrm{rk}\,\mathrm{Ker}(\psi_W) = \dim \mathfrak{h}^H$. □

Comparing Eq. (2.1) and Theorem 2.1 provides the following necessary condition:

Corollary 2.2. *If the action $(G : V)$ is stable and the $\Bbbk[V]$-module $\mathrm{Ker}(\phi)$ is generated by G-equivariant morphisms, then $\mathfrak{h} = \mathfrak{h}^H$ (i.e., the adjoint representation of H is trivial). In particular, $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{g}$.*

There are several cases in which this condition on $\mathfrak{h}$ is also sufficient.

• If $(G : V)$ is the isotropy representation of a symmetric variety, then the condition that $\mathfrak{h}$ is toral does imply that $\mathrm{Ker}(\phi)$ is a free $\Bbbk[V]$-module generated by G-equivariant morphisms, see [12, Theorem 5.8].

• If H is finite, then $\mathrm{Ker}(\phi)$ is a trivial $\Bbbk[V]$-module.

Next, we provide one more good case. For $F \in \mathrm{Mor}(V, N)$, let $\mathcal{V}(F)$ denote the set of zeros of F. If $\dim N = 1$, then F is a polynomial function on V and $\mathcal{V}(F)$ is a divisor.

Proposition 2.3. *Suppose that G is semisimple and $\mathbf{g.i.g.}(G : V)$ is a one-dimensional torus. Then $\mathrm{Ker}(\phi)$ is a free $\Bbbk[V]$-module of rank 1 generated by a G-equivariant morphism.*

Proof. Since G is semisimple and $\mathbf{g.i.g.}$ is reductive, the action $(G : V)$ is stable [22, Theorem 7.15]. Hence $\mathrm{rk}\,\mathrm{Ker}(\phi_G) = 1$ in view of Theorem 2.1. Then we can pick a nonzero homogeneous primitive element $F \in \mathrm{Ker}(\phi_G)$, i.e., F cannot be written as $f\check{F}$, where $\check{F} \in \mathrm{Ker}(\phi_G)$ and $f \in \Bbbk[V]^G$ with $\deg f > 0$. Then F is also primitive as element of $\mathrm{Mor}(V, \mathfrak{g})$. Indeed, assume that $F = f\check{F}$, where $\check{F} \in \mathrm{Mor}(V, \mathfrak{g})$, $f \in \Bbbk[V]$ and $\deg f > 0$. Because F is a G-equivariant morphism, $\mathcal{V}(F)$ is G-stable. Since $\mathcal{V}(f) \subset \mathcal{V}(F)$ and $\mathcal{V}(f)$ is a divisor, $\mathcal{V}(f)$ is necessarily a G-stable divisor in V. Because G is semisimple, $f \in \Bbbk[V]^G$. It follows that $\check{F} \in \mathrm{Mor}_G(V, \mathfrak{g})$. The relation $F = f\check{F}$ shows that $\check{F}(v) \in \mathfrak{g}_v$ for any $v \in V \setminus \mathcal{V}(f)$. Hence $\check{F}(v) \in \mathfrak{g}_v$ for any $v \in V$, and this contradicts the primitivity of F in $\mathrm{Ker}(\phi_G)$.

Let $\tilde{F} \in \mathrm{Ker}(\phi)$ be an arbitrary homogeneous element. Since $\mathrm{rk}\,\mathrm{Ker}(\phi) = 1$, there are coprime homogenous $f, \tilde{f} \in \Bbbk[V]$ such that $fF = \tilde{f}\tilde{F}$. If $\deg \tilde{f} > 0$, then $\mathcal{V}(\tilde{f}) \subset \mathcal{V}(F)$ and, as in the previous paragraph, this leads to a contradiction. Thus, $\tilde{f}$ is invertible, and we are done. □

Using the theory to be developed in Section 3, we provide a number of non-trivial examples of representations with toral generic stabilisers such that $\mathrm{Ker}(\phi)$ is generated by G-equivariant morphisms, see Sections 5 and 6.

3. Semi-direct products with good invariant-theoretic properties

In this section, we describe a class of representations $(G : V)$ such that $\mathrm{Ker}(\phi)$ is generated by G-equivariant morphisms, $\mathfrak{q} = \mathfrak{g} \ltimes V^*$ satisfies the Kostant criterion, and $(Q : \mathfrak{q}^*)$ has nice invariant-theoretic properties.

For $F \in \mathrm{Mor}(V, \mathfrak{g})$ and $\eta = (\xi, v) \in \mathfrak{q}^* = \mathfrak{g}^* \times V$, we define $\hat{F} \in \Bbbk[\mathfrak{q}^*]$ by $\hat{F}(\eta) := \langle F(v), \xi \rangle$, where $\langle\ ,\ \rangle$ denote the pairing of dual spaces.

Lemma 3.1. *We have $\hat{F} \in \Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$ if and only if $F(v) \cdot v = 0$ for all $v \in V$, i.e., $F \in \mathrm{Ker}(\phi)$.*

Proof. By (1.2), the invariance with respect to $1 \ltimes V^*$ means that

$$\langle F(v), \xi \rangle = \langle F(v), \xi + \mu(v, \zeta) \rangle, \quad (\xi, v) \in \mathfrak{q}^*,$$

for any $\zeta \in V^*$. Hence $0 = \langle F(v), \mu(v, \zeta) \rangle = \langle F(v) \cdot v, \zeta \rangle$, and we are done. $\quad\square$

Thus, any $F \in \mathrm{Ker}(\phi)$ gives rise to $\hat{F} \in \Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$. Moreover, it is clear that if F is G-equivariant, then $\hat{F} \in \Bbbk[\mathfrak{q}^*]^Q$. It follows from Eq. (1.2) that if $\zeta \in V^*$ is regarded as a linear function on $\mathfrak{q}^* = \mathfrak{g}^* \times V$, then ζ is $1 \ltimes V^*$-invariant. Hence

- both $\mathcal{S}(V^*) = \Bbbk[V]$ and $\{\hat{F} \mid F \in \mathrm{Ker}(\phi)\}$ lie in $\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$;
- both $\Bbbk[V]^G$ and $\{\hat{F} \mid F \in \mathrm{Ker}(\phi_G)\}$ lie in $\Bbbk[\mathfrak{q}^*]^Q$.

We provide below certain conditions that guarantee us that $\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$ and $\Bbbk[\mathfrak{q}^*]^Q$ are generated by the respective subsets.

Recall some properties to the symmetric invariants of semi-direct products:

(i) The decomposition $\mathfrak{q}^* = \mathfrak{g}^* \oplus V$ yields a bi-grading of $\Bbbk[\mathfrak{q}^*]^Q$ [12, Theorem 2.3(i)]. The same argument proves that the algebra $\Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$ is also bi-graded.

(ii) The algebra $\Bbbk[V]^G$ is contained in $\Bbbk[\mathfrak{q}^*]^Q$. Moreover, a minimal generating system for $\Bbbk[V]^G$ is a part of a minimal generating system of $\Bbbk[\mathfrak{q}^*]^S$ [12, Sect. 2 (A)]. In particular, if $\Bbbk[\mathfrak{q}^*]^Q$ is a polynomial ring, then so is $\Bbbk[V]^G$.

Remark 3.2. Note that $\hat{F}$ associated with $F \in \mathrm{Ker}(\phi)$ has degree 1 w.r.t. $\mathfrak{g}$. Conversely, it can be shown that if $f \in \Bbbk[\mathfrak{q}^*]^{1 \ltimes V^*}$ has degree 1 w.r.t. $\mathfrak{g}$, then $f = \hat{F}$ for some $F \in \mathrm{Ker}(\phi)$, see [24, Lemma 2.1]. In other words, there is a natural bijection $\mathrm{Ker}(\phi) \overset{1:1}{\longleftrightarrow} (\mathfrak{g} \otimes \Bbbk[V])^{1 \ltimes V^*}$. It is also true that $\mathrm{Ker}(\phi_G) \overset{1:1}{\longleftrightarrow} (\mathfrak{g} \otimes \Bbbk[V])^{G \ltimes V^*}$.

If $G \subset GL(V)$ is reductive, then $\Bbbk[V]^G$ is finitely generated and $q(V /\!\!/ G)$ stands for the minus degree of the Poincaré series of the graded algebra $\Bbbk[V]^G$. More precisely, $\Bbbk[V]^G = \bigoplus_{j \in \mathbb{N}} \Bbbk[V]_j^G$ and its Poincaré series is

$$\mathcal{F}(\Bbbk[V]^G; t) = \sum_{j \in \mathbb{N}} \dim \Bbbk[V]_j^G t^j.$$

Here $\mathcal{F}(\Bbbk[V]^G; t) = P(t)/\tilde{P}(t)$ is a rational function and, by definition, $q(V /\!\!/ G) = \deg \tilde{P} - \deg P$. In particular, if $\Bbbk[V]^G$ is a polynomial ring, then $q(V /\!\!/ G)$ equals the sum of degrees of the basic invariants. By [5, Korollar 5], if G is semisimple, then $q(V /\!\!/ G) \leqslant q(V) = \dim V$. The arbitrary representations of simple algebraic groups and the irreducible representations of semisimple groups such that $q(V /\!\!/ G) < \dim V$ are classified in [6].

Recall some properties of the linear actions of semisimple groups. If $G \subset GL(V)$ is semisimple, then

- $\Bbbk(V)^G$ is the quotient field of $\Bbbk[V]^G$, hence $\max_{v \in V} \dim G{\cdot}v = \dim V - \dim V /\!\!/ G$ [22];
- $(G : V)$ is stable if and only if $\mathsf{g.i.g.}(G : V)$ is reductive [22, Theorem 7.15].

Theorem 3.3. *Let $G \subset GL(V)$ be semisimple and $l = \min_{v \in V} \dim \mathfrak{g}_v = \dim \mathsf{g.i.g.}(G : V)$ be positive. Suppose also that $\mathrm{codim}\,(V \setminus V_{\mathsf{reg}}) \geqslant 2$ and there are linearly independent (over $\Bbbk[V]$) homogeneous morphisms $F_1, \dots, F_l \in \mathrm{Ker}(\phi)$ such that*

$$\sum_{i=1}^{l} \deg F_i + q(V /\!\!/ G) = \dim V. \tag{3.1}$$

Then

(i) *$F_1(v), \dots, F_l(v) \in \mathfrak{g}$ are linearly independent for all $v \in V_{\mathsf{reg}}$ and $\bigwedge_{i=1}^{l} F_i : V \to \wedge^m \mathfrak{g}$ is G-equivariant;*

(ii) *$\mathrm{Ker}(\phi)$ is a free $\Bbbk[V]$-module of rank l, with basis $F_1, \dots, F_l$;*

(iii) *$\Bbbk[\mathfrak{q}^*]^{R_u(Q)} = \Bbbk[V][\hat{F}_1, \dots, \hat{F}_l]$, that is, $\mathfrak{q}^* /\!\!/ R_u(Q) \simeq V \times \mathbb{A}^l$;*

(iv) *The $\Bbbk$-linear span of $F_1, \dots, F_l$ (resp. $\hat{F}_1, \dots, \hat{F}_l$) is a G-stable subspace of $\mathrm{Mor}(V, \mathfrak{g})$ (resp. $\Bbbk[\mathfrak{q}^*]$).*

Proof. (i) Since a generic isotropy group is l-dimensional, $\max_{v \in V} \dim G{\cdot}v = \dim \mathfrak{g} - l =: m$. By [5, Satz 1 & Korollar 4], there is a G-equivariant map $c : V \to \wedge^m \mathfrak{g}^* \simeq \wedge^l \mathfrak{g}$ such that $\deg c = \dim V - q(V /\!\!/ G)$ and if $v \in V_{\mathsf{reg}}$, then $0 \neq c(v) \in \wedge^l(\mathfrak{g}_v) \subset \wedge^l \mathfrak{g}$. On the other hand, the map $\tilde{c} = \bigwedge_{i=1}^{l} F_i : V \to \wedge^l \mathfrak{g}$ has the same degree and also $\tilde{c}(v) \in \wedge^l(\mathfrak{g}_v) \subset \wedge^l \mathfrak{g}$ for almost all $v \in V_{\mathsf{reg}}$. In other words, $c(v)$ and $\tilde{c}(v)$ are proportional for almost all $v \in V$. Consequently, there are coprime homogeneous $f, \tilde{f} \in \Bbbk[V]$ such that $fc = \tilde{f}\tilde{c}$. Since $\deg c = \deg \tilde{c}$, we have $\deg f = \deg \tilde{f}$ as well. If $\deg \tilde{f} > 0$, then there is $v \in V_{\mathsf{reg}}$ such that $\tilde{f}(v) = 0$ and $f(v) \neq 0$. Then $c(v) = 0$, a contradiction! Hence $f, \tilde{f} \in \Bbbk^\times$, $\bigwedge_{i=1}^{l} F_i : V \to \wedge^m \mathfrak{g}$ is G-equivariant, and $F_1(v), \dots, F_l(v) \in \mathfrak{g}$ are linearly independent for all $v \in V_{\mathsf{reg}}$.

(ii) As $\mathrm{codim}\,(V \setminus V_{\mathsf{reg}}) \geqslant 2$, the last property also implies that $(F_1, \dots, F_l)$ is a basis for the $\Bbbk[V]$-module $\mathrm{Ker}(\phi)$. Indeed, recall that $\mathrm{rk}\,\mathrm{Ker}(\phi) = \min_{v \in V} \dim \mathfrak{g}_v = l$. If

$F \in \mathrm{Ker}(\phi)$, then there are $f, f_i \in \Bbbk[V]$ such that $fF = \sum_{i=1}^{l} f_i F_i$. Again, if $\deg f > 0$, then there is $v \in V_{\mathrm{reg}}$ such that $f(v) = 0$ and $f_i(v) \ne 0$ for all (some) i. This contradicts the linear independence of $\{F_i(v)\}$ for all $v \in V_{\mathrm{reg}}$. Hence $f \in \Bbbk^\times$, and we are done.

(iii) Recall that now $R_u(Q) = 1 \ltimes V^*$, $\mu : V \times V^* \to \mathfrak{g}^*$ is the moment mapping, and the $R_u(Q)$-orbits in $\mathfrak{q}^*$ are

$$R_u(Q){\cdot}(\xi, v) = (\xi + \mu(v, V^*), v) = (\xi + (\mathfrak{g}_v)^\perp, v).$$

Hence $\dim R_u(Q){\cdot}(\xi, v) = \dim(\mathfrak{g}_v)^\perp$ and $\max_{\eta \in \mathfrak{q}^*} \dim R_u(Q){\cdot}\eta = \dim \mathfrak{g} - l$. Therefore $\mathrm{tr.deg}\, \Bbbk[\mathfrak{q}^*]^{R_u(Q)} = \dim V + l$. Let $(\zeta_1, \ldots, \zeta_n)$, $n = \dim V$, be a basis of V^* (We regard the ζ_i's as linear functions on $\mathfrak{q}^*$.) Then $\zeta_1, \ldots, \zeta_n, \hat{F}_1, \ldots, \hat{F}_l$ are algebraically independent and belong to $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$. Consider the map $\pi : \mathfrak{q}^* = \mathfrak{g}^* \oplus V \to V \times \mathbb{A}^l$ given by

$$\big(\eta = (\xi, v) \in \mathfrak{q}^*\big) \mapsto \big((v, \hat{F}_1(\eta), \ldots, \hat{F}_l(\eta) \in V \times \mathbb{A}^l\big). \tag{3.2}$$

By the Igusa Lemma [22, Theorem 4.12], in order to prove that π is the quotient morphism by $R_u(Q)$, it suffices to verify the following two conditions:

($\diamondsuit_1$) The closure of $(V \times \mathbb{A}^l) \setminus \mathrm{Im}\,(\pi)$ does not contain divisors;

($\diamondsuit_2$) There is a dense open subset $\Psi \subset V \times \mathbb{A}^l$ such that $\pi^{-1}(b)$ contains a dense $R_u(Q)$-orbit for all $b \in \Psi$.

For ($\diamondsuit_1$): If $v \in V_{\mathrm{reg}}$, then $\{F_i(v)\}$ are linearly independent in view of (i). Therefore, the system of linear equations $\langle F_i(v), \xi \rangle = a_i$, $1 \leqslant i \leqslant l$, has a solution ξ for any $(a_1, \ldots, a_l) \in \mathbb{A}^l$. Therefore, $\mathrm{Im}\,(\pi) \supset V_{\mathrm{reg}} \times \mathbb{A}^l$.

For ($\diamondsuit_2$): Suppose that $v \in V_{\mathrm{reg}}$, $\bar{a} = (a_1, \ldots, a_l) \in \mathbb{A}^l$, and ξ_0 is a solution to the system $\langle F_i(v), \xi \rangle = a_i$. Then $\pi^{-1}(v, \bar{a}) = (\xi_0 + (\mathfrak{g}_v)^\perp, v)$, which is a sole $R_u(Q)$-orbit.

Thus, $\Bbbk[\mathfrak{q}^*]^{R_u(Q)} = \Bbbk[\zeta_1, \ldots, \zeta_n, \hat{F}_1, \ldots, \hat{F}_l]$ and the morphism $\pi_{\mathfrak{q}^*, R_u(Q)}$ is given by (3.2).

(iv) Since $\mathfrak{q}^* /\!\!/ R_u(Q) \simeq V \times \mathbb{A}^l$, $G = Q/R_u(Q)$ acts on $\mathfrak{q}^* /\!\!/ R_u(Q)$, and V is a G-module, the explicit form of the free generators of $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ shows that the $\Bbbk$-linear span $\langle \hat{F}_1, \ldots, \hat{F}_l \rangle$ is a G-stable subspace of $\Bbbk[\mathfrak{q}^*]$. Using the definition of $\hat{F}_i$, one readily verifies that $g{\cdot}\hat{F}_i = \widehat{g{*}F_i}$. This means that

$$\langle \hat{F}_1, \ldots, \hat{F}_l \rangle \text{ is a } G\text{-stable subspace} \Leftrightarrow \langle F_1, \ldots, F_l \rangle \text{ is a } G\text{-stable subspace.} \quad \square$$

Note that part (ii) of this theorem is a direct consequence of (i), and our proof of (ii), i.e., essentially the proof of the implication (i)⇒(ii), appears already in the proof of Theorem 1.9 in [9].

The condition (3.1) is rather strong, and all known to us instances of such a phenomenon occur only if $\mathbf{g.s.}(\mathfrak{g} : V)$ is abelian, see examples in Sections 5 and 6. As a by-product of our proof of part (i) in Theorem 3.3, we also obtain the following assertion:

Lemma 3.4. *Suppose that $G \subset GL(V)$ is semisimple, $\operatorname{codim}(V \setminus V_{\mathsf{reg}}) \geqslant 2$, $\dim \mathsf{g.i.g.}(G : V) = l$, and $F_1, \ldots, F_l \in \operatorname{Ker}(\phi)$ are homogeneous and linearly independent. Then $\sum_{i=1}^{l} \deg F_i \geqslant \deg c = \dim V - q(V/\!/G)$.*

Remark 3.5. The interest of Theorem 3.3 is in the case, where $l = \dim \mathsf{g.s.}(\mathfrak{g} : V) > 0$, i.e., there **are** certain morphisms $\{F_i\}$. If $l = 0$, then the *codimension*-2 *condition* for $(G : V)$ implies that $q(V/\!/G) = q(V)$ [5, Korollar 4]. i.e., formally, Eq. (3.1) holds. Then parts (i), (ii), (iv) become vacuous, but part (iii) still makes sense and remains true. For, in this case $\Bbbk[\mathfrak{q}^*]^{R_u(Q)} \simeq \Bbbk[V]$, see [11, Theorem 6.4].

Theorem 3.6. *Let $G, V, F_1, \ldots, F_l$ be as in Theorem 3.3. Suppose also that the identity component of $H = \mathsf{g.i.g.}(G : V)$ is a torus, H is not contained in a proper normal subgroup of G, and $\mathfrak{h}^H = \mathfrak{h}$. Then*

(i) *If the $\mathbf{C}\cdot n\cdot\mathbf{C}$ holds for $(G : V)$ with $n \geqslant 2$, then it also holds for $(Q : \mathfrak{q}^*)$;*
(ii) *The morphisms $F_1, \ldots, F_l$ are G-equivariant, the corresponding polynomials $\hat{F}_1, \ldots, \hat{F}_l$ are G-invariant, and hence $\Bbbk[\mathfrak{q}^*]^Q = \Bbbk[V]^G[\hat{F}_1, \ldots, \hat{F}_l]$, i.e., $\mathfrak{q}^*/\!/Q \simeq V/\!/G \times \mathbb{A}^l$;*
(iii) *$\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ is a maximal Poisson-commutative subalgebra of $\Bbbk[\mathfrak{q}^*]$;*
(iv) *If $\Bbbk[V]^G$ is a polynomial algebra, then the Kostant criterion holds for $\mathfrak{q}$.*

Proof. (i) Since a generic stabiliser is abelian, the standard deformation argument shows that $\mathfrak{g}_v$ is abelian for any $v \in V_{\mathsf{reg}}$. It then follows from [11, Prop. 5.5] that (ξ, v) is Q-regular for any $\xi \in \mathfrak{g}^*$. Hence $\mathfrak{q}^*_{\mathsf{reg}} \supset \mathfrak{g}^* \times V_{\mathsf{reg}}$.

(ii) By Theorem 3.3(iv), the space $\langle F_1, \ldots, F_l \rangle$ is G-stable and therefore

$$g * F_i = \sum_{j=1}^{l} a_{ij}(g) F_j \qquad \forall g \in G.$$

Recall that $g(F_i(g^{-1}v)) = (g * F_i)(v)$. If $g \in G_v$, then $F_i(g^{-1}v) = F_i(v) \in \mathfrak{g}_v$. Moreover, if $G_v \sim H$ (i.e., G_v is conjugate to H), then $g(F_i(g^{-1}v)) = F_i(v)$ in view of the assumption $\mathfrak{h}^H = \mathfrak{h}$. Therefore $F_i(v) = \sum_{j=1}^{l} a_{ij}(g) F_j$ for all v such that $G_v \sim H$ and $g \in G_v$. By Theorem 3.3(i), $\{F_i(v)\}$ are linearly independent. Hence $a_{ij}(g) = \delta_{ij}$ for any $g \in G_v$ and $G_v \sim H$. Hence the kernel of the representation $\rho : G \to GL(\langle F_1, \ldots, F_l \rangle)$ contains the normal subgroup generated by all generic isotropy subgroups. Under our assumption, this implies $\operatorname{Ker}(\rho) = G$. Therefore, each F_i is G-equivariant and thereby each $\hat{F}_i$ is G-invariant and also Q-invariant. Hence G acts trivially on $\mathbb{A}^l$ and

$$\mathfrak{q}^*/\!/Q = (\mathfrak{q}^*/\!/R_u(Q))/\!/G \simeq (V \times \mathbb{A}^l)/\!/G \simeq V/\!/G \times \mathbb{A}^l.$$

(iii) This is a particular case of more general results of [24]. However, using the G-equivariance of $\{F_i\}$ one can verify directly that the basic invariants in $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ pairwise commute w.r.t. the Poisson bracket $\{\ ,\ \}$ (cf. the proof of Theorem 3.3 in [13]).

(iv) If $\Bbbk[V]^G$ is a polynomial algebra, then so is $\Bbbk[\mathfrak{q}^*]^Q$ (in view of (ii)), and the sum of degrees of the basic invariants in $\Bbbk[\mathfrak{q}^*]^Q$ equals $q(V/\!\!/G) + \sum_{i=1}^l \deg \hat{F}_i = q(V/\!\!/G) + (\sum_{i=1}^l \deg F_i) + l = \dim V + l = b(\mathfrak{q})$. Together with the **C·2·C** for $(Q:\mathfrak{q}^*)$, this implies that the Kostant criterion holds for $\mathfrak{q}$, see [12, Theorem 1.2]. $\square$

Corollary 3.7. *Under the assumptions of Theorems 3.3 and 3.6, the $\Bbbk[V]$-module $\mathrm{Ker}(\phi)$ is free and is generated by G-equivariant morphisms. Therefore, the $\Bbbk[V]^G$-module $\mathrm{Ker}(\phi_G)$ is also free, with the "same basis" $F_1, \ldots, F_l$. That is, $\mathrm{Ker}(\phi) \simeq \mathrm{Ker}(\phi_G) \otimes_{\Bbbk[V]^G} \Bbbk[V]$.*

Example 3.8. Let $\mathfrak{g}$ be a semisimple Lie algebra of rank l. Then $\mathfrak{g} \simeq \mathfrak{g}^*$ as G-module, $\mathrm{ind}\,\mathfrak{g} = l$, and $\Bbbk[\mathfrak{g}]^G = \Bbbk[f_1, \ldots, f_l]$ is a graded polynomial algebra. Set $d_i = \deg f_i$. Then $q(\mathfrak{g}/\!\!/G) = \sum_{i=1}^l d_i = b(\mathfrak{g})$ is the dimension of a Borel subalgebra. Here $\mathbf{g.i.g.}(G:\mathfrak{g}) = \mathbb{T}_l$ is a maximal torus. It is known that $\mathrm{Mor}_G(\mathfrak{g}, \mathfrak{g})$ is a free $\Bbbk[\mathfrak{g}]^G$-module generated by the differentials $df_i =: F_i$, $i = 1, \ldots, l$. (This is a special case of a general result of Vust [23, Ch. III, § 2], see also [11, Theorem 4.5].) Here $\deg F_i = d_i - 1$ and hence Eq. (3.1) holds. Thus, Theorems 3.3 and 3.6 apply to $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{g} \ltimes \mathfrak{g}$. A specific feature of this case is that here $\phi_G \equiv 0$ and $\mathrm{Ker}(\phi_G) = \mathrm{Mor}_G(\mathfrak{g}, \mathfrak{g})$.

Remark 3.9. The semisimplicity of G is assumed in Theorems 3.3 and 3.6, because Knop's results in [5] heavily rely on this assumption. Using those results and Eq. (3.1), we then prove that $\bigwedge_{i=1}^l F_i(v) \neq 0$ for all $v \in V_{\mathsf{reg}}$ and so on... But, if one can directly verify that $Z = \{v \in V \mid \bigwedge_{i=1}^l F_i(v) = 0\}$ does not contain divisors, then the proof of Theorem 3.3(iii), (iv) goes through with $V \setminus Z$ in place of V_{reg} and without the semisimplicity condition. (See Example 5.3 below.)

Furthermore, if we know somehow that $\{F_i\}$ are G-equivariant (i.e., $F_i \in \mathrm{Ker}(\phi_G)$), then $F_i(v) \in (\mathfrak{g}_v)^{G_v}$ for all $v \in V$. For $v \in (V \setminus Z) \cap V_{\mathsf{reg}}$, this implies that $\dim \mathfrak{g}_v = \dim(\mathfrak{g}_v)^{G_v}$. Hence a generic stabiliser is abelian and the **C·2·C** for $(G:V)$ implies that for $(Q:\mathfrak{q}^*)$, cf. Theorem 3.6(i). In this situation, we also have $\mathfrak{q}^*/\!\!/Q \simeq V/\!\!/G \times \mathbb{A}^l$, and $\{F_1, \ldots, F_l\}$ is a basis for both $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\phi_G)$.

Remark 3.10. The assumptions of Theorem 3.6 that the adjoint representation of $H = \mathbf{g.i.g.}(G:V)$ is trivial and that H is not contained in a proper normal subgroup of G are essential. We will see in Example 5.1 that if this is not the case, then the morphisms $F_1, \ldots, F_l$ satisfying (3.1) can be not G-equivariant and $\langle F_1, \ldots, F_l \rangle$ affords a nontrivial representation of (a simple factor of) G.

On the other hand, if $l = \dim \mathbf{g.i.g.}(G:V) = 1$, then the assumptions of both theorems can be simplified, and one also obtains stronger results.

Theorem 3.11. *Suppose that $G \subset GL(V)$ is semisimple, $\mathrm{codim}\,(V \setminus V_{\mathsf{reg}}) \geqslant 2$, and $\mathbf{g.s.}(\mathfrak{g} : V) = \mathfrak{t}_1$. As usual, $\mathfrak{q} = \mathfrak{g} \ltimes V^*$. Then*

(i) _The_ **C·2·C** _holds for_ $(Q : \mathfrak{q}^*)$;

(ii) $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ _is freely generated by a basis of_ V^* _and one more polynomial_ $\hat{F}$ _such that_ $\deg \hat{F} = \dim V - q(V/\!/G) + 1$. _In particular,_ $\mathfrak{q}^*/\!/ R_u(Q) \simeq V \times \mathbb{A}^1$;

(iii) $\mathfrak{q}^*/\!/ Q \simeq V/\!/G \times \mathbb{A}^1$;

(iv) _If_ $\Bbbk[V]^G$ _is a polynomial ring, then_ $\mathfrak{q}$ _satisfies the Kostant criterion;_

(v) _Furthermore, if_ $\pi_{V,G} : V \to V/\!/G$ _is equidimensional and_

$(*)$ _each irreducible component of_ $\mathcal{N}_G(V) := \pi_{V,G}^{-1}(\pi_{V,G}(0))$ _contains a_ G-_regular point,_

then $\pi_{\mathfrak{q}^*,Q} : \mathfrak{q}^* \to \mathfrak{q}^*/\!/ Q$ _is also equidimensional and the enveloping algebra_ $\mathcal{U}(\mathfrak{q})$ _is a free module over its centre_ $\mathcal{Z}(\mathfrak{q})$.

Proof. Since $l = 1$, we have $b(\mathfrak{q}) = \dim V + 1$. Here we need only one morphism $F : V \to \wedge^{\dim \mathfrak{g}-1} \mathfrak{g}^* \simeq \mathfrak{g}$ of degree $\dim V - q(V/\!/G)$ such that $0 \neq F(v) \in \mathfrak{g}_v$ for all $v \in V_{\mathrm{reg}}$. The existence of such a G-equivariant morphism follows from Knop's theory [5]. As the morphism F is G-equivariant and $F \in \mathrm{Ker}(\phi)$, the corresponding polynomial $\hat{F}$ lies in $\Bbbk[\mathfrak{q}^*]^Q$. Then the proofs of Theorems 3.3 and 3.6 apply and yield parts (i)–(iv).

(v) The equidimensionality of $\pi_{V,G}$ is equivalent to that $\dim \mathcal{N}_G(V) = \dim V - \dim V/\!/G$, see [22, Eq. (8.1)]. And for the equidimensionality of $\pi_{\mathfrak{q}^*,Q}$, it suffices to prove that

$$\dim \mathcal{N}_Q(\mathfrak{q}^*) = \dim \mathfrak{q} - \dim \mathfrak{q}^*/\!/ Q = \dim V + \dim \mathfrak{g} - \dim V/\!/G - 1 = \dim \mathcal{N}_G(V) + \dim \mathfrak{g} - 1,$$

where $\mathcal{N}_Q(\mathfrak{q}^*) = \pi_{\mathfrak{q}^*,Q}^{-1}(\pi_{\mathfrak{q}^*,Q}(0))$. It follows from (iii) that

$$\mathcal{N}_Q(\mathfrak{q}^*) = \{(\xi, v) \mid v \in \mathcal{N}_G(V) \ \& \ \hat{F}(\xi, v) = \langle F(v), \xi \rangle = 0\}.$$

In other words, $\mathcal{N}_Q(\mathfrak{q}^*) = \big(\mathfrak{g}^* \times \mathcal{N}_G(V)\big) \cap \{\hat{F} = 0\}$. Under assumption $(*)$, we have $\dim \mathcal{N}_Q(\mathfrak{q}^*) = \dim \mathcal{N}_G(V) + \dim \mathfrak{g} - 1$, as required. Then $\mathcal{S}(\mathfrak{q}) = \Bbbk[\mathfrak{q}^*]$ is a free $\mathcal{S}(\mathfrak{q})^Q$-module; and, by a standard deformation argument, this implies that $\mathcal{U}(\mathfrak{q})$ is a free module over $\mathcal{Z}(\mathfrak{q}) \simeq \mathcal{S}(\mathfrak{q})^Q$. $\square$

Note that if $\mathcal{N}_G(V)$ contains finitely many G-orbits, then $\pi_{V,G}$ is equidimensional [22, § 5.2] and hence condition $(*)$ is satisfied.

Remark. If $l \geqslant 1$ and $\mathcal{N}_G(V)$ contains finitely many G-orbits, then there is a general criterion for the equidimensionality of $\pi_{\mathfrak{q}^*,Q}$ in terms of the stratification of $\mathcal{N}_G(V)$ determined by the covariants $F_1, \dots, F_l$. Namely,

$$\mathcal{N}_Q(\mathfrak{q}^*) = \{(\xi, v) \mid v \in \mathcal{N}_G(V) \ \& \ \langle F_i(v), \xi \rangle = 0 \quad i = 1, \dots, l\}$$

and using the projection $\mathcal{N}_Q(\mathfrak{q}^*) \to \mathcal{N}_G(V)$, $(\xi, v) \mapsto v$, one proves that

$$\pi_{\mathfrak{q}^*,Q} \text{ is equidimensional} \iff \dim \mathfrak{g}_v + \dim \langle F_1(v), \dots, F_l(v) \rangle \geqslant 2l$$

for any $v \in \mathcal{N}_G(V)$. However, this condition is not easily verified in specific examples, if $l > 1$. If $(G : V)$ is the isotropy representation of a symmetric variety such that g.i.g. is a torus, then a version of this condition is verified in [12, Sect. 5].

4. The codimension-2 condition for representations

In this section, we provide some sufficient conditions for the **C**·2·**C** to hold for $(G : V)$. A G-stable divisor $D \subset V$ is said to be *bad*, if $\max_{v \in D} \dim G \cdot v < \max_{v \in V} \dim G \cdot v$. That is, if

$$\min_{v \in D} \dim G_v > \min_{v \in V} \dim G_v.$$

Hence the **C**·2·**C** holds for $(G : V)$ if and only if V contains no bad divisors.

Proposition 4.1. *Suppose that G is reductive, the action $(G : V)$ is stable, and $\mathcal{N}_G(V)$ contains finitely many G-orbits. Then the* **C**·2·**C** *holds for $(G : V)$.*

Proof. Since $\mathcal{N}_G(V)$ has finitely many orbits, $\pi_{V,G} : V \to V /\!\!/ G$ is equidimensional and each fibre of $\pi_{V,G}$ also has finitely many orbits [22, §5.2, Cor. 3]. Assume that D is a (G-stable) bad divisor in V. Then $\pi_{V,G}(D)$ is a proper (closed) subvariety of $V /\!\!/ G$, see e.g. [21, Theorem 1], and since $\pi_{V,G}$ is equidimensional, $\pi_{V,G}(D)$ is actually a divisor in $V /\!\!/ G$. Hence $\dim D \cap \pi_{V,G}^{-1}(\xi) = \dim \pi_{V,G}^{-1}(\xi)$ for any $\xi \in \pi_{V,G}(D)$ and therefore $D \cap \pi_{V,G}^{-1}(\xi)$ contains G-regular elements. Hence D cannot be bad. $\square$

Example 4.2 *(Vinberg's ϑ-groups [20]).* Let ϑ be an automorphism of $\mathfrak{g}$ of finite order k. If $\varsigma = \sqrt[k]{1}$ is primitive, then $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the eigenspace of ϑ corresponding to ς^i. The above decomposition is also called a *periodic grading* of $\mathfrak{g}$. Here $\mathfrak{g}_0$ is reductive and each $\mathfrak{g}_i$ is a $\mathfrak{g}_0$-module. If G_0 is the connected subgroup of G with $\mathsf{Lie}\, G_0 = \mathfrak{g}_0$, then the linear group $G_0 \to GL(\mathfrak{g}_1)$ is called a *ϑ-group*. A fundamental invariant-theoretic property of ϑ-groups is that $\mathcal{N}_{G_0}(\mathfrak{g}_1)$ contains finitely many G_0-orbits and $\Bbbk[\mathfrak{g}_1]^{G_0}$ is a polynomial ring. If $k = 2$, then $(G_0 : \mathfrak{g}_1)$ is always stable. There are also many interesting examples of stable ϑ-groups, if $k \geqslant 3$, see e.g. [20, §9].

Example 4.3 *(reduced θ-groups).* Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading. Then $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ for a unique semisimple element $h \in \mathfrak{g}(0)$. Let $\mathfrak{g}(0)'$ be the orthocomplement to $\Bbbk h$ in $\mathfrak{g}(0)$ w.r.t. the Killing form on $\mathfrak{g}$ and $G(0)' \subset G(0)$ the corresponding connected subgroups of G. Here the reductive group $G(0)$ has finitely many orbits in each $\mathfrak{g}(i)$ with $i \neq 0$ [20], while there is a dichotomy for $G(0)'$-orbits. Either the $G(0)'$-orbits in $\mathfrak{g}(1)$ coincide with the $G(0)$-orbits, or $\dim \mathfrak{g}(1) /\!\!/ G(0)' = 1$ and the $G(0)'$-orbits in $\mathcal{N}_{G(0)'}\mathfrak{g}(1)$ coincide with the $G(0)$-orbits [4, Theorem 2.9]. In the latter case, the action $(G(0)' : \mathfrak{g}(1))$ is also stable. The linear groups of the form $G(0)' \to GL(\mathfrak{g}(1))$ are called *reduced ϑ-groups*.

In the following assertion G is not necessarily reductive.

Theorem 4.4. *Let $G : V_1 \oplus V_2 = V$ be a reducible representation. Suppose that generic isotropy groups $S_i = \mathsf{g.i.g.}(G : V_i)$, $i = 1, 2$, exist and the $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds for both $(S_1 : V_2)$ and $(S_2 : V_1)$. Then the $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds also for $(G : V)$.*

Proof. Assume that $D \subset V$ is a bad divisor. Consider the projections $p_i : D \to V_i$, $i = 1, 2$.

• If p_1 is dominant and p_2 is not, then $D = V_1 \times D_2$, where $D_2 \subset V_2$ is a G-stable divisor. Take a generic point $x_1^o \in V_1$ such that $G_{x_1^o} = S_1$. The fact that D is bad means that

$$\min_{x_2 \in D_2} \dim (G_{x_1^o})_{x_2} = \min_{\bar{x} \in D} \dim G_{\bar{x}} > \min_{\bar{v} \in V} \dim G_{\bar{v}} = \min_{x_2 \in V_2} \dim (G_{x_1^o})_{x_2}.$$

That is, D_2 appears to be a bad divisor for $(S_1 : V_2)$. Thus, this case is impossible.

• If p_2 is dominant and p_1 is not, then $D = D_1 \times V_2$ and the argument is "symmetric".

• If both p_1, p_2 are dominant, then we again can take a point $\bar{x} = (x_1, x_2) \in D$ such that $G_{x_1} = S_1$. Here $p_1^{-1}(x_1) = \{x_1\} \times D_2$ and the similar argument shows that D_2 is a bad divisor for $(S_1 : V_2)$. □

Notation 4.5. In specific examples and the tables in Appendix A, we identify the representations V of semisimple groups with their highest weights, using the *multiplicative* notation and the Vinberg–Onishchik numbering of the fundamental weights. For instance, if $\varphi_1, \ldots, \varphi_n$ are the fundamental weights of a simple algebraic group G, then $V = \varphi_1^2 + \varphi_{n-1}$ stands for the direct sum of two simple G-modules with highest weights $2\varphi_1$ and φ_{n-1}. If $G = G_1 \times G_2 \times \ldots$ is semisimple, then the fundamental weights of the first (resp. second) factor are denoted by $\{\varphi_i\}$ (resp. $\{\varphi_i'\}$) and so on... The dual G-module for ψ is denoted by ψ^*. We omit the index for the unique fundamental weight of SL_2.

Example 4.6. We provide several cases, where the last theorem allows us to check the codimension-2 condition.

1°. $G = SL_n$, $V_1 = \varphi_1^2$, and $V_2 = \varphi_2$. Here $S_1 = SO_n$ and $(S_1 : V_2)$ is equivalent to the adjoint representation of SO_n modulo a trivial summand. If n is even, then $S_2 = Sp_n$ and $(S_2 : V_1)$ is equivalent to the adjoint representation of Sp_n modulo a trivial summand. This already shows that $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds if n is even. For n odd, S_2 is not reductive and the only a priori possible bad divisor is $D_1 \times V_2$, where D_1 consists of the symmetric matrices with $\det = 0$. Here a direct calculation of stabilisers shows that this divisor is not bad. Thus, the $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds for all n.

2°. $G = SL_n$, $V_1 = \varphi_1^2$, and $V_2 = \varphi_2^* = \varphi_{n-2}$. This is similar to 1°.

3°. $G = SL_n \times SL_n$ and $V_1 = V_2 = \varphi_1 \varphi_1'$. Here $S_1 = S_2 = \Delta_{SL_n} \simeq SL_n$ and $(S_1 : V_2)$ is equivalent to the adjoint representation of SL_n modulo a trivial summand.

4°. $G = Sp_6$ and $V_1 = V_2 = \varphi_2$. Here $S_1 = S_2 = (SL_2)^3$ and $(S_1 : V_2)$ is equivalent to $(SL_2 \times SL_2 \times SL_2 : \varphi\varphi' + \varphi\varphi'' + \varphi'\varphi'')$ modulo a 2-dimensional trivial summand. Applying Theorem 4.4 to the last representation, one readily obtains the **C·2·C**.

Below is a variation of Theorem 4.4 that concerns the case in which $V_1 \simeq V_2$.

Theorem 4.7. *For any representation $G \to GL(V)$, one naturally defines the representation of $\hat{G} = G \times SL_2$ in $\hat{V} = V \otimes \Bbbk^2$. Let G_* be a generic isotropy group for $(G : V)$. If **C·2·C** holds for $(G_* : V)$ and $\mathsf{g.s.}(\mathfrak{g} : V) = \mathsf{g.s.}(\hat{\mathfrak{g}} : \hat{V})$, then **C·2·C** also holds for $(\hat{G} : \hat{V})$.*

Proof. Since $\hat{V}|_G = V \oplus V$, Theorem 4.4 shows that the **C·2·C** holds for $(G : V \oplus V)$. Let $\hat{D} \subset \hat{V}$ be a $\hat{G}$-stable divisor. As above, consider the G-equivariant projections $p_i : \hat{D} \to V^{(i)}$, where $V^{(i)}$ is the i-th copy of V and $i = 1, 2$. Since $\hat{D}$ is SL_2-stable, both projections must be dominant. Take $(x_1, x_2) \in \hat{D}$ such that $G_{x_1} = G_*$. Since x_1 is G-generic and **C·2·C** holds for $(G_* : V)$, there is $x_2 \in p_2(p_1^{-1}(x_1))$ such that

$$\dim(G_{x_1})_{x_2} = \min_{v_2 \in V^{(2)}} \dim(G_{x_1})_{v_2} = \min_{\bar{v} \in V \oplus V} \dim G_{\bar{v}} = \min_{\hat{v} \in \hat{V}} \dim G_{\hat{v}}.$$

This means that $\hat{D}$ cannot be bad. $\square$

Example 4.8. Theorem 4.7 applies, if we add the factor SL_2 to G in Example 4.6, 3° & 4°.

Theorem 4.9. *Suppose that the **C·2·C** holds for $(G_1 \times G_2 : V_1 \otimes V_2 = V)$ and $\mathsf{g.s.}(\mathfrak{g}_1 \times \mathfrak{g}_2 : V_1 \otimes V_2) = \mathsf{g.s.}(\mathfrak{g}_1 : V_1^{\oplus d})$, where $d = \dim V_2$. Then **C·2·C** also holds for $(G_1 : V_1^{\oplus d})$.*

Proof. Assume that $D \in V$ is a bad divisor for $(G_1 : V_1^{\oplus d})$. Then $\dim G_1 \cdot u < \max_{v \in V} \dim G_1 \cdot v$ for all $u \in D$. The coincidence of generic stabilisers implies that D is also $G_1 \times G_2$-stable and then

$$\dim G \cdot u < \max_{v \in V} \dim G_1 \cdot v + \dim G_2 = \max_{v \in V} \dim G \cdot v \quad \text{for all } u \in D.$$

Hence D is bad for $(G : V)$, too. A contradiction! $\square$

Example 4.10. The representation $(SL_6 \times SL_3 : \varphi_2 \varphi_1')$ is the ϑ-group associated with an automorphism of order 3 of $\mathbf{E}_7$, see item 5 in the table in [20, §9]. A generic isotropy group H here is reductive (namely, $\mathsf{Lie}\, H = \mathfrak{t}_1$). Therefore, this action is stable and hence **C·2·C** holds here (use Proposition 4.1). All assumptions of Theorem 3.11 are satisfied here, and therefore $\mathfrak{q} = (\mathfrak{sl}_6 \times \mathfrak{sl}_3) \ltimes (\varphi_2 \varphi_1')^*$ satisfies the Kostant criterion and $\mathcal{U}(\mathfrak{q})$ is a free module over $\mathcal{Z}(\mathfrak{q})$.

Forgetting about SL_3, we obtain the representation $(SL_6 : 3\varphi_2)$. Since both have the same generic stabilisers (namely $\mathfrak{t}_1$), the **C·2·C** also holds for the latter in view of Theorem 4.9. Here the algebra $\Bbbk[3\varphi_2]^{SL_6}$ is still polynomial [1,17], but the equidimensionality

of the quotient morphism fails [18]. Hence $\mathfrak{q}' = \mathfrak{sl}_6 \ltimes 3\varphi_2^*$ satisfies the Kostant criterion, but $\mathcal{U}(\mathfrak{q}')$ is **not** a free $\mathcal{Z}(\mathfrak{q}')$-module.

5. Constructing covariants for semi-direct products, I

If an action $(G : V)$ is associated with a periodic or $\mathbb{Z}$-grading of a simple Lie algebra, then usually most of the assumptions of Theorems 3.3 and 3.6 are automatically satisfied for it. The most appealing and non-trivial task is to produce linearly independent morphisms $\{F_i\}$ in $\text{Ker}(\phi)$ such that (3.1) holds.

Example 5.1. $G = SL(V_1) \times SL(V_2) \times SL(V_3)$ and $V = V_1 \otimes V_2 \otimes V_3$, where $\dim V_1 = \dim V_2 = n$ and $\dim V_3 = 2$. In other words, $G = SL_n \times SL_n \times SL_2$ and $V = \varphi_1 \varphi_1' \varphi'' \simeq \Bbbk^n \otimes \Bbbk^n \otimes \Bbbk^2$.

Upon the restriction to $\tilde{G} := SL(V_1) \times SL(V_2)$, the space V splits in two copies of $V_1 \otimes V_2$. We regard the $\tilde{G}$-module $V_1 \otimes V_2$ as the space n by n matrices, equipped with the action $(g_1, g_2) \cdot A = g_1 A g_2^{-1}$, where $g_i \in SL(V_i)$. The corresponding action of $(s_1, s_2) \in \tilde{\mathfrak{g}}$ is given by $(s_1, s_2) \cdot A = s_1 A - A s_2$. We think of elements of V as pairs (A, B) of n by n matrices, where the action of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2 = SL(V_3)$ is given by $(A, B) \mapsto (\alpha A + \beta B, \gamma A + \delta B)$. By Examples 4.6($3^o$) and 4.8, the **C**·2·**C** holds for both $(G : V)$ and $(\tilde{G} : V)$. The algebra $\Bbbk[V]^{\tilde{G}}$ is polynomial and its basic invariants are the coefficients of the characteristic polynomial

$$\det(A + \lambda B) = \sum_{i=0}^{n} f_i(A, B)\lambda^i, \text{ where } f_0(A, B) = \det A \text{ and } f_n(A, B) = \det B,$$

see e.g. [14, Theorem 4]. Since $\deg f_i(A, B) = n$ for all i, $q(V /\!\!/ \tilde{G}) = n(n+1)$. Looking at the weights of the polynomials $\{f_i(A, B)\}_{i=0}^n$ w.r.t. a maximal torus in SL_2, one realises that $V /\!\!/ \tilde{G}$ is isomorphic to $(\varphi'')^n$ (the space of binary forms of degree n) as an SL_2-module. (We also write R_n for this SL_2-module.) It is known that $q(\mathsf{R}_n /\!\!/ SL_2) = \dim \mathsf{R}_n = n + 1$ for $n \geqslant 3$. In our case, the coordinates in $\mathsf{R}_n = V /\!\!/ \tilde{G}$ are of degree n w.r.t. the initial grading of $\Bbbk[V]$. Therefore,

$$q(V /\!\!/ G) = n \cdot q(\mathsf{R}_n /\!\!/ SL_2) = n(n+1) = q(V /\!\!/ \tilde{G}) \text{ if } n \geqslant 3.$$

It is easily verified that $\tilde{H} := \mathsf{g.i.g.}(\tilde{G} : V) \simeq \mathbb{T}_{n-1}$ for any $n \geqslant 2$, where the torus $\mathbb{T}_{n-1}$ is diagonally embedded in $\tilde{G} \simeq SL_n \times SL_n$. Furthermore, the identity component of $H = \mathsf{g.i.g.}(G : V)$ is the same torus for $n \geqslant 3$. In other words, $\mathfrak{h} = \mathsf{g.s.}(\mathfrak{g}:V) = \mathsf{g.s.}(\tilde{\mathfrak{g}}:V) = \tilde{\mathfrak{h}}$ for $n \geqslant 3$. (See Example 5.2 for $(G : V)$ with $n = 2$.) However, H can be disconnected. Using the isomorphism $V /\!\!/ \tilde{G} \simeq \mathsf{R}_n$, one verifies that H/H^0 is isomorphic to $\mathsf{g.i.g.}(SL_2 : \mathsf{R}_n)$, and the latter is isomorphic to

• $\mathbb{Z}_3$, if $n = 3$; • $\mathbb{Z}_2 \ltimes \mathbb{Z}_4$, if $n = 4$; • $\{1\}$, if $n \geqslant 5$ is odd; • $\mathbb{Z}_2$, if $n \geqslant 6$ is even.

We will compare below the coadjoint representations of the Lie algebras $\mathfrak{q} = \mathfrak{g} \ltimes V^*$ and $\tilde{\mathfrak{q}} = \tilde{\mathfrak{g}} \ltimes V^*$ for $n \geqslant 3$. Accordingly, we consider the corresponding connected groups Q and $\tilde{Q}$, two morphisms of $\Bbbk[V]$-modules

$$\phi : \mathrm{Mor}(V, \mathfrak{g}) \to \mathrm{Mor}(V, V) \ \ \text{and} \ \ \tilde{\phi} : \mathrm{Mor}(V, \tilde{\mathfrak{g}}) \to \mathrm{Mor}(V, V)$$

and the corresponding morphisms ϕ_G and $\tilde{\phi}_{\tilde{G}}$ of modules of covariants (see Section 2). Clearly, $\mathrm{Mor}(V, \tilde{\mathfrak{g}}) \subset \mathrm{Mor}(V, \mathfrak{g})$ and $\tilde{\phi} = \phi|_{\mathrm{Mor}(V,\tilde{\mathfrak{g}})}$. Note also that $R_u(Q) = R_u(\tilde{Q})$.

For $A \in \mathfrak{gl}_n$, let A^* denote the *adjugate* of A, i.e., the transpose of its cofactor matrix. (Hence $AA^* = A^*A = (\det A)I$.) Note that $A \mapsto A^*$ is a polynomial mapping of degree $n - 1$. Let $A \mapsto \bar{A} = A - \frac{\mathrm{tr}\,(A)}{n}I$ denote the projection from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$.

Consider the morphism $F \in \mathrm{Mor}(V, \tilde{\mathfrak{g}})$, where $F(A, B) = (\overline{BA^*}, \overline{A^*B}) \in \tilde{\mathfrak{g}} \subset \mathfrak{g}$. Here $\overline{BA^*}$ (resp. $\overline{A^*B}$) is regarded as an element of $\mathfrak{sl}(V_1)$ (resp. $\mathfrak{sl}(V_2)$). One readily verifies that $F(A, B){\cdot}(A, B) = 0$, cf. the proof of Theorem 5.1.1(i). Hence $F \in \mathrm{Ker}(\tilde{\phi}) \subset \mathrm{Ker}(\phi)$. Since the map $A \mapsto A^*$ has degree $n - 1$, we obtain $\deg F = n$. We will see below that the morphism F is $\tilde{G}$-equivariant. However, it is not SL_2-equivariant, hence not G-equivariant. Still, F is a lowest weight vector in a simple SL_2-module R_{n-2}. Indeed, for any γ we have

$$F(A, \gamma A + B) = (\overline{(\gamma A + B)A^*}, \overline{A^*(\gamma A + B)}) = (\overline{\gamma AA^* + BA^*}, \overline{\gamma A^*A + A^*B}) = F(A, B),$$

i.e., the subgroup $\left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mid \gamma \in \Bbbk \right\} \subset SL_2$ stabilises F. By a direct calculation, we also have $g * F = t^{2-n}F$ for $g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Having at hand one suitable covariant, we perform a "polarisation". Consider

$$\mathbb{F}_\lambda(A, B) := F(A + \lambda B, B) = (\overline{B(A + \lambda B)^*}, \overline{(A + \lambda B)^*B}) = \sum_{i=0}^{n-1} F_i(A, B)\lambda^i.$$

Note that $F_0 = F$ and $F_{n-1}(A, B) = (\overline{BB^*}, \overline{B^*B}) = 0$. That is, we obtain only the morphisms $F_0, F_1, \ldots, F_{n-2}$ in $\mathrm{Mor}(V, \tilde{\mathfrak{g}})$. It follows from the previous observation that the $\Bbbk$-linear span $\langle F_0, F_1, \ldots, F_{n-2} \rangle$ is an SL_2-module isomorphic to R_{n-2}.

Theorem 5.1.1. *We have*

(i) $\mathbb{F}_\lambda$ *is a $\tilde{G}$-equivariant morphism for any $\lambda \in \Bbbk$. Therefore, all $\{F_i\}$ are $\tilde{G}$-equivariant;*
(ii) $F_i \in \mathrm{Ker}(\tilde{\phi})$ *for all i.*

Proof. (i) By definition,

$$\mathbb{F}_\lambda(g_1 A g_2^{-1}, g_1 B g_2^{-1}) = \left(\overline{g_1 B g_2^{-1}{\cdot}(g_1(A + \lambda B)g_2^{-1})^*}, \overline{(g_1(A + \lambda B)g_2^{-1})^*{\cdot}g_1 B g_2^{-1}} \right).$$

If $A + \lambda B$ is invertible, then the first component is being transformed as follows:

$$\overline{g_1 B g_2^{-1}(g_1(A + \lambda B)g_2^{-1})^*} = \det(A + \lambda B)\overline{g_1 B g_2^{-1}}\cdot(g_1(A + \lambda B)g_2^{-1})^{-1}$$

$$= \det(A + \lambda B)\overline{g_1 B(A + \lambda B)^{-1}g_1^{-1}} = g_1\overline{B(A + \lambda B)^*}g_1^{-1}.$$

Likewise, for the second component, we obtain $g_2\big(\overline{(A + \lambda B)^* B}\big)g_2^{-1}$. Thus,

$$\mathbb{F}_\lambda((g_1, g_2)\cdot(A, B)) = \mathbb{F}_\lambda(g_1 A g_2^{-1}, g_1 B g_2^{-1}) = (g_1, g_2)\cdot\mathbb{F}_\lambda(A, B)$$

whenever $A + \lambda B$ is invertible. Since $\mathbb{F}_\lambda$ is a polynomial mapping that is $\tilde{G}$-equivariant on the open subset of triples (A, B, λ) such that $A + \lambda B$ is invertible, it is always equivariant.

(ii) It suffices to verify that $\mathbb{F}_\lambda(A, B)\cdot(A, B) = 0$ for any λ. The first component in the LHS equals

$$\overline{B(A+\lambda B)^*}A - A\overline{(A+\lambda B)^* B}$$

$$= B(A+\lambda B)^* A - \frac{\operatorname{tr} B(A+\lambda B)^*}{n}A - A(A+\lambda B)^* B + \frac{\operatorname{tr}(A+\lambda B)^* B}{n}A. \quad (5.1)$$

Now, if both A and $A + \lambda B$ are invertible, then

$$B(A + \lambda B)^* A = \det(A+\lambda B)\cdot B(A+\lambda B)^{-1}A = \det(A+\lambda B)\cdot B(A(I+A^{-1}B))^{-1}A$$

$$= \det(A+\lambda B)\cdot B(I+A^{-1}B)^{-1} = \det(A+\lambda B)\cdot(B-\lambda BA^{-1}B+\lambda^2 BA^{-1}BA^{-1}B-\cdots).$$

A similar transform yields the very same formula for $A(A+\lambda B)^* B$. Since the difference in (5.1) vanishes on the open subset of triples (A, B, λ), where A and $A + \lambda B$ are invertible, it is identically zero. And likewise for the second component in $\mathbb{F}_\lambda(A, B)\cdot(A, B)$. □

Remark. Permuting A and B in the definition of $F = F_0$, one defines the companion morphism $\hat{F} \in \operatorname{Mor}(V, \tilde{\mathfrak{g}})$ by $\hat{F}(A, B) = (\overline{AB^*}, \overline{B^* A})$. Then we can prove that $\hat{F} = -F_{n-2}$.

Note that $\sum_{i=0}^{n-2} \deg F_i = n(n - 1) = \dim V - q(V/\!/\tilde{G}) = \dim V - q(V/\!/G)$. Hence $\tilde{G}, V, \tilde{\mathfrak{q}}$, and the covariants $F_0, \ldots, F_{n-2}$ satisfy all the assumptions of Theorems 3.3 and 3.6. Hence

- $\tilde{\mathfrak{q}}^*/\!/R_u(\tilde{Q}) \simeq V \times \mathbb{A}^{n-1}$ *and* $\tilde{\mathfrak{q}}^*/\!/\tilde{Q} \simeq V/\!/\tilde{G} \times \mathbb{A}^{n-1} \simeq \mathbb{A}^{2n}$;
- $\operatorname{Ker}(\tilde{\phi})$ *(resp.* $\operatorname{Ker}(\tilde{\phi}_{\tilde{G}})$*) is a free* $\Bbbk[V]$ *(resp.* $\Bbbk[V]^{\tilde{G}}$*)-module with basis* $F_0, \ldots, F_{n-2}$;
- *the Kostant criterion holds for* $\tilde{\mathfrak{q}} = (\mathfrak{sl}_n \times \mathfrak{sl}_n) \ltimes (\Bbbk^n \otimes \Bbbk^n \otimes \Bbbk^2)^*$.

However, G, V, and $\mathfrak{q} = (\mathfrak{sl}_n \times \mathfrak{sl}_n \times \mathfrak{sl}_2) \ltimes (\Bbbk^n \otimes \Bbbk^n \otimes \Bbbk^2)^*$ do **not** satisfy all the assumptions of Theorem 3.6. For, either $\mathfrak{h} \neq \mathfrak{h}^H$ ($n = 3, 4$) or H is contained in a proper normal
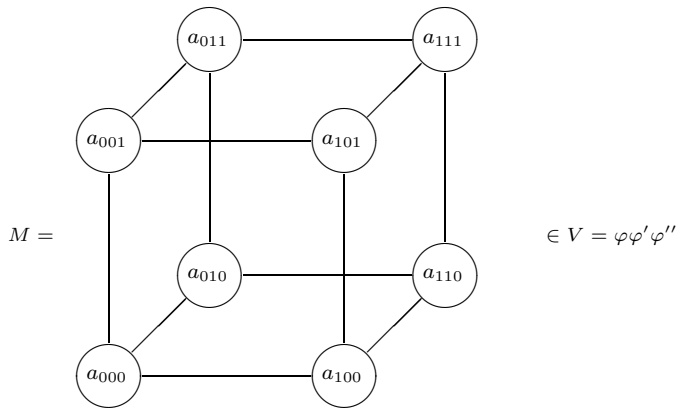
Fig. 1. A cubic 2-matrix.

subgroup of G ($n \geqslant 5$). But Theorem 3.3 still applies, and we have $\mathfrak{q}^* /\!\!/ R_u(Q) \simeq V \times \mathbb{A}^{n-1}$. Then $\mathfrak{q}^* /\!\!/ \tilde{Q} \simeq V /\!\!/ \tilde{G} \times \mathbb{A}^{n-1}$, and the last variety is isomorphic to $\mathsf{R}_n \oplus \mathsf{R}_{n-2}$ as $Q/\tilde{Q}$-module, i.e., SL_2-module. Therefore, $\mathfrak{q}^* /\!\!/ Q \simeq (\mathsf{R}_n \oplus \mathsf{R}_{n-2}) /\!\!/ SL_2$, which is not an affine space for $n \geqslant 3$. In other words, $\Bbbk[\mathfrak{q}^*]^Q$ is not a polynomial ring for $n \geqslant 3$. For instance, it is a hypersurface for $n = 3, 4$, see e.g. [19, 3.4.3].

Remark 5.1.2. Since $\mathsf{g.s.}(\mathfrak{g} : V) = \mathsf{g.s.}(\tilde{\mathfrak{g}} : V) = \mathfrak{t}_{n-1}$, we have $\mathrm{rk}\,\mathrm{Ker}(\phi) = \mathrm{rk}\,\mathrm{Ker}(\tilde{\phi}) = n - 1$ by Eq. (2.1). Moreover, because $\tilde{H} = \mathsf{g.i.g.}(\tilde{G} : V)$ is abelian and connected, we also get $\mathrm{rk}\,\mathrm{Ker}(\tilde{\phi}) = \mathrm{rk}\,\mathrm{Ker}(\tilde{\phi}_{\tilde{G}})$.

But the situation for ϕ and ϕ_G is different. If $n = 3, 4$, then the component group $H/\tilde{H}$ acts nontrivially on $\mathfrak{h}$ and, actually, $\mathfrak{h}^H = \{0\}$. Therefore, $\mathrm{rk}\,\mathrm{Ker}(\phi_G) = 0$. On the other hand, if $n \geqslant 5$, then $\mathfrak{h}^H = \mathfrak{h}$, hence $\mathrm{rk}\,\mathrm{Ker}(\phi) = \mathrm{rk}\,\mathrm{Ker}(\phi_G)$. However, even if $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\phi_G)$ have the same rank, the free generators of the former are not G-equivariant (they are only $\tilde{G}$-equivariant). In fact, we do not know the generators of the $\Bbbk[V]^G$-module $\mathrm{Ker}(\phi_G)$ if $n \geqslant 5$.

Example 5.2. The case of $n = 2$ in Example 5.1 does not fit into the general picture with $n \geqslant 3$, so we consider it separately. Now $G = (SL_2)^3$ and $V = \varphi \varphi' \varphi''$. This is a reduced ϑ-group (see Example 4.3) related to a $\mathbb{Z}$-grading of $\mathbf{D}_4$. Therefore $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds here. We have $V /\!\!/ G = \mathbb{A}^1$, $q(V /\!\!/ G) = 4$, and $\mathsf{g.i.g.}(G : V) \simeq \mathbb{T}_2$. More precisely, if the elements of a maximal torus

$$\mathbb{T} = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, \begin{pmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{pmatrix}, \begin{pmatrix} t_3 & 0 \\ 0 & t_3^{-1} \end{pmatrix} \mid t_i \in \Bbbk^\times \right\} \subset G$$

are represented as triples (t_1, t_2, t_3), then $\mathsf{g.i.g.}(G : V) = \{(t_1, t_2, t_3) \mid t_1 t_2 t_3 = 1\}$.

The elements of V can be regarded as cubic 2-matrices with entries a_{ijk}, see Fig. 1, where the i-th factor of G acts along the i-th coordinate, $i = 1, 2, 3$.

We provide below three morphisms from V to $\mathfrak{sl}_2$ that are thought of as morphisms to the consecutive factors of $\mathfrak{g}$, where the column $\begin{bmatrix} m \\ n \\ p \end{bmatrix}$ represents the matrix $\left(\begin{smallmatrix} n & m \\ p & -n \end{smallmatrix} \right)$:

$$\tilde{F}_1(M) = \begin{bmatrix} a_{111}a_{100} - a_{101}a_{110} \\ a_{111}a_{000} + a_{011}a_{100} - a_{001}a_{110} - a_{101}a_{010} \\ a_{011}a_{000} - a_{001}a_{010} \end{bmatrix}$$

$$\tilde{F}_2(M) = \begin{bmatrix} a_{111}a_{010} - a_{011}a_{110} \\ a_{111}a_{000} - a_{011}a_{100} - a_{001}a_{110} + a_{101}a_{010} \\ a_{101}a_{000} - a_{001}a_{100} \end{bmatrix}$$

$$\tilde{F}_3(M) = \begin{bmatrix} a_{111}a_{001} - a_{101}a_{011} \\ a_{111}a_{000} - a_{011}a_{100} + a_{001}a_{110} - a_{101}a_{010} \\ a_{110}a_{000} - a_{100}a_{010} \end{bmatrix}$$

Letting $\mathbb{F}_{\lambda,\mu,\nu}(M) = (\lambda \tilde{F}_1(M), \mu \tilde{F}_2(M), \nu \tilde{F}_3(M))$ with $\lambda, \mu, \nu \in \Bbbk$, we obtain a 3-dimensional subspace of $\mathrm{Mor}(V, \mathfrak{g})$, and one verifies directly that $\mathbb{F}_{\lambda,\mu,\nu} \in \mathrm{Ker}(\phi)$ if and only if $\lambda + \mu + \nu = 0$. Then $F_1 = \mathbb{F}_{\lambda,-\lambda,0}$ and $F_2 = \mathbb{F}_{0,\mu,-\mu}$ satisfy (3.1) and Theorems 3.3 and 3.6 apply.

Hence $\mathrm{Ker}(\phi)$ *(resp.* $\mathrm{Ker}(\phi_G)$*) is a free* $\Bbbk[\varphi\varphi'\varphi'']$*-module (resp.* $\Bbbk[\varphi\varphi'\varphi'']^{(SL_2)^3}$*-module) and* $\mathfrak{q} = (\mathfrak{sl}_2)^3 \ltimes \varphi\varphi'\varphi''$ *satisfies the Kostant criterion. Furthermore, using the explicit classification of* G*-orbits in* V*, one can prove that* **C·3·C** *holds for* $(G : V)$ *and hence for* $(Q : \mathfrak{q}^*)$*, and also that* $\mathcal{U}(\mathfrak{q})$ *is a free* $\mathcal{Z}(\mathfrak{q})$*-module.*

Example 5.3. $G = \prod_{i=1}^{k} GL(\mathbb{U}_i)$ and $V = \bigoplus_{i=1}^{k} \mathbb{U}_i \otimes \mathbb{U}_{i+1}^*$, where $\mathbb{U}_{k+1} = \mathbb{U}_1$.

Assume that $\dim \mathbb{U}_i = n$ for all i. Then $(G : V)$ is a ϑ-group related to an automorphism of order k of $\tilde{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V}) = \mathfrak{gl}_{nk}$, where $\mathbb{V} = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_k$. Namely, if $\varsigma = \sqrt[k]{1}$ and

$$t = \mathsf{diag}(\underbrace{\varsigma^{k-1}, \ldots, \varsigma^{k-1}}_{n}, \ldots, \underbrace{\varsigma, \ldots, \varsigma}_{n}, \underbrace{1, \ldots, 1}_{n}),$$

then $\vartheta = \mathrm{Ad}\,(t)$, $G = \tilde{G}_0$, and $V = \tilde{\mathfrak{g}}_1$. In the matrix form, we have

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{gl}(\mathbb{U}_1) & 0 & \cdots & 0 \\ 0 & \mathfrak{gl}(\mathbb{U}_2) & & \\ \vdots & & \ddots & \\ 0 & & & \mathfrak{gl}(\mathbb{U}_k) \end{pmatrix}, \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 0 & M_1 & \cdots & 0 \\ \cdots & 0 & \ddots & \\ & & \ddots & M_{k-1} \\ M_k & \cdots & & 0 \end{pmatrix}$$

is a typical element of $V = \tilde{\mathfrak{g}}_1$. We also write $\mathcal{M} \doteq (M_1, \ldots, M_k)$. Here $\dim \mathfrak{g} = kn^2 = \dim V$, $\mathsf{g.s.}(\mathfrak{g} : V) = \mathfrak{t}_n$, and $V /\!/ G \simeq \mathbb{A}^n$. The centre of $\tilde{G} = GL(\mathbb{V})$ belongs to G and acts trivially on everything. Therefore, without any harm, we can replace $\tilde{\mathfrak{g}} = \mathfrak{gl}_{nk}$ with $\mathfrak{sl}_{nk}$. But, it is notationally simpler to deal with $\mathfrak{gl}_{nk}$.

If $g_i \in GL(\mathbb{U}_i)$, $\boldsymbol{g} = (g_1, \ldots, g_k) \in G$, and $\mathcal{M} \doteq (M_1, \ldots, M_k)$, then the G-action on V is given by

$$\boldsymbol{g}\cdot\mathcal{M} \doteq (g_1 M_1 g_2^{-1}, g_2 M_2 g_3^{-1}, \ldots, g_k M_k g_1^{-1}).$$

Accordingly, for $\boldsymbol{s} = (s_1, \ldots, s_k) \in \mathfrak{g}$, we have

$$\boldsymbol{s}\cdot\mathcal{M} \doteq (s_1 M_1 - M_1 s_2, \ s_2 M_2 - M_2 s_3, \ldots, \ s_k M_k - M_k s_1). \tag{5.2}$$

Vinberg's theory (Example 4.2) implies that here $\Bbbk[V]^G$ is a polynomial ring and $\mathcal{N}_G(V)$ contains finitely many G-orbits. But in this case, one can explicitly describe the basic invariants and thereby compute $q(V /\!/ G)$. The representation $(G : V)$ is a quiver representation related to the extended Dynkin quiver $\tilde{\mathbf{A}}_{nk-1}$, and the algebra $\Bbbk[V]^G$ is well known. But we prefer an "elementary" invariant-theoretic point of view in our exposition.

Theorem 5.3.1. *The algebra $\Bbbk[V]^G$ is freely generated by the coefficients of the characteristic polynomial of the matrix $M_1 \cdots M_k$ (or any cyclic permutation of this product). In particular, the degrees of the basic invariants are $k, 2k, \ldots, nk$ and $\dim V - q(V /\!/ G) = k\binom{n}{2}$.*

Proof. Using the First Fundamental Theorem of Invariant Theory or the Igusa lemma [22, Theorem 4.12], one readily verifies that the quotient of V by $G' = \prod_{i=2}^{k} GL(\mathbb{U}_i)$ is given by the mapping $\mathcal{M} \mapsto M_1 \cdots M_k \in \mathsf{Mat}_n(\Bbbk)$. Since $g_1 \cdot (M_1 \cdots M_k) = g_1 M_1 \cdots M_k g_1^{-1}$, the induced action of $GL(\mathbb{U}_1) = G/G'$ on $V /\!/ G' \simeq \mathsf{Mat}_n(\Bbbk)$ is equivalent to the adjoint representation. $\square$

Define the morphism $F_i \in \mathrm{Mor}(V, \mathfrak{g})$ by $F_i(\mathcal{M}) = \mathcal{M}^{ki}$ (the usual matrix power in $\mathfrak{gl}_{nk}$).

Theorem 5.3.2. *We have*

(i) *each F_i is G-equivariant, lies in $\mathrm{Ker}(\phi)$, and $\sum_{i=0}^{n-1} \deg F_i = k\binom{n}{2}$;*
(ii) *For $Z = \{\mathcal{M} \in V \mid \bigwedge_{i=0}^{n-1} F_i(\mathcal{M}) = 0\}$, we have $\mathrm{codim}_V Z \geqslant 2$.*

Proof. (i) It is clear from the definition that all F_i are G-equivariant. Next, $\mathcal{M}^k$ is a block-diagonal matrix, where the first block is $M_{[1,k]} := M_1 \cdots M_k$ and the subsequent blocks are cyclic permutations of this product. The equality $F_1(\mathcal{M})\cdot\mathcal{M} = 0$ readily follows from this observation and (5.2). And likewise for F_i ($i \geqslant 2$). The case of $i = 0$ is obvious.

(ii) We have the commutative diagram

If $M_{[1,k]}$ is a G/G'-regular ($=non\text{-}derogatory$) matrix, then $\{F_i(\mathcal{M})\}_{i=0}^{n-1}$ are linearly independent. Let Y denote the variety of all derogatory matrices in $\mathsf{Mat}_n(\Bbbk)$. Then $Z \subset \pi_{G'}^{-1}(Y)$, and it suffices to prove that $\operatorname{codim} \pi_{G'}^{-1}(Y) \geqslant 2$. Consider the matrices $\mathcal{M}(1) \doteq (I_n, \ldots, I_n, A)$ and $\mathcal{M}(2) \doteq (I_n, \ldots, I_n, E, I_n)$, where $A = \mathsf{diag}(a_1, \ldots, a_n)$ with

$a_i \neq a_j$ and $E = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & 0 & \ddots \\ & & \ddots & 1 \\ 0 & \cdots & & 0 \end{pmatrix}$ is a regular nilpotent element of $\mathfrak{gl}_n$. The plane

$\mathcal{P} = \{\alpha\mathcal{M}(1) + \beta\mathcal{M}(2) \mid \alpha, \beta \in \Bbbk\}$ has the property that, for any nonzero $\mathcal{M} \in \mathcal{P}$, the corresponding matrix $M_{[1,k]}$ is non-derogatory. Hence $\mathcal{P} \cap \pi_{G'}^{-1}(Y) = \{0\}$, and we are done. $\square$

Remark. If we work with $\tilde{G} = SL(\mathbb{V})$ in place of $GL(\mathbb{V})$, then a generic stabiliser becomes $\mathfrak{t}_{n-1}$. Here the constant morphism F_0 should be omitted and the matrices $\mathcal{M}^{ki}$, $i \geqslant 1$, should be replaced with their projections to $\mathfrak{sl}(\mathbb{V})$.

Thus, *by Remark 3.9 and Theorem 5.3.2, the proof of Theorems 3.3 and 3.6 can be adjusted to the present case. Therefore, $\operatorname{Ker}(\phi)$ (resp. $\operatorname{Ker}(\phi_G)$) is a free $\Bbbk[V]$-module (resp. $\Bbbk[V]^G$-module) with basis $F_0, F_1, \ldots, F_{n-1}$ and $\mathfrak{q} = (\prod_{i=1}^{k} \mathfrak{gl}(\mathbb{U}_i)) \ltimes (\bigoplus_{i=1}^{k} \mathbb{U}_i^* \otimes \mathbb{U}_{i+1})$ satisfies the Kostant criterion.*

It is worth noting that the special case of the involutive automorphism ϑ (i.e., if $k = 2$) has already been settled in [12, Sect. 5].

6. Constructing covariants for semi-direct products, II

Example 6.1. $G = SL_n = SL(\mathbb{U})$, $V = \varphi_1^2 + \varphi_2^* = \mathsf{S}^2(\mathbb{U}) \oplus \wedge^2(\mathbb{U}^*)$.

We regard V as the space of pairs of matrices: $V = \{(A, B) \mid A^t = A \And B^t = -B\}$, where the action of $g \in G$ is given by

$$g{\cdot}(A, B) = (gAg^t, (g^t)^{-1}Bg^{-1}) \tag{6.1}$$

and the corresponding action of $s \in \mathfrak{g} = \mathfrak{sl}_n$ is

$$s{\cdot}(A, B) = (sA + As^t, -s^tB - Bs). \tag{6.2}$$

In what follows, one has to distinguish the cases of even or odd n. The algebra $\Bbbk[V]^G$ is (bi)graded polynomial and the (bi)degrees of the basic invariants are [1,17]:

$$\begin{cases} (2,2), (4,4), \ldots, (n-2, n-2), (n,0), (0, n/2), & \text{if } n = 2k, \\ (2,2), (4,4), \ldots, (n-1, n-1), (n,0) & \text{if } n = 2k+1. \end{cases}$$ Here the invariant

of degree $(n,0)$ is $\det A$, and the invariant of degree $(0, n/2)$ is $\operatorname{Pf} B$. While the invariants of degree $(2i, 2i)$ are just $\operatorname{tr}(AB)^{2i}$, $2i < n$.

A generic isotropy group is $H \simeq \mathbb{T}_{[n/2]}$ [2, Table 2]. For instance, one can take

$$H = \begin{cases} \mathsf{diag}(t_1, \ldots, t_k, t_1^{-1}, \ldots, t_k^{-1}) \mid t_i \in \Bbbk^{\times}\}, & \text{if } n = 2k \\ \mathsf{diag}(t_1, \ldots, t_k, 1, t_1^{-1}, \ldots, t_k^{-1}) \mid t_i \in \Bbbk^{\times}\}, & \text{if } n = 2k + 1. \end{cases}$$

We have to construct $[n/2]$ morphisms $\{F_i\}$ in $\mathrm{Ker}(\phi)$. To begin with, take $F_1(A, B) = AB$. Since $(AB)^t = -BA$, we have $\mathrm{tr}\,(AB) = 0$, and it follows from (6.1) that $g{\cdot}AB = g(AB)g^{-1}$. Hence $F_1 \in \mathrm{Mor}_G(V, \mathfrak{g})$. We continue by letting $F_i(A, B) = (AB)^{2i-1}$, $i = 1, 2, \ldots, [n/2]$. To ensure that the resulting matrix is traceless, we must consider only the odd powers of AB. Using (6.2), one verifies that $F_i(A, B){\cdot}(A, B) = 0$, hence $F_i \in \mathrm{Ker}(\phi)$. The corresponding Q-invariants in $\Bbbk[\mathfrak{q}^*]$ are $\hat{F}_i(\xi, A, B) = \mathrm{tr}\,(\xi(AB)^{2i-1})$. Let $I_k(\bar{d})$ denote the diagonal k by k matrix with diagonal entries $\bar{d} = (d_1, \ldots, d_k)$. Taking $A = \begin{pmatrix} 0_k & I_k(\bar{d}) \\ I_k(\bar{d}) & 0_k \end{pmatrix}$ and $B = \begin{pmatrix} 0_k & I_k(\bar{c}) \\ -I_k(\bar{c}) & 0_k \end{pmatrix}$ shows that the matrices $(AB)^{2i-1}$, $1 \leqslant i \leqslant k$, are linearly independent whenever the elements $\{c_j d_j\}$ are different. Hence $F_1, \ldots, F_k$ are linearly independent for $n = 2k$. This construction can easily be adjusted to $n = 2k + 1$.

Having the degrees of all basic invariants and covariants, one verifies that $\sum_{i=1}^{[n/2]} \deg F_i + q(V/\!\!/G) = \dim V$ if n is odd; while for n even one obtains $\sum_{i=1}^{[n/2]} \deg F_i = \dim V - q(V/\!\!/G) + [n/2]$. Since the **C·2·C** holds here (Example 4.6(2^0)), we have

if n is odd, then the assumptions of Theorems 3.3 and 3.6 are satisfied. Therefore, $\mathrm{Ker}(\phi)$ (resp. $\mathrm{Ker}(\phi_G)$) is a free $\Bbbk[V]$-module (resp. $\Bbbk[V]^G$-module) with basis $F_1, F_2, \ldots, F_{[n/2]}$ and $\mathfrak{q} := \mathfrak{sl}_n \ltimes (\varphi_1^2 + \varphi_2^*)^*$ satisfies the Kostant criterion.

If n is even, then the same conclusion is still true, but one have to modify the constructed covariants $\{F_i\}$ in order to obtain a new family such that Equality (3.1) to be satisfied. This will appear in a forthcoming paper.

Example 6.2. If we slightly change V of Example 6.2, i.e., take $G = SL_n$ and $\tilde{V} = \varphi_1^2 + \varphi_2 = \mathcal{S}^2(\mathbb{U}) \oplus \wedge^2(\mathbb{U})$, then the action $(G : \tilde{V})$ has similar properties. Namely, $\Bbbk[\tilde{V}]^G$ is polynomial [1,17] and **g.i.g.**$(G : \tilde{V}) = \mathbb{T}_{[n/2]}$ [2, Table 2]. However, the construction of covariants in $\mathrm{Ker}(\phi)$ becomes totally different and more involved. We regard $\tilde{V}$ as the space of pairs of matrices: $\tilde{V} = \{(A, B) \mid A^t = A \ \& \ B^t = -B\}$, where the action of $g \in G$ is given by

$$g{\cdot}(A, B) = (gAg^t, gBg^t) \tag{6.3}$$

and the corresponding action of $s \in \mathfrak{g} = \mathfrak{sl}_n$ is

$$s{\cdot}(A, B) = (sA + As^t, sB + Bs^t). \tag{6.4}$$

Consider the "characteristic polynomial"

$$\mathcal{F}(\lambda) = \det(A + \lambda B) = \sum_{i=0}^{n} f_i(A, B)\lambda^i.$$

Since $(A + \lambda B)^t = A - \lambda B$, we have $\mathcal{F}(\lambda) = \mathcal{F}(-\lambda)$, i.e., $\mathcal{F}(\lambda) = P(\lambda^2)$ and $f_i(A, B) \equiv 0$ unless i is even. If n is odd, then $\Bbbk[\tilde{V}]^G$ is freely generated by the f_{2i}'s. For n even, the only correction is that $f_n(A, B) = \det B$ should be replaced with $\mathrm{Pf}\, B$ [1,17]. Therefore,

$$\dim \tilde{V} - q(\tilde{V} /\!\!/ G) = \begin{cases} 2k^2 - k, & \text{if } n = 2k \\ 2k^2 + k, & \text{if } n = 2k + 1 \end{cases}.$$

We provide below a construction of the required covariants in $\mathrm{Ker}(\phi)$. As in Example 5.1, let A^* be the adjugate of A. Consider the morphism $F : \tilde{V} \to \mathfrak{gl}_n$, $F(A, B) = BA^*$.

Lemma 6.2.1. *We have*

(a) $\mathrm{tr}\,(BA^*) = 0$, *i.e.,* $F(A, B) \in \mathfrak{g} = \mathfrak{sl}_n$;
(b) F *is* G-*equivariant;*
(c) $F \in \mathrm{Ker}(\phi)$.

Proof. (a) Since $A^t = A$, we have $(A^*)^t = A^*$. Hence $(BA^*)^t = -A^*B$.

(b) By definition, $F(g{\cdot}(A, B)) = gBg^t\,(gAg^t)^*$. If $\det A \neq 0$, then the RHS equals

$$\det(gAg^t){\cdot}gBg^t{\cdot}(gAg^t)^{-1} = \det A{\cdot}gBA^{-1}g^{-1} = gBA^*g^{-1}.$$

Hence F is a G-equivariant mapping from $\tilde{V}$ to $\mathfrak{g} = \mathfrak{sl}_n$ on the dense open subset of $\tilde{V}$, where A is invertible. Since F is a polynomial morphism, this holds on the whole of $\tilde{V}$.

(c) We have $F(A, B){\cdot}(A, B) = (BA^*A + A(BA^*)^t, BA^*B + B(BA^*)^t) =$
$$(BA^*A - AA^*B, BA^*B - BA^*B) = 0. \quad \square$$

Having constructed one suitable covariant, we perform a "polarisation". Consider

$$\mathbb{H}_\lambda(A, B) := F(A + \lambda B, B) = B(A + \lambda B)^* = \sum_{i=0}^{n-1} F_i(A, B)\lambda^i.$$

Clearly $F_0(A, B) = F(A, B)$ and if n is odd, then $\det B = 0$ and the coefficient of λ^{n-1} equals $BB^* = (\det B)I = 0$.

Theorem 6.2.2. *We have*

(a) $\mathrm{tr}\, F_{2i}(A, B) = 0$ *for all* i;
(b) $\mathbb{H}_\lambda$ *is a* G-*equivariant mapping from* $\tilde{V}$ *to* $\mathfrak{gl}_n$. *In particular,* $F_{2i} \in \mathrm{Mor}_G(\tilde{V}, \mathfrak{g})$ *for all* i;
(c) $F_{2i} \in \mathrm{Ker}(\phi)$ *for all* i.

Proof. (a) If both A and $A + \lambda B$ are invertible, then

$$B(A + \lambda B)^* = \det(A + \lambda B) \cdot B(A + \lambda B)^{-1}$$

$$= \det(A + \lambda B) \cdot B\big(A(I + \lambda A^{-1}B)\big)^{-1} = \det(A + \lambda B) \cdot B(I + \lambda A^{-1}B)^{-1}A^{-1} =$$

$$\det(A + \lambda B) \cdot \big(BA^{-1} + \lambda(BA^{-1})^2 + \lambda^2(BA^{-1})^3 + \dots\big)$$

Since A is symmetric, so is A^{-1} and therefore $(BA^{-1})^{2i+1}$ is a product of a symmetric and a skew-symmetric matrices. Hence $\operatorname{tr}(BA^{-1})^{2i+1} = 0$. As $\det(A + \lambda B) = P(\lambda^2)$, the total coefficient of λ^{2i} is a traceless matrix. Since this is true for a dense open subset of triples (A, B, λ) such that A and $A + \lambda B$ are invertible, and $\mathbb{H}_\lambda$ is a polynomial mapping, this holds for all triples.

(b) The proof is the same as in Lemma 6.2.1(b).

(c) We prove that $\mathbb{H}_\lambda^{even} := \sum_i F_{2i}\lambda^{2i} \in \operatorname{Ker}(\phi)$ for all λ. Equivalently, only odd powers of λ survive in $\mathbb{H}_\lambda(A, B) \cdot (A, B)$. By definition,

$$\mathbb{H}_\lambda(A, B) \cdot (A, B) = (B(A + \lambda B)^* A + A(B(A + \lambda B)^*)^t, B(A + \lambda B)^* B + B(B(A + \lambda B)^*)^t).$$

Let us transform the first component in the RHS. Again, assuming first that A and $A + \lambda B$ are invertible, one obtains:

$$(\mathbf{F1}) = B(A + \lambda B)^* A = \det(A + \lambda B) B(A + \lambda B)^{-1} A$$

$$= \det(A + \lambda B) B\big(A(I + \lambda A^{-1}B)\big)^{-1} A = \det(A + \lambda B) B(I + \lambda A^{-1}B)^{-1}$$

$$= \det(A + \lambda B)(B - \lambda(BA^{-1}B) + \lambda^2(BA^{-1}BA^{-1}B) - \dots)$$

and

$$(\mathbf{F2}) = A(B(A + \lambda B)^*)^t = -\det(A + \lambda B)(A(A - \lambda B)^{-1}B)$$

$$= -\det(A + \lambda B)A\big((I - \lambda BA^{-1})A\big)^{-1}B = -\det(A + \lambda B)(I - \lambda BA^{-1})^{-1}B$$

$$= -\det(A + \lambda B)(B + \lambda(BA^{-1}B) + \lambda^2(BA^{-1}BA^{-1}B) + \dots)$$

Because $\det(A + \lambda B) = P(\lambda^2)$, the sum $(\mathbf{F1}) + (\mathbf{F2})$ contains only odd powers of λ. Again, using the polynomiality of $\mathbb{H}_\lambda$, we conclude that this property holds for all A, B, λ.

The argument for the second component is similar. $\square$

Thus, we have constructed $[n/2]$ covariants F_{2i} ($0 \leqslant 2i \leqslant n - 2$) in $\operatorname{Ker}(\phi)$. These covariants are linearly independent, because their bi-degrees are different. Since $\deg F_{2i} = n$ for all i, we have $\sum_{i=0}^{[n/2]-1} \deg F_{2i} - \dim V + q(V /\!/ G) = 0$ if n is odd (and $= [n/2]$ if n is even). Since the $\mathbf{C} \cdot 2 \cdot \mathbf{C}$ holds here (Example 4.6(1°)), we conclude that

if n is odd, then the assumptions of Theorems 3.3 and 3.6 are satisfied. Therefore, $\operatorname{Ker}(\phi)$ (resp. $\operatorname{Ker}(\phi_G)$) is a free $\Bbbk[V]$-module (resp. $\Bbbk[V]^G$-module) with basis $F_0, F_2, \dots, F_{2[n/2]-2}$ and $\mathfrak{q} := \mathfrak{sl}_n \ltimes (\varphi_1^2 + \varphi_2)^*$ satisfies the Kostant criterion.

If n is even, then the same conclusion is still true, but one have to modify the constructed covariants $\{F_i\}$ in order to obtain a new family such that Equality (3.1) to be satisfied. This will appear in a forthcoming paper.

Example 6.3. $G = Sp(\mathbb{U}) \times SO(\mathbb{V})$, $V = \mathbb{U} \otimes \mathbb{V}$.

This representation is a ϑ-group associated with an outer automorphism of order 4 of $\mathfrak{sl}(\mathbb{U} \oplus \mathbb{V})$. Therefore $\Bbbk[V]^G$ is polynomial and $\mathcal{N}_G(V)$ contains finitely many G-orbits, cf. Example 4.2. Furthermore, a generic stabiliser is reductive if and only if either $\dim \mathbb{U} \geqslant \dim \mathbb{V}$ or $\dim \mathbb{V} - \dim \mathbb{U} \in 2\mathbb{N}$. In these cases, the action is stable and hence **C·2·C** holds.

Set $2m = \dim \mathbb{U}$ and $n = \dim \mathbb{V}$. Let $\mathcal{I}$ (resp. $\mathcal{J}$) be a symmetric (resp. skew-symmetric) matrix of order n (resp. $2m$) such that $\mathcal{I}^2 = I$ (resp. $\mathcal{J}^2 = -I$). We regard $SO(\mathbb{V})$ (resp. $Sp(\mathbb{U})$) as the group that preserves the bilinear form with matrix $\mathcal{I}$ (resp. $\mathcal{J}$). Then

$$
\begin{aligned}
\mathfrak{sp}(\mathbb{U}) &= \mathfrak{sp}_{2m} = \{X \in \mathsf{Mat}_{2m} \mid X^t \mathcal{J} + \mathcal{J}X = 0\} = \{X \mid (\mathcal{J}X)^t = \mathcal{J}X\}; \\
\mathfrak{so}(\mathbb{V}) &= \mathfrak{so}_n = \{Y \in \mathsf{Mat}_n \mid Y^t \mathcal{I} + \mathcal{I}Y = 0\} = \{Y \mid (\mathcal{I}Y)^t = -\mathcal{I}Y\}.
\end{aligned}
\tag{6.5}
$$

We identify $\mathbb{U} \otimes \mathbb{V}$ with the space of $2m$ by n matrices, where the action of $\mathfrak{sp}(\mathbb{U}) \times \mathfrak{so}(\mathbb{V})$ is given by $(s_1, s_2) \cdot M = s_1 M - M s_2$. Here a generic isotropy group is a torus if and only if $0 \leqslant \dim \mathbb{V} - \dim \mathbb{U} \leqslant 2$ (more precisely, if $n \geqslant 2m$, then $\mathsf{g.i.g.}(G : V) \simeq SO_{n-2m} \times \mathbb{T}_m$). The corresponding possibilities are considered below. (Whenever it is convenient, we may assume that $\mathcal{I} = I$; and then $\mathfrak{so}(\mathbb{V})$ consists of the usual skew-symmetric matrices.)

(i) <u>Assume that $\dim \mathbb{U} = \dim \mathbb{V} = 2m$.</u> Here $\mathsf{g.i.g.}(G : V) = \mathbb{T}_m$ and this torus is embedded diagonally in $Sp(\mathbb{U}) \times SO(\mathbb{V})$. The degrees of the basic invariants of $\Bbbk[V]^G$ are $4, 8, \ldots, 4m - 4, 2m$ [7]. Hence $\dim V - q(V /\!\!/ G) = 4m^2 - 2m - 2m(m-1) = 2m^2$.

Define the covariant $F_1 : \mathbb{U} \otimes \mathbb{V} \to \mathsf{Mat}_n \times \mathsf{Mat}_n$ by $F_1(M) = (M\mathcal{I}M^t \mathcal{J}, \mathcal{I}M^t \mathcal{J}M)$. Using Eq. (6.5), one verifies that $F_1(M) \in \mathfrak{g} = \mathfrak{sp}(\mathbb{U}) \times \mathfrak{so}(\mathbb{V})$. Moreover, F_1 is G-equivariant, and $F_1(M) \cdot M = 0$, i.e., $F_1 \in \mathrm{Ker}(\phi)$. If a matrix R is either symplectic or orthogonal, then so is R^{2i-1} for any i. Therefore, the covariants

$$
F_i(M) = F_1(M)^{2i-1} = \left((M\mathcal{I}M^t \mathcal{J})^{2i-1}, (\mathcal{I}M^t \mathcal{J}M)^{2i-1} \right).
$$

are well-defined. Moreover, $F_1, \ldots, F_m$ are linearly independent. (Assuming for simplicity that $\mathcal{I} = I$, one easily verifies that $F_1(D), \ldots, F_m(D)$ are linearly independent for a generic diagonal matrix D.) Here $\deg F_i = 2(2i-1)$. Hence $\sum_{i=1}^m \deg F_i = 2m^2$, so that (3.1) holds. Thus, Theorems 3.3 and 3.6 apply here.

(ii) <u>Assume that $\dim \mathbb{V} = 2m + 1 = \dim \mathbb{U} + 1$.</u> Here again $\mathsf{g.i.g.}(G : V) = \mathbb{T}_m$ and this torus is embedded diagonally in $Sp(\mathbb{U}) \times SO(\mathbb{V})$. The degrees of the basic invariants of $\Bbbk[V]^G$ are $4, 8, \ldots, 4m$ [7]. Hence $\dim V - q(V /\!\!/ G) = 2m(2m+1) - 2m(m+1) = 2m^2$, as in (i). The formulae for F_i, $i = 1, \ldots, m$, also remain the same. Note only that now $\mathcal{I}$ and $\mathcal{J}$ have different order and therefore the matrices $M\mathcal{I}M^t \mathcal{J}$ and $\mathcal{I}M^t \mathcal{J}M$ are of order $2m$ and $2m + 1$, respectively.

(iii) <u>Assume that $\dim \mathbb{V} = 2m + 2 = \dim \mathbb{U} + 2$.</u> Here $\mathsf{g.i.g.}(G : V) = \mathbb{T}_{m+1}$, but only an m-dimensional subtorus is embedded diagonally in G, whereas a complemen-

tary 1-dimensional torus belongs to $SO(\mathbb{V})$. (This is not surprising, since $\operatorname{rk} Sp(\mathbb{U}) = m$.) The degrees of the basic invariants of $\Bbbk[V]^G$ are $4, 8, \dots, 4m$, as in (ii). Hence $\dim V - q(V /\!\!/ G) = 2m(2m+2) - m(2m+2) = 2m^2 + 2m$.

As in (i), we construct the linearly independent covariants $F_1, \dots, F_m$ with $\sum_{i=1}^{m} \deg F_i = 2m^2$, but this is not sufficient now. These m covariants take a generic G-regular element $M \in V$ to the diagonally embedded m-dimensional torus in the stabiliser $\mathfrak{g}_M \simeq \mathfrak{t}_{m+1}$. We need one more covariant (of degree $2m$) that takes M to a 1-dimensional subtorus sitting only in $\mathfrak{so}(\mathbb{V})$. In other words, starting with a $2m$ by $2m+2$ matrix M, we wish to construct, in a natural way, a skew-symmetric matrix of order $2m+2$. Here is the solution: Let M_{ij} be the square matrix of order $2m$ obtained by deleting the i-th and j-th columns from M, $1 \leqslant i < j \leqslant 2m+2$. We then set

$$a_{ij} = \begin{cases} (-1)^{i+j} \det M_{ij}, & \text{if } i < j \\ 0, & \text{if } i = j \\ -a_{ji}, & \text{if } i > j. \end{cases} \text{. Clearly, } A_M = (a_{ij}) \text{ is a skew-symmetric matrix}$$

of order $2m+2$, and we define $F_{m+1}(M) = (0, A_M) \in \mathfrak{sp}(\mathbb{U}) \times \mathfrak{so}(\mathbb{V})$. It is easily seen that F_{m+1} is equivariant, $\deg F_{m+1} = 2m$, and $F_{m+1}(M){\cdot}M = (0, -M A_M) = 0$. Thus,

if $0 \leqslant \dim \mathbb{V} - \dim \mathbb{U} \leqslant 2$, then the assumptions of [Theorems 3.3 and 3.6](#) are satisfied. Therefore, $\operatorname{Ker}(\phi)$ (resp. $\operatorname{Ker}(\phi_G)$) is a free $\Bbbk[V]$-module (resp. $\Bbbk[V]^G$-module) with basis $\{F_i\}$ and $\mathfrak{q} = (\mathfrak{sp}(\mathbb{U}) \times \mathfrak{so}(\mathbb{V})) \ltimes (\mathbb{U} \otimes \mathbb{V})^*$ satisfies the Kostant criterion.

Example 6.4. $G = SO(\mathbb{V}) = SO_{n+2}$ and $V = n\mathbb{V}$, the sum of n copies of the defining representation of SO_{n+2}.

Here $\mathsf{g.i.g.}(G : V) = SO_2 \simeq \mathbb{T}_1$, $V /\!\!/ G \simeq \mathbb{A}^{(n+1)n/2}$, and $q(V /\!\!/ G) = (n+1)n$. The explicit construction of the unique covariant of degree $\dim V - q(V /\!\!/ G) = n$ in $\operatorname{Ker}(\phi)$ is similar to the construction of F_{m+1} in [Example 6.3](#)(iii). We regard an element of V as $n+2$ by n matrix M and consider its minors of order n, $\det M_{ij}$, where $1 \leqslant i < j \leqslant n+2$. Then $F(M) = (a_{ij})$, where $a_{ij} = (-1)^{i+j} \det M_{ij}$ for $i < j$, etc.

Appendix A. Tables of representations with toral generic stabilisers

Using classification results of Elashvili [2,3], one can write down the arbitrary representations of simple algebraic groups or the irreducible representations of semisimple groups whose generic stabiliser is toral. The subsequent four tables include **all** such representations. But their content is not disjoint. Recall that $\mathfrak{q} = \mathfrak{g} \ltimes V^*$ and we are interested in the symmetric invariants of $\mathfrak{q}$.

In [Table 1](#), we gather all representations with 1-dimensional generic stabiliser. The column (FA) (resp. (Eq)) refers to the presence of the property that $\Bbbk[V]^G$ is a polynomial ring (resp. $\pi_{V,G} : V \to V /\!\!/ G$ is equidimensional). This information is inferred from tables in [1,7,17,18]. Results of Section [4](#) imply that **C**·**2**·**C** holds for all these representations. By [Theorem 3.11](#), $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ is a polynomial ring for all items of this table. Furthermore, if (FA) holds, then $\Bbbk[\mathfrak{q}^*]^Q$ is a polynomial ring. Finally, if (Eq) holds, then $\mathcal{U}(\mathfrak{q})$ is a free module over $\mathcal{Z}(\mathfrak{q})$.

Table 1
Representations $(G : V)$ with $\mathsf{g.s.}(\mathfrak{g} : V) = \mathfrak{t}_1$.

№	G	V	$\dim V$	$\dim V /\!/ G$	$q(V /\!/ G)$	(FA)	(Eq)
1a	$SO_{n+2} \times SO_n$	$\varphi_1 \varphi_1'$	$n(n+2)$	n	$n(n+1)$	*yes*	*yes*
1b	SO_{n+2}	$n\varphi_1$	$n(n+2)$	$\binom{n+1}{2}$	$n(n+1)$	*yes*	*yes*, if $n \leqslant 3$
2	SL_6	$2\varphi_2 + \varphi_2^*$	45	11	36	*yes*	*no*
3a	$SL_6 \times SL_3$	$\varphi_2 \varphi_1'$	45	3	36	*yes*	*yes*
3b	$SL_6 \times SL_2$	$\varphi_2 \varphi'^{\,2}$	45	8	36	*no*	*no*
3c	SL_6	$3\varphi_2$	45	11	36	*yes*	*no*
4a	$Sp_6 \times SL_2$	$\varphi_2 \varphi'$	28	5	22	*no*	*no*
4b	Sp_6	$2\varphi_2$	28	8	22	*yes*	*yes*
5	$SL_4 \times SL_2$	$\varphi_2 \varphi'^{\,3}$	24	7	20	*no*	*no*

Remarks.

1) In № 1(a,b), we have $V /\!/ SO_{n+2} \simeq S^2\varphi_1'$ as SO_n-module.
2) In № 3(a,b,c), we have $V /\!/ SL_6 \simeq \mathsf{R}_6 + \mathsf{R}_2 + \mathsf{R}_0$ as SL_2-module.
3) In № 4(a,b), we have $V /\!/ Sp_6 \simeq \mathsf{R}_3 + \mathsf{R}_2 + \mathsf{R}_0$ as SL_2-module.
4) In № 5, we have $V /\!/ SL_4 \simeq \wedge^2(\mathsf{R}_3) = \mathsf{R}_6 + \mathsf{R}_2$ as SL_2-module.

This explains why $\Bbbk[V]^G$ is not polynomial in 3b, 4a, 5.

In Table 2, we gather all representations with a toral generic stabiliser that are ϑ-groups in the sense of Vinberg (Example 4.2) and related restrictions. Namely, items 6a, 7a, and 8a (which are not ϑ-groups!) are obtained from the genuine ϑ-groups (items 6,7,8) by omitting the second factor of G_0 and we say that these are "restrictions" of ϑ-groups. It appears that this passage does not change generic isotropy groups, which are always contained in the first factor of G_0. Moreover, the number $q(\mathfrak{g}_1 /\!/ G_0)$ is not affected, too. Therefore the covariants $\{F_i\}$ produced for these ϑ-groups, as described in the respective examples, are also suitable for their "restrictions". The symbol $(\mathbf{X}_n^{(k)}, m)$ in column "ϑ" represents the following information on the automorphism ϑ of $\mathfrak{g}$. Here $\mathbf{X}_n$ is the Cartan type of $\mathfrak{g}$, m is the order of ϑ, and k is the minimal integer such that ϑ^k is inner (this number is omitted if it equals 1).
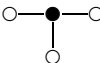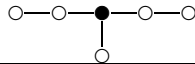
The theory of ϑ-groups implies that, for all items of Tables 2 and 3, $\Bbbk[V]^G$ is a polynomial ring. Our description of the corresponding covariants shows that, for all items except №5 in Table 2 and №2 in Table 3, $\Bbbk[\mathfrak{q}^*]^Q$ is a polynomial ring, the Kostant criterion is always satisfied for $\mathfrak{q}$, and $\mathrm{Ker}(\phi)$ is a free $\Bbbk[V]$-module generated by G-equivariant morphisms. The explicit construction of covariants $F_i \in \mathrm{Ker}(\phi)$ is given in the examples mentioned in the column "Ref."

The column (FA) in Table 4 refers to the presence of the property that $\Bbbk[V]^G$ is a polynomial ring. For items 1, 2, 3a, $\Bbbk[\mathfrak{q}^*]^Q$ is a polynomial ring, while in case 3, only $\Bbbk[\mathfrak{q}^*]^{R_u(Q)}$ is a polynomial ring. (A more precise information can be found in the respective examples.)

Table 2
ϑ-groups with toral generic stabiliser and their "restrictions".

№	G_0	$\mathfrak{g}_1$	$\mathfrak{h}$	ϑ	Ref.
1	$SO_{2m} \times Sp_{2m}$	$\varphi_1 \varphi_1'$	$\mathfrak{t}_m$	$(\mathbf{A}_{4m-1}^{(2)}, 4)$	Example 6.3(i)
2	$SO_{2m+1} \times Sp_{2m}$	$\varphi_1 \varphi_1'$	$\mathfrak{t}_m$	$(\mathbf{A}_{4m}^{(2)}, 4)$	Example 6.3(ii)
3	$SO_{2m+2} \times Sp_{2m}$	$\varphi_1 \varphi_1'$	$\mathfrak{t}_{m+1}$	$(\mathbf{A}_{4m+1}^{(2)}, 4)$	Example 6.3(iii)
4	$(SL_n)^k \times \mathbb{T}_{k-1}$	$\sum\limits_{i=1}^{n} \varphi_1^{(i)} \varphi_{n-1}^{(i+1)}$	$\mathfrak{t}_{n-1}$	$(\mathbf{A}_{kn-1}, k)$	Example 5.3
5	$SL_4 \times SL_4 \times SL_2$	$\varphi_1 \varphi_1' \varphi''$	$\mathfrak{t}_3$	$(\mathbf{E}_7, 4)$	Example 5.1
6	$SL_6 \times SL_3$	$\varphi_2 \varphi_1'$	$\mathfrak{t}_1$	$(\mathbf{E}_7, 3)$	Example 4.10
6a	SL_6	$3\varphi_2$	$\mathfrak{t}_1$	–	Example 4.10
7	$SL_6 \times SL_2$	$\varphi_3 \varphi'$	$\mathfrak{t}_2$	$(\mathbf{E}_6, 2)$	[12, Sect. 5]
7a	SL_6	$2\varphi_3$	$\mathfrak{t}_2$	–	
8	$SO_{n+2} \times SO_n$	$\varphi_1 \varphi_1'$	$\mathfrak{t}_1$	$(\mathbf{D}_{n+1}, 2)$	[12, Sect. 5]
8a	SO_{n+2}	$n\varphi_1$	$\mathfrak{t}_1$	–	Example 6.4

Table 3
Reduced ϑ-groups with toral generic stabilisers.

№	$G(0)'$	$\mathfrak{g}(1)$	$\mathfrak{h}$	$\mathbb{Z}$-grading of $\mathfrak{g}$	Ref.
1	$SL_2 \times SL_2 \times SL_2$	$\varphi \varphi' \varphi''$	$\mathfrak{t}_2$	$(\mathbf{D}_4, \alpha_2)$	Example 5.2
2	$SL_3 \times SL_3 \times SL_2$	$\varphi_1 \varphi_1' \varphi''$	$\mathfrak{t}_2$	$(\mathbf{E}_6, \alpha_3)$	Example 5.1

Table 4
The remaining representations with toral generic stabiliser.

№	G	V	$\mathfrak{h}$	Rem.	(FA)	Ref.
1	SL_n	$\varphi_1^2 + \varphi_2^*$	$\mathfrak{t}_{[n/2]}$	$n \geqslant 4$	yes	Example 6.1
2	SL_n	$\varphi_1^2 + \varphi_2$	$\mathfrak{t}_{[n/2]}$	$n \geqslant 4$	yes	Example 6.2
3	$SL_n \times SL_n \times SL_2$	$\varphi_1 \varphi_1' \varphi''$	$\mathfrak{t}_{n-1}$	$n \geqslant 5$	no	Example 5.1
3a	$SL_n \times SL_n$	$\varphi_1 \varphi_1' + \varphi_1 \varphi_1'$	$\mathfrak{t}_{n-1}$	$n \geqslant 3$	yes	Example 5.1
4	SL_8	$\varphi_3 + \varphi_7$	$\mathfrak{t}_2$	–	yes	
5	SL_8	$\varphi_3 + \varphi_1$	$\mathfrak{t}_2$	–	yes	
6	$Sp_4 \times SO_7$	$\varphi_1 \varphi_3'$	$\mathfrak{t}_2$	–	yes	

We do not know whether it is possible to construct covariants $F_1, F_2 \in \mathrm{Ker}(\phi)$ for items 4–6 of Table 4 such that $\deg F_1 + \deg F_2 = \dim V - q(V /\!\!/ G)$ and whether the $\Bbbk[V]$-module $\mathrm{Ker}(\phi)$ is free or $\Bbbk[\mathfrak{q}^*]^Q$ is a polynomial ring in these cases. Nevertheless, using Theorem 2.8 in [24], Remark 3.2, and the fact that $H = \mathsf{g.i.g.}(G : V) \simeq \mathbb{T}_2$ is connected, one can prove that there do exist certain linearly independent G-equivariant morphisms $F_1, F_2 \in \mathrm{Ker}(\phi)$. However, this existence assertion says nothing about their degrees.

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