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Maximal subgroups of $SL_n(D)$

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ABSTRACT

Given a non-commutative F-central division ring D, N a subnormal subgroup of $GL_n(D)$ and M a non-abelian maximal subgroup of N, if M is a non-abelian soluble maximal subgroup of N, then, $n = 1$ and D is cyclic of prime degree p with a maximal cyclic subfield K/F such that the groups $Gal(K/F)$ and $M/(K^* \cap M)$ are isomorphic. Furthermore, for any $x \in M \setminus K^*$, we have $x^p \in F^*$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.

1. Introduction

Throughout this paper, all division rings are non-commutative. Let D be an F-central division ring with multiplicative group D^* and take the unit group $GL_n(D)$ of the full $n \times n$ matrix ring $M_n(D)$. The subgroup structure of $GL_n(D)$ has attracted the attention of several researchers. See for example, [1], [4], [7], [9], [11], [12], [15], [23], [25], [31], [32] and [33]. A good reference for the most important results concerning the subgroups of

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this unit group, namely the skew linear groups, can be found in [14] and [31], as well as [32] particularly for soluble skew linear groups.

The problem of whether the multiplicative group of D contains non-cyclic free subgroups seems to be posed first by Lichtman in [22]. In [11] and [12] stronger versions of this problem have been investigated which deal with the existence of non-cyclic free subgroups in normal or subnormal subgroups of $GL_1(D)$. Also, the question on the existence of non-cyclic free subgroups in linear groups over a field was studied by Tits in [33] which asserts that every subgroup of the general linear group over F either contains a non-cyclic free subgroup or is soluble-by-finite (assuming finitely generated if $CharF > 0$). This result of Tits is now referred as the Tits Alternative. Lichtman in [22] showed that there exists a finitely generated group which is not soluble-by-finite and does not contain a non-cyclic free subgroup.

As showed in [2], there is a similarity between the group theory properties of normal or subnormal subgroups of $GL_1(D)$ and the maximal ones. So it is natural to ask if there exists a non-cyclic free subgroup in a maximal subgroup of $GL_n(D)$ or $SL_n(D)$ in general. The structure of maximal subgroups of $GL_n(D)$ and $SL_n(D)$ is studied in several articles. See for example, [1], [4], [5], [7], [9], [10], [15], [21], [23] and [25].

The following theorem was recently proved in [9].

Theorem. *Let D be a finite dimensional division ring over its center F. Assume that M is a non-abelian maximal subgroup of $GL_n(D)$. Then, either M contains a non-cyclic free subgroup or there exists a unique maximal subfield K of $M_n(D)$ such that $N_{GL_n(D)}(K^*) = M$, $K^* \lhd M$, K/F is Galois with $Gal(K/F) \cong M/K^*$, and $Gal(K/F)$ is a finite simple group and $F[M] = M_n(D)$.*

In addition, in this direction the following theorem was proved in [18].

Theorem. *Let D be a finite dimensional division ring over its center F. Suppose that N be a non-central normal subgroup of $GL_n(D)$ with $n \geqslant 1$. Given a maximal subgroup M of N, then either M contains a non-cyclic free subgroup or there exists an abelian subgroup A such that M/A is locally finite (finite if $CharF = 0$).*

The structure of nilpotent maximal subgroups of $SL_n(D)$ is studied in [25] and [26]. It is shown that the nilpotent maximal subgroups of $SL_n(D)$ are abelian.

The purpose of this article is to prove that for a given F-central division ring D, N a subnormal subgroup of $GL_n(D)$ and M a non-abelian maximal subgroup of N, if M is a non-abelian soluble maximal subgroup of N, then, $n = 1$ and D is cyclic of prime degree p with a maximal cyclic subfield K/F such that the groups $Gal(K/F)$ and $M/(K^* \cap N)$ are isomorphic. Furthermore, for any $x \in M \setminus K^*$, we have $x^p \in F^*$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.

It should be pointed out that when $n > 1$ every subnormal subgroup of $GL_n(D)$ contains $SL_n(D)$ and is thus normal ([6, p. 138]).

2. Notations and conventions

We recall here some of the notations that we will need throughout this article. Given a subset S and a subring K of a ring R, the subring generated by K and S is denoted by $K[S]$. The unit group of R is written as R^*. For a group G and subset $S \subset G$, $Z(G)$ and $C_G(S)$ are the center and the centralizer of S in G, and the same notations are applied for R. $N_G(S)$ is used for the normalizer of S in G and G' for the derived subgroup.

Let R be a ring, S a subring of R and G a group of units of R normalizing S such that $R = S[G]$. Suppose that $N = S \cap G$ is a normal subgroup of G and $R = \bigoplus_{t \in T} tS$, where T is some transversal of N to G. Set $E = G/N$. Then, we say that R is a crossed product of S by E and we denote it by (R, S, G, E).

A field of positive characteristic is called locally finite if every finite subset of the field is contained in a finite subfield.

Given a division ring D with center F and subgroup G of $GL_n(D)$, the space of column n-vectors $V = D^n$ over D is a G-D bimodule. G is called irreducible (resp. completely reducible, reducible) if V is irreducible (resp. completely reducible, reducible) as G-D bimodule. Furthermore, G is absolutely irreducible if $F[G] = M_n(D)$.

An irreducible group G is said to be imprimitive if for some integer $m \geq 2$, there exist subspaces $V_1, \cdots, V_m$ of V such that $V = \bigoplus_{i=1}^m V_i$ and for any $g \in G$ the mapping $V_i \to gV_i$ is a permutation of the set $\{V_1, \cdots, V_m\}$; otherwise G is called primitive.

Here, we define the wreath product of a skew linear group and a permutation group. (See [32, pp. 106–109])

Let U be a linear space over a division ring D, G_1 a subgroup of $GL(U)$, and Γ a subgroup of a symmetric group S_k on $\{1, \ldots, k\}, k > 1$. The Cartesian product $U^k = V_1$ can be regarded as a linear space over D, and we write any vector $v \in V_1$ in the form $v = (u_1, \ldots, u_k)$, $u_j \in U$. For any $f_1, \ldots, f_k \in G_1$ and $s \in \Gamma$, we define a mapping $f : V_1 \to V_1$, $f = \langle f_1, \ldots, f_k, s\rangle$, by setting

$$f(v) = f(u_1, \ldots, u_k) = \overline{v} \in V_1,$$

where the $s(v)$th component of $\overline{v}$ is $f_v(u_v)$, $v = 1, \ldots, k$. Obviously f is an automorphism of V_1. The group of all such automorphisms is called the wreath product of the skew linear group G_1 and the permutation group Γ, and is denoted by $G_1 \wr \Gamma$. The group $G_1 \wr \Gamma$ is imprimitive. In addition, by Lemma 5 of [32, p. 108], any imprimitive subgroup P of $GL_n(D)$ is conjugate to a subgroup of $(GL_r(D) \wr S_k)$, when $n = rk$ for some natural numbers r and k with $k > 1$.

For example, recall that a monomial matrix is a square matrix with exactly one non-zero entry in each row and column. It is not hard to see that the set of all $n \times n$ monomial matrices over D is conjugate to $D^* \wr S_n$, when $n > 1$.

Let $n = rk$, for some natural numbers r and k, when $k > 1$. Obviously, $(GL_r(D) \wr S_k) \subseteq GL_n(D)$. Let $\mathcal{A}$ be the set of all $k \times k$ monomial matrices with entries in D and choose $A \in \mathcal{A}$. We construct a new matrix in $GL_n(D)$ as follows. We replace each nonzero entry in A with a matrix from $GL_r(D)$. Also, we replace each zero entry in A with the zero matrix from $M_r(D)$. Denote the set of these new matrices by $\mathcal{B}$. Hence $\mathcal{B} \subseteq GL_n(D)$. It is not hard to see that $\mathcal{B} = (GL_r(D) \wr S_k)$.

3. Preliminary results

This section contains some preliminary results that we use throughout this article.

Theorem A *([34, Theorem 1.1]). Let $D = E(G)$ be a division ring generated as such by its nilpotent subgroup G of class at most 2 and its division subring E. Assume $E \leq C_D(G)$ and suppose that E also contains the center $Z(G)$ of G. Set $H = N_{D^*}(G)$ and let $T = \tau(G)$, the maximal periodic normal subgroup of G.*

(1) *If T is not abelian, then $Char(D) = 0$ and T has a unique quaternion subgroup $Q = \langle i, j \rangle$ of order 8 and $H = Q^+GE^*$, where $Q^+ = \langle Q, 1 + j, -(1 + i + j + ij)/2 \rangle$. Also, Q is normal in Q^+ and $Q^+/\langle -1, 2 \rangle \cong Aut(Q) \cong S_4$.*
(2) *If T is abelian with a non-central (in D) element x of order 4, then $Char(D) \neq 2$ and $H = \langle 1 + x \rangle GE^*$. Also, $(1 + x)^2 = 2x \in GE^*$.*
(3) *In all other cases, $H = GE^*$.*

Theorem B *([34, Proposition 4.1]). Let $D = E(M)$ be a division ring generated as such by its metabelian subgroup M and its division subring E such that $E \subseteq C_D(M)$. Set $K = N_{D^*}(M)$, $G = C_M(M')$, $T = \tau(G), F = E(Z(G))$, $L = N_{F^*}(M) = K \cap F$. If M has a quaternion subgroup Q of order 8 with $M = QC_M(Q)$, then $K = Q^+ML$. If T is abelian and contains an element x of order 4 not in the center of G, then $K = \langle 1 + x \rangle ML$ and $K = ML$ in all other cases. In addition, G is nilpotent of class at most 2, $C_D(M') = E(G)$.*

Theorem C *([35, Corollary 24]). Let A be a one-sided Artinian ring. Suppose S is a right Goldie subring of A and G is a locally soluble subgroup of the group of units of A normalizing S. Set $R = S[G] \subseteq A$ and assume R is prime. Then R is right Goldie.*

Theorem D *([20, Theorem 2]). Let R be a prime ring with 1, $Z = Z(R)$ be the center of R containing at least five elements, and $\overline{U}$ the Z-subalgebra of R generated by R^*. Assume that $\overline{U}$ contains a nonzero ideal of R. If N is a soluble normal subgroup of R^*, then either R is a domain or $N \subseteq Z$.*

Theorem E *([5]). Let N be normal in a primitive subgroup M of $GL_n(D)$. Then, we have:*

(1) *$F[N]$ is a prime ring;*
(2) *$C_{M_n(D)}(N)$ is a simple Artinian ring;*

(3) *If $C_{M_n(D)}(N)$ is a division ring, then N is irreducible.*

Theorem F *([24, p. 104]). Let K be a field of characteristic 0 and let A denote an Abelian subgroup of a group G.*

(1) *If $[G : A] < \infty$, then KG satisfies a polynomial identity of degree $2[G : A]$.*
(2) *If KG satisfies a polynomial identity of degree n, then G has such a subgroup A with $[G : A]$ bounded by a fixed function of n.*

Theorem G *([28, p. 36]). Suppose R is a primitive ring satisfying a polynomial identity of degree d. Then R has some PI-degree $n \leqslant [d/2]$, and $R \cong M_t(D)$ for a division ring D (unique up to isomorphism) with $n^2 = [R : Z(R)] = t^2[D : Z(D)]$.*

Let D be a division ring with center F. Let N be a non-central subnormal subgroup of $GL_n(D)$ with a maximal subgroup M. For each $d \in D^*$, denote by $A_d \in GL_n(D)$ the matrix obtained from the unit matrix by replacing the $(1, 1)$th and (n, n)th entries with d and d^{-1}, respectively. In Lemma 1 of [17] and Lemma 1 of [18] it is shown that either M is irreducible or there exists $P \in GL_n(D)$ such that $P^{-1}A_dP \in M$ for all $d \in D^*$. Therefore, the authors conclude that either M is irreducible or it contains an isomorphic copy of D^*. This result is used in several papers, for example in [17], [18] and [26]. Set $S = \{A_d \mid d \in D^*\}$. This set is not closed under multiplication. Hence, S is not a group. But in fact what the proof in [17] shows is that M contains a copy of $D_1 = \{diag(a, b) : a, b \in D^*\} \cap SL_2(D) = \{diag(a, b) : ab \in D'\}$. This is a group, which fits into the short exact sequence $1 \to D' \to D_1 \to D^* \to 1$, so a copy of D' is immediate. Consequently, we have the following:

Lemma 3.1. *Given a division ring D, let N be a non-central subnormal subgroup of $GL_n(D)$ with $n \geqslant 1$. Assume that M is a maximal subgroup of N. Then, either M is primitive or contains an isomorphic copy of D'.*

Lemma 3.2. *Let D be a finite dimensional division ring over its center F. Suppose that K is a subfield of $M_n(D)$ containing F, when $n \geqslant 1$. If $G \subseteq N_{GL_n(D)}(K^*)$ is a subgroup of $GL_n(D)$ such that $C_{GL_n(D)}(G) = F^*$, then K/F is Galois. Also, $G/C_G(K^*) \cong Gal(K/F)$.*

Proof. Consider the homomorphism $f : G \to Gal(K/F)$ given by $f(a) = f_a$, where $f_a(x) = axa^{-1}$, for any $x \in K$. It is clear that $ker(f) = C_G(K^*)$. We have $F \subseteq Fix(Gal(K/F)) \subseteq Fix(im(f)) = F$, which implies that K/F is a Galois extension. By Proposition 2.14 of [27], f is surjective. Therefore, we conclude that $G/C_G(K^*) \cong Gal(K/F)$, as desired. □

Lemma 3.3. *Let D be a finite dimensional division ring over its center F. Assume that G be an absolutely irreducible primitive subgroup of $GL_n(D)$. If N be a non-central normal* *abelian subgroup of G, then $K = F[N]$ is a subfield of $M_n(D)$ containing F such that K/F is Galois and $G/C_G(N) \cong Gal(K/F)$. Thus, G has a normal subgroup of finite index.*

Proof. We know that G is a primitive skew linear group. Hence, using Theorem E, we obtain that $F[N]$ is a commutative prime ring and so a commutative integral domain. We know that a finite dimensional integral domain is a field, hence we conclude that $K = F[N]$ is a subfield of $M_n(D)$. Consequently, $G \subset N_{GL_n(D)}(K^*)$. Using Lemma 4.2, we can conclude that K/F is Galois and $G/C_G(N) \cong Gal(K/F)$, as desired. □

The following lemma is a direct generalization of a similar lemma that appears in [8].

Lemma 3.4. *Let D be a finite dimensional division ring over its center F. Suppose that G be an absolutely irreducible subgroup of $GL_n(D)$ with $n \geqslant 1$. If K is a subfield of $M_n(D)$ containing F such that $[G : C_G(K^*)] = [K : F]$, then $C_{M_n(D)}(K) = F[C_G(K^*)]$.*

Proof. Set $A = C_{M_n(D)}(K)$ and $B = F[C_G(K^*)]$. Since $B \subseteq A$, we conclude that $[B : F] \leqslant [A : F]$. Using Double Centralizer Theorem (see [6, p. 42]), we obtain that $M_n(D) \otimes_F K \cong M_m(F) \otimes_F C_{M_n(D)}(K)$, when $m = [K : F]$. This means that $[A : F][K : F] = [M_n(D) : F]$. On the other hand, $F[G] = M_n(D)$. We supposed that $[G : C_G(K^*)] = [K : F]$, so

$$[M_n(D) : F] \leqslant [F[C_G(K^*)] : F][G : C_G(K^*)] = [B : F][K : F].$$

Hence, $[A : F] \leqslant [B : F]$. Thus $A = B$, as we claimed. □

Using Double Centralizer Theorem, Lemma 3.2 and Lemma 3.4, we have following result.

Corollary 3.5. *Let D be a finite dimensional division ring over its center F. Assume that G be an absolutely irreducible subgroup of $GL_n(D)$ with $n \geqslant 1$. If K is a subfield of $M_n(D)$ containing F such that $G \subseteq N_{GL_n(D)}(K^*)$, then $F[C_G(K^*)]$ is a simple ring.*

4. Soluble maximal subgroups in $SL_n(D)$

The following lemma will play a key role in the proof of our main theorem.

Lemma 4.1. *Let R be a ring and $K \subseteq R$ a subfield. Let $M \subseteq N_{R^*}(K^*)$ such that $C_M(K^*) = K^* \cap M$. Then, for any subgroup H of M such that $H/(H \cap K^*)$ is a nontrivial finite group, we have $(K[H], K, H, H/(K^* \cap H))$ is a crossed product. In particular, $dim_K K[H] = |H/(H \cap K^*)|$.*

Proof. Given a set $\{m_i\}$ of distinct representatives of the cosets of $K^* \cap H$ in H, it is enough to show that $\{m_i\}$ is a linearly independent set over K. To do this, assume that $k_1m_1 + \cdots + k_sm_s = 0$ is a nontrivial relation with s minimal. Since $C_M(K^*) = K^* \cap M$, there exists an element $x \in K^*$ such that $x_1 = m_1xm_1^{-1} \neq m_2xm_2^{-1} = x_2$. Therefore, $0 = (k_1m_1 + \cdots + k_sm_s)x - x_1(k_1m_1 + \cdots + k_sm_s) = (x_2 - x_1)k_2m_2 + \cdots + (x_s - x_1)k_sm_s$ with $x_i = m_ixm_i^{-1}$, which is a nontrivial relation with a smaller number of nonzero summands and hence the representatives m_i are linearly independent. $\square$

By using the above results, we are now in the position to prove the following:

Theorem 4.2. *Given an F-central division algebra D and N a subnormal subgroup of $GL_n(D)$, let M be a non-abelian maximal subgroup of N. If M is soluble then $F[M] = M_n(D)$.*

Proof. We may assume $F^* \subseteq M$. Otherwise, we may replace M and N by F^*M and F^*N, respectively.

First, assume that $n > 1$. Let $L = F[M] \cap N$. By maximality of M in N, we have either $L = N$ or $L = M$. If the first case occurs, we conclude that $N \subseteq F[M]$. By Lemma 2.3 of [12], $SL_n(D) \subseteq N$ and thus N is a normal subgroup of $GL_n(D)$. Hence, $SL_n(D) \subseteq F[M]^*$. By Corollary 1 of [29], we have $F[M] = M_n(D)$. If the second case happens, then M is a normal subgroup of $F[M]^*$. Assume that M is imprimitive. By Lemma 3.1, M contains an isomorphic copy of D', which contradicts the Hua's Theorem of [16], which asserts that D^* is insoluble. Consider that M is primitive, then $F[M]$ is a prime ring by Theorem E and Goldie by Theorem C. The $Z(F[M])$-subalgebra of $F[M]$ generated by $F[M]^*$ is $F[M]$. Using Theorem D, we conclude that either M is abelian or $F[M]$ is an ore domain. The first case cannot happen. Finally, assume $F[M]$ is an ore domain. By Theorem 5.7.8 of [31], the ring Q of quotients of $F[M]$ is naturally embedded in $M_n(D)$. Then Q is a division ring. The same argument as above conclude that either $Q = M_n(D)$ or M is a soluble normal subgroup of Q^*. Since $n > 1$, the first case cannot occur. The second case contradicts Theorem 14.4.4 of [30].

Consider now the case $n = 1$. We shall prove the remaining part of proof of the theorem in the following cases.

(1) M **is metabelian.** By Theorem 3.3 of [13], we conclude that $F[M] = D$.
(2) M **contains a characteristic subgroup** G **which is nilpotent of class two.** Set $A = F(G)$. We claim that $A = D$. We have $M \subseteq N_N(A^*) \subseteq N$. If $M = N_N(A^*)$, then $A^* \cap N \subseteq M$. This means that $A^* \cap N$ is a soluble subnormal subgroup of A^*. Using Theorem 14.4.4 of [30], we obtain that G is abelian, which is a contradiction. Hence, by the maximality of M in N, we may consider $N = N_N(A^*)$. By Theorem 14.3.8 of [30], we conclude that $D = A = F(G)$, as we claimed. Set $H = N_{D^*}(G)$, and so $M \subseteq H$. Next, we shall use the conclusions of Theorem A. If the case (1) occurs, then $B = F[Q]$ is a division ring. On the other hand, any group has a unique maximal

periodic normal subgroup. Therefore, T is a characteristic subgroup of G and Q is a characteristic subgroup of T. Since the characteristic property is transitive, we conclude that Q is a normal subgroup of M. By a similar argument as above, we have either Q is abelian or $D = F[Q]$. But, $F[Q] \subseteq F[M]$ and Q is a quaternion group of order 8. Consequently, both cases cannot happen. If the case (2) happens, then $\langle x \rangle$ is characteristic in G and hence normal in M. Therefore, $C = F[x]$ is a field which is normalized by M. As x is not central, we have $C_D(x) \subsetneqq D$ which in turn implies that $M \subseteq N_N(C_D(x)^*) \subseteq N$. By a similar argument as above and the fact that $C_D(x) \subsetneq D$, we have $C_D(x)^* \cap N \subseteq M$. On the other hand, x is a non-central FC-element. By Theorem 3.2 of [13], we obtain that $C_D(x) = F(x) = F[x]$. Using Double Centralizer Theorem of [6], we conclude that $[D : F]$ is finite, i.e., $F[M]$ is a division ring. By maximality of M in N, we have either $M = F[M]^* \cap N$ or $N \subseteq F[M]^*$. Using Theorems 14.4.4 and 14.3.8 of [30], we conclude that both cases cannot occur. Finally, if the case (3) of Theorem A occurs, then $H = F^*G$, and so H and M are nilpotent which contradicts Theorem 2.3 of [26].

(3) **Other cases.** Let r be the least number such that $M^{(r)} \subseteq F^*$ and $M^{(r-1)} \nsubseteq F^*$. If $M^{(r-1)}$ is not abelian, then $M^{(r-1)}$ is a non-abelian nilpotent characteristic subgroup of class two, and the proof follows from the case (2). Otherwise, $M^{(r-1)}$ is abelian and set $L := M^{(r-2)}$ which is a non-abelian metabelian characteristic subgroups of M. Consider now $G = C_L(L')$ which is a nilpotent characteristic subgroup of M of class at most two. If G is of class two, by the case (2) we are done, otherwise we may assume G abelian. Thus, by Theorem B, we have the following three cases to consider.
   (a) $L = QC_L(Q)$**.** Then $C_L(Q) \lhd L$ and so we conclude that $L/C_L(Q) \cong Q/Q \cap C_L(Q) = Q/Z(Q) \cong C_2 \times C_2$ is abelian. Thus, $L' \subseteq C_L(Q)$ and so $Q \subseteq C_L(L') = G$, which is a contradiction since G is abelian.
   (b) The case (2) of Theorem B cannot occur since G is abelian.
   (c) $H = LF(G)^*$**.** In this case $H/F(G)^* \cong L/F(G)^* \cap L$ is abelian because $L' \subseteq F(G)^* \cap L$, and so is $H' \subseteq F(G)^*$. This means that H is metabelian. But, $M \subseteq H$ and hence M is metabelian, which reduces to the case (1). □

Theorem 4.3. *Given an F-central division algebra D and N a subnormal subgroup of $GL_n(D)$, if M is a non-abelian absolutely irreducible soluble maximal subgroup of N, then, $n = 1$ and there exists a non-central maximal normal abelian subgroup A of M such that $K = F[A]$ is a maximal subfield of D. Also, D is cyclic of prime degree p such that the groups $Gal(K/F)$ and $M/(K^* \cap N)$ are isomorphic. Furthermore, for any $x \in M \setminus K^*$, we have $x^p \in F^*$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.*

Proof. As in the proof of Theorem 4.2, we may assume $F^* \subset M$ and M primitive. Since M is a soluble absolutely irreducible skew linear group, by Corollary 5.6.8 of [31], it is abelian-by-locally finite. Let A be a maximal abelian normal subgroup of M such that M/A is locally finite. We consider the following two cases.

(1) A **is non-central.** We claim that A is irreducible. Since $M \subseteq N_{GL_n(D)}(C_{M_n(D)}(A)^*)$ and A is not central, by Theorem 14.3.8 of [30] and Corollary 1 of [29], we obtain $C_{M_n(D)}(A)^* \cap N \subset M$, and using Theorem E, we see that $C_{M_n(D)}(A)$ is a simple Artinian ring. If $C_{M_n(D)}(A)^*$ is finite, then F^* and A become finite. Therefore, by Theorem E, $F[A]$ is a finite simple ring. Since $C_{M_n(D)}(C_{M_n(D)}(F[A])) = F[A]$, using Double Centralizer Theorem, we conclude that D is algebraic over F. Hence, by Jacobson's Theorem [19, p. 208], we have $D = F$, which is a contradiction. Now, assume that $C_{M_n(D)}(A)$ is infinite. We know that $C_{M_n(D)}(A) \cong M_s(D_1)$, where D_1 is a division ring. First, assume $s = 1$. As $C_{M_n(D)}(A)$ is a division ring, by Theorem E, A is irreducible, as claimed. Now, assume $s > 1$. By Lemma 2.3 of [12], $SL_n(D) \subseteq N$. Now, we have $C_{M_n(D)}(A)^* \cap N \subset M$, which means $SL_s(D_1)$ is soluble. Therefore, $C_{M_n(D)}(A)^*$ is the multiplicative group of a field. Therefore, by Theorem E, A is irreducible.

Since M is a primitive skew linear group, by Theorem E, we conclude that $F[A]$ is a commutative prime ring and hence a commutative integral domain. Now, by Theorem 5.7.8 of [31], the field of fractions of $F[N]$ is embedded in $M_n(D)$. Therefore, there exists a subfield K of $M_n(D)$ such that $F[A] \subset K$. It is clear that $M \subset N_{GL_n(D)}(K^*)$ and $C_M(K^*) = K^* \cap M = A$. We next claim that M/A is simple. To do this, assume that L is a subgroup of M such that $A \subsetneq L \lhd M$ and set $R = F[L] = K[L]$. Since L/A is locally finite, we may write $K[L] = \cup_H K[H]$, where the union is taken over all subgroups H of L containing A such that H/A is finite. As we saw, A is irreducible and so is any subgroup containing it. Thus, by Theorem 1.1.14 of [31], $K[H] = F[H]$ is a prime ring that is of finite dimension over K and hence a simple Artinian ring. Therefore, $K[A]$ is the union of simple Artinian rings. Now, since that $M \subseteq N_{GL_n(D)}(R^*)$. If $M = N_N(R^*)$, then $R^* \cap N \subseteq M$ and hence $K[H]^* \cap N \subseteq M$. Thus, a similar argument as above leads to the commutativity of $K[H]^* \cap N$ and H. Therefore, for any $x, y \in L$, we have $xy = yx$, which contradicts the maximality of A, and consequently $N_N(R^*) = N$. Using Theorem 14.3.8 of [30] and Corollary 1 of [29], we obtain $M_n(D) = F[A]$, i.e., $F[M] = F[A]$. Now, to complete the proof of the simplicity of M/A, it is enough to verify that $L = M$. To do so, given $x \in M$, there exists a subgroup H of L with $A = K^* \cap M \subseteq H \subseteq L$ such that H/A is finite and $x \in K[H]$, and also by Lemma 4.1, $(K[H], K, H, H/A)$ is a crossed product central simple algebra with center E, say. Setting $C = K[H]$, the Skolem-Noether Theorem gives us $H/A \subseteq N_{C^*}(K)/A \cong Gal(K/E)$. Therefore, $|H/A| \leq |Gal(K/E)| = dim_E K = dim_K C = |H/A|$, which implies that $H/A = N_{C^*}(K)/A$. But $x \in N_{C^*}(K) = H$ which says that $x \in L$, and hence $L = M$. Therefore, M/A is simple as well as soluble, i.e., $M/A \cong C_p$, for some prime p. This also implies that $p = dim_K K[M] = dim_K F[M] = dim_K M_n(D)$, and hence $[D : F]$ is finite, by Lemma 6 of [1]. Finally, the Double Centralizer Theorem yields $dim_F M_n(D) = dim_K M_n(D)^2 = p^2$, i.e., $n = 1$, $[D : F] = p^2$. Therefore, K is a maximal subfield of $M_n(D)$ and $F[M] = \sum_{i=1}^{p} Kx^i$. Thus the equality $[D : K] = [K : F] = p$ implies that $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$. Now, if $x \in M \setminus K^*$, then $x^p \in K^*$.

Therefore, $K \subseteq C_D(x^p)$. Since $x \in C_D(x^p)$ and by the fact that $[K : F] = p$, we obtain that $C_D(x^p) = D$, and so $x^p \in F^*$. By a similar argument as above, we have $A = K^* \cap M = K^* \cap N$.

(2) A **is central.** Since $F[M] = M_n(D)$, we conclude that $Z(M) = F^*$. So we have M/F^* is locally finite. This implies that $M/Z(M)$ is a locally finite group. Therefore, by Lemma 3 of [1], we obtain that M' is locally finite. Since M is primitive, by the same argument as that used in the previous case, we conclude that either $F[M'] = M_n(D)$ or $M' \subseteq Z(M) = F^*$.

Suppose that $M' \subseteq Z(M) = F^*$. Now, given $x, y \in M \backslash F$ such that $x \neq y$, we have $xyx^{-1}y^{-1} \in F^*$. Therefore, $F^* \langle x, y\rangle \lhd M$, which means $M \subseteq N_N(F[\langle x, y\rangle]^*)$. Now, by maximality of M, we obtain either $N_N(F[\langle x, y\rangle]^*) = M$ or $N_N(F[\langle x, y\rangle]^*) = N$. Since x, y are algebraic over F, we have $F[\langle x, y\rangle] = F[x, y]$. Because M/F^* is locally finite, we obtain that $F[M]$ is locally finite dimensional. Therefore, $[F[x, y] : F] < \infty$. By a similar argument as above, we have $F[\langle x, y\rangle] = M_n(D)$. This means that $[M_n(D) : F] < \infty$. Since M is irreducible, we find that M is completely reducible. Consequently, M is a completely reducible linear group and nilpotent, so by Corollary 6.5 of [3], we have $M/Z(M)$ is finite, which contradicts Theorem 1 of [18].

We may assume $F[M'] = M_n(D)$. Assume that K is a maximal locally finite subfield of F. By Theorem 1.1.12 of [31] and Theorem E, $K[M']$ is a simple Artinian ring, and hence $K[M'] \cong M_s(D_1)$, where D_1 is a division ring. If $a \in U(K[M'])$, then there exist $n_1, ..., n_k$ in M' and $a_1, ..., a_k$ in K such that $a = a_1n_1 + ... + a_kn_k$. Since M' is a locally finite group, we conclude that $\mathbb{F}_p[\langle a_1, ..., a_k\rangle][\langle n_1, ..., n_k\rangle]$ is a finite ring. So, a must be torsion. Thus $U(K[M'])$ is a torsion group. Set $F_1 = Z(D_1)$, hence by Jacobson's Theorem [19, p. 208], we have $F_1 = D_1$. If $s = 1$, then $K[M'] = F_1$. Thus, M' is abelian. On the other hand, we have $F[M'] = M_n(D)$. Therefore, $n = 1$ and $D = F$, which is a contradiction. So, we may assume that $s > 1$. In that case we have $U(K[M']) \cong GL_s(F_1)$. This means that $GL_s(F_1)$ is torsion and so F_1 is a torsion group. Thus, F_1 is a locally finite field. Now, we have, $K^* \subseteq Z(U(K[M'])) \subseteq C_{M_n(D)}(F[M']) = F$. Therefore, $Z(U(K[M'])) \cup \{0\}$ is a locally finite field. By maximality of K, we conclude that $K^* = Z(U(K[M']))$. It is clearly seen that $K[M'] \cong M_s(K)$. We conclude that $GL_n(D)$ contains an isomorphic copy of $GL_s(K)$. Since $M_n(D)$ contains an isomorphic copy of $M_s(K)$, by Theorem 1.1.9 of [31], we conclude that $s \leqslant n$. On the other hand, $F[K[M']] = F[M'] = M_n(D)$. Using the fact that $[K[M'] : K] = s^2$, we obtain $[M_n(D) : F] \leq s^2$. Therefore $[D : F]n^2 \leq s^2$. So, $D = F$, which is a contradiction.

Next assume that $Char(F) = 0$. Since M' is a locally finite normal subgroup of M, by Corollary 5.4.6 of [31], $M/C_M(M')$ is locally finite and it has a metabelian normal subgroup of finite index. Since $F[M'] = M_n(D)$, we have $C_M(M') = F^*$. Thus, M/F^* has a metabelian normal subgroup of finite index. Suppose G is a normal subgroup of M such that G/F^* is a metabelian normal subgroup of M/F^* and $[M/F^* : G/F^*] < \infty$. Hence, we obtain that $[M : G] < \infty$ and $G'' \subseteq F$. Since M is primitive, arguing as in the previous case, we may conclude that either $F[G] =$

$M_n(D)$ or $G \subseteq Z(M) = F^*$. Consider that $G \subseteq F^*$. Then $[M : F^*] < \infty$. Using Theorem 1 of [18], we arrive at a contradiction. So, G is non-central and $F[G] = M_n(D)$.

First, Assume that $G' \nsubseteq F$. Since $G' \subseteq M'$ and M' is a locally finite group, we obtain that G' is a locally finite group. By Theorem 1.1.14 of [31] and Theorem E, we conclude that $F[G']$ is a simple Artinian ring. We have $M \subseteq N_N(F[G']^*)$, by maximality of M, two cases may occur, i.e., either $M = N_N(F[G']^*)$ or $N = N_N(F[G']^*)$.

If $M = N_{GL_n(D)}(F[G']^*)$, then $F[G']^* \cap N \subseteq M$. Since M is soluble, we obtain that G' is non-central abelian normal subgroup of M, which reduces to the previous case.

Consider $N = N_{GL_n(D)}(F[G']^*)$. Using Theorem 14.3.8 of [30] and Corollary 1 of [29], we have either $G' \subseteq F$ or $F[G'] = M_n(D)$. By our assumption the first case cannot happen. Now, assume that $F[G'] = M_n(D)$. Since $G'' \subseteq F$ we have $G'' \subseteq Z(G')$. This implies that G' is nilpotent. Therefore, by 2.5.2 of [31], G' is abelian-by-finite. Thus, by Theorem F, the group ring FG' satisfies a polynomial identity. We conclude that $F[G']$ satisfies a polynomial identity, and hence D satisfies a polynomial identity. Now, by Theorem G, we have $[D : F] < \infty$. Since M is an absolutely irreducible skew linear group, we conclude that M is an irreducible linear group (cf. [32, p. 100]). Therefore, by Theorem 6 of [32, p. 135], M contains an abelian normal subgroup H, say, of finite index. If $H \subseteq F^*$, then M/F^* is finite. Using Theorem 1 of [18], we arrive at a contradiction. So, H is non-central, which reduces to the previous case.

Now, consider that $G' \subseteq F$. Thus, G is nilpotent. With a similar argument as before, we obtain that either $M' \cap G \subseteq F^*$ or $F[M' \cap G] = M_n(D)$. $M' \cap G$ is a locally finite nilpotent group. As above, the second case cannot happen. Now, assume $M' \cap G \subseteq F^*$. Since $F[M'] = M_n(D)$ and $[M' : M' \cap G] < \infty$, we conclude that $[D : F] < \infty$. A similar argument as above holds. □

Combining Theorems 4.2 and 4.3, we obtain our final result as follows:

Theorem 4.4. *Let D be an F-central division algebra and N a subnormal subgroup of $GL_n(D)$. If M is a non-abelian soluble maximal subgroup of N, then, $n = 1$ and D is cyclic of prime degree p with a maximal cyclic subfield K/F such that the groups $Gal(K/F)$ and $M/(K^* \cap N)$ are isomorphic. Furthermore, for any $x \in M \setminus K^*$, we have $x^p \in F^*$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.*

As another application, we obtain the following result.

Corollary 4.5. *Let D be an F-central division algebra and M is a non-abelian soluble maximal subgroup of $SL_n(D)$. Then, $n = 1$ and D is cyclic of prime degree p with a maximal cyclic subfield K/F such that the groups $Gal(K/F)$ and $M/(K^* \cap N)$ are isomorphic. Furthermore, for any $x \in M \setminus K^*$, we have $x^p \in F^*$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$.*

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References

[1] S. Akbari, R. Ebrahimian, H. Momenaee Kermani, A. Salehi Golsefidy, Maximal subgroups of $GL_n(D)$, J. Algebra 259 (2003) 201–225.
[2] S. Akbari, M. Mahdavi-Hezavehi, M.G. Mahmudi, Maximal subgroups of $GL_1(D)$, J. Algebra 217 (1999) 422–433.
[3] J.E. Dixon, The Structure of Linear Groups, Van Nostrand, London, 1971.
[4] H. Dorbidi, R. Fallah-Moghaddam, M. Mahdavi-Hezavehi, Existence of non-abelian free subgroups in the maximal subgroups of $GL_n(D)$, Algebra Colloq. 21 (3) (2014) 483–496.
[5] H. Dorbidi, R. Fallah-Moghaddam, M. Mahdavi-Hezavehi, Soluble maximal subgroups of $GL_n(D)$, J. Algebra Appl. 9 (6) (2010) 921–932.
[6] P.K. Draxl, Skew Fields, Cambridge Univ. Press, Cambridge, 1983.
[7] R. Ebrahimian, Nilpotent maximal subgroups of $GL_n(D)$, J. Algebra 280 (2004) 244–248.
[8] R. Ebrahimian, D. Kiani, M. Mahdavi-Hezavehi, Supersoluble crossed product criterion for division algebras, Israel J. Math. 145 (1) (2005) 325–331.
[9] R. Fallah-Moghaddam, M. Mahdavi-Hezavehi, Free subgroups in maximal subgroups of $GL_n(D)$, Comm. Algebra 45 (2017) 3724–3729.
[10] R. Fallah-Moghaddam, M. Mahdavi-Hezavehi, Locally finite conditions on maximal subgroups of $GL_n(D)$, Algebra Colloq. 19 (1) (2012) 73–86.
[11] J.Z. Goncalves, A. Mandel, Are there free groups in division rings, Israel J. Math. 51 (1) (1986) 69–80.
[12] J.Z. Goncalves, Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings, Canad. Math. Bull. 27 (1982) 365–370.
[13] B.X. Hai, N.A. Tu, On multiplicative subgroups in division rings, J. Algebra Appl. 15 (3) (2016) 1–16.
[14] R. Hazrat, M. Mahdavi-Hezavehi, M. Motiee, Multiplicative groups of division rings, Math. Proc. R. Ir. Acad. 114 (1) (2014) 37–114.
[15] R. Hazrat, A.R. Wadsworth, On maximal subgroups of the multiplicative group of a division algebra, J. Algebra 322 (2009) 2528–2543.
[16] L.K. Hua, On the multiplicative group of a sfield, Sci. Rec. Acad. Sin. 3 (1950) 1–6.
[17] D. Kiani, Polynomial identities and maximal subgroups of $GL_n(D)$, Manuscripta Math. 124 (2007) 2354–2363.
[18] D. Kiani, M. Mahdavi-Hezavehi, Tits alternative for maximal subgroups of skew linear groups, Comm. Algebra 38 (2010) 2354–2363.
[19] T.Y. Lam, A First Course in Non-commutative Rings, Grad. Texts in Math., vol. 131, Springer-Verlag, Berlin, 2001.
[20] C. Lanski, Solvable subgroups in prime rings, Proc. Amer. Math. Soc. 82 (1981) 533–537.
[21] S. Li, The maximality of monomial subgroups of linear groups over division rings, J. Algebra 127 (1) (1989) 22–39.
[22] A.E. Lichtman, On subgroups of the multiplicative group of skew fields, Proc. Amer. Math. Soc. 63 (1977) 15–16.
[23] M. Mahdavi-Hezavehi, Free subgroups in maximal subgroups of $GL_1(D)$, J. Algebra 241 (2001) 720–730.
[24] D.S. Passman, Group ring satisfying a polynomial identity, J. Algebra 20 (1972) 103–117.
[25] M. Ramezan-Nassab, D. Kiani, (Locally soluble)-by-(locally finite) maximal subgroups of $GL_n(D)$, J. Algebra 376 (2013) 1–9.
[26] M. Ramezan-Nassab, D. Kiani, Nilpotent and polycyclic-by-finite maximal subgroups of skew linear groups, J. Algebra 399 (2014) 269–276.
[27] P. Morandi, Fields and Galois Theory, Springer-Verlag, New York, 1996.
[28] L.H. Rowen, Ring Theory II, Cambridge Univ. Press, Cambridge, 1980.
[29] A. Rosenberg, The Cartan-Bruer-Hua Theorem for matrix and local matrix rings, Proc. Amer. Math. Soc. 7 (1956) 891–898.
[30] W.R. Scott, Group Theory, Dover, New York, 1987.

[31] M. Shirvani, B.A.F. Wehrfritz, Skew linear groups, London Math. Soc. Lecture Note Ser., vol. 118, 1986.
[32] D.A. Suprunenko, Matrix Groups, Amer. Math. Soc., Providence, RI, 1976.
[33] J. Tits, Free subgroups in linear groups, J. Algebra 20 (1972) 250–270.
[34] B.A. Wehrfritz, Normalizers of nilpotent subgroups of division rings, Quart. J. Math. 58 (2007) 127–135.
[35] B.A. Wehrfritz, Goldie subrings of Artinian rings generated by groups, Quart. J. Math. 40 (1989) 501–512.