

# Unique Tensor Factorization of Loop-Resistant Algebras over a Field of Finite Characteristic

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Tensor product decomposition of algebras is known to be non-unique in many cases. But we know that a  $\oplus$ -indecomposable, finite-dimensional  $\mathbb{C}$ -algebra  $A$  has an essentially unique tensor factorization

$$A = A_1 \otimes \cdots \otimes A_r$$

into non-trivial,  $\otimes$ -indecomposable factors  $A_i$ . Thus the semiring of isomorphism classes of finite-dimensional  $\mathbb{C}$ -algebras is a polynomial semiring  $\mathbb{N}[\mathcal{X}]$ . Moreover, the field  $\mathbb{C}$  of complex numbers can be replaced by an arbitrary (not necessarily algebraically closed) field of characteristic zero if we restrict ourselves to split algebras.

Here, we show that the above result still holds in finite characteristics if we only consider loop-resistant algebras. © 2002 Elsevier Science (USA)

*Key Words:* tensor product of algebras; unique factorization.

## 1. INTRODUCTION

The unique factorization theorem for algebras over  $\mathbb{C}$  (or some other field of characteristic zero) was proved in Nüsken [5, 6] starting with three other unique factorization results. Namely, the unique factorization of natural numbers and a unique factorization theorem for graphs with respect to the cartesian product by Sabidussi [7] were combined with an extension of a unique factorization result for local algebras due to Horst [3]. In fact, a part of that proof could have been done over a field  $k$  of arbitrary characteristics, but the methods for local algebras definitely fail in finite characteristics. Here we show that, for a still large class of algebras,



no results on local algebras are needed. Additionally, we replace the inner transform used in Nüsken [5, 6] by the more powerful quotient transform, so we access even more algebras.

Among fields of finite characteristic, finite fields are of special interest. They are not algebraically closed, and one source of non-unique factorization comes up in examples like the following: The tensor product  $\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n}$  of the  $\mathbb{F}_p$ -algebra  $\mathbb{F}_{p^n}$  with itself is isomorphic to the tensor product  $\mathbb{F}_{p^n} \otimes n \cdot \mathbb{F}_p$  of  $\mathbb{F}_{p^n}$  with the direct sum of  $n$  copies of  $\mathbb{F}_p$ . More generally, given a Galois extension  $E|F$  of degree  $n$ , we have  $E \otimes E \simeq E \otimes nF$  as  $F$ -algebras. This shows that an extra condition is necessary to enable unique factorization unless the field is algebraically closed: we require all algebras to be split. By Wedderburn's theorem, the semisimple residue algebra  $A/\text{rad } A$  of  $A$  modulo its Jacobson radical  $\text{rad } A$  is a sum of matrix algebras over  $k$ -division algebras. An algebra  $A$  is *split* iff the residue algebra  $A/\text{rad } A$  is a sum of matrix algebras over the ground field  $k$  rather than over arbitrary  $k$ -division algebras. If  $k$  is algebraically closed, every  $k$ -algebra is split. Furthermore, this condition is necessary for the essential tools to work well. (In particular, Graph Lemma 3.2(ii) only holds for split algebras; see Frontier A.4 in [5].)

We call a finite-dimensional,  $\oplus$ -indecomposable algebra  $A$  *loop-resistant* iff there is an idempotent  $e \in A$  with  $A/\langle 1 - e \rangle = k^{n \times n}$  for some  $n$ . An arbitrary finite-dimensional algebra is *loop-resistant* iff each  $\oplus$ -indecomposable direct summand is *loop-resistant*. We obtain the best result we could have hoped for:

**UNIQUE FACTORIZATION 1.1.** *Let  $k$  be a field. The set  $\mathcal{M}$  of isomorphism classes of  $\oplus$ -indecomposable, loop-resistant, split  $k$ -algebras endowed with the operation induced by the tensor product is a free commutative monoid over the set  $\mathcal{X}$  of isomorphism classes of  $\{\oplus, \otimes\}$ -indecomposable, loop-resistant, split algebras.*

*The semiring  $\mathcal{U}$  of isomorphism classes of loop-resistant, split algebras is the positive cone  $\mathbb{N}[\mathcal{X}]$  of the polynomial ring  $\mathbb{Z}[\mathcal{X}]$ .*

Since a semiring  $\mathbb{N}[\mathcal{X}]$  is not factorial unless  $\mathcal{X}$  is empty, more than unique factorization for  $\oplus$ -indecomposable algebras could not have been expected. However, the result completely characterizes tensor decompositions of arbitrary loop-resistant, split  $k$ -algebras. As noted above, in characteristic zero the result is true without the restriction to loop-resistant algebras.

The central tool in the proof is the valued graph of an algebra. It does not only nicely reflect direct sums and tensor products, but it also offers a sort of inverse construction leading to algebras closely connected with the original one. This will be used to isolate tensor factors.

This text continues the theory started on V. Strassen's suggestion of the characteristic zero result. His demand for clarity and precision still guides me.

## 2. PRELIMINARIES

Background material on algebras is provided by Drozd and Kirichenko [2, Chaps. 3 and 8]. (Note that notation differs slightly.) Also Auslander, Reiten, and Smalø [1] cover all needed material but the Morita transformation.

### 2.1. Algebras, Sum, and Product

Throughout this paper, an *algebra*  $A$  is a finite-dimensional, associative, unitary, and split  $k$ -algebra. *Split* means that the residue algebra  $A/\text{rad } A$  is a sum of matrix algebras over the ground field  $k$  (rather than over arbitrary  $k$ -division algebras). (In [5, 6], this was called *schurian*.) This condition has no effect if  $k$  is algebraically closed. Unless otherwise stated, morphisms of algebras are unitary, i.e., they map the unit element to the unit element.

The *sum*  $A_1 \oplus A_2$  of algebras  $A_1, A_2$  is also known as the direct product. Here, however, by the *product*  $A_1 \otimes A_2$  we mean their tensor product, i.e., the tensor product of the underlying vector spaces (over  $k$ ) with the multiplication induced by  $(a'_1 \otimes a'_2) \cdot (a_1 \otimes a_2) = a'_1 a_1 \otimes a'_2 a_2$ . An algebra is  *$\otimes$ -indecomposable* ( *$\oplus$ -indecomposable*) iff it cannot be written as a proper product (sum).

### 2.2. Valued Graphs, Disjoint Union, and Cartesian Product

A *graph*  $\Gamma$  is a pair  $(V(\Gamma), E(\Gamma))$ , where the *vertex set*  $V(\Gamma)$  is a finite set, and the *edge set*  $E(\Gamma)$  is a set of unordered pairs of vertices. A graph morphism  $\sigma: \Gamma_1 \rightarrow \Gamma_2$  is a map  $\sigma: V(\Gamma_1) \rightarrow V(\Gamma_2)$  that respects edges; i.e., for each edge  $e_1 = \{\alpha, \beta\}$  in  $\Gamma_1$ , the image  $\sigma(e_1) = \{\sigma(\alpha), \sigma(\beta)\}$  is an edge in  $\Gamma_2$ . The smallest non-empty graph is a *point* consisting of a single vertex and no edges.

A *multiplicity*  $m$  on a graph  $\Gamma$  is any function  $m: V(\Gamma) \rightarrow \mathbb{N}$ . A *morphism*  $\sigma: m_1 \rightarrow m_2$  of multiplicities is an injective graph morphism  $\underline{\sigma}: \Gamma_1 \rightarrow \Gamma_2$  and a family of injective(!) maps  $\sigma_{\alpha_1}: \{1, \dots, m_1(\alpha_1)\} \rightarrow \{1, \dots, m_2(\underline{\sigma}(\alpha_1))\}$  for  $\alpha_1 \in \Gamma_1$ . Of course, multiplicities on the same graph can be added (we write  $m_1 + m_2$ ) and a multiplicity can be scaled by a rational number provided all new values are still natural numbers (we write  $c \cdot m$ ).

A *valued graph*  $\Gamma'$  is a graph  $\Gamma$  with a multiplicity  $m_{\Gamma'}$  on it. The smallest non-empty example of a valued graph is a *sincere simple point*, that is, a valued graph with one vertex whose multiplicity is 1; a *simple point* is any

valued graph whose multiplicity is 1 at exactly one vertex and zero otherwise. A *morphism*  $\sigma: \Gamma'_1 \rightarrow \Gamma'_2$  of valued graphs is a morphism  $\sigma: m_{\Gamma_1} \rightarrow m_{\Gamma_2}$  of their multiplicities. Instead of considering valued subgraphs, we also consider multiplicities on valued graphs by ignoring the original multiplicity; a vertex with multiplicity zero can be interpreted as not being in the subgraph ...

The *support*  $\text{supp } m$  of a multiplicity  $m$  on  $\Gamma$  is the valued graph whose vertex set is the set of vertices  $\alpha$  with  $m(\alpha) > 0$ , whose edge set consists of all edges of  $\Gamma$  with both vertices in the new vertex set, and whose multiplicity is the restriction of  $m$  to the new vertex set. A multiplicity  $m$  on  $\Gamma$  is called *sincere* iff its support coincides with  $\Gamma$ ,  $\text{supp } m = \Gamma$ ; in other words, iff 0 is not a value. A valued graph is *sincere* iff its multiplicity is sincere. A support is sincere.

The *disjoint union*  $\Gamma_1 \uplus \Gamma_2$  of graphs  $\Gamma_1, \Gamma_2$  is obtained by taking the disjoint union of the vertex and of the edge sets. The vertex set of the *cartesian product*  $\Gamma_1 \times \Gamma_2$  is the cartesian product  $V(\Gamma_1) \times V(\Gamma_2)$ ; two vertices  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2)$  are connected by an edge iff either  $\alpha_2 = \beta_2$  and  $\alpha_1$  is connected to  $\beta_1$  in  $\Gamma_1$ , or  $\alpha_1 = \beta_1$  and  $\alpha_2$  is connected to  $\beta_2$  in  $\Gamma_2$ . For example, the product of the *one edge graph*  $\Theta$ , whose vertices are (1) and (2) and whose only edge is  $\{(1), (2)\}$ , with itself is a *square*:  $\Theta \times \Theta$  has four vertices and four edges,

$$\bullet - \bullet \quad \times \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad = \quad \begin{array}{cc} \bullet & - & \bullet \\ | & & | \\ \bullet & - & \bullet \end{array} .$$

A graph is  $\times$ -*indecomposable* iff it cannot be written as a proper cartesian product. For instance, the one edge graph  $\Theta$  is  $\times$ -indecomposable but neither a point nor the square is. A graph is *connected* ( $\uplus$ -*indecomposable*) iff it cannot be written as a proper disjoint union.

Suppose we have multiplicities  $m_1, m_2$  on graphs  $\Gamma_1, \Gamma_2$ , respectively. The *disjoint union*  $m_1 \uplus m_2$  is the obvious multiplicity on the disjoint union  $\Gamma_1 \uplus \Gamma_2$ . The *cartesian product*  $m_1 \times m_2$  is the multiplicity on  $\Gamma_1 \times \Gamma_2$  defined by  $(m_1 \times m_2)(\alpha_1, \alpha_2) = m_1(\alpha_1) \cdot m_2(\alpha_2)$ .

The operations *disjoint union* and *cartesian product* carry over to valued graphs. Finally, it is easy to compose isomorphisms by these operations.

### 2.3. Natural Numbers

We note a consequence of unique factorization of natural numbers.

LEMMA 2.1. *Let  $m: \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{N}$  be the product of  $m_i: \Lambda_i \rightarrow \mathbb{N}$ . If all values of  $m = m_1 \times m_2$  are divisible by a fixed  $n \in \mathbb{N}$ , then there is a decomposition  $n = n_1 \cdot n_2$  such that all values of  $m_i$  are divisible by  $n_i$ .*

## 2.4. Cartesian Product

In order to state the basic results on the cartesian product of graphs, we need two further notions. A  $\Gamma_1$ -slice in a product  $\Gamma_1 \times \Gamma_2$  is an induced subgraph  $\Gamma_1 \times \alpha_2$  with vertex set  $V(\Gamma_1) \times \{\alpha_2\}$  for some vertex  $\alpha_2 \in \Gamma_2$ . (An induced subgraph inherits all edges from the parent that are contained in the new vertex set. By abuse of notation,  $\alpha_2$  also denotes the graph with only vertex  $\alpha_2$ .) By definition of the product, such a  $\Gamma_1$ -slice is isomorphic to  $\Gamma_1$ . Similarly, we define  $\Gamma_2$ -slices. A slice in a product  $\Gamma_1 \times \cdots \times \Gamma_r$  is an induced subgraph of the form  $\Delta_1 \times \cdots \times \Delta_r$  such that, for each  $i$ , either  $\Delta_i = \Gamma_i$  or  $\Delta_i$  is a point. A  $\Gamma_1 \times \Gamma_2$ -rectangle is an induced subgraph  $\Delta_1 \times \Delta_2$  of  $\Gamma_1 \times \Gamma_2$ , i.e., an induced subgraph whose vertex set is a product of subsets of the vertex sets of the  $\Gamma_i$ .

**CARTESIAN PRODUCT LEMMA 2.2** (Imrich [4]). *Let  $\delta: \Gamma_1 \times \Gamma_2 \rightarrow \Delta_1 \times \Delta_2$  be an isomorphism of connected graphs. Then:*

- (i) *Every  $\Gamma_1$ -slice is mapped onto a  $\Delta_1 \times \Delta_2$ -rectangle.*
- (ii) *If every  $\Gamma_1$ -slice is mapped into a  $\Delta_1$ -slice or into a  $\Delta_2$ -slice, then all  $\Gamma_1$ -slices are mapped into the same kind of slices: either all  $\Gamma_1$ -slices are mapped into  $\Delta_1$ -slices or all  $\Gamma_1$ -slices are mapped into  $\Delta_2$ -slices.*
- (iii) *If every  $\Gamma_1$ -slice is mapped onto a  $\Delta_1$ -slice, then  $\delta = \delta_1 \times \delta_2$ , where  $\delta_j: \Gamma_j \rightarrow \Delta_j$  are graph isomorphisms.*

*Proof.* See Imrich [4]: Satz 1 is (i), the proof of Satz 2 contains (ii), and Lemma 4 is (iii). ■

Moreover, isomorphisms of cartesian products are products:

**COROLLARY 2.3.** *Suppose*

$$\delta: \Gamma_1 \times \cdots \times \Gamma_r \longrightarrow \Delta_1 \times \cdots \times \Delta_s$$

*is a graph isomorphism and all graphs  $\Gamma_i, \Delta_j$  are  $\times$ -indecomposable and connected. Then  $r = s$  and, up to a permutation of the factors,  $\delta$  is a product of isomorphisms. Precisely: there are a bijection  $\sigma: \{1, \dots, r\} \rightarrow \{1, \dots, s\}$  and isomorphisms  $\delta_i: \Gamma_i \rightarrow \Delta_{\sigma(i)}$  such that  $\tilde{\sigma}\delta = \delta_1 \times \cdots \times \delta_r$ , where  $\tilde{\sigma}(\alpha_1, \dots, \alpha_s) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(r)})$  for  $\alpha_j \in \Delta_j$ .*

Imrich [4] proves this, and it also follows directly from Cartesian Product Lemma 2.2. We give a proof of this since the same strategy is reused with algebras later on.

*Proof.* First, we show that there exists an index  $j$  such that  $\delta$  maps every  $\Gamma_1$ -slice  $\Gamma_1 \times \alpha'$  into a  $\Delta_j$ -slice for some  $j$  that might depend on the vertex  $\alpha' \in \Gamma' := \Gamma_2 \times \cdots \times \Gamma_r$ . For every  $j$ , by Cartesian Product Lemma 2.2(i), each  $\Gamma_1$ -slice is mapped to a rectangle in  $\Delta_j \times \Delta'$ , where

$\Delta' = \Delta_1 \times \cdots \times \Delta_{j-1} \times \Delta_{j+1} \times \cdots \times \Delta_s$ . Since  $\Gamma_1$  is  $\times$ -indecomposable, this rectangle must degenerate. So either for some  $j$ , the image of the  $\Gamma_1$ -slice  $\Gamma_1 \times \alpha'$  is contained in a  $\Delta_j$ -slice, or for any index  $j$ , it is contained in  $\Delta_1 \times \cdots \times \Delta_{j-1} \times \beta_j \times \Delta_{j+1} \times \cdots \times \Delta_s$  for some vertex  $\beta_j \in \Delta_j$ . But the latter means that the  $\Gamma_1$ -slice is mapped to the point  $\beta_1 \times \cdots \times \beta_s$ , contradicting that  $\Gamma_1$  is not a point.

Next, note that by Cartesian Product Lemma 2.2(ii), the index  $j$  does actually not depend on  $\alpha'$ .

Now, we prove the assertion by induction on  $r$ . The case  $r = 0$  being evident, we suppose  $r > 0$ . Fix  $j$  according to the claim we just proved. Then we have the following situation: Let  $\Gamma'$  and  $\Delta'$  be, as above, the product of the factors different from  $\Gamma_1$  or  $\Delta_j$ , respectively, and the isomorphism

$$\delta: \Gamma_1 \times \Gamma' \longrightarrow \Delta_j \times \Delta'$$

is defined by  $\hat{\delta}(\alpha_1, \dots, \alpha_s) = (\beta_j, \beta_1, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_s)$  for  $\delta(\alpha_1, \dots, \alpha_s) = (\beta_1, \dots, \beta_s)$ , the graphs  $\Gamma_1$  and  $\Delta_j$  are  $\times$ -indecomposable, and  $\hat{\delta}$  maps  $\Gamma_1$ -slices into  $\Delta_j$ -slices. Take any  $\alpha' \in \Gamma'$  and choose  $\beta' \in \Delta'$  with  $\hat{\delta}(\Gamma_1 \times \alpha') \subseteq \Delta_j \times \beta'$ . Then  $\hat{\delta}^{-1}(\Delta_j \times \beta')$  also is a degenerate rectangle that further contains  $\Gamma_1 \times \alpha'$ . Since  $\Gamma_1$  is non-trivial, we obtain  $\hat{\delta}(\Gamma_1 \times \alpha') = \Delta_j \times \beta'$ . Thus we have that any  $\Gamma_1$ -slice is mapped onto a  $\Delta_j$ -slice and, by Cartesian Product Lemma 2.2(iii), we obtain isomorphisms  $\delta_1: \Gamma_1 \rightarrow \Delta_j$  and  $\delta': \Gamma' \rightarrow \Delta'$ , with  $\hat{\delta} = \delta_1 \times \delta'$ . By induction, we have a bijection  $\sigma': \{2, \dots, r\} \rightarrow \{1, \dots, j-1, j+1, \dots, s\}$  and isomorphisms  $\delta_j: \Gamma_j \rightarrow \Delta_{\sigma'j}$  such that  $\sigma'\delta' = \delta_2 \times \cdots \times \delta_r$ . Extending  $\sigma'$  to  $\sigma: \{1, \dots, r\} \rightarrow \{1, \dots, s\}$  by  $\sigma(1) = j$ , we obtain the assertion. ■

Now we can read off the following stronger version of Cartesian Product Lemma 2.2(ii).

**COROLLARY 2.4.** *Let  $\delta: \Gamma' \times \Gamma'' \rightarrow \Delta' \times \Delta''$  be an isomorphism of connected graphs. If one  $\Gamma'$ -slice is mapped into a  $\Delta'$ -slice, then every  $\Gamma'$ -slice is mapped into a  $\Delta'$ -slice.*

### 3. ALGEBRAS AND GRAPHS

In this section, we study the valued graph of an algebra and the mentioned inverse construction, the quotient transform. We work out their nice structure for use in the next section.

The *valued graph*  $\Delta'(A)$  of an algebra  $A$  is a valued graph consisting of the graph  $\Delta(A)$  of the algebra  $A$  and the *multiplicity*  $\text{mult}(A)$  of  $A$ : Its vertices are the isomorphism classes  $[P]$  of indecomposable projective  $A$ -modules  $P$ . Next, two different vertices  $[P_1], [P_2]$  are connected by an edge iff there

is a projectively irreducible morphism  $P_1 \rightarrow P_2$ , or  $P_2 \rightarrow P_1$ . Here, a morphism  $\psi: P_1 \rightarrow P_2$  of indecomposable projectives is called *projectively irreducible* iff it is no isomorphism and cannot be written as a sum  $\sum_\nu \psi_{2\nu} \psi_{1\nu}$  of compositions of non-isomorphisms  $\psi_{1\nu}: P_1 \rightarrow R_\nu$  and  $\psi_{2\nu}: R_\nu \rightarrow P_2$  via indecomposable projectives  $R_\nu$ . This is very close to the definition of the Gabriel quiver: this directed graph additionally measures the direction and frequency of projectively irreducible morphisms and this also from a vertex to itself. In other words, the graph of an algebra is the Gabriel quiver without directions, loops, and multiple edges. Finally, the multiplicity  $\text{mult}(A)$  is defined by decomposing  $A \simeq \bigoplus_\nu \text{mult}(A)([P_\nu]) P_\nu$  as a module with pairwise non-isomorphic indecomposable projectives  $P_\nu$ .

Consider an isomorphism  $\varphi: A_1 \rightarrow A_2$  of algebras. It induces an isomorphism  $\Delta'(\varphi): \Delta'(A_1) \rightarrow \Delta'(A_2)$  as follows:  $\Delta'(\varphi)$  maps the class  $[P_1]$  to the class  $[P_1^{\varphi^{-1}}]$  of the scalar restriction of  $P_1$  via  $\varphi^{-1}$ . Note that, if  $P_1$  is embedded in  $A_1$  as an  $A_1$ -module, then the scalar restriction  $P_1^{\varphi^{-1}}$  is isomorphic to the  $A_2$ -module  $\varphi P_1$  via  $\varphi$ . We let  $\Delta'(\varphi)_{[p_1]}$  be the identity of  $\{1, \dots, \text{mult}(A_1)([P_1])\}$ . Then  $\Delta'(\varphi)$  is an isomorphism of valued graphs.

EXAMPLE 3.1. Consider the algebra  $T_n$  of upper triangular  $n \times n$ -matrices over  $k$ . Let  $E_{\nu\mu}$  denote the matrix with the only non-zero entry at position  $(\nu, \mu)$  and equal to 1. Then  $1 = E_{11} + \dots + E_{nn}$  is a maximal 1-decomposition; i.e., each  $E_{\nu\nu}$  is idempotent, they are pairwise orthogonal ( $E_{\mu\mu} E_{\nu\nu} = 0$  for  $\mu \neq \nu$ ), and maximal with these properties. Thus, as a module, the algebra decomposes as  $T_n = T_n E_{11} \oplus \dots \oplus T_n E_{nn}$ ; the summands correspond to columns. Let  $(\nu) := [T_n E_{\nu\nu}]$  denote the isomorphism class of the projective module  $T_n E_{\nu\nu}$ . Since  $\dim_k T_n E_{\nu\nu} = \nu$ , they are pairwise different. Hence,  $\{(\nu) | \nu \in \{1, \dots, n\}\}$  is the vertex set of the valued graph of  $T_n$  and the multiplicity is all 1. Further, the morphisms

$$\varphi_{\nu\mu}: \begin{array}{l} T_n E_{\mu\mu} \longrightarrow T_n E_{\nu\nu}, \\ x \longmapsto x E_{\mu\nu}, \end{array}$$

for  $\nu \geq \mu$ , are up to scalar multiple all morphisms between the given modules. Obviously,  $\varphi_{\nu\mu} = \varphi_{\nu\lambda} \varphi_{\lambda\mu}$  if  $\nu \geq \lambda \geq \mu$ . Consequently,  $\varphi_{\nu, \nu+1}$  is a projectively irreducible morphism and there is an edge between  $(\nu+1)$  and  $(\nu)$ . Thus

$$\Delta'(T_n) = \left( \begin{array}{cccc} \begin{array}{c} \bullet \\ (1) \end{array} - \begin{array}{c} \bullet \\ (2) \end{array} - \begin{array}{c} \bullet \\ (3) \end{array} & \cdots & \begin{array}{c} \bullet \\ (n-1) \end{array} - \begin{array}{c} \bullet \\ (n) \end{array} \end{array} \right),$$

where we have attached the names below and the multiplicities above the bullets. Observe that the vertices correspond to the diagonal positions and the edges correspond to the near diagonal positions.

**GRAPH LEMMA 3.2.** *Suppose  $A_i$  is an algebra for  $i \in \{0, 1, 2\}$ . Let  $A = A_0$ . Then the above construction defines a sincere valued graph  $\Delta'(A)$  consisting of a graph  $\Delta(A)$  and a multiplicity  $\text{mult}(A)$  on it. It is a functor for algebra isomorphisms. Further we have:*

- (i)  $\Delta'(A_1 \oplus A_2) = \Delta'(A_1) \uplus \Delta'(A_2)$ .  $\Delta'(\varphi_1 \oplus \varphi_2) = \Delta'(\varphi_1) \uplus \Delta'(\varphi_2)$ .
- (ii)  $\Delta'(A_1 \otimes A_2) = \Delta'(A_1) \times \Delta'(A_2)$ .  $\Delta'(\varphi_1 \otimes \varphi_2) = \Delta'(\varphi_1) \times \Delta'(\varphi_2)$ .
- (iii) *The algebra  $A$  is  $\oplus$ -indecomposable iff the graph  $\Delta'(A)$  is connected.*
- (iv) *The tensor product of two  $\oplus$ -indecomposable algebras is again  $\oplus$ -indecomposable.*
- (v) *The algebra  $A$  is local iff the graph  $\Delta'(A)$  is a (sincere) simple point.*

*Proof.* This combines Graph Lemma 7, Corollary 8, and a subsequent remark from Nüsken [6]. ■

**EXAMPLE 3.1 (continued).** (i)  $\Delta'(T_2 \oplus T_1) = \begin{pmatrix} 1 & & 1 \\ \bullet & - & \bullet \end{pmatrix} = \Delta'(T_2) \uplus \Delta'(T_1)$ .

$$(ii) \quad \Delta'(T_2 \otimes T_2) = \begin{pmatrix} 1 & & 1 \\ \bullet & - & \bullet \\ | & & | \\ \bullet & - & \bullet \end{pmatrix} = \Delta'(T_2) \times \Delta'(T_2).$$

(iii) The algebra  $T_n$  is  $\oplus$ -indecomposable; its valued graph  $\Delta'(T_n)$  is  $\uplus$ -indecomposable. (Be not misled by the module decomposition of  $T_n$ .)

(iv) The algebra  $T_2$  is  $\oplus$ -indecomposable, as is  $T_2 \times T_2$ .

(v) Among the algebras  $T_n$ , only  $T_1 = k$  is local.

**EXAMPLE 3.3.** As a non-trivial example of a local algebra, consider the algebra  $A = k[x]/\langle x^2 \rangle$ . Its valued graph is  $\Delta'(A) = \begin{pmatrix} 1 \\ \bullet \end{pmatrix}$ . (By the way, the Gabriel quiver of  $A$  is non-trivial:  $Q(A) = (\bullet \circ \bullet)$ !)

Next, we construct the inner transform. To any algebra  $A$  with a multiplicity  $m$  on its graph, define the projective  $A$ -module  $V_{A,m} = \bigoplus_{P \in \mathcal{P}_A} m([P]) P$ , where  $\mathcal{P}_A$  is a fixed system of representatives for the isomorphism classes of indecomposable projective  $A$ -modules. Note that as  $A$ -modules,  $V_{A, \text{mult}(A)} \simeq A$ . Consider an isomorphism  $\varphi: A_1 \rightarrow A_2$  of algebras, and a morphism  $\sigma: m_1 \rightarrow m_2$  of multiplicities with underlying graph morphism  $\underline{\sigma} = \Delta(\varphi)$ . Note that  $V_{A_2, m_2}$  is an  $A_1$ -module via  $\varphi$ . Choose an  $A_1$ -linear projection  $\pi_{\varphi, \sigma}: V_{A_2, m_2} \rightarrow V_{A_1, m_1}$  and an  $A_1$ -linear injection  $\iota_{\varphi, \sigma}: V_{A_1, m_1} \rightarrow V_{A_2, m_2}$  such that  $\pi_{\varphi, \sigma} \iota_{\varphi, \sigma} = \text{id}_{V_{A_1, m_1}}$ . Now, the



inner transform is given by

$$\begin{aligned} M(A, m) &:= \text{End}_A^{\text{op}} V_{A, m}, \\ M(\varphi, \sigma): \quad M(A_1, m_1) &\longrightarrow M(A_2, m_2), \\ \eta &\longmapsto \iota_{\varphi, \sigma} \eta \pi_{\varphi, \sigma}. \end{aligned}$$

Clearly,  $M(A, m)$  is an algebra and  $M(\varphi, \sigma)$  is a not necessarily unitary algebra morphism. Since  $\pi_{\varphi, \sigma} \iota_{\varphi, \sigma} = \text{id}_{V_{A_1, m_1}}$ , the transformed morphism  $M(\varphi, \sigma)$  is always injective. If  $\sigma$  is an isomorphism, then  $\pi_{\varphi, \sigma}$  and  $\iota_{\varphi, \sigma}$  are isomorphisms, and so is  $M(\varphi, \sigma)$ . In case  $m$  is sincere,  $M(A, m)$  is in fact a Morita transform of  $A$ .

*Small Case.* It might be helpful to consider a special case: if  $m$  is pointwise less than or equal to  $\text{mult}(A)$ , then the projective  $V_{A, m}$  is isomorphic to  $Ae$  for some idempotent  $e \in A$ . So the inner transform  $M(A, m)$  is isomorphic to  $eAe$ , then. This already shows that we “cut off” vertices outside  $\text{supp } m$ , as is formally stated in Inner Transform Lemma 3.4(i).

EXAMPLE 3.1 (continued). On  $\Delta'(T_2)$ , consider the multiplicity  $m_2$  given by  $m_2((1)) = 1$ ,  $m_2((2)) = 2$ . Then

$$B_2 := M(T_2, m_2) = \left[ \begin{array}{c|cc} k & k & k \\ \hline & k & k \\ & k & k \end{array} \right].$$

Indeed,  $V_{T_2, m_2} = T_2 E_{11} \oplus T_2 E_{22} \oplus T_2 E_{22}$ . Its opposite endomorphism algebra is

$$B_2 = \left[ \begin{array}{ccc} H_{11} & H_{21} & H_{21} \\ H_{12} & H_{22} & H_{22} \\ H_{12} & H_{22} & H_{22} \end{array} \right],$$

and we already observed that  $H_{\nu\mu} := \text{Hom}_{T_2}(T_2 E_{\mu\mu}, T_2 E_{\nu\nu})$  is  $k$  if  $\nu \geq \mu$  and 0 otherwise.

Similarly, on  $\Delta'(T_3)$ , the multiplicity  $m_3$  with  $m_3((1)) = 1$ ,  $m_3((2)) = 0$ ,  $m_3((3)) = 1$  yields

$$B_3 := M(T_3, m_3) = \left[ \begin{array}{cc} k & k \\ & k \end{array} \right] \simeq T_2.$$

Note that the non-diagonal entry stems from  $\text{Hom}_{T_3}(T_3 E_{11}, T_3 E_{33})$ , which only contains projectively reducible morphisms. Yet, in  $B_3$ , their counterparts are projectively irreducible.

Some properties of the inner transform are collected in the following result.

**INNER TRANSFORM LEMMA 3.4.** *Suppose  $A_i$  is an algebra, and  $m_i$  is a multiplicity on its graph  $\Delta'(A_i)$ , for  $i \in \{0, 1, 2\}$ . Let  $A = A_0$ , and  $m = m_0$ . To any algebra  $A$  and any multiplicity  $m$  on  $\Delta'(A)$ , the above construction associates an algebra  $M(A, m)$ , the inner transform of  $A$  by  $m$ . For an algebra isomorphism  $\varphi: A_1 \rightarrow A_2$  and a morphism  $\sigma: m_1 \rightarrow m_2$  of multiplicities, with  $\underline{\sigma} = \Delta(\varphi)$  the transformed morphism,  $M(\varphi, \sigma): M(A_1, m_1) \rightarrow M(A_2, m_2)$  is a not necessarily unitary algebra morphism. It is always injective. And if  $\sigma$  is an isomorphism,  $M(\varphi, \sigma)$  is an isomorphism as well. Further, we have:*

(i) *There is a bijective valued graph morphism  $\vartheta_{A, m}: \text{supp } m \rightarrow \Delta'(M(A, m))$  which is an isomorphism of valued sets (in other words, the valued graph  $\Delta'(M(A, m))$  of the transformed algebra may have more edges but all other data are given by  $\text{supp } m$ ), and such that*

(ii)  *$\Delta'(M(\varphi, \sigma))\vartheta_{A_1, m_1} = \vartheta_{A_2, m_2}\Delta'(\varphi)$  for any pair of isomorphisms  $\varphi: A_1 \rightarrow A_2$  and  $\sigma: m_1 \rightarrow m_2$ , with  $\underline{\sigma} = \Delta(\varphi)$ .*

(iii)  *$M(A, \text{mult}(A)) \simeq A$ .*

(iv)  *$M(A_1 \otimes A_2, m_1 \times m_2) \simeq M(A_1, m_1) \otimes M(A_2, m_2)$ .*

*Proof.* This is Transform Lemma 9 in Nüsken [6]. The fact that  $\vartheta_{A, m}$  maps edges to edges is proven in Nüsken [5]. ■

*Small Case.* For better understanding, we sketch parts of the proof in case that  $V_{A, m} \simeq Ae$  for some idempotent  $e \in A$ .

(i) Write  $e = \sum_{\nu} e_{\nu}$  as a sum of primitive, pairwise orthogonal idempotents  $e_{\nu}$ . Then  $V_{A, m} \simeq \bigoplus_{\nu} Ae_{\nu}$  and  $B := M(A, m) = \text{End}_A^{\text{op}} V_{A, m} = eAe = \bigoplus_{\nu} V_e Ae_{\nu}$ . Since  $e_{\nu}$  is again primitive in  $B$ ,  $B$ 's projectives correspond to  $A$ 's projectives; their multiplicity is given by  $m$ . An edge  $\{\nu, \mu\}$  in  $\Delta'(B)$  indicates positive dimension of  $e_{\mu}(\text{rad } B)e_{\nu}/e_{\mu}(\text{rad}^2 B)e_{\nu}$ , or its counterpart with  $\nu$  and  $\mu$  exchanged. Now,  $\text{rad } B = \text{rad } eAe = e(\text{rad } A)e$ ,  $\text{rad}^2 B = \text{rad}^2 eAe = e(\text{rad } A)e(\text{rad } A)e \subseteq e(\text{rad}^2 A)e$ . Together this implies that  $e_{\mu}(\text{rad } B)e_{\nu}/e_{\mu}(\text{rad}^2 B)e_{\nu}$  is at least as large as  $e_{\mu}(\text{rad } A)e_{\nu}/e_{\mu}(\text{rad}^2 A)e_{\nu}$ .

(iii) Here  $e = 1$ .

(iv) Note  $(e_1 \otimes e_2)(A_1 \otimes A_2)(e_1 \otimes e_2) = e_1 A_1 e_1 \otimes e_2 A_2 e_2$ . ■

**EXAMPLE 3.1 (continued).** (i) The inner transform of  $T_3$  by  $m_3$  shows that  $\vartheta_{A, m}$  needs not to be an isomorphism of valued graphs:

$$\vartheta_{T_3, m_3}: \begin{pmatrix} 1 & 1 \\ \bullet & \bullet \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ \bullet & - \bullet \end{pmatrix}.$$

It clearly is an isomorphism of the underlying valued sets.

(iv) Let  $m_1 = \text{mult}(T_2)$ . Then

$$\begin{aligned} M(T_2 \otimes T_2, m_2 \times m_1) &= M(T_2, m_2) \otimes M(T_2, m_1) \\ &= M(T_2, m_2) \otimes T_2 \\ &= \begin{bmatrix} k & k & k & k & k & k \\ & k & k & & k & k \\ & & k & k & & k \\ & & & k & k & k \\ & & & & k & k \\ & & & & & k \end{bmatrix}. \end{aligned}$$

**COROLLARY 3.5.** *Let  $A$  be an arbitrary algebra. If  $A$  is  $\otimes$ -indecomposable, then  $\text{mult}(A)$  is primitive (i.e., its values do not have a non-trivial common divisor), or  $A$  is a matrix algebra over  $k$  with prime multiplicity.*

*Proof.* Suppose that  $\text{mult}(A)$  is not primitive. Then  $\text{mult}(A) = p \cdot m$ , with  $p \in \mathbb{N}_{>1}$  prime for some multiplicity  $m$  on  $\Delta(A)$ . Reading  $p$  as a multiplicity on the point  $\Delta'(k)$ , we have

$$A \simeq M(A, p \cdot m) \simeq M(k, p) \otimes M(A, m),$$

with  $M(k, p) \not\simeq k$ . Since  $A$  is  $\otimes$ -indecomposable, we must have  $M(A, m) \simeq k$ . But then the equation reads  $A \simeq M(k, p)$ . Thus  $A$  is a matrix algebra over  $k$  with prime multiplicity. ■

Now, we aim at constructing the quotient transform. As with the inner transform, we want to “cut off” some vertices of the graph or change their multiplicities. However, we want to eliminate the traces of the “cut off” vertices more effectively. We start with an algebra  $A$  and a multiplicity  $m$  on its graph  $\Delta(A)$ . We say that  $\hat{m}$  *complements*  $m$  iff  $m + \hat{m}$  is sincere and the supports of  $m$  and  $\hat{m}$  are disjoint. Denote by  $\tilde{m}$  the smallest complementary multiplicity; i.e.,  $\tilde{m}(\alpha) = 1$  if  $m(\alpha) = 0$  and  $\tilde{m}(\alpha) = 0$  otherwise. Furthermore, let  $\tilde{\varepsilon}: \tilde{m} \rightarrow m + \tilde{m}$  be the canonical embedding, i.e., the morphism with  $\tilde{\varepsilon} = \Delta(\text{id}_A)$  and  $\tilde{\varepsilon}_\alpha(x) = m(\alpha) + x$  for  $\alpha \in \Delta(A)$ ,  $x \in \{1, \dots, \tilde{m}(\alpha)\}$ . Applying the inner transform to  $\tilde{\varepsilon}$  gives us a morphism  $M(A, \tilde{m}) \rightarrow M(A, m + \tilde{m})$  whose cokernel in the category of algebras (i.e., the quotient of  $M(A, m + \tilde{m})$  by the ideal generated by the image of this morphism) shall be the *quotient transform*

$$Q(A, m) := \text{coker } M(\text{id}_A, \tilde{\varepsilon})$$

of  $A$  by  $m$ . Note that—up to isomorphism—this does not depend on the choice of  $\tilde{e}$ . We can also describe the quotient transform as

$$Q(A, m) = \text{End}_A^{\text{op}} V_{A, m+\tilde{m}} / \hat{I} \simeq \text{End}_A^{\text{op}} V_{A, m} / I,$$

where the ideal  $\hat{I}$  (or  $I$ , respectively) is generated by all endomorphisms of  $V_{A, m+\tilde{m}}$  (or  $V_{A, m}$ , respectively) that factor via  $V_{A, \tilde{m}}$ . The ideals are also generated by those endomorphisms that factor via some indecomposable projective  $P$  with  $\tilde{m}([P]) > 0$ . We see that only the support of  $\tilde{m}$  really matters. Furthermore, we have a surjection  $M(A, m) \twoheadrightarrow Q(A, m)$ . We add that, in case  $m$  is sincere, the quotient transform  $Q(A, m)$  is isomorphic to the inner transform  $M(A, m)$ .

*Small Case.* If, as above,  $V_{A, m} \simeq Ae$  for some idempotent  $e \in A$ , then  $V_{A, \tilde{m}} \simeq A\tilde{e}$  for some idempotent  $\tilde{e}$  orthogonal to  $e$ . So the quotient transform of  $A$  by  $m$  is isomorphic to  $(e + \tilde{e})A(e + \tilde{e})/\langle \tilde{e} \rangle$  and  $eAe/eA\tilde{e}Ae$ . If  $\tilde{e}$  can be chosen to be  $1 - e$ , in other words, if  $m(\alpha) = \text{mult}(A)(\alpha)$  or  $m(\alpha) = 0$  for all  $\alpha$ , then this can also be read as  $A/\langle 1 - e \rangle$ . This shows that the quotient transform also “cuts off” the vertices outside  $\text{supp } m$ .

EXAMPLE 3.1 (continued). Consider the quotient transform of  $T_3$  by  $m_3$ : We already have calculated  $M(T_3, m_3) \simeq T_2$ . The smallest complementary multiplicity  $\tilde{m}_3$  is given by  $\tilde{m}_3((1)) = 0$ ,  $\tilde{m}_3((2)) = 1$ ,  $\tilde{m}_3((3)) = 0$ . So  $M(T_3, \tilde{m}_3) = k$  and  $M(T_3, m_3 + \tilde{m}_3) = M(T_3, \text{mult}(T_3)) = T_3$ . Since  $M(T_3, \tilde{m}_3)$  is mapped to  $k \cdot E_{22}$  and we must divide by the thereof generated ideal, we obtain

$$Q(T_3, m_3) = T_3 / \langle E_{22} \rangle = \begin{bmatrix} k & \\ & k \end{bmatrix}.$$

Comparing this to the inner transform of  $T_3$  by  $m_3$  shows that the quotient transform is indeed the stronger concept, meaning that the latter can be strictly smaller.

EXAMPLE 3.6. For the algebra  $k(\bullet \xrightarrow{b} \bullet) / \langle aba, bab \rangle$ , we even find that the quotient transform by any simple point is  $k$ , whereas the inner transform is different, namely,  $k[x]/\langle x^2 \rangle$ .

Next, we extend the quotient transform to isomorphisms. Consider an isomorphism  $\varphi: A_1 \rightarrow A_2$  of algebras and an isomorphism  $\sigma: m_1 \rightarrow m_2$  of multiplicities with underlying graph morphism  $\underline{\sigma} = \Delta(\varphi)$ . We construct an isomorphism  $Q(\varphi, \sigma): Q(A_1, m_1) \xrightarrow{\sim} Q(A_2, m_2)$ . Since  $\sigma$  is an isomorphism, we have an isomorphism  $\tilde{\sigma}: \tilde{m}_1 \rightarrow \tilde{m}_2$ , with  $\tilde{\sigma} = \underline{\sigma} = \Delta(\varphi)$ . These two isomorphisms induce an isomorphism  $\sigma + \tilde{\sigma}: m_1 + \tilde{m}_1 \rightarrow m_2 + \tilde{m}_2$ , and applying the inner transform and taking cokernels yields the desired

isomorphism  $Q(\varphi, \sigma)$  :

$$\begin{array}{ccccc}
 M(A_1, \tilde{m}_1) & \xrightarrow{M(\text{id}_{A_1}, \tilde{e}_1)} & M(A_1, m_1 + \tilde{m}_1) & \longrightarrow & Q(A_1, m_1) \\
 \downarrow M(\varphi, \tilde{\sigma}) & & \downarrow M(\varphi, \sigma + \tilde{\sigma}) & & \downarrow Q(\varphi, \sigma) \\
 M(A_2, \tilde{m}_2) & \xrightarrow{M(\text{id}_{A_2}, \tilde{e}_2)} & M(A_2, m_2 + \tilde{m}_2) & \longrightarrow & Q(A_2, m_2)
 \end{array}$$

We only remark that some more functoriality could be achieved.

**QUOTIENT TRANSFORM LEMMA 3.7.** *Suppose  $A_i$  is an algebra, and suppose  $m_i$  is a multiplicity on its graph  $\Delta'(A_i)$  for  $i \in \{0, 1, 2\}$ . Let  $A = A_0$ , and  $m = m_0$ . To any algebra  $A$  and any multiplicity  $m$  on  $\Delta'(A)$ , the above construction associates an algebra  $Q(A, m)$ , the quotient transform of  $A$  by  $m$ . For an algebra isomorphism  $\varphi: A_1 \rightarrow A_2$  and an isomorphism  $\sigma: m_1 \rightarrow m_2$  of multiplicities, with  $\underline{\sigma} = \Delta(\varphi)$ , the transformed morphism  $Q(\varphi, \sigma): Q(A_1, m_1) \rightarrow Q(A_2, m_2)$  is an isomorphism, too. Further we have:*

- (i) *There is a graph isomorphism  $\theta_{A, m}: \text{supp } m \rightarrow \Delta'(Q(A, m))$  such that*
- (ii)  *$\Delta'(Q(\varphi, \sigma))\theta_{A_1, m_1} = \theta_{A_2, m_2}\Delta'(\varphi)$  for any pair of isomorphisms  $\varphi: A_1 \rightarrow A_2$  and  $\sigma: m_1 \rightarrow m_2$ , with  $\underline{\sigma} = \Delta(\varphi)$ .*
- (iii)  *$Q(A, \text{mult}(A)) \simeq A$ .*
- (iv)  *$Q(A_1 \otimes A_2, m_1 \times m_2) \simeq Q(A_1, m_1) \otimes Q(A_2, m_2)$ .*

*Proof.* (i) Let  $P$  be a projective  $A$ -module. Then  $P' := \text{Hom}_A(V_{A, m+\tilde{m}}, P)$  is a projective module over  $A' := M(A, m + \tilde{m}) = \text{End}_A^{\text{op}}(V_{A, m+\tilde{m}})$ . Let  $I_P \subseteq P'$  be the submodule of all morphisms  $V_{A, m+\tilde{m}} \rightarrow P$  factoring via a multiple of  $V_{A, \tilde{m}}$ . Then  $\bar{P} := P'/I_P$  is a projective module over  $\bar{A} := Q(A, m) = A'/I_{V_{A, m+\tilde{m}}}$ . Indeed, if  $V_{A, m+\tilde{m}} = \bigoplus P_\nu$ , then  $A' = \bigoplus P'_\nu$  and  $I_{V_{A, m+\tilde{m}}} = \bigoplus I_{P_\nu}$ , and hence  $\bar{A} = \bigoplus \bar{P}_\nu$ . So  $P'$  and  $\bar{P}$  are projective as claimed. Clearly,  $\bar{P} = 0$  if  $P$  is an indecomposable projective with  $m([P]) = 0$ . We shall prove below that  $\bar{P}$  is indecomposable if  $P$  is indecomposable and  $[P] \in \text{supp } m$ . Thus each indecomposable projective over  $\bar{A}$  occurs up to isomorphism, and we can define a surjective map

$$\begin{aligned}
 \theta: \text{supp } m &\longrightarrow \Delta'(\bar{A}) = \Delta'(Q(A, m)), \\
 [P] &\longmapsto [\bar{P}].
 \end{aligned}$$

Furthermore, we will show that  $\theta$  is injective and a graph isomorphism. Then  $\theta$  is an isomorphism of valued graphs, since by  $\bar{A} = \bigoplus \bar{P}_\nu$  it is clear that also the multiplicities behave well:  $\text{mult}(Q(A, m))([\bar{P}]) = m([P])$ .

Since indecomposability and the edges of  $\Delta'(-)$  can be recognized via morphism spaces, we will deal with module morphisms. A morphism

$\psi: P_1 \rightarrow P_2$  of projective  $A$ -modules  $P_1, P_2$  defines a morphism  $\psi: P'_1 \rightarrow P'_2$ ,  $\eta \mapsto \psi \circ \eta$ . If  $\eta \in I_{P_1}$ , i.e.,  $\eta$  factors via a multiple of  $V_{A, \tilde{m}}$ , then so does  $\psi \circ \eta$ , and  $\psi \circ \eta \in I_{P'_2}$ . Hence,  $\psi'$  induces a morphism  $\bar{\psi}: \bar{P}_1 \rightarrow \bar{P}_2$ . Altogether we have a (linear) map

$$\begin{aligned} - : \quad \text{Hom}_A(P_1, P_2) &\longrightarrow \text{Hom}_{\bar{A}}(\bar{P}_1, \bar{P}_2), \\ \psi &\longmapsto \bar{\psi}, \end{aligned}$$

for each pair  $P_1, P_2$  of indecomposable projectives. These maps behave well with respect to concatenation:  $\text{id}_{\bar{P}} = \text{id}_{\bar{P}}$ , and if  $\psi = \psi_2 \psi_1$ , then  $\bar{\psi} = \bar{\psi}_2 \bar{\psi}_1$ . Thus altogether they define a functor.

Before continuing, we define two subspaces of  $\text{Hom}_A(P_1, P_2)$  for indecomposable projectives  $P_1, P_2$  over an arbitrary algebra  $A$ , namely, the *projective functor radical*  $\text{Rad}_A(P_1, P_2)$  and its *square*: Let  $\text{Rad}_A(P_1, P_2)$  be the space(!) of all non-isomorphisms from  $P_1$  to  $P_2$ . Note that in case  $P_1 \not\simeq P_2$ , this is just  $\text{Hom}_A(P_1, P_2)$  itself, and in case  $P_1 = P_2$ , it is the maximal ideal in the local algebra  $\text{End}_A(P_1)$ . Second, let  $\text{Rad}_A^2(P_1, P_2)$  be the space generated by compositions  $\psi_2 \psi_1$  of two non-isomorphisms  $\psi_1: P_1 \rightarrow R$ ,  $\psi_2: R \rightarrow P_2$  between indecomposable projectives. Thus formally  $\text{Rad}_A^2(P_1, P_2) = \sum_R \text{Rad}_A(R, P_2) \circ \text{Rad}_A(P_1, R)$ , where  $R$  runs over  $\mathcal{P}_A$ . Clearly,  $\text{Rad}_A^2(P_1, P_2) \subseteq \text{Rad}_A(P_1, P_2) \subseteq \text{Hom}_A(P_1, P_2)$ . Let  $\text{Hom}_A^{(0)}(P_1, P_2) := \text{Hom}_A(P_1, P_2) / \text{Rad}_A(P_1, P_2)$ . By definition,  $P_1 \simeq P_2$  iff  $\text{Hom}_A^{(0)}(P_1, P_2)$  is non-zero. Recall that in  $\Delta'(A)$ , there is an edge between  $[P_1]$  and  $[P_2]$  iff there is a projectively irreducible morphism  $P_1 \rightarrow P_2$ , or  $P_2 \rightarrow P_1$ . But a projectively irreducible morphism  $P_1 \rightarrow P_2$  is an element of  $\text{Rad}_A(P_1, P_2) \setminus \text{Rad}_A^2(P_1, P_2)$ . Hence there is an edge iff  $\text{Hom}_A^{(1)}(P_1, P_2) := \text{Rad}_A(P_1, P_2) / \text{Rad}_A^2(P_1, P_2)$  or  $\text{Hom}_A^{(1)}(P_2, P_1)$  is non-zero.

(1) The map  $-$  is surjective.

Indeed, suppose  $\zeta: \bar{P}_1 \rightarrow \bar{P}_2$  is an  $\bar{A}$ -module morphism. By scalar restriction, this is also an  $A'$ -module morphism which can be lifted to  $\xi: P'_1 \rightarrow P'_2$ , since  $P'_1$  is a projective  $A'$ -module. By Yoneda's Lemma,  $\xi = \psi'$  for some  $\psi: P_1 \rightarrow P_2$ . Thus  $\bar{\psi} = \zeta$ , and  $-$  is surjective.

(2) The kernel of  $-$  is contained in  $\text{Rad}_A^2(P_1, P_2)$  if  $P_1, P_2$  are in the support of  $m$ , i.e.,  $m([P_\nu]) \neq 0$  for both  $\nu$ .

Indeed, suppose  $\bar{\psi} = 0$ . Then for any  $\eta: V_{A, m+\tilde{m}} \rightarrow P_1$ , the composition  $\psi \circ \eta$  factors via a multiple of  $V_{A, \tilde{m}}$ . Choose  $P_1 \xrightarrow{\iota} V_{A, m+\tilde{m}} \xrightarrow{\pi} P_1$  with  $\pi \iota = \text{id}_{P_1}$  and consider  $\eta = \pi$ . Then  $\psi = (\psi \eta) \iota$  factors via a multiple of  $V_{A, \tilde{m}}$ . But this means that  $\psi$  is a sum of compositions  $\psi_2 \psi_1$ , where  $\psi_1: P_1 \rightarrow R$ , and  $\psi_2: R \rightarrow P_2$ , and  $R$  is an indecomposable direct summand of  $V_{A, \tilde{m}}$ . Since  $R$  is outside the support of  $m$ ,  $R$  is not isomorphic to  $P_1$  or  $P_2$ . So  $\psi_1 \in \text{Rad}_A(P_1, R)$ ,  $\psi_2 \in \text{Rad}_A(R, P_2)$ , and  $\psi \in \text{Rad}_A^2(P_1, P_2)$ .

By (1),  $-$  is a surjective algebra morphism in case  $P_1 = P_2 = P$ . If  $P$  is indecomposable,  $\text{End}_A(P)$  is local by Fitting's Lemma, and so is its image  $\text{End}_{\bar{A}}(\bar{P})$ . Thus  $\bar{P}$  is indecomposable, too. Clearly,  $\bar{P} \neq 0$  iff  $m([P]) \neq 0$ , since only in this case there are morphisms  $V_{A, m+\tilde{m}} \rightarrow P$  not factoring via  $V_{A, \tilde{m}}$ . Also, if  $P_1 \simeq P_2$ , then  $\bar{P}_1 \simeq \bar{P}_2$ . Thus  $\theta$  is well-defined and surjective (as argued above).

By (2), we obtain isomorphisms  $\text{Hom}_A^{(\ell)}(P_1, P_2) \rightarrow \text{Hom}_{\bar{A}}^{(\ell)}(\bar{P}_1, \bar{P}_2)$ , for  $\ell \in \{0, 1\}$ . Indeed, since  $-$  is surjective and well-behaved with respect to concatenation, also its restrictions to  $\text{Rad}_A(P_1, P_2) \rightarrow \text{Rad}_{\bar{A}}(\bar{P}_1, \bar{P}_2)$  and  $\text{Rad}_A^2(P_1, P_2) \rightarrow \text{Rad}_{\bar{A}}^2(\bar{P}_1, \bar{P}_2)$  are surjective. By (2), this induces isomorphisms as claimed.

So if  $\bar{P}_1 \simeq \bar{P}_2$ , then  $\text{Hom}_A^{(0)}(P_1, P_2) \neq 0$ , hence  $P_1 \simeq P_2$ , proving that  $\theta$  is injective. From  $\text{Hom}_A^{(1)}(P_1, P_2) \simeq \text{Hom}_{\bar{A}}^{(1)}(\bar{P}_1, \bar{P}_2)$ , we see that  $\theta$  is a graph isomorphism.

(ii) Suppose  $P_1 \subseteq A_1$  as a direct summand, calculate  $Q(\varphi, \sigma)(\bar{P}_1)$ , and compare with  $\varphi(P_1)$ .

(iii) Take  $m := \text{mult}(A)$ ; then  $\tilde{m} = 0$ . Hence  $Q(A, m) \simeq M(A, m) \simeq A$ , since then  $V_{A, m} \simeq A$  and  $A \simeq \text{End}_A^{\text{op}} A$ .

(iv) Compare the kernels of the defining projections using the isomorphism from Inner Transform Lemma 3.4(iv). Indeed, note first that  $(m_1 + \tilde{m}_1) \times (m_2 + \tilde{m}_2) = m_1 \times m_2 + \hat{m}$ , with  $\hat{m}$  complementary to  $m_1 \times m_2$ , i.e.,  $\text{supp } \hat{m} = \text{supp } \overline{m_1 \times m_2}$ , which suffices for representing the quotient transform,

$$\begin{array}{ccc}
 I & & I_1 + I_2 \\
 \downarrow & & \downarrow \\
 M(A_1 \otimes A_2, (m_1 + \tilde{m}_1) \times (m_2 + \tilde{m}_2)) & \simeq & M(A_1, m_1 + \tilde{m}_1) \otimes M(A_2, m_2 + \tilde{m}_2) \\
 \downarrow & & \downarrow \\
 Q(A_1 \otimes A_2, m_1 \times m_2) & & Q(A_1, \tilde{m}_1) \otimes Q(A_2, \tilde{m}_2).
 \end{array}$$

By definition,  $I$  is generated by endomorphisms of  $V_{A_1, m_1 + \tilde{m}_1} \otimes V_{A_2, m_2 + \tilde{m}_2}$  factoring via  $V_{A_1, \tilde{m}_1} \otimes V_{A_2, m_2 + \tilde{m}_2}$  or  $V_{A_1, m_1 + \tilde{m}_1} \otimes V_{A_2, \tilde{m}_2}$ . Since  $\text{Hom}_{A_1 \otimes A_2}(V_1 \otimes V_2, W_1 \otimes W_2) = \text{Hom}_{A_1}(V_1, W_1) \otimes \text{Hom}_{A_2}(V_2, W_2)$ , it is already generated by  $\psi_1 \otimes \psi_2$ , where  $\psi_1$  factors via  $V_{A_1, \tilde{m}_1}$  or  $\psi_2$  factors via  $V_{A_2, \tilde{m}_2}$ . But this describes exactly  $I_1 + I_2$ . ■

EXAMPLE 3.1 (continued). For the quotient transform of  $T_3$  by  $m_3$ , we observe that  $\theta_{T_3, m_3}: \begin{pmatrix} 1 & 1 \\ \bullet & \bullet \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ \bullet & \bullet \end{pmatrix}$  is an isomorphism of valued graphs contrary to  $\vartheta_{T_3, m_3}$ .

Combining (i) with Graph Lemma 3.2(v), we immediately obtain the following: If  $m$  is the multiplicity of a simple point, then  $Q(A, m)$  is local. For ease of notation, for a vertex  $\alpha \in \Delta'(A)$  we write  $Q(A, \alpha)$  instead of  $Q(A, e_\alpha)$ , where  $e_\alpha$  is the simple point multiplicity on  $\Delta(A)$ , with  $e_\alpha(\alpha) = 1$  and  $e_\alpha(\beta) = 0$  for  $\beta \neq \alpha$ .

We will always apply the quotient transform as formulated in the following result and then use the other parts of Quotient Transform Lemma 3.7 for further simplification.

**COROLLARY 3.8.** *Let  $\varphi: B \rightarrow C$  be an isomorphism of algebras and let  $\sigma: \ell \rightarrow m$  be an isomorphism of multiplicities with underlying graph morphism  $\underline{\sigma} = \Delta(\varphi)$ . Then*

$$Q(\varphi, \sigma): Q(B, \ell) \rightarrow Q(C, m)$$

*is an isomorphism of algebras whose corresponding valued graph isomorphism essentially is a restriction of  $\Delta'(\varphi)$ , namely,  $\Delta'(Q(\varphi, \sigma))\theta_{B, \ell}(\beta) = \theta_{C, m}\Delta'(\varphi)(\beta)$  for  $\beta \in \text{supp } \ell$ . ■*

#### 4. LOOP-RESISTANT ALGEBRAS

In this section, we prove the main result. The proofs differ only slightly from those in characteristic zero. Several times we will restrict to slices; thus a given tensor factor is restricted to a point. If the result is trivial, i.e., the ground field  $k$ , that takes us one step further. Otherwise we would have to deal with a local factor in a tensor product. At the time of writing, I do not know a way to get rid of such a factor in finite characteristic. Thus we restrict to a class of algebras where this can always be avoided, namely, loop-resistant algebras. From this it should also be clear why we prefer the quotient transform to the inner transform.

We call a  $\oplus$ -indecomposable algebra *loop-resistant* iff one of the quotient restrictions to a simple point multiplicity of its graph is  $k$ , i.e.,  $Q(A, \alpha) = k$  for some vertex  $\alpha \in \Delta'(A)$ . (Then the Gabriel quiver of  $A$  is resistant to attempts to cover its points with loops, and vice versa.) An arbitrary algebra is *loop-resistant* iff each  $\oplus$ -indecomposable direct summand is *loop-resistant*. Yet, for all but the last sentence of Unique Factorization 1.1, we only deal with  $\oplus$ -indecomposable algebras. Note that this definition coincides with the one given in the Introduction by considering an idempotent  $e$  such that all direct summands of  $Ae$  are in the isomorphism class of indecomposable projectives given by  $\alpha$ , whereas all summands of  $A(1 - e)$  are not. Due to Quotient Transform Lemma 3.7(iv), the tensor product  $A_1 \otimes A_2$  is loop-resistant iff both factors,  $A_1$  and  $A_2$ , are so. In particular, a loop-resistant algebra does not have a non-trivial local factor.



The crucial lemma now reads as follows:

LEMMA 4.1. *Let  $\varphi: B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$  be an isomorphism of  $\oplus$ -indecomposable, loop-resistant, split algebras.*

(i) *Suppose that  $B_1$  is  $\otimes$ -indecomposable and  $\Delta(B_1)$  is not a point. Then either every  $\Delta(B_1)$ -slice is mapped into a  $\Delta(C_1)$ -slice or every  $\Delta(B_1)$ -slice is mapped into a  $\Delta(C_2)$ -slice.*

(ii) *Assume that the first case of (i) applies and that  $C_1$  is also  $\otimes$ -indecomposable. Then  $\Delta'(\varphi) = \delta_1 \times \delta_2$ , for some  $\delta_i: \Delta'(B_i) \rightarrow \Delta'(C_i)$ .*

(iii) *If  $\Delta'(\varphi) = \delta_1 \times \delta_2$ , where  $\delta_i: \Delta'(B_i) \rightarrow \Delta'(C_i)$ , then there are isomorphisms  $\varphi_i: B_i \rightarrow C_i$  such that  $\Delta'(\varphi_i) = \delta_i$  for  $i \in \{1, 2\}$ .*

*Proof.* If  $m$  is a multiplicity on  $\Gamma$ , and  $\Lambda \subseteq \Gamma$  is a subgraph, then we say  $m'$  is the projection of  $m$  to  $\Lambda$  iff  $m'(\alpha) = m(\alpha)$ , for  $\alpha \in \Lambda$ , and  $m'(\alpha) = 0$  otherwise.

(i) We obtain an isomorphism  $\Delta'(\varphi): \Delta'(B_1) \times \Delta'(B_2) \rightarrow \Delta'(C_1) \times \Delta'(C_2)$  by Graph Lemma 3.2(ii). Choose  $\beta_2$  such that  $Q(B_2, \beta_2) = k$ . By Cartesian Product Lemma 2.2(i), the slice  $\Delta(B_1) \times \beta_2$  is mapped onto a rectangle  $\Lambda_1 \times \Lambda_2$ . We show that either  $\Lambda_1$  or  $\Lambda_2$  is just a point, i.e., a one vertex graph. Let  $\ell_2$  be the projection of  $B_2$ 's multiplicity to  $\beta_2$ , and write  $m_j$  for the projection of  $C_j$ 's multiplicity on  $\Lambda_j$ . Since  $\Delta'(\varphi)$  is an isomorphism of valued graphs, the value of  $\ell_2$  divides the values of  $m_1 \times m_2$ . By Lemma 2.1, we can write  $\ell_2 = n_1 n_2 \cdot e_{\beta_2}$  such that  $n_j$  divides all values of  $m_j$ . Let  $m'_j := (1/n_j)m_j$ ; then  $\text{mult}(B_1) \times e_{\beta_2} = (m'_1 \times m'_2)\Delta'(\varphi)$ . Using Quotient Transform Lemma 3.7, we get  $B_1 \simeq B_1 \otimes Q(B_2, \beta_2) \simeq Q(C_1, m'_1) \otimes Q(C_2, m'_2)$ . Since  $B_1$  is  $\otimes$ -indecomposable, one of the factors on the right is trivial, say  $Q(C_2, m'_2)$ . So its graph is a simple point. By Quotient Transform Lemma 3.7(i), this graph is  $\Lambda_2$ . And so the rectangle is contained in a slice. Consequently, the  $\Delta(B_1)$ -slice  $\Delta(B_1) \times \beta_2$  is mapped into a  $\Delta(C_1)$ -slice or into a  $\Delta(C_2)$ -slice. Corollary 2.4 completes the first part.

(ii) Since every  $\Delta(B_1)$ -slice is mapped into a  $\Delta(C_1)$ -slice,  $\Delta(C_1)$  cannot be a point and thus, by the first part, all  $\Delta(C_1)$ -slices are mapped into  $\Delta(B_1)$ - or  $\Delta(B_2)$ -slices. The latter case cannot occur, since  $\Delta(\varphi)$  is bijective and  $\Delta(B_1)$  is not a point. Hence every  $\Delta(B_1)$ -slice is mapped onto a  $\Delta(C_1)$ -slice. By Cartesian Product Lemma 2.2(iii),  $\Delta(\varphi)$  is a product.

By Corollary 3.5,  $\text{mult}(B_1)$  and  $\text{mult}(C_1)$  are primitive. Since  $\Delta(\varphi) = \delta_1 \times \delta_2$ , we obtain  $\text{mult}(B_1) \times \text{mult}(B_2) = (\text{mult}(C_1)\delta_1) \times (\text{mult}(C_2)\delta_2)$ . Since the left factors are primitive, the right ones coincide, and so do the left ones. (The values of a primitive multiplicity can be linearly combined to 1.)

(iii) Choose a vertex  $(\beta_1, \beta_2) \in \Delta'(B_1) \times \Delta'(B_2)$  with  $Q(B_1 \otimes B_2, (\beta_1, \beta_2)) = k$  and set  $(\gamma_1, \gamma_2) := (\delta_1\beta_1, \delta_2\beta_2)$ . Abbreviate  $\ell_i := \text{mult}(B_i)$ ,

$m_j := \text{mult}(C_j)$ . Then  $\ell_1 \times \ell_2 = (m_1 \times m_2)\Delta'(\varphi)$ . In particular,  $\Delta'(\varphi)$  restricts to an isomorphism  $\delta_1 \times \varepsilon_2: \ell_1 \times e_{\beta_2} \rightarrow m_1 \times e_{\gamma_2}$ . (Here,  $\varepsilon_2$  is the unique morphism  $e_{\beta_2} \rightarrow e_{\gamma_2}$ .) Thus  $Q(\varphi, \delta_1 \times \varepsilon_2)$  defines an isomorphism

$$\varphi_1: B_1 \otimes Q(B_2, \beta_2) \longrightarrow C_1 \otimes Q(C_2, \gamma_2),$$

using Quotient Transform Lemma 3.7. Moreover,  $\Delta'(\varphi_1) = \delta_1 \times \varepsilon_2$ . Similarly, we obtain

$$\varphi_2: Q(B_1, \beta_1) \otimes B_2 \longrightarrow Q(C_1, \gamma_1) \otimes C_2,$$

with  $\Delta'(\varphi_2) = \varepsilon_1 \times \delta_2$ . Further,  $Q(C_1 \otimes C_2, (\gamma_1, \gamma_2)) = k$ , too, due to the isomorphism  $Q(\varphi, \varepsilon_1 \times \varepsilon_2)$ . Since  $Q(B_i, \beta_i) = k$  and  $Q(C_j, \gamma_j) = k$ , we can identify  $B_i \otimes k \simeq B_i$  and  $C_j \otimes k \simeq C_j$ , and then the morphisms  $\varphi_i$  are the desired ones. ■

**LEMMA 4.2.** *Let  $\varphi: B_1 \otimes B' \rightarrow C_1 \otimes \cdots \otimes C_s$  be an isomorphism of loop-resistant algebras and suppose that  $B_1, C_1, \dots, C_s$  are  $\{\oplus, \otimes\}$ -indecomposable and  $\Delta(B_1)$  is not a point. Then there are an index  $j$  and isomorphisms  $\varphi_1: B_1 \rightarrow C_j$  and  $\varphi: B' \rightarrow C' := C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_s$  such that  $\Delta'(\varphi) = \Delta'(\xi_j \circ (\varphi_1 \otimes \varphi'))$ , where  $\xi_j: C_j \otimes C' \rightarrow C_1 \otimes \cdots \otimes C_s$  is defined by  $\xi_j(c_j \otimes c_1 \otimes \cdots \otimes c_{j-1} \otimes c_{j+1} \otimes \cdots \otimes c_s) = c_1 \otimes \cdots \otimes c_s$ .*

*Proof.* First, we show that there exists an index  $j$  such that  $\Delta(\varphi)$  maps every  $\Delta(B_1)$ -slice into a  $\Delta(C_j)$ -slice of  $\Delta(C_1) \times \cdots \times \Delta(C_s)$ . Otherwise, fix any  $\beta' \in \Delta(B')$ . By Lemma 4.1(i), for every  $j$ , the slice  $\Delta(B_1) \times \beta'$  is mapped into a slice  $\Delta(C_1) \times \cdots \times \Delta(C_{j-1}) \times \gamma_j \times \Delta(C_{j+1}) \times \cdots \times \Delta(C_s)$ , where  $\gamma_j \in \Delta(C_j)$ . Therefore  $\Delta(B_1) \times \beta'$  is mapped to the point  $(\gamma_1, \dots, \gamma_s)$ . But  $\Delta(\varphi)$  is bijective, hence  $\Delta(B_1)$  is a point, which contradicts the assumption.

We may assume that  $j = 1$ . (If necessary, replace  $\varphi$  by  $\xi_j^{-1} \circ \varphi$ .) Then we have the following situation:  $\varphi: B_1 \otimes B' \rightarrow C_1 \otimes C'$  is an isomorphism,  $B_1, C_1$  are  $\otimes$ -indecomposable, and  $\Delta(B_1)$  is not a point. Moreover,  $\Delta(\varphi)$  maps  $\Delta(B_1)$ -slices into  $\Delta(C_1)$ -slices; hence by Lemma 4.1(ii) we have  $\Delta'(\varphi) = \delta_1 \times \delta'$  with  $\delta_1: \Delta'(B_1) \rightarrow \Delta'(C_1)$  and  $\delta': \Delta'(B') \rightarrow \Delta'(C')$ . Now, Lemma 4.1(iii) gives isomorphisms  $\varphi_1: B_1 \rightarrow C_1$  and  $\varphi': B' \rightarrow C'$  such that  $\Delta'(\varphi) = \delta_1 \times \delta' = \Delta'(\varphi_1) \times \Delta'(\varphi') = \Delta'(\varphi_1 \otimes \varphi')$ , using Graph Lemma 3.2(ii) for the last equality. This is the assertion of the present lemma in case  $j = 1$ . ■

**THEOREM 4.3.** *Let  $B_1, \dots, B_r$  and  $C_1, \dots, C_s$  be  $\{\oplus, \otimes\}$ -indecomposable, loop-resistant algebras and let  $\varphi: B_1 \otimes \cdots \otimes B_r \rightarrow C_1 \otimes \cdots \otimes C_s$  be an isomorphism. Then there are a bijection  $\sigma: \{1, \dots, r\} \rightarrow \{1, \dots, s\}$  and isomorphisms  $\psi_i: B_i \rightarrow C_{\sigma i}$ . Moreover, it can be assured that  $\Delta'(\varphi) = \Delta'(\xi \circ (\psi_1 \otimes \cdots \otimes \psi_r))$ , where the algebra isomorphism  $\xi: C_{\sigma 1} \otimes \cdots \otimes C_{\sigma r} \rightarrow C_1 \otimes \cdots \otimes C_s$  is defined by  $\xi(c_{\sigma 1} \otimes \cdots \otimes c_{\sigma r}) = c_1 \otimes \cdots \otimes c_s$ .*

*Proof.* We prove this by induction on  $r$ . The case  $r = 0$  being evident, we suppose  $r > 0$ . For convenience, sort the  $B_i$  by descending graph size and, whenever the graphs are just points, by descending multiplicities. We claim that, for some  $j$ , there are isomorphisms  $\psi_1: B_1 \rightarrow C_j$  and  $\varphi': B_2 \otimes \cdots \otimes B_r \rightarrow C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_s$  such that  $\Delta'(\varphi) = \Delta'(\zeta_j \circ (\psi_1 \otimes \varphi'))$ , with  $\zeta_j$  as in Lemma 4.2. If  $\Delta(B_1)$  is not a point, this is the statement of Lemma 4.2. Otherwise, all graphs  $\Delta(B_i)$  are points; in particular, the statement about the graph maps is trivial. If  $\text{mult}(B_1)$  is not 1, then, by Corollary 3.5,  $B_1 \simeq Q(k, p)$  for some prime  $p$ . By Lemma 2.1, there is an index  $j$  such that  $\text{mult}(C_j)$  is divisible by  $p$ . Using Corollary 3.5 again,  $C_j \simeq Q(k, p)$ , too, and we therefore have an isomorphism  $\psi_1: B_1 \rightarrow C_j$ . But then, using Quotient Transform Lemma 3.7, we have  $\varphi': B_2 \otimes \cdots \otimes B_r \simeq Q(B_1, 1) \otimes B_2 \otimes \cdots \otimes B_r \simeq C_1 \otimes \cdots \otimes C_{j-1} \otimes Q(C_j, 1) \otimes C_{j+1} \otimes \cdots \otimes C_s \simeq C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_s$ . In the remaining case, all graphs and multiplicities are trivial; in other words, the algebras are local. Since they are also loop-resistant, they are even trivial, i.e., isomorphic to  $k$ . So the claim is proven.

By induction hypothesis, we get a bijection  $\sigma': \{2, \dots, r\} \rightarrow \{1, \dots, j-1, j+1, \dots, s\}$ , isomorphisms  $\psi_i: B_i \rightarrow C_{\sigma' i}$ , and a permuting morphism  $\zeta'$  defined by  $\sigma'$  such that  $\Delta'(\varphi') = \Delta'(\zeta' \circ (\psi_2 \otimes \cdots \otimes \psi_r))$ . Extend  $\sigma'$  to a bijection  $\sigma: \{1, \dots, r\} \rightarrow \{1, \dots, s\}$  by setting  $\sigma(1) = j$ , and define  $\zeta := \zeta_j \circ (\text{id}_{C_1} \otimes \zeta')$ . This completes the proof. ■

This now implies Unique Factorization 1.1.

*Proof* (Unique Factorization 1.1). By Graph Lemma 3.2(iv), the set  $\mathcal{M}$  together with the multiplication induced by the tensor product is a commutative monoid. By Theorem 4.3, we can define an isomorphism from  $\mathcal{M}$  to the free commutative monoid with basis  $\mathcal{X}$ .

Since the monoid  $\mathcal{M}$  is free over  $\mathcal{X}$ , it suffices to show that the additive monoid of  $\mathcal{U}$  is free over  $\mathcal{M}$ . In other words: the additive decomposition of algebras should be unique. This is well known (see [2, Theorem 2.5.1]). ■

It remains an open problem whether the monoid of isomorphism classes of *local* split algebras in finite characteristics has unique factorization. There are examples that show that the proof technique and some intermediate results of Horst [3] do not hold in finite characteristics. Yet, they do not provide evidence for non-unique factorization.

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