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Clifford classes for some overgroups of the special linear groups

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Dedicated to J.G. Thompson on his 70th birthday

Abstract

In [A. Turull, J. Algebra 235 (2001), 275–314], we calculated the Schur index of each of the irreducible characters of the finite special linear groups. In the present paper, we calculate the Schur index of all the irreducible characters of some overgroups of the special linear groups. The overgroups in question are the special linear groups extended by diagonal automorphisms, and the subgroups of the general linear group that contain the special linear group. To each conjugacy class of irreducible characters of the special linear group in each overgroup is associated a Clifford class. The Clifford class controls all the irreducible characters of the overgroup and intermediate subgroups that are related to the given irreducible by Clifford theory. Knowing only the Clifford class, we can parametrize all the irreducible characters of the intermediate subgroups, and compute, for each parametrized irreducible character, its field of character values, as well as its Schur index over each field. We explicitly compute the Clifford class in each case, and deduce from it the information on the Schur index of all the irreducible characters of the overgroups.

Keywords: Brauer group; Schur index; Special linear groups; Representations

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Introduction

Among the many things we learn from the work of John G. Thompson is a sense of the importance of the study of the finite simple groups. It is through them, and their closely related quasi-simple, and almost simple groups, that we can hope to better understand finite groups. The classification of finite simple groups tells us that many of the covering groups of the finite simple groups are classical groups. Among these, the special linear groups are an important example. The present paper is a contribution to our understanding of the representations over fields in characteristic zero of the finite special linear groups and certain of their overgroups. It is dedicated to John G. Thompson on the occasion of his 70th birthday.

Work of Deligne and Lusztig and others has advanced our understanding of the characters of the finite groups of Lie type, giving us, in particular, a good parametrization of the irreducible complex characters of many finite groups of Lie type. Our understanding of the representations of such groups over arbitrary fields of characteristic zero is less complete. During the last few years, the author developed tools that allow us to give systematic answers to the question of the Schur indices for some large families of characters, and hence make the problem of explicitly describing the Schur index of all the irreducible characters of some families of groups tractable. The present paper uses these techniques to describe the Schur index of all the irreducible characters of some classes of overgroups of the special linear groups.

The complex irreducible characters of the finite general linear groups were calculated by Green [2]. In [4], Lehrer parametrized the irreducible complex characters of the finite special linear groups. In [9], we calculated the Schur indices of all the irreducible characters of the finite special linear groups. In the present paper, we calculate the Schur indices of all the irreducible characters of the groups in two families. The first family of groups is that of the subgroups of the general linear group that contain the special linear group. The second family is that of any extension of the special linear group by a group of diagonal automorphisms.

The irreducible characters of the groups in these families are related by Clifford theory. The standard tools of Clifford theory, namely induction and restriction, not to mention multiplication by irreducible characters of a quotient group, give straightforward information on the character values, but the information they give on the Schur indices is more complicated. However, the types of relationships in whole families can be classified. Consider a finite group H and a normal subgroup J, and an irreducible character $\chi \in \mathrm{Irr}(H)$. Now, if $L \supseteq J$ is a subgroup of H, then L has a set of irreducible characters $\mathrm{Irr}(L, \chi)$ related to χ by Clifford theory, see below after the statement of Theorem 1.1. Let F be a field characteristic zero, which, for convenience, we assume contains all the values of the restriction of χ to J, but we do not assume that F is algebraically closed, or even a splitting field.

Setting $G = H/J$, there is a set $\mathrm{Clif}(G, F)$, whose elements are called Clifford classes. This set is briefly described at the beginning of Section 1. A Clifford class $[\![\chi]\!] \in \mathrm{Clif}(G, F)$ is associated naturally with χ. Knowing $[\![\chi]\!]$ alone allows us to parametrize all $\mathrm{Irr}(L, \chi)$, and to know what the field of values and the Schur index of each character is. Hence, the calculation of the Schur indices of all the irreducible characters of all the subgroups L can be obtained from the much simpler calculation of the element $[\![\chi]\!]$ for each irreducible character $\chi \in \mathrm{Irr}(H)$. This is the approach we take in the present paper. This approach is analogous to the one taken in [7] for the double covers of the symmetric and alternating groups, but here, instead of working with a quotient group of order 2, we are working with a cyclic quotient group.

Suppose G is a finite cyclic group, and F is an arbitrary field of characteristic zero. The elements of $\mathrm{Clif}(G, F)$, that is the Clifford classes, can be characterized easily, see Theorem 1.1 below. Given a Clifford class $[\![\chi]\!] \in \mathrm{Clif}(G, F)$, the irreducible characters of $L \supseteq J$ any subgroup of H, can be parametrized from $[\![\chi]\!]$, see Theorem 1.2 below. Recall that, associated to each irreducible character $\psi \in \mathrm{Irr}(L)$, is an element $[\psi]$ the Brauer group $\mathrm{Br}(F(\psi))$ of $F(\psi)$, as follows. Let M be any module over $F(\psi)$ affording as character a multiple of ψ. Then, $\mathrm{End}_{F(\psi)L}(M)$ is a central simple algebra over $F(\psi)$, and its equivalence class in $\mathrm{Br}(F(\psi))$ is $[\psi]$. If $J \subseteq L \subseteq H$, and $\psi \in \mathrm{Irr}(L, \chi)$, then Theorem 5.3 in [8] gives an explicit formula for $[\psi]$ in terms of the Clifford class $[\![\chi]\!]$, and the parameters that describe ψ. As is well known, once $[\psi]$ is known, the Schur index of ψ over every field can be calculated.

The main results in the present paper are Theorem 3.3, which calculates $[\![\chi]\!]$ for all characters when $H = \mathbf{GL}(n, q)$, and Theorem 4.6, which calculates $[\![\chi]\!]$ for all characters of any extension of $\mathbf{SL}(n, q)$ by its diagonal automorphisms. As consequences, we have Corollaries 3.8 and 4.9, which calculate $[\psi]$ for their corresponding subgroups.

For the convenience of the reader, we have tried to limit the amount of prerequisites for this paper. The required general facts about $\mathrm{Clif}(G, F)$ when G is cyclic are taken from [8], with the most important facts described in Section 1. The technique to calculate $[\![\chi]\!]$ relies on results of [10]. The parametrization used for the irreducible characters of $\mathbf{SL}(n, q)$ is that of [9], which is very close to the standard one, and is described in Section 2. A few results from [9] are also used. However, the present calculation of $[\![\chi]\!]$ does not require the prior calculation of the $[\psi]$ for $\psi \in \mathrm{Irr}(\mathbf{SL}(n, q))$. In fact, one can use our present calculations to obtain $[\psi]$ for each $\psi \in \mathrm{Irr}(\mathbf{SL}(n, q))$, thus recovering some results of [9]. Conversely, Corollaries 3.8 and 4.9 show how one can use the values of $[\psi]$ for $\psi \in \mathrm{Irr}(\mathbf{SL}(n, q))$ to obtain $[\zeta]$ for all $\zeta \in \mathrm{Irr}(L)$. For the most part, the computation of the Schur indices is only implicitly given in the present paper. The process to obtain the Schur indices of ψ over every field containing F from $[\zeta] \in \mathrm{Br}(F(\zeta))$ is standard, and number theoretical in nature. The type of element

[ζ] that arises in the present paper is closely related to the type studied in [9], so we do not repeat these algorithms here.

Our maps are consistently composed from right to left, so that, in particular, we may compose characters with other functions, such as Galois automorphisms. In the present paper, if n is a positive integer, we denote by n_2 the 2-part of n, that is the largest power of 2 dividing n. Furthermore, if n and m are integers, not both 0, we denote by (n, m) their greatest common divisor. We assume throughout that $q > 1$ is a power of a prime p.

1. Clifford theory for cyclic quotient groups

Let F be a field of characteristic zero, let $\overline{F}$ be its algebraic closure and let G be a finite group. For the action of an element $g \in G$ on an element $a \in A$, A a G-algebra, we use reverse exponential notation ${}^g a$. We now review some of the notation, terminology, and results from [8,10].

The set $\mathrm{Clif}(G, F)$ is a generalization of the Brauer group $\mathrm{Br}(F)$. Its elements are equivalence classes of central simple G-algebras over F. First we define what is meant by a *central simple* G-algebra A. *Simple* means, of course, that it has exactly two G-invariant two sided ideals. *Central* here means that $\mathrm{C}_{\mathrm{Z}(A)}(G) = F$. We then define which, among the central simple G-algebras, are *trivial* G-algebras. These are the G-algebras E which are just the full F-endomorphism algebra of a non-zero FG-module, with the natural action of G on E. We say that two central simple G-algebras A and B are *equivalent* if there exist trivial G-algebras E and E' such that

$$A \otimes E \simeq B \otimes E',$$

as G-algebras, where the tensor products are over F. This is an equivalence relation [10] and the set of equivalence classes is denoted by $\mathrm{Clif}(G, F)$. In the case where $G = 1$, this is just the set of elements of the Brauer group of F.

From [10], we furthermore have the following. Let H be a finite group and J be a normal subgroup of H such that $H/J = G$. Let χ be an irreducible character of H. Let F be a field containing all the values of χ on elements of J. Then, there exist non-zero modules M for H over F affording a character ψ such that $\mathrm{Res}^H_J(\psi)$ is a rational multiple of $\mathrm{Res}^H_J(\chi)$. Any such module M is called χ-quasi-homogeneous. For each χ-quasi-homogeneous module M, $\mathrm{End}_{FJ}(M)$ is naturally a central simple G-algebra over F, and its equivalence class in $\mathrm{Clif}(G, F)$ depends on χ but does not depend on the chosen M. The equivalence class of $\mathrm{End}_{FJ}(M)$ is denoted $[\![\chi]\!]$. Furthermore, given $[\![\chi]\!] \in \mathrm{Clif}(G, F)$, we can calculate the Clifford theory of χ with respect to J, including the Schur indices of all the characters involved.

A description of the elements of $\mathrm{Clif}(G, F)$ is particularly simple in the case when G is cyclic. The following theorem appears in [8] as Theorem A, and is mentioned in [10] as Theorem 1.3.

Theorem 1.1. *Let F be a field of characteristic zero and let G be a finite cyclic group, with preferred generator g_0. Given any triple $[Z, \alpha, b]$, where Z is a central simple commutative G-algebra, $\alpha \in F^\times$ and $b \in \mathrm{Br}(F)$, we may construct a central simple G-algebra which we denote $[Z, \alpha, b]$. Then, for every central simple G-algebra A over F, there exists some $[Z, \alpha, b]$ which is equivalent to A. In addition, two G-algebras $[Z, \alpha, b]$ and $[Z_0, \alpha_0, b_0]$ are equivalent if and only if both of the following hold*:

(1) *$Z \simeq Z_0$ as G-algebras.*
(2) *Setting $I = \mathrm{C}_G(Z) = \mathrm{C}_G(Z_0)$ and $m = |I|$, we have that $\alpha\alpha_0^{-1} = \beta^m$ for some $\beta \in F^\times$ such that $b = b_0\,\mathrm{br}(G/I, Z, \beta)$.*

Here, $\mathrm{Br}(F)$ is the Brauer group of the field F, $\mathrm{br}(G/I, Z, \beta) \in \mathrm{Br}(F)$ and $b_0\,\mathrm{br}(G/I, Z, \beta)$ represents the product in the Brauer group $\mathrm{Br}(F)$. The element $\mathrm{br}(G/I, Z, \beta)$ is constructed by a slight generalization of the usual crossed product, see [8]. The equivalence class of the crossed product defines an element of the Brauer group which we denote by

$$\mathrm{br}(G/I, Z, \beta) \in \mathrm{Br}(F).$$

The subgroup $I = \mathrm{C}_G(Z)$ of G depends only on the Clifford class (as can be seen from Theorem 1.1 in the case when G is cyclic), and is called the *inertia group* of the Clifford class of A. The central simple G-algebra $[Z, \alpha, b]$ of Theorem 1.1 is an uncomplicated representative of its class in $\mathrm{Clif}(G, F)$. We will use $[Z, \alpha, b]$ to denote both the G-algebra and its equivalence class in $\mathrm{Clif}(G, F)$.

Assume now that $H/J = G$ is cyclic with preferred generator g_0, that $\chi \in \mathrm{Irr}(H)$, and that we are given that $[\![\chi]\!] = [Z, \alpha, b]$. If ψ is any irreducible character of J contained in the restriction of χ to J, then the usual Clifford theoretical inertia group of ψ modulo J is $I = \mathrm{C}_G(Z)$, the inertia group of $[Z, \alpha, b]$.

In [8], there is an explicit description the characters associated to χ by Clifford theory. For each subgroup L, with $J \leqslant L \leqslant H$, the set of relevant irreducible characters is

$$\mathrm{Irr}(L, \chi) = \left\{\psi \in \mathrm{Irr}(L) \mid \left(\mathrm{Res}_J^L(\psi), \mathrm{Res}_J^H(\chi)\right)_J \neq 0\right\}.$$

The commutative G-algebra Z allows for a parametrization of all the irreducible characters of each subgroup of H that contains J and are related to χ via Clifford theory. The elements Z and α allow us to give to each parametrized irreducible character its field of values. Finally, Z, α and b allow us to assign to each of the parametrized characters an element of the Brauer group of its field of values, and

hence its Schur index. We proceed to briefly describe this parametrization and results.

Associated with the Clifford class $[\![\chi]\!] \in \mathrm{Clif}(G,F)$ is its *centroid* $\Delta = \Delta([\![\chi]\!])$, which is defined up to isomorphism. Δ is an I-graded G-algebra. In our case Δ is easy to describe. Consider the ring $R = Z[X]$ of polynomials with coefficients in Z. Then, R is a $\mathbf{N}$-graded infinite dimensional G-algebra over F. We can also consider R to be graded by I, where the grading of X is the smallest power i_0 of g_0 which is in I (that is i_0 is the preferred generator of I), and powers of X are graded by the corresponding powers of i_0. Let $\mathfrak{I}$ be the principal ideal of R generated by $X^{|I|} - \alpha$. Then, $\mathfrak{I}$ is a I-graded G-invariant ideal of R. Hence, $R/\mathfrak{I}$ is a G-algebra over F, which is graded by I. We have $\Delta \simeq R/\mathfrak{I}$ as I-graded G-algebras. In particular,

$$\Delta \simeq Z \otimes_F F[X]/\big(X^{|I|} - \alpha\big)$$

as G-algebras.

Using the centroid, we can parametrize, from $[\![\chi]\!]$ alone, the irreducible characters in the Clifford theory of χ. For each subgroup L, with $J \leqslant L \leqslant H$, the set of relevant irreducible characters is $\mathrm{Irr}(L,\chi)$. The elements of $\mathbf{Irr}(L,\chi)$ can be parametrized, using Δ only, as follows. We get the field of character values of each irreducible character in $\mathrm{Irr}(L,\chi)$ at the same time.

Theorem 1.2. *Let $J \leqslant L \leqslant H$ and set $\bar{L} = L/J \leqslant G$. Set S to be the sum of Δ_ℓ for $\ell \in \bar{L} \cap I$ and $Z_L = \mathrm{C}_S(\bar{L})$. Then*

$$Z_L \simeq \mathrm{C}_Z(\bar{L}) \otimes_F F[X]/\big(X^{|I\cap\bar{L}|} - \alpha\big).$$

Here Z_L is a G-algebra over F, graded by $\bar{L} \cap I$, and $F[X]$ denotes the F-algebra of polynomials in one variable, and $(X^{|I\cap\bar{L}|} - \alpha)$ the ideal in it generated by the given polynomial. The algebra $F[X]/(X^{|I\cap\bar{L}|} - \alpha)$ is graded by $\bar{L} \cap I$, assigning to the class of X as grading the smallest power of g_0 in $\bar{L} \cap I$. Furthermore, $\mathrm{Irr}(L,\chi)$ *is in one-to-one correspondence with the pairs (e,ϕ), where e is a primitive idempotent of Z_L, and $\phi : eZ_L \to \overline{F}$ is an F-monomorphism. We denote by $\psi_{(e,\phi)}$ the character corresponding to (e,ϕ). The field of values of $\psi_{(e,\phi)}$ is simply the image $\phi(eZ_L)$.*

Proof. See Theorem 4.1 in [8].

For a parametrized character $\psi \in \mathrm{Irr}(L,\chi)$, Theorem 5.3 in [8] describes explicitly $[\psi] \in \mathrm{Br}(F(\psi))$, and hence describes the Schur index of ψ. The reader can use this result, in conjunction with the calculation of the Clifford classes below, to obtain the exact value of $[\psi]$ in terms of these parameters, for each $\psi \in \mathrm{Irr}(L,\chi)$, for L any subgroup of $\mathbf{GL}(n,q)$ that contains $\mathbf{SL}(n,q)$, or for L any subgroup of any extension of $\mathbf{SL}(n,q)$ by its diagonal automorphisms. Our next theorem is a direct consequence of Theorem 5.3 in [8]. It calculates $[\psi]$ as

far as possible without relying on the details of the parametrization. This will be sufficient for us in this paper.

Theorem 1.3. *Let H be a finite group and let J be a normal subgroup of H. Assume that $H/J = G$ is cyclic and let g_0 be a fixed generator for G. Let $\chi \in \mathrm{Irr}(H)$ and let F be a field of characteristic zero that contains all the values of $\mathrm{Res}_J^H(\chi)$. Let $[\![\chi]\!] \in \mathrm{Clif}(G, F)$ be the Clifford class associated with χ. By Theorem* 1.1, *$[\![\chi]\!]$ is represented by $[Z, \alpha, b]$, for an appropriate triple. Let K be the field of values over F of any irreducible character in $\mathrm{Res}_J^H(\chi)$, and let $I = \mathrm{C}_G(Z)$ be the inertia group. Let L be any subgroup between J and H, and let $\psi \in \mathrm{Irr}(L, \chi)$. Then, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α (with $\bar{I} = (L/J) \cap I$), and*

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$ and $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$.

Furthermore, if λ is any linear character of G, then the β' associated with the irreducible character $\mathrm{Res}_L^G(\lambda)\psi \in \mathrm{Irr}(L, \chi)$ can be taken to be

$$\beta' = \lambda(i_0)\beta$$

where i_0 is the smallest power of g_0 in $\bar{I}$.

Proof. See Theorem B in [8].

In the situations of the present paper, the field of values over F of an irreducible character in $\mathrm{Res}_J^H(\chi)$ is always a very small extension of F itself. Our next two corollaries describe what happens in the cases relevant for this paper.

Corollary 1.4. *Assume the hypotheses of Theorem* 1.3. *Assume, furthermore, that the field of values over F of any irreducible character in $\mathrm{Res}_J^H(\chi)$ is F, or, equivalently, assume that F is a direct summand of Z. Then, Z is isomorphic to the direct product of $[G : I]$ copies of F permuted transitively by G/I. Furthermore, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α (with $\bar{I} = (L/J) \cap I$), and $F(\psi) = F(\beta)$. In addition, $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$.*

Proof. The structure of Z is calculated for example in Theorem 1.4 in [10]. In particular, the direct summands of Z are the fields of values of the irreducible characters in $\mathrm{Res}_J^H(\chi)$, and the structure of Z is as stated. It follows that, by Theorem 1.2, Z_L is isomorphic to a direct sum of copies of $F[X]/(X^{|\bar{I}|} - \alpha)$. It then follows from Theorem 1.2 that, since $F(\psi)$ is a homomorphic image of Z_L, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and $F(\psi) = F(\beta)$. Since $K(\psi) = F(\psi)$ in this case, for every $\beta' \in F(\psi)^\times$, we

have $\mathrm{br}(K(\psi)/F(\psi), \beta') = 1$. Hence, the value of $[\psi]$ follows immediately from Theorem 1.3. This completes the proof of the corollary.

Corollary 1.5. *Assume the hypotheses of Theorem* 1.3. *Assume, furthermore, that the field of values over F of any irreducible character in $\mathrm{Res}_J^H(\chi)$ is a quadratic extension K of F, or, equivalently, assume that Z contains a direct summand K which is a quadratic extension of F. Set $r = [G : I]$. Then, r is even, and Z is isomorphic to the direct product of $r/2$ copies of K permuted transitively by G/I, where the subgroup of order* 2 *of G/I acts as the Galois automorphism of K/F on each copy of K. The remaining information depends on whether or not r_2 (the* 2*-part of r) divides $[G : L]$. As before, we set $\bar{I} = (L/J) \cap I$.*

(1) *Suppose r_2 divides $[G : L]$. Then, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and $F(\psi) = K(\beta)$. In addition, $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$.*

(2) *Suppose r_2 does not divide $[G : L]$. Then, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, $F(\psi) = F(\beta)$. Furthermore, there exists some $\beta_0 \in F(\psi)^\times$ such that β_0 is an $|I|_2$-th root of α, and*

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$, and $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$.

Proof. Again, the structure of Z is calculated for example in Theorem 1.4 in [10]. In particular, the direct summands of Z are the fields of values of the irreducible characters in $\mathrm{Res}_J^H(\chi)$, and the structure of Z is as stated. Suppose first that r_2 divides $[G : L]$. It follows that, by Theorem 1.2, Z_L is isomorphic to a direct sum of copies of $K[X]/(X^{|\bar{I}|} - \alpha)$. Hence, from Theorem 1.2, we obtain that, since $F(\psi)$ is a homomorphic image of Z_L, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and $F(\psi) = K(\beta)$. Now since $K \subseteq F(\psi)$, we have $F(\psi) = K(\psi)$, and it follows from Theorem 1.3 that $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$, as desired.

Suppose now that r_2 does not divide $[G : L]$. It follows that, by Theorem 1.2, Z_L is isomorphic to a direct sum of copies of $F[X]/(X^{|\bar{I}|} - \alpha)$. Hence, from Theorem 1.2, we obtain that, since $F(\psi)$ is a homomorphic image of Z_L, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and $F(\psi) = F(\beta)$. By Theorem 1.3, there exists some $\beta_1 \in F(\psi)^\times$ such that β_1 is an $|\bar{I}|$-th root of α, and

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta_1^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$ and $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$. The degree of the extension $K(\psi)/F(\psi)$ is at most 2, so that, for example, by Proposition a in [6, p. 260], the order of $\mathrm{br}(K(\psi)/F(\psi), \beta_0^{-1})$ as an element of the Brauer group is at most 2. Furthermore, since G is cyclic and r_2 does not divide $[G : L]$, we must have $|I|_2 = |\bar{I}|_2$. Let d be the odd part of $|\bar{I}|$. Then, setting $\beta_0 = \beta_1^d$, we have that β_0 is a $|I|_2$-th root of α, and $\mathrm{br}(K(\psi)/F(\psi), \beta_1^{-1}) = \mathrm{br}(K(\psi)/F(\psi), \beta_0^{-1})$. The result follows.

2. Parametrizing the characters of $\mathbf{SL}(n, q)$ and $\mathbf{GL}(n, q)$

The irreducible characters of $\mathbf{GL}(n, q)$ have been described by J.A. Green [2]. We will use the notation of [9], which was suggested by the results of Lehrer [4] and Karkar and Green [3], to parameterize the characters of $\mathbf{GL}(n, q)$. We now describe this parametrization.

We let p be a prime number and we let q be a power of p. We have the finite field $\mathbf{F}_p$, with exactly p elements. We fix an algebraic closure of $\mathbf{F}_p$, which we denote $\overline{\mathbf{F}_p}$. We think of each finite field in characteristic p as a subfield of $\overline{\mathbf{F}_p}$. In particular, $\mathbf{F}_q$ is a subfield of $\overline{\mathbf{F}_p}$. For each positive integer d, we denote by $F_d = \mathbf{F}_{q^d}^{\times}$ the multiplicative group of the field $\mathbf{F}_{q^d}$. We denote by $\widehat{F_d}$ the character group of F_d, that is $\widehat{F_d}$ is the group of group homomorphisms $F_d \to \mathrm{C}^{\times}$. We define $\sigma_q : \widehat{F_d} \to \widehat{F_d}$ by $\sigma_q(\theta) = \theta^q$.

Definition 2.1.

(1) Two characters θ and ϕ in $\widehat{F_d}$ are *conjugate* if $\sigma_q^k(\theta) = \phi$ for some integer k. This yields an equivalence relation.
(2) A *d-simplex* s is a conjugacy class of size d in $\widehat{F_d}$. If θ is some element of s, we write $s = \langle\theta\rangle$.
(3) The *degree* of a d-simplex s is $d(s) = d$.
(4) We denote by $\mathcal{G}_d$ the union of all the d-simplexes, and by $\mathcal{G}$ the union $\bigcup_{d=1}^{\infty} \mathcal{G}_d$.
(5) Let $\mathcal{P}$ be the set of all partitions. For each $\nu \in \mathcal{P}$, we denote, as usual, by $|\nu|$ the sum of its parts.
(6) We let $\mathcal{F}$ be the set of all functions $\lambda : \mathcal{G} \to \mathcal{P}$, which assign the empty partition to almost all elements of $\mathcal{G}$ and have the property that, for every $\theta \in \mathcal{G}$, we have $\lambda(\sigma_q(\theta)) = \lambda(\theta)$.
(7) For each $\lambda \in \mathcal{F}$, we define its *degree* to be

$$\mathbf{deg}(\lambda) = \sum_{\theta \in \mathcal{G}} \big|\lambda(\theta)\big|.$$

(8) We denote by $\mathcal{F}_n$ the set of all elements of $\mathcal{F}$ of degree n.
(9) For each $\lambda \in \mathcal{F}_n$, we denote by χ_λ the irreducible character $(\cdots \langle\theta\rangle^{\lambda(\theta)} \cdots)$, where we take $\langle\theta\rangle^{\lambda(\theta)}$ once for each simplex. Hence, $\chi_\lambda \in \mathrm{Irr}(\mathbf{GL}(n,q))$.

Theorem 2.2. *The map $\lambda \mapsto \chi_\lambda$ is a bijection $\mathcal{F}_n \to \mathrm{Irr}(\mathbf{GL}(n,q))$.*

Proof. See Theorem 2.4 in [9].

Two types of action on the set of irreducible characters of $\mathbf{GL}(n,q)$ play a key role in this paper. They are the Galois action, and the multiplication of irreducible characters by linear characters. Both these actions correspond to easily described actions on the parameter set. We now proceed to describe them in turn.

Definition 2.3.

(1) For each $\lambda \in \mathcal{F}$, we denote by $\mathbf{Q}(\lambda)$ the field $\mathbf{Q}$ extended by the values of all the θ in the support of λ. Hence, $\mathbf{Q}(\lambda)$ is $\mathbf{Q}$ extended by a primitive m-th root of 1, where m is the least common multiple of all the $|\theta(F_d)|$ for $\theta \in \mathcal{G}_d$ such that $\lambda(\theta)$ is not the empty partition.
(2) Let $\lambda \in \mathcal{F}_n$, and let $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. Then, we define $\sigma\lambda : \mathcal{G} \to \mathcal{P}$, by, for $\theta \in \mathcal{G}$, setting $\sigma\lambda(\theta) = \lambda(\sigma^{-1}\theta)$ if θ is such that $\mathbf{Q}(\theta) \subseteq \mathbf{Q}(\lambda)$, and $\sigma\lambda(\theta)$ is the empty partition otherwise. Naturally, here $\sigma^{-1}\theta$ denotes function composition. We have that $\sigma\lambda \in \mathcal{F}_n$.
(3) Let $\lambda \in \mathcal{F}_n$. We set

$$\mathrm{Galg}(\lambda) = \{\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) \colon \sigma\lambda = \lambda\}.$$

(4) For any subgroup H of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, we denote by H' the fixed field of H.

Lemma 2.4. *Let $\lambda \in \mathcal{F}_n$. Then $\mathbf{Q}(\chi_\lambda) \subseteq \mathbf{Q}(\lambda)$ and, for each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, we have that the composition of the character with the Galois automorphism is simply $\sigma\chi_\lambda = \chi_{\sigma\lambda}$, see Definition* 2.3.

Proof. See Lemma 2.7 in [9].

The second type of action that plays a major role in this paper is the action on $\mathrm{Irr}(\mathbf{GL}(n,q))$ by linear characters of $\mathbf{GL}(n,q)$, that is by linear characters of $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$. It too corresponds to some easily described action of the parameter set which we now proceed to describe. This action was first described in [3].

We let n be a positive integer. Then $\mathbf{F}_{q^n}$ can be viewed as a vector space over $\mathbf{F}_q$ of dimension n. We set $\mathbf{GL}(n,q)$ to be the group of all invertible linear transformations of the vector space $\mathbf{F}_{q^n}$ over $\mathbf{F}_q$ onto itself. Each element of F_n acts on the vector space $\mathbf{F}_{q^n}$ by left multiplication, so we think of F_n as a subgroup

of $\mathbf{GL}(n,q)$. There is the determinant function **det** which provides a surjective homomorphism

$$\mathbf{det}:\mathbf{GL}(n,q)\to F_1,$$

whose kernel is denoted $\mathbf{SL}(n,q)$, and which provides a fixed isomorphism from $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$ onto F_1. Hence, the determinant function **det** provides a fixed isomorphism from $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$ onto F_1. If $\alpha\in\widehat{F_1}$, then $\alpha\,\mathbf{det}$ is a linear character of $\mathbf{GL}(n,q)$. Building on some results of Lehrer [4], Karkar and Green [3] described the multiplication action of this character on $\mathrm{Irr}(\mathbf{GL}(n,q))$. We now recall the notation introduced in [9] to describe their result, and to study certain aspects of the multiplication action, and its interaction with the Galois action introduced earlier.

Definition 2.5. Let $\alpha\in\widehat{F_1}$. Then, we wish to define the actions of α on various objects.

(1) By abuse of notation, we may also view α as a linear character of $\mathbf{GL}(n,q)$ (strictly speaking as the composition $\alpha\,\mathbf{det}$), or as a linear character of F_d for any positive integer d (strictly speaking as the composition $\alpha\,\mathrm{Norm}$, where Norm is the norm homomorphism $\mathrm{Norm}:\mathbf{F}_{q^d}^{\times}\to\mathbf{F}_q^{\times}$). The context will determine which version of α needs to be used.
(2) If $\chi\in\mathrm{Irr}(\mathbf{GL}(n,q))$, then $\alpha\chi$ is simply the product of the two characters of $\mathbf{GL}(n,q)$.
(3) If $\theta\in\widehat{F_d}$, for some positive integer d, then $\alpha\theta$ is simply the product of the two elements of $\widehat{F_d}$.
(4) If $\lambda\in\mathcal{F}$, then we define $\alpha\lambda:\mathcal{G}\to\mathcal{P}$ by $\alpha\lambda(\theta)=\lambda(\alpha^{-1}\theta)$. It is easy to see that $\alpha\lambda\in\mathcal{F}$ and $\mathbf{deg}(\alpha\lambda)=\mathbf{deg}(\lambda)$.
(5) If $\lambda\in\mathcal{F}$, then we define the following subgroup of F_1:

$$\mathcal{I}(\lambda)=\bigcap\bigl\{\ker(\alpha):\ \alpha\in\widehat{F_1}\text{ and }\alpha\lambda=\lambda\bigr\}.$$

(6) Let $\lambda\in\mathcal{F}_n$. We set

$$\mathrm{Galr}(\lambda)=\bigl\{\sigma\in\mathrm{Gal}\bigl(\mathbf{Q}(\lambda)/\mathbf{Q}\bigr):\ \text{for some }\alpha\in\widehat{F_1}\text{ we have }\sigma\lambda=\alpha\lambda\bigr\}.$$

Theorem 2.6. *For each $\alpha\in\widehat{F_1}$ and each $\lambda\in\mathcal{F}_n$, $\alpha\chi_\lambda$ is an irreducible character of $\mathbf{GL}(n,q)$, and in fact*

$$\alpha\chi_\lambda=\chi_{\alpha\lambda}.$$

Proof. See Theorem 3.4 in [9].

Finally, we set up notation for the irreducible characters of $\mathbf{SL}(n,q)$. The characters of $\mathbf{SL}(n,q)$ were originally parameterized by Lehrer [4]. We now set up notation for them as in [9].

Definition 2.7. Let $\lambda \in \mathcal{F}_n$, then we denote by ψ_λ any irreducible character contained in

$$\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda).$$

Remark. The character ψ_λ is only defined up to conjugation by some element of $\mathbf{GL}(n,q)$.

Proposition 2.8. *Each irreducible character of* $\mathbf{SL}(n,q)$ *is* $\mathbf{GL}(n,q)$ *conjugate to some* ψ_λ, *for some* $\lambda \in \mathcal{F}_n$. *Furthermore, if* $\lambda, \lambda' \in \mathcal{F}_n$, *then the* $\mathbf{GL}(n,q)$*-orbit of characters conjugate to* $\psi_{\lambda'}$ *is the same as that of characters conjugate to* ψ_λ *if and only if there exists some* $\alpha \in \widehat{F_1}$ *such that* $\lambda' = \alpha\lambda$.

Proof. See Proposition 4.2 in [9].

3. Clifford classes for the general linear group

In this section, we calculate the Clifford class $[\![\chi_\lambda]\!] \in \mathrm{Clif}(G,F)$ for each irreducible character χ_λ of $\mathbf{GL}(n,q)$ in terms of its parameter λ. From this, we easily obtain, for each irreducible character ψ of some subgroup of $\mathbf{GL}(n,q)$ that contains $\mathbf{SL}(n,q)$, its field of values $F(\psi)$, its element in the Brauer group $[\psi] \in \mathrm{Br}(F(\psi))$, and its Schur index. Here we take $H = \mathbf{GL}(n,q)$, $J = \mathbf{SL}(n,q)$, and $G = H/J$. The determinant provides a standard isomorphism between G and $\mathbb{F}_q^\times$. G is cyclic of order $q-1$, and we may apply the results of Section 1. We will take F to be any field containing all the values of the restriction of χ_λ to J. The field $\mathbf{Q}(\mathrm{Res}^H_J(\chi_\lambda))$ is contained in the field of values of each of the irreducible characters involved. Hence, the requirement that F contain all the values of χ_λ on J does not affect the later calculation of fields of values and Schur indices of the characters related to χ by Clifford theory. For completeness, we begin by calculating the field $\mathbf{Q}(\mathrm{Res}^H_J(\chi_\lambda))$ itself.

Proposition 3.1. *Set* $H = \mathbf{GL}(n,q)$, $J = \mathbf{SL}(n,q)$, *and* $G = H/J$. *Let* $\chi \in \mathrm{Irr}(H)$ *be such that* $\chi = \chi_\lambda$, *where* $\lambda \in \mathcal{F}_n$, *see Theorem* 2.2. *Then, we have*

$$\mathbf{Q}\big(\mathrm{Res}^H_J(\chi)\big) = \mathrm{Galr}(\lambda)'$$

where $\mathrm{Galr}(\lambda)$ *is defined in Definition* 2.5, *and the prime in Definition* 2.3.

Proof. By Lemma 2.4, we know that $\mathbf{Q}(\chi) \subseteq \mathbf{Q}(\lambda)$. Since $\mathbf{Q}(\lambda)/\mathbf{Q}$ is a Galois extension, our result will follow once we prove that, for $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, we have $\sigma\,\mathrm{Res}^H_J(\chi) = \mathrm{Res}^H_J(\chi)$ if and only if $\sigma \in \mathrm{Galr}(\lambda)$. By Theorem 2.4, $\sigma\chi = \chi_{\sigma\lambda}$. The restriction $\mathrm{Res}^H_J(\chi)$ is the sum of a single $\mathbf{GL}(n,q)$-conjugacy class of irreducible characters of J. Hence, our result follows immediately from Proposition 2.8.

Two irreducible characters of $\mathbf{GL}(n,q)$ have the same Clifford theory if and only if one of them is the other times some character of $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$.

Proposition 3.2. *Set $H=\mathbf{GL}(n,q)$, $J=\mathbf{SL}(n,q)$, and $G=H/J$. Let $\chi_\lambda,\chi_\mu\in \mathrm{Irr}(\mathbf{GL}(n,q))$. Then the following are equivalent*:

(1) *There exists a linear character $\alpha\in\mathrm{Irr}(\mathbf{GL}(n,q)/\mathbf{SL}(n,q))$ such that $\alpha\chi_\lambda=\chi_\mu$.*
(2) *There exists some $\alpha\in\widehat{F_1}$ such that $\alpha\lambda=\mu$.*
(3) *The restriction of χ_λ and χ_μ to $\mathbf{SL}(n,q)$ are equal, i.e.,*

$$\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)=\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\mu).$$

Proof. That (1) and (2) are equivalent follows immediately from Theorem 2.6, since $\widehat{F_1}$ is identified with $\mathrm{Irr}(\mathbf{GL}(n,q)/\mathbf{SL}(n,q))$. It is immediate that (1) implies (3). The converse follows from standard Clifford theory, using the fact that, since $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$ is cyclic, the restriction of any irreducible character of $\mathbf{GL}(n,q)$ to $\mathbf{SL}(n,q)$ will be the sum of some irreducible character and its $\mathbf{GL}(n,q)$ conjugates, each with multiplicity one.

Hence, it would be enough to calculate $[\![\chi_\lambda]\!]$ only for a set of representatives of the classes of irreducible characters of $\mathbf{GL}(n,q)$ under the equivalence of the proposition. However, we simply calculate $[\![\chi_\lambda]\!]$ for all $\lambda\in\mathcal{F}_n$, and notice that, as expected, our results are unchanged when λ is replaced by $\alpha\lambda$. Our next theorem describes the Clifford classes for the general linear group.

Theorem 3.3. *We set $H=\mathbf{GL}(n,q)$, $J=\mathbf{SL}(n,q)$, and $G=H/J$. Let $\chi\in\mathrm{Irr}(H)$ be such that $\chi=\chi_\lambda$, where $\lambda\in\mathcal{F}_n$, see Theorem* 2.2. *We let F be a field containing the values of χ on J. We fix a generator g_0 of the cyclic group G. Then, $[\![\chi]\!]\in\mathrm{Clif}(G,F)$, and, by Theorem* 1.1, *$[\![\chi]\!]=[Z,\alpha,b]$, for appropriate Z, α, and b, which are as follows.*

We take η to be a primitive p-th root of unity. If $2\leqslant n_2\leqslant(p-1)_2$, we set $s=2(p-1)_2/n_2$, otherwise we set $s=1$. Let r be the size of the stabilizer of the action of $\widehat{F_1}$ on λ. Set $i_0=g_0^r$, and I to be the subgroup of G generated by i_0. There necessarily exists some $j_0\in i_0$ such that $\chi(j_0)\neq 0$. We set $\alpha_0=\chi(j_0)^{|I|}$, for any one such j_0. If p is odd, q is not a square, $2\leqslant n_2\leqslant(p-1)_2$, and, for any element $\beta\in\widehat{F_1}$ of order n_2, we have $\beta\lambda=\lambda$, then we set $K=F(\sqrt{\varepsilon p})$, where $\varepsilon\in\{1,-1\}$ and $p\equiv\varepsilon \pmod 4$. Otherwise, we set $K=F$. With this notation, we have:

(1) *Z is isomorphic to the direct sum of $r/[K:F]$ copies of K, which are permuted transitively by G/I, and where the subgroup of G/I of order $[K:F]$ acts on each copy of K as the Galois group of K/F.*

(2) *α can be taken to be the product of $\alpha_0 \in F^\times$ times the $|I|$-th power of any element of $F^\times$.*
(3) $b = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$.

In addition, we always have $b^2 = 1$. Furthermore, if either $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $b = 1 \in \mathrm{Br}(F)$.

Remark 3.4. There are, of course, formulas to calculate the value of $\chi(j_0)$. Alternatively, we note that the value of $\chi(j_0)$ up to multiplication by some element of $F^\times$ is determined by the action of $\mathrm{Galr}(\lambda)$. Indeed, if $\sigma \in \mathrm{Galr}(\lambda)$, then, by Definition 2.5, there exists some $\alpha_\sigma \in \widehat{F_1}$ such that $\sigma\lambda = \alpha_\sigma\lambda$. By Theorem 2.6, it follows that $\sigma(\chi(j_0))/\chi(j_0) = \alpha_\sigma(i_0)$. Even though α_σ may not be uniquely determined by σ, this equation implies that $\alpha_\sigma(i_0)$ is uniquely determined by σ. Hence, the map $\sigma \mapsto \alpha_\sigma(i_0)$ is uniquely defined from the action of $\mathrm{Galr}(\lambda)$. Furthermore, it determines $\chi(j_0)$ up to multiplication by $F^\times$ in the following sense. Suppose $\gamma \in \mathbf{Q}(\lambda)^\times$ is such that, for each $\sigma \in \mathrm{Galr}(\lambda)$, we have $\sigma(\gamma)/\gamma = \alpha_\sigma(i_0)$. Then, it follows that $\chi(j_0)/\gamma \in \mathbf{Q}(\mathrm{Res}_J^H(\chi))^\times \subseteq F^\times$.

Proof of Theorem 3.3. ψ_λ is an irreducible character of J contained in $\mathrm{Res}_J^H(\chi)$. By Proposition 4.2 of [9], the ordinary Clifford inertia group of ψ_λ in H is the set of all elements of H whose determinant is in $\mathcal{I}(\lambda)$. Hence, by Definition 2.5, it is the subgroup of H that contains J and has index r in H. Hence, I is the image in G of the inertia group in H of ψ_λ. As explained at the beginning of Section 1, the image in G of the ordinary Clifford theoretic inertia group of any irreducible summand of $\mathrm{Res}_J^H(\chi)$ is the inertia group of the Clifford class $[\![\chi]\!]$. Hence, I is the inertia group of $[\![\chi]\!]$. In addition, it follows from Theorem 4.8 in [9] that in every case $K = F(\psi_\lambda)$.

Theorem 1.4 in [10] now tells us that j_0 exists, that $\alpha_0 = \chi(j_0)^{|I|} \in F^\times$, and that Z and α are as given. We take $\beta \in F^\times$ to be any element such that $\beta^{|I|}\alpha_0 = \alpha$.

Suppose the theorem holds over some field F_0. Then, the invariants over a larger field F such that $F_0 \subseteq F$ are obtained simply by extending the scalars appropriately. Hence, by, for example, Corollary c in page 278 of [6], the theorem also holds over F. Hence, we assume, without loss, that $F = \mathbf{Q}(\mathrm{Res}_J^H(\chi)) = \mathrm{Galr}(\lambda)'$, the smallest possible field given by Proposition 3.1. In particular, we have $[F(\eta) : F] = p - 1$.

Let U be the Sylow p-subgroup of $\mathbf{GL}(n, q)$ of unipotent upper triangular matrices. Then, by Zelevinsky's theorem, see Theorem 4.5 in [9], there exists a linear character $\theta = \theta_\lambda$ of U such that

$$\left(\mathrm{Res}_U^{\mathbf{GL}(n,q)}(\chi_\lambda), \theta\right)_U = 1,$$

with some further special properties. In particular, $\theta = 1$ if and only if χ is a character whose kernel contains J. Let $\nu \in F_1$ be an element of order $p-1$, and let $x \in \mathbf{GL}(n, q)$ be the diagonal matrix whose entries are $\nu^{n-1}, \nu^{n-2}, \ldots, 1$.

Then, by Lemma 4.6 in [9], $\mathbf{Q}(\theta)$ is contained in the field of p-th roots of 1, the element x normalizes U, its determinant is $\mathbf{det}(x) = \nu^{\binom{n}{2}}$, and $\theta^x = \tau_0\theta$, where τ_0 is a generator of the group $\mathrm{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$. This implies that the hypotheses of Theorem 4.3 in [10] are satisfied. Accordingly, we pick some γ in the algebraic closure of F such that $\gamma^r = \beta\chi(j_0)$. Notice that since $r|I| = |G|$, we have $\alpha = \beta^{|I|}\alpha_0 = \gamma^{|G|} \in F^\times$.

Let $\phi : \mathrm{Gal}(F(\theta)/F) \to H$ be the group homomorphism that assigns to τ_0 the element $x \in H$. Let L be a finite Galois extension of F such that $F(\theta) \subseteq L$ and $\gamma \in L$. Let $\phi' : \mathrm{Gal}(L/F) \to H$ be ϕ preceded by restriction to $\mathrm{Gal}(F(\theta)/F)$. For each element $h \in H$ set $\varepsilon(h)$ to be the smallest non-negative integer such that $h \in g_0^{\varepsilon(h)}$. Then, by Theorem 4.3 in [10], the element $b \in \mathrm{Br}(F)$ is represented as an element of $H^2(\mathrm{Gal}(L/F), L^\times)$ by the 2-cocycle f defined as follows:

$$f : \mathrm{Gal}(L/F) \times \mathrm{Gal}(L/F) \to L^\times$$

is such that

$$f(\sigma,\tau) = \left(\frac{\sigma(\gamma)}{\gamma}\right)^{-\varepsilon(\phi'(\tau))}.$$

Consider the function $d : \mathrm{Gal}(L/F) \to L^\times$ defined by $d(\tau) = \gamma^{\varepsilon(\phi'(x))}$. It shows that the function

$$f_1 : \mathrm{Gal}(L/F) \times \mathrm{Gal}(L/F) \to L^\times$$

defined by

$$f_1(\sigma,\tau) = \gamma^{\varepsilon(\phi'(\sigma))-\varepsilon(\phi'(\sigma\tau))}\sigma(\gamma)^{\varepsilon(\phi'(\tau))}$$

is a coboundary. Now setting $f_2 = ff_1$ we see that f_2 also represents the element $b \in \mathrm{Br}(F)$, and

$$f_2(\sigma,\tau) = \gamma^{-\varepsilon(\phi'(\sigma\tau))+\varepsilon(\phi'(\sigma))+\varepsilon(\phi'(\tau))}.$$

Since ϕ' followed by the projection to G is a group homomorphism, we have that, for each $\sigma, \tau \in \mathrm{Gal}(L/F)$, the integer $-\varepsilon(\phi'(\sigma\tau)) + \varepsilon(\phi'(\sigma)) + \varepsilon(\phi'(\tau))$ is divisible by $|G|$. Since $\gamma^{|G|} \in F^\times$, f_2 has values in $F^\times$. Since the value of $f_2(\sigma,\tau)$ depends only on the restriction of σ and τ to $\mathrm{Gal}(F(\theta)/F)$, by inflation, b is also represented as an element of $H^2(\mathrm{Gal}(F(\theta)/F), F(\theta)^\times)$ by the 2-cocycle f_3 such that

$$f_3(\sigma,\tau) = \gamma^{-\varepsilon(\phi(\sigma\tau))+\varepsilon(\phi(\sigma))+\varepsilon(\phi(\tau))}.$$

Now $\mathrm{Gal}(F(\theta)/F)$ is cyclic, and so $b = \mathrm{br}(F(\theta)/F, \delta)$, for some appropriate $\delta \in F(\theta)^\times$. The δ can easily be calculated from f_3, see, for example, the argument of Corollary 5.4 in [10], and we obtain $\delta = \gamma^{m|\mathrm{Gal}(F(\theta)/F)|}$, where $m = \varepsilon(\phi(\tau_0)) = \varepsilon(x)$.

Since the determinant of x is $\mathbf{det}(x) = \nu^{\binom{n}{2}}$, the order of the image of x in G is $(p-1)/(p-1, \binom{n}{2})$. It follows that the order of g_0^m is $(p-1)/(p-1, \binom{n}{2})$.

Suppose that $\theta = 1$. Then, by the above, $b = 1$. Furthermore, the restriction of χ to $\mathbf{SL}(n,q)$ is the trivial character, χ is a linear character of G, and only one element of $\mathcal{G}$ is assigned a non-empty partition under λ. In this case, $I = G$, and $\beta'\lambda \neq \lambda$ for each $\beta' \in \widehat{F_1}$. Since $\alpha_0 = \chi(j_1)^{|I|} = 1$, $\beta^{|I|} = \alpha$, and, as $p-1$ divides $|I|$, and s divides $p-1$, it follows that $\alpha^{(p-1)/s}$ is the $(p-1)$-th power of some element of $F^\times$. Hence, $1 = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$, as $[F(\eta):F] = p-1$. Hence, the theorem holds in this case. We assume henceforth that $\theta \neq 1$. Hence, θ has order p, $F(\theta) = F(\eta)$, and $[F(\theta):F] = p-1$.

By the argument of Remark 3.4, we see that if $y, z \in \mathbf{GL}(n,q)$ have the same determinant and $\chi(z) \neq 0$, then $\chi(y)/\chi(z) \in F$. Hence, whenever $y \in i_0$ then $\chi(y)/\gamma^r \in F$. Set $k = |\mathbf{GL}(n,q)|/|\mathrm{C}_{\mathbf{GL}(n,q)}(j_0)|$ to be the number of conjugates of j_0. Then, $\chi(j_0)k/\chi(1)$ is the value of the central character associated with χ on the conjugacy class sum S of j_0. It follows that, for each positive integer c, $\chi(j_0)^c k^c/\chi(1)^c$ is the value of the central character on S^c. Since all summands of S^c are in g_0^{rc}, it follows that for each $y \in g_0^{rc}$, we have $\chi(y)/\chi(j_0)^c \in F$, which implies that $\chi(y)/\gamma^{rc} \in F$. If c is not divisible by r, then $\chi(y) = 0$ for each $y \in g_0^c$ since $y \notin I$. It follows that, for each positive integer c, we have $\chi(x)/\gamma^c \in F$ for each $x \in g_0^c$.

Suppose that either $p = 2$, or n is odd, or $n_2 > (p-1)_2$. Then, $s = 1$, and it suffices to show that $b = 1$. Let y be the central matrix in $\mathbf{GL}(n,q)$ whose diagonal entries are all ν. Then, $\mathbf{det}(y) = \nu^n$ has multiplicative order $(p-1)/(p-1,n)$. By the above, the determinant of g_0^m has order $(p-1)/(p-1,\binom{n}{2})$. Hence, there is some power y^a of y that has the same determinant as g_0^m. The value of χ on y^a is an $F^\times$ multiple of a $(p-1)$-th root of unity. By the previous paragraph, γ^m is also an $F^\times$ multiple of a $(p-1)$-th root of unity. Hence, $\delta = \gamma^{m|\mathrm{Gal}(F(\eta)/F)|} = \gamma^{m(p-1)}$ is the $(p-1)$-th power of some element of $F^\times$. It then follows that

$$b = \mathrm{br}\big(F(\eta)/F, \delta\big) = \mathrm{br}\big(F(\eta)/F, 1\big) = 1,$$

and the theorem holds in this case.

We assume, henceforth, that $2 \leqslant n_2 \leqslant (p-1)_2$. Hence, we also have $s = 2(p-1)_2/n_2$. We set $i_1 \in G$ to be the 2-part of g_0^m. Then, i_1 is an element of order s. Let y be again the central matrix in $\mathbf{GL}(n,q)$ whose diagonal entries are ν. The order of $\mathbf{det}(y) = \nu^n$ is $(p-1)/(p-1,n)$. Since $(p-1)/(p-1,n)$ is a multiple of the order of $\mathbf{det}(g_0^{2m})$, for some positive integer c we have $\mathbf{det}(y^c) = \mathbf{det}(g_0^{2m})$. Hence, we have that $\mathbf{det}(y^c) = \mathbf{det}(g_0^{2m})$, and this implies that γ^{2m} is a $(p-1)$-th root of unity up to multiplication by some element of $F^\times$. Therefore,

$$b^2 = \mathrm{br}\big(F(\theta)/F, \gamma^{2m(p-1)}\big) = 1.$$

Since $2 \leqslant s$, and s is the 2-part of the order of g_0^m, we have $m_2 = (q-1)_2/s$. Let o be the odd part of m. Now we have

$$b = \mathrm{br}\big(F(\eta)/F, \gamma^{m(p-1)}\big)$$
$$= \mathrm{br}\big(F(\eta)/F, \gamma^{(q-1)_2(p-1)/s}\big)^o = \mathrm{br}\big(F(\eta)/F, \gamma^{(q-1)(p-1)/s}\big)^o,$$

the last equation because, since $b^2 = 1$, any odd power of b is equal to b. However, we have $\alpha = \gamma^{q-1}$, so we get

$$b = \mathrm{br}\big(F(\eta)/F, \alpha^{(p-1)/s}\big)^o.$$

Since the order of $\mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$ divides s, it must be a power of 2, and, since o is odd, it follows that the order of $\mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$ is actually 2, and $b = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$. This concludes the proof of the theorem.

Remark 3.5. In effect, Theorem 3.3 describes all the characters of all the subgroups of $\mathbf{GL}(n,q)$ that contain $\mathbf{SL}(n,q)$, including their fields of values, and the element of the Brauer group associated with each of them. Indeed, for each $\chi \in \mathrm{Irr}(\mathbf{GL}(n,q))$, we are given $[Z,\alpha,b]$, and then Theorem 1.2 describes a parametrization for all the characters in $\mathrm{Irr}(L,\chi)$, which yields in particular their field of values, and Theorem 5.3 in [8] then describes the element of the Brauer group associated with each character. Once the element of the Brauer group corresponding to each character is known, it is straight forward to calculate the Schur index, and the local Schur indices. In fact, the techniques used in [9] apply directly to elements of the Brauer group as given in the forms that they arise here.

We close this section with some general results on the Schur indices and the element of the Brauer group associated with the irreducible characters of any subgroup of $\mathbf{GL}(n,q)$ that contains $\mathbf{SL}(n,q)$.

Corollary 3.6. *Let $L \supseteq \mathbf{SL}(n,q)$ be a subgroup of $\mathbf{GL}(n,q)$, and let $\psi \in \mathrm{Irr}(L)$. Then the Schur index of ψ at most* 2.

Proof. Let $\chi_\lambda \in \mathrm{Irr}(\mathbf{GL}(n,q))$ be such that it contains ψ in its restriction to L. We set $F = \mathrm{Galr}(\lambda)'$ to be the smallest base field, as given in Proposition 3.1. We will work, without loss, over F as our base field. Then Theorem 3.3 gives us $[\![\chi_\lambda]\!] = [Z,\alpha,b] \in \mathrm{Clif}(G,F)$ explicitly. We have that $b^2 = 1$. Furthermore, in the notation of Theorem 3.3, we have the field K, and, either $K = F$ or K is a quadratic extension of F. Hence, by the structure of Z given in Theorem 3.3, the element of the Brauer group $[\psi] \in \mathrm{Br}(F(\psi))$ associated with ψ is given either by Corollary 1.4 or by Corollary 1.5. We have that either $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$, or there exists some $\beta_0 \in F(\psi)^\times$ such that

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$. Since $[K:F] \leqslant 2$, by, for example, Proposition a in page 260 of [6], we have that $\mathrm{br}(K(\psi)/F(\psi), \beta_0^{-1})$

has order 1 or 2 in $\mathrm{Br}(F(\psi))$. It then follows that $[\psi]$ has likewise order 1 or 2 in $\mathrm{Br}(F(\psi))$, which implies that the Schur index of ψ is at most 2, as desired.

Corollary 3.7. *Let $L \supseteq \mathbf{SL}(n,q)$ be a subgroup of $\mathbf{GL}(n,q)$, and let $\psi \in \mathrm{Irr}(L)$. If $2 \leqslant n_2 \leqslant (p-1)_2$, we set $s = 2(p-1)_2/n_2$, otherwise we set $s = 1$. Assume that s divides $[L : \mathbf{SL}(n,q)]$. Then the Schur index of ψ is* 1.

Proof. ψ will be contained in the restriction of some irreducible character $\chi = \chi_\lambda \in \mathrm{Irr}(\mathbf{GL}(n,q))$. We set $F = \mathrm{Galr}(\lambda)'$ to be the smallest base field, as given in Proposition 3.1. We will work, without loss, over F as our base field. Then Theorem 3.3 gives us $[\![\chi_\lambda]\!] = [Z, \alpha, b] \in \mathrm{Clif}(G,F)$ explicitly, and we use the notation of this theorem. We set $\bar{I} = (L/J) \cap I$, where I is the inertia group of $[\![\chi]\!]$.

Suppose first that s divides $|I|$. Then, sr divides $q-1$. Suppose, in this case, that p is odd, q is not a square, $2 \leqslant n_2 \leqslant (p-1)_2$, and, for any element $\beta \in \widehat{F}_1$ of order n_2, we have $\beta\lambda = \lambda$. Then $s = 2(p-1)_2/n_2$, and n_2 divides r, and $(q-1)_2 = (p-1)_2$. This implies that $2(p-1)_2$ divides $(p-1)_2$, a contradiction. Hence, our second assumption does not hold, and it follows from Theorem 3.3 that $K = F$. This implies that Z is a direct sum of copies of F. By Corollary 1.4, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and $F(\psi) = F(\beta)$. In addition, $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$. By Theorem 3.3, we have $b = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$. By, for example, Corollary c in page 278 in [6], we have that

$$\bar{b} = \mathrm{br}\big(F(\psi,\eta)/F(\psi), \alpha^{(p-1)/s}\big).$$

Since s divides $[L : \mathbf{SL}(n,q)]$, and s divides $|I|$, we also have that s divides $|\bar{I}| = |(L/J) \cap I|$. Hence, α is the s-th power of some element of $F(\psi)$, which implies that $\alpha^{(p-1)/s}$ is the $(p-1)$-th power of some element of $F(\psi)$. This implies that $\bar{b} = 1$, as desired.

Now suppose that s does not divide $|I|$. This implies that $s \neq 1$, which yields that $2 \leqslant n_2 \leqslant (p-1)_2$, and that $s = 2(p-1)_2/n_2$. Since s is a power of 2, we have that $|I|_2$ divides $s/2 = (p-1)_2/n_2$. Considering the determinant of 2-elements of the center of $\mathbf{GL}(n,q)$, we see that $(q-1)_2/n_2$ divides $|I|_2$. It follows that q is not a square, and $|I|_2 = (p-1)_2/n_2$. Since s divides $[L : \mathbf{SL}(n,q)]$, this further implies that $|\bar{I}|_2 = |I|_2 = s/2$. Furthermore, since $r|I| = q-1$, this yields that $r_2 = n_2$. By the definition of r in Theorem 3.3, it follows that, for any element $\beta \in \widehat{F}_1$ of order n_2, we have $\beta\lambda = \lambda$. Hence, we have $K = F(\sqrt{\varepsilon p})$, where $\varepsilon \in \{1,-1\}$ and $p \equiv \varepsilon \pmod 4$. Since s divides $[L : \mathbf{SL}(n,q)]$, we have that $[G : L]_2$ divides $\frac{1}{2}(q-1)_2 n_2/(p-1)_2 = n_2/2$. Hence, r_2 does not divide $[G : L]$ and we are in the second case of Corollary 1.5. Hence, there exists some

$\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α, and, $F(\psi) = F(\beta)$. Furthermore, there exists some $\beta_0 \in F(\psi)^\times$ such that β_0 is an $s/2$-th root of α, and, we have

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$ and $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$. By Theorem 3.3, $b = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$. From Theorem 3.3, we know that $\alpha_0 = \chi(j_0)^{|I|}$, and that α is α_0 times the $|I|$-th power of some element of $F^\times$. Since $|\bar{I}|$ divides $|I|$, the values of χ are contained in F extended by p'-th roots of unity, and p does not divide $|I|$, it follows that $F(\psi)$ is contained in the field of p'-th roots of unity. Hence, $K(\psi) \subseteq F(\eta, \psi)$ and $[F(\eta, \psi) : K(\psi)] = (p-1)/2$. By, for example, Corollary b in page 277 of [6], we have

$$\mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big) = \mathrm{br}\big(F(\eta, \psi)/F(\psi), \beta_0^{-(p-1)/2}\big).$$

Since $\beta_0^{s/2} = \alpha$, and $s/2$ divides $(p-1)/2$, we have $\beta_0^{-(p-1)/2} = \alpha^{-(p-1)/s}$. Hence,

$$\begin{aligned}[\psi] &= \mathrm{br}\big(F(\eta, \psi)/F(\psi), \alpha^{-(p-1)/s}\big)\,\mathrm{br}\big(F(\eta, \psi)/F(\psi), \alpha^{(p-1)/s}\big) \\ &= 1 \in \mathrm{Br}\big(F(\psi)\big).\end{aligned}$$

Hence, $[\psi] = 1$, and the Schur index of ψ is 1, as desired.

Corollary 3.8. *Let $L \supseteq \mathbf{SL}(n, q)$ be a subgroup of $\mathbf{GL}(n, q)$, let $\psi \in \mathrm{Irr}(L)$, and let F be a field of characteristic zero. Then, $[\psi] \in \mathrm{Br}(F(\psi))$ can be calculated as follows. If $2 \leqslant n_2 \leqslant (p-1)_2$, we set $s = 2(p-1)_2/n_2$, otherwise we set $s = 1$. Then*

(1) *If s divides $[L : \mathbf{SL}(n, q)]$, then $[\psi] = 1$.*
(2) *Suppose s does not divide $[L : \mathbf{SL}(n, q)]$. Let $\lambda \in \mathcal{F}_n$ be such that*

$$\big(\mathrm{Res}^{L}_{\mathbf{SL}(n,q)}(\psi), \mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)\big) \neq 0.$$

Let ψ_λ be any irreducible character of $\mathbf{SL}(n, q)$ contained in the restriction of χ_λ. Then, $F(\psi_\lambda) \subseteq F(\psi)$, and $[\psi] \in \mathrm{Br}(F(\psi))$ is simply the image of $[\psi_\lambda] \in \mathrm{Br}(F(\psi_\lambda))$ under extension of scalars.

Proof. If s divides $[L : \mathbf{SL}(n, q)]$, then the result follows immediately from Corollary 3.7. Hence, we assume that s does not divide $[L : \mathbf{SL}(n, q)]$, and we are in the situation of (2). There is no loss in assuming that F contains all the values of the restriction of χ_λ to $\mathbf{SL}(n, q)$, so we do. Theorem 3.3 gives us $[\![\chi_\lambda]\!] = [Z, \alpha, b] \in \mathrm{Clif}(G, F)$ explicitly, and we use the notation of this theorem. We set $\bar{I} = (L/J) \cap I$, where I is the inertia group of $[\![\chi]\!]$. It follows from Theorem 1.3 that $[\psi_\lambda] = \bar{b}$, where $\bar{b}$ is the extension of scalars of b to $\bar{b} \in \mathrm{Br}(F(\psi_\lambda))$, because we have, in this case $K = F(\psi_\lambda)$. Notice that the 2 part of the order of the center

of $\mathbf{GL}(n,q)$ is divisible by $(p-1)_2/n_2$, which implies, since s does not divide $[L:\mathbf{SL}(n,q)]$, that $[L:\tilde{I}]$ is odd. It follows that $F(\psi)$ and $[\psi]$ are given by Corollary 1.4 or Corollary 1.5(1). In either case, we have that $K \subseteq F(\psi)$, and $[\psi]$ is the extension of scalars of the element $b \in \mathrm{Br}(F)$. Hence, $[\psi]$ is the extension of scalars of $[\psi_\lambda]$, as desired.

Remark 3.9. We refer the reader to [9], where $[\psi_\lambda]$ is calculated explicitly for each λ, and its local Schur indices analyzed.

4. Clifford classes for extensions by diagonal automorphisms

The group $J = \mathbf{SL}(n,q)$ has a group of diagonal automorphisms D, which is cyclic of order $d = (q-1,n)$, see, for example, [1]. Let $\widetilde{H}$ be an extension of J by D. As we see below, such a group always exists. However, the group $\widetilde{H}$ is not uniquely determined by this information, and could be any one of a family of isoclinic groups, see [1]. We construct below each of the possibilities. The Clifford classes and Schur indices of the characters of $\widetilde{H}$ depend on which isoclinic group is chosen.

Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{H})$. Our goal is to calculate the Clifford class of $\widetilde{\chi}$, that is the element $[\![\widetilde{\chi}]\!] \in \mathrm{Clif}(D,F)$, where F is any field that contains all the values of the restriction of $\widetilde{\chi}$ to J. The argument of Proposition 3.2 shows that the restriction of $\widetilde{\chi}$ to J is the sum of a $\mathbf{GL}(n,q)$ conjugacy class of irreducible characters of $\mathbf{SL}(n,q)$. Hence, we have $\mathrm{Res}_J^{\widetilde{H}}(\widetilde{\chi}) = \mathrm{Res}_J^H(\chi_\lambda)$ for an appropriate element $\lambda \in \mathcal{F}_n$, where we keep the notation $H = \mathbf{GL}(n,q)$. By Proposition 3.2, $\widetilde{\chi}$ determines uniquely λ, up to multiplication by any element $\alpha \in \widehat{F_1}$. Our goal is to calculate $[\![\widetilde{\chi}]\!] \in \mathrm{Clif}(D,F)$ as a function of λ and the particular $\widetilde{H}$. Of course, $[\![\widetilde{\chi}]\!]$ will remain the same if $\widetilde{\chi}$ is replaced by its product times a linear character of D, or if λ is replaced by $\alpha\lambda$ for some $\alpha \in \widehat{F_1}$.

We begin by constructing the groups in question. Fix a special linear group $J = \mathbf{SL}(n,q)$, and let D be its group of diagonal automorphisms. We wish to construct an extension $\widetilde{H}$ of $\mathbf{SL}(n,q)$ by D. We refer the reader to [1] for standard results on D. In particular, we have that D is cyclic, D is naturally isomorphic to $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)\,\mathrm{Z}(\mathbf{GL}(n,q))$, and the order of D is $d := (q-1,n)$. We keep the notation $H = \mathbf{GL}(n,q)$, so that we have $D \simeq H/J\,\mathrm{Z}(H)$. The group $(\mathbb{F}_q^\times)^d$ of d-th powers of $\mathbb{F}_q^\times$ is a group of order $(q-1)/d$, and it follows that taking n/d-th powers in it is an automorphism. Hence, there exists some automorphism $\phi_1 : (\mathbb{F}_q^\times)^d \to (\mathbb{F}_q^\times)^d$ such that, for all $x \in (\mathbb{F}_q^\times)^d$, we have $x\phi_1(x)^{n/d} = 1$. The automorphism ϕ_1 can be extended, in various ways, to an endomorphism ϕ_2 of the cyclic group $\mathbb{F}_q^\times$. (The various extensions give rise to different isoclinic groups, see Proposition 4.1 below.) Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. Let C be the subgroup of order $d(q-1)$ of $\overline{\mathbb{F}}_q^\times$. We can view $H = \mathbf{GL}(n,q)$ and $\overline{\mathbb{F}}_q^\times$ as

subgroups of $\mathbf{GL}(n,\overline{\mathbb{F}}_q)$, where $\overline{\mathbb{F}}_q^\times$ is identified with the group of scalar matrices. Then $C\cap J$ is the subgroup of order d of $\overline{\mathbb{F}}_q^\times$. Furthermore, the map $C\to\mathbb{F}_q^\times$ given by $x\mapsto x^d$ is a surjective homomorphism with kernel $C\cap J$. We let $\phi_3:\mathbb{F}_q^\times\to C/(C\cap\mathbf{SL}(n,q))$ be the map ϕ_2 followed by the induced isomorphism $\mathbb{F}_q^\times\to C/(C\cap\mathbf{SL}(n,q))$. Hence, ϕ_3 is an homomorphism, and for all $x\in(\mathbb{F}_q^\times)^d$, if $c\in\phi_3(x)$, then we have $xc^n=1$. Finally, set $\phi:\mathbf{GL}(n,q)\to C/(C\cap J)$ to be **det** followed by ϕ_3. Let $\pi:C\to C/(C\cap J)$ be the projection homomorphism. We define

$$\widetilde{H}=\big\{hc\mid h\in H,\ c\in C \text{ and } \phi(h)=\pi(c)\big\}.$$

Both H and $\widetilde{H}$ are normal subgroups of HC, and $J\subseteq H\cap\widetilde{H}$. We set $\widetilde{G}=\widetilde{H}/J$. Furthermore, there is natural surjective group homomorphism $G\to\widetilde{G}$ given by $hJ\mapsto h\phi(h)J$. We set $g_1\in\widetilde{G}$ to be the image of g_0, the preferred generator of $G=H/J$, under this homomorphism. g_1 is a generator for $\widetilde{G}$.

Proposition 4.1. *$\widetilde{H}$ is an extension of $\mathbf{SL}(n,q)$ by D, which is uniquely determined by a choice of ϕ_2. Furthermore, the different choices of ϕ_2 give rise to all the different extensions of $\mathbf{SL}(n,q)$ by D up to isomorphism.*

Proof. By the above, $J\subseteq\widetilde{H}$, and $\widetilde{H}/J=\widetilde{G}$ is a natural homomorphic image of $G=H/J$. In particular, $\widetilde{G}$ is generated by g_1, the image of any generator g_0 under the homomorphism. Let $hc\in g_1^d$. Then, $hc\in\mathbf{GL}(n,q)$, $\mathbf{det}(h)\in(\mathbb{F}_q^\times)^d$, and $\phi(h)=\pi(c)$. It follows, by our choice of ϕ, that $\mathbf{det}(h)c^n=1$. This tells us that $\mathbf{det}(hc)=1$, in other words, that $hc\in\mathbf{SL}(n,q)$. Hence, $[\widetilde{H}:J]\leqslant d$. The cyclic group D is generated by the action of g_0 on J, which is the same as the action of g_1 on J. Hence, $[\widetilde{H}:J]=d$, and $\widetilde{H}$ is an extension of $\mathbf{SL}(n,q)$ by D.

Given $\mathbf{SL}(n,q)$, our conditions uniquely determine ϕ_1. The group $\widetilde{H}$ is uniquely determined by ϕ, which in turn is uniquely determined by ϕ_3, which itself is uniquely determined by ϕ_2. However, ϕ_2 is just any extension of ϕ_1. If θ is any homomorphism from $\mathbb{F}_q^\times$ to itself such that $\theta^d=1$, then, $\phi_2'=\theta\phi_2$ is another extension of ϕ_1, and furthermore, all extensions of ϕ_1 have this form. Let ϕ', $\widetilde{H}'$, and g_1' be the map, the group and the generator arising from ϕ_2' which correspond to ϕ, $\widetilde{H}$, and g_1, respectively. Let $h\in g_0$, and choose $c,c'\in C$ such that $hc\in\widetilde{H}$ and $hc'\in\widetilde{H}'$. Then, by the definition of $\widetilde{H}$ and $\widetilde{H}'$, we have that $(hc)^d=h^d\phi_2(\mathbf{det}(h))$, and

$$(hc')^d=h^d\phi_2'\big(\mathbf{det}(h)\big)=h^d\phi_2\big(\mathbf{det}(h)\big)\theta\big(\mathbf{det}(h)\big).$$

Comparing this result with the isoclinisms described, for example, in [1], we see that the different θ give rise to all extensions of $\mathbf{SL}(n,q)$ by D. This completes the proof of the proposition.

Remark 4.2. Our calculations of the Clifford classes, therefore, involve a particular choice of ϕ or ϕ_2. The reader may note that the relationship among

the Clifford classes of the various isoclinic groups in our case is exactly the one predicted in general by the results in [11].

Lemma 4.3. *Let $\lambda \in \mathcal{F}_n$, let F be a field containing the values of χ_λ on $\mathbf{SL}(n,q)$, and let Z be the commutative central simple G-algebra over F defined in Theorem* 3.3. *Then, the action of G on Z factors through the natural homomorphism $G \to D$. Hence, there is a commutative central simple D-algebra $\widetilde{Z}$ whose elements and algebra structure are those of Z, and whose action from D is that induced from the action of G on Z and the natural homomorphism. Furthermore, the inertia group $\tilde{I}$ of $\widetilde{Z}$ is simply the projection of I in D, and has order $|\tilde{I}| = |I|d/(q-1)$.*

Proof. We have

$$G = \mathbf{GL}(n,q)/\mathbf{SL}(n,q) \quad \text{and} \quad D = \mathbf{GL}(n,q)/\mathbf{SL}(n,q)\,\mathrm{Z}\big(\mathbf{GL}(n,q)\big).$$

There is a natural projection homomorphism $G \to D$. The projection of $\mathrm{Z}(\mathbf{GL}(n,q))$ in G is the kernel of this projection, and the order of this kernel is $(q-1)/d$. As remarked in the proof of Theorem 3.3, the inertia group I is the image in G of the inertia group I_0 in $\mathbf{GL}(n,q)$ of ψ_λ. Since $\mathrm{Z}(\mathbf{GL}(n,q))$ will certainly be contained in the inertia group I_0, it follows that the projection of $\mathrm{Z}(\mathbf{GL}(n,q))$ in G is contained in I. Since in Theorem 3.3 I is acting trivially on Z, the action of G on Z does factor through the natural homomorphism $G \to D$. Hence, $\widetilde{Z}$ is a central simple commutative D-algebra over F. As recalled in Section 1, the inertia group $\tilde{I}$ is the centralizer of the action of D on $\widetilde{Z}$. Hence, its preimage in G is simply the inertia group I of Z. The result follows.

Lemma 4.4. *Let ϕ_2 be as in Proposition* 4.1. *Let $\lambda \in \mathcal{F}_n$, and set $\chi = \chi_\lambda$. The group G has a preferred generator g_0. Set*

$$\mu = \frac{\chi(\phi_2(\mathbf{det}(g_0)))}{\chi(1)},$$

where we view $\phi_2(\mathbf{det}(g_0))$ as a scalar matrix in the center of $\mathbf{GL}(n,q)$. Then μ is a $(q-1)$-th root of 1, *and $\mu^{(1-q)/d} \in \mathbf{Q}(\mathrm{Res}^H_J(\chi))$. Furthermore, the* 2*-part of the multiplicative order of $\mu^{(1-q)/d}$ in $F^\times$ divides d_2, and it is d_2 unless d is even and either the central involution of H is in the kernel of χ, or the* 2*-part of the multiplicative order of ϕ_2 is not $(q-1)_2$.*

Proof. The map ϕ_2 is an endomorphism $\phi_2 : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, and $\mathbb{F}_q^\times$ is identified with the center $\mathrm{Z}(\mathbf{GL}(n,q))$. Hence χ can be evaluated at any value of ϕ_2, and $\phi_2(\mathbf{det}(g_0))$ is an element of order a divisor of $q-1$. Furthermore, since $\mathbf{det}(g_0)$ is a generator for $\mathbb{F}_q^\times$, the multiplicative order of ϕ_2 is the order of $\phi_2(\mathbf{det}(g_0))$. The restriction of χ to the center is a multiple of some linear character. Hence

$$\mu = \frac{\chi(\phi_2(\mathbf{det}(g_0)))}{\chi(1)}$$

is some $(q-1)$-th root of 1. Furthermore, the subgroup of order d of $\mathrm{Z}(H)$ is contained in J, and since the value of χ at some appropriate element of this subgroup is a non-zero rational multiple of $\mu^{(1-q)/d}$, we have $\mu^{(1-q)/d} \in \mathbf{Q}(\mathrm{Res}_J^H(\chi))$. The 2-part of the order of $\mu^{(1-q)/d}$ always divides d_2. Suppose, it is not d_2. Then, d is even, and the 2-part of the order of μ divides $(q-1)_2/2$, which implies that either the central involution of H is in the kernel of χ, or the 2-part of the order of $\phi_2(\mathbf{det}(g_0))$ is not $(q-1)_2$. Hence, the result holds.

Lemma 4.5. *Let $\widetilde{H}$ be the group $\mathbf{SL}(n,q)$ extended by its group D of diagonal automorphisms, as described in Proposition* 4.1*, and use the notation of the beginning of this section. Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{H})$, and let F be a field containing the values of the restriction of $\widetilde{\chi}$ to J. Let $\lambda \in \mathcal{F}_n$ be such that $\mathrm{Res}_J^H(\chi_\lambda) = \mathrm{Res}_J^{\widetilde{H}}(\widetilde{\chi})$. Set $\chi = \chi_\lambda$. Let $\widetilde{Z}$ and $\tilde{I}$ be as in Proposition* 4.3*. By Theorem* 3.3*, and in its notation, there exists some $j_0 \in i_0$ such that $\chi(j_0) \neq 0$. Let μ be as in Lemma* 4.4*. Then, $\mu\chi(j_0)^{|\tilde{I}|} \in F^\times$, and we set $\widetilde{\alpha}_0 = \mu\chi(j_0)^{|\tilde{I}|} \in F^\times$. Take any $\widetilde{\alpha} \in F^\times$ such that, up to $|\tilde{I}|$-th powers in $F^\times$, it is $\widetilde{\alpha}_0$. Then, setting $\alpha = \mu^{(1-q)/d}\widetilde{\alpha}^{(q-1)/d}$, we have that $[\![\chi]\!] = [Z, \alpha, b] \in \mathrm{Clif}(G,F)$ for some explicit unique $b \in \mathrm{Br}(F)$ (see Theorem* 3.3*), and furthermore*

$$[\![\widetilde{\chi}]\!] = [\widetilde{Z}, \widetilde{\alpha}, b] \in \mathrm{Clif}(D,F).$$

Proof. Both H and $\widetilde{H}$ are normal subgroups of the group HC. Since $\chi \in \mathrm{Irr}(H)$, and C is cyclic and central, there exists some extension $\rho \in \mathrm{Irr}(HC)$ of χ to HC. Let $\widetilde{\chi}' = \mathrm{Res}_{\widetilde{H}}^{HC}(\rho)$. Then $\mathrm{Res}_J^{\widetilde{H}}(\widetilde{\chi}') = \mathrm{Res}_J^H(\chi) = \mathrm{Res}_J^{\widetilde{H}}(\widetilde{\chi})$. It follows that $\widetilde{\chi}'$ is irreducible, and it is equal to $\widetilde{\chi}$ times some linear character of D. Hence, $[\![\widetilde{\chi}']\!] = [\![\widetilde{\chi}]\!] \in \mathrm{Clif}(D,F)$. Hence, we assume, without loss, that $\widetilde{\chi}' = \widetilde{\chi}$.

Theorem 3.3 calculates $[\![\chi]\!]$, and we adopt its notation. By, for example, Theorem 1.4 in [10], we see that the first entry of $[\![\widetilde{\chi}]\!] \in \mathrm{Clif}(D,F)$ is $\widetilde{Z}$, as given in Lemma 4.3. In the notation of Theorem 3.3, we have $j_0 \in i_0 = g_0^r$, and $r = [G:I] = [\widetilde{G}:\tilde{I}]$. Let $c_0 \in \phi(g_0)$. Then $j_0 c_0^r \in \widetilde{H}$, and furthermore $j_0 c_0^r \in g_1^r$. $c_0 \in C$, so that it is in the center of HC, and it follows that $\rho(c_0)/\chi(1)$ is a root of unity, and we set $\mu' = \rho(c_0)/\chi(1)$. Hence, $\widetilde{\chi}(j_0 c_0^r) = (\mu')^r \chi(j_0) \neq 0$. Since $r|\tilde{I}| = d$, it then follows from Theorem 1.4 in [10] that $\widetilde{\alpha}_0 := (\mu')^d \chi(j_0)^{|\tilde{I}|} \in F^\times$, and that the second coordinate of $[\![\widetilde{\chi}]\!]$ is equal to $\widetilde{\alpha}_0$, up to $|\tilde{I}|$-th powers in $F^\times$. However, $c_0^d = \phi_2(\mathbf{det}(g_0))$ by our choice of ϕ, so $(\mu')^d = \mu$, and $\widetilde{\alpha}_0 = \mu\chi(j_0)^{|\tilde{I}|}$.

Now pick $\widetilde{\alpha} = \beta^{|\tilde{I}|}\tilde{\alpha}_0$, for some $\beta \in F^\times$. We take $\alpha = \beta^{|I|}\alpha_0$, and $\alpha \in F^\times$ is allowable in Theorem 3.3, so that there is a unique $b \in \mathrm{Br}(F)$, such that $[\![\chi]\!] = [Z, \alpha, b] \in \mathrm{Clif}(G,F)$. Since by Lemma 4.3, $|\tilde{I}| = |I|d/(q-1)$, we have that

$$\alpha = \big(\beta\chi(j_0)\big)^{|I|} = \Big(\big(\beta\chi(j_0)\big)^{|\tilde{I}|}\Big)^{(q-1)/d} = \mu^{(1-q)/d}\widetilde{\alpha}^{(q-1)/d}.$$

Choose γ to be an r-th root of $\beta\chi(j_0)$, so that $\gamma^r = \beta\chi(j_0)$. Let L be a finite Galois extension of F which is a splitting field for ρ, and such that $\gamma \in L$. Let M be a HC-module over L affording the character ρ. Let $g \in g_0$ be any representative of g_0, and set $A = \mathrm{End}_{FJ}(M)$, and set $B = \mathrm{C}_A(g\gamma^{-1})$. By Lemma 3.1 in [10], B is a central simple algebra representing the element $b \in \mathrm{Br}(F)$, where $[\![\chi]\!] = [Z, \alpha, b]$. However, M affords the character $\widetilde{\chi}$ when restricted to $\widetilde{H}$. Hence, we may apply again Lemma 3.1 in [10], with $\widetilde{G}$ in place of G, g_1 in place of g_0, gc_0 in place of g, and $\mu'\gamma$ in place of γ. Since the action of c_0 on M is multiplication by μ', we have that the action of $gc_0(\mu'\gamma)^{-1}$ on M is the same as that of $g\gamma^{-1}$, it follows that $B = \mathrm{C}_A(gc_0(\mu'\gamma)^{-1})$. Hence,

$$[\![\widetilde{\chi}]\!] = \big[\widetilde{Z}, \widetilde{\alpha}, b\big].$$

This concludes the proof of the lemma.

Theorem 4.6. *Let $\widetilde{H}$ be the group $\mathbf{SL}(n,q)$ extended by its group D of diagonal automorphisms, as described in Proposition* 4.1*, and use the notation of the beginning of this section. In particular, we have $J = \mathbf{SL}(n,q)$, $H = \mathbf{GL}(n,q)$, $\widetilde{H}/J = D$, and $H/J = G$. Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{H})$, and let F be a field containing the values of the restriction of $\widetilde{\chi}$ to J. Let $\lambda \in \mathcal{F}_n$ be such that $\mathrm{Res}_J^H(\chi_\lambda) = \mathrm{Res}_J^{\widetilde{H}}(\widetilde{\chi})$. Set $\chi = \chi_\lambda$. We fix a generator g_0 of G, and the corresponding generator g_1 of the cyclic group D. Then, $[\![\widetilde{\chi}]\!] \in \mathrm{Clif}(D,F)$, and, by Theorem* 1.1*, $[\![\widetilde{\chi}]\!] = [\widetilde{Z}, \widetilde{\alpha}, \tilde{b}]$, for appropriate $\widetilde{Z}$, $\widetilde{\alpha}$, and $\tilde{b}$, which are as follows.*

We take η to be a primitive p-th root of unity. Let μ be as in Lemma 4.4*. Let r be the size of the stabilizer of the action of $\widehat{F_1}$ on λ. Set $i_0 = g_0^r$, and I to be the subgroup of G generated by i_0, and $\tilde{I}$ to be the subgroup of D generated by g_1^r. There necessarily exists some $j_0 \in i_0$ such that $\chi(j_0) \neq 0$. We set $\widetilde{\alpha}_0 = \mu\chi(j_0)^{|\tilde{I}|}$, for any one such j_0. If q is not a square, $2 \leqslant n_2 \leqslant (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then we set $K = F(\sqrt{\varepsilon p})$, where $\varepsilon \in \{1, -1\}$ and $p \equiv \varepsilon \pmod 4$. Otherwise, we set $K = F$. If $2 \leqslant n_2 \leqslant (p-1)_2$, the central involution of H is not in the kernel of χ, and $(q-1)_2$ divides the multiplicative order of ϕ_2, then we set $\mu_1 = -1$. Otherwise, we set $\mu_1 = 1$. Finally, if $2 \leqslant n_2 \leqslant (p-1)_2$, and q is not a square, then we set $m = 1$. We set $m = 0$, otherwise. With this notation, we have*:

(1) *$\widetilde{Z}$ is isomorphic to the direct sum of $r/[K:F]$ copies of K, which are permuted transitively by $D/\tilde{I}$, and where the subgroup of $D/\tilde{I}$ of order $[K:F]$ acts on each copy of K as the Galois group of K/F.*
(2) *$\widetilde{\alpha}$ can be taken to be the product of $\widetilde{\alpha}_0 \in F^\times$ times the $|\tilde{I}|$-th power of any element of $F^\times$.*
(3) *$\tilde{b} = \mathrm{br}(F(\eta)/F, \mu_1(\widetilde{\alpha}^m)^{(p-1)/2})$.*

In particular, if either $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $\tilde{b} = 1$. Furthermore, if $2 \leqslant n_2 \leqslant (p-1)_2$, and q is a square, then $\tilde{b} = \mathrm{br}(F(\eta)/F, \mu_1)$.

Proof. The structure of $\widetilde{Z}$, and the possible values for $\widetilde{\alpha}$ follow immediately from Lemma 4.5. If either $p=2$, or n is odd, or $n_2 > (p-1)_2$, then, by Lemma 4.5 and Theorem 3.3, we have $b = 1 \in \mathrm{Br}(F)$. Furthermore, our notation gives us that, in this case, $\mu_1 = 1$, and $m = 0$, so that the theorem holds in this case. Hence, assume that $2 \leqslant n_2 \leqslant (p-1)_2$. By Lemma 4.5 and Theorem 3.3, we have that $b = \mathrm{br}(F(\eta)/F, \alpha^{(p-1)/s})$, where $\alpha = \mu^{(1-q)/d}\widetilde{\alpha}^{(q-1)/d}$, and $s = 2(p-1)_2/n_2$. Since $n_2 \leqslant (p-1)_2$, we have $d_2 = n_2$. Set

$$m_0 = \frac{q-1}{d_{2'}(p-1)_2} \quad \text{where } d_{2'} \text{ is the odd part of } d, \text{ and}$$

$$\mu_2 = \mu^e \quad \text{where } e = \frac{(1-q)(p-1)n_2}{2d(p-1)_2}.$$

Then, we have

$$b = \mathrm{br}\big(F(\eta)/F, \mu_2(\widetilde{\alpha}^{m_0})^{(p-1)/2}\big).$$

Clearly, m_0 is an integer, and it is even or odd according to whether or not q is a square. Hence, $m_0 \equiv m \pmod 2$. Since $\mathrm{br}(F(\eta)/F, a) = 1$ whenever a is the $(p-1)$-th power of some element of $F^\times$, it follows that

$$b = \mathrm{br}\big(F(\eta)/F, \mu_2\big(\widetilde{\alpha}^{m}\big)^{(p-1)/2}\big).$$

Set $\rho = \mu^{(1-q)/d}$, and $f = ((p-1)n_2)/(2(p-1)_2)$, so that $\mu_2 = \rho^f$. By Lemma 4.4, $\rho \in F^\times$, and ρ is a d-th root of 1. Write $\rho = \rho_1\rho_2$, where both ρ_1 and ρ_2 are powers of ρ, and ρ_1 has multiplicative odd order in $F^\times$, and ρ_2 has order a power of 2. By Lemma 4.4, the order of ρ_2 is $d_2 = n_2$ if the central involution is not in the kernel of χ, and $(q-1)_2$ divides the multiplicative order of ϕ_2, and the order of ρ_2 divides $n_2/2$ otherwise. Hence, in all cases, we have $\rho_2^{n_2/2} = \rho_2^f = \mu_1$. Now ρ_1^f is the $(p-1)$-th power of some element of $F^\times$. Hence, μ_1 and μ_2 are equal up to $(q-1)$-th powers in $F^\times$, and, in particular, we may replace one by the other in the formula for b. This completes the proof of the theorem.

Remark 4.7. From our calculation in Theorem 4.6 of the Clifford classes for the characters of the groups $\widetilde{H} = \mathbf{SL}(n,q)D$, we can describe all the characters of every subgroup of $\widetilde{H}$ that contains $\mathbf{SL}(n,q)$, including their fields of values, and the element of the Brauer group associated with them. We can furthermore calculate their Schur indices, including their local Schur indices. The method to do this is completely analogous to the one for the case of $\mathbf{GL}(n,q)$, as described in Remark 3.5.

We now describe some general results on Schur indices and elements of the Brauer group associated with irreducible characters of subgroups of $\widetilde{H}$.

Corollary 4.8. *Let $L \supseteq \mathbf{SL}(n,q)$ be a subgroup of $\widetilde{H}$, and let $\psi \in \mathrm{Irr}(L)$. Then the Schur index of ψ at most* 2.

Proof. This can be proved, mutatis mutandis, the way Corollary 3.6 was. Our corollary also follows easily from the next one.

Corollary 4.9. *Let $\widetilde{H}$ be as in Proposition* 4.1. *Let $L \supseteq \mathbf{SL}(n,q)$ be a subgroup of $\widetilde{H}$, and let $\psi \in \mathrm{Irr}(L)$, and let F be a field of characteristic zero. Then, the element $[\psi] \in \mathrm{Br}(F(\psi))$ can be calculated as follows. We let $\psi_\lambda \in \mathrm{Irr}(\mathbf{SL}(n,q))$ be an irreducible character contained in the restriction of ψ to $\mathbf{SL}(n,q)$, and we let η be a primitive p-th root of* 1. *Furthermore, we set $\mu_1 = -1$ if q is odd, $(q-1)_2$ divides the order of ϕ_2, and the central involution of $\widetilde{H}$ is not in the kernel of ψ; and we set $\mu_1 = 1$ otherwise.*

(1) *If either $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $[\psi] = 1$, and the Schur index of ψ is* 1.
(2) *Suppose $2 \leqslant n_2 \leqslant (p-1)_2$, and $[L : \mathbf{SL}(n,q)]$ is even. Then*

$$[\psi] = \mathrm{br}\big(F(\psi,\eta)/F(\psi), \mu_1\big) \in \mathrm{Br}\big(F(\psi)\big).$$

(3) *Suppose $2 \leqslant n_2 \leqslant (p-1)_2$, and $[L : \mathbf{SL}(n,q)]$ is odd. Then, $F(\psi_\lambda) \subseteq F(\psi)$, and $[\psi] \in \mathrm{Br}(F(\psi))$ is simply the image of $[\psi_\lambda] \in \mathrm{Br}(F(\psi_\lambda))$ under extension of scalars.*

Proof. Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{H})$ be such that its restriction to L contains ψ. Let $\lambda \in \mathcal{F}_n$ be such that it parametrizes an irreducible character ψ_λ contained in the restriction of ψ to $J = \mathbf{SL}(n,q)$. Then, $\mathrm{Res}^H_J(\chi_\lambda) = \mathrm{Res}^{\widetilde{H}}_J(\widetilde{\chi})$. If the corollary holds over some field contained in $\mathbf{Q}(\psi)$, then it follows that it holds over any field of characteristic zero by simple extension of scalars. Notice that $\mathrm{Res}^{\widetilde{H}}_J(\widetilde{\chi})$ is a sum of conjugates of $\mathrm{Res}^L_J(\psi)$, and it follows that $\mathbf{Q}(\mathrm{Res}^{\widetilde{H}}_J(\widetilde{\chi})) \subseteq \mathbf{Q}(\psi)$. We set, without loss,

$$F = \mathbf{Q}\big(\mathrm{Res}^{\widetilde{H}}_J(\widetilde{\chi})\big) = \mathbf{Q}\big(\mathrm{Res}^H_J(\chi_\lambda)\big).$$

Then, by Proposition 3.1, $F = \mathrm{Galr}(\lambda)'$. Assume first that $p = 2$, or n is odd, or $n_2 > (p-1)_2$. By Theorem 4.6, $[\![\widetilde{\chi}]\!] = [\widetilde{Z}, \widetilde{\alpha}, 1] \in \mathrm{Clif}(D, F)$, where $\widetilde{Z}$ is a direct sum of copies of F. Hence, by Corollary 1.4, $[\psi] = 1$, and the Schur index of ψ is 1, as desired. We assume henceforth that $2 \leqslant n_2 \leqslant (p-1)_2$. Now, by Theorem 4.6, and in its notation, $[\![\widetilde{\chi}]\!] = [\widetilde{Z}, \widetilde{\alpha}, \tilde{b}]$, with

$$\tilde{b} = \mathrm{br}\big(F(\eta)/F, \mu_1\big(\widetilde{\alpha}^m\big)^{(p-1)/2}\big).$$

Suppose now that $\widetilde{Z}$ is the direct sum of copies of F. By Corollary 1.4, $F = F(\psi_\lambda)$, $[\psi_\lambda] = \tilde{b} \in \mathrm{Br}(F)$, and there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of α (with $\bar{I} = (L/J) \cap \tilde{I}$), and $F(\psi) = F(\beta)$. In addition, $[\psi] = \bar{b} \in \mathrm{Br}(F(\psi))$, where $\bar{b}$ is the image of $\tilde{b}$ under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$. It follows that if $[L : \mathbf{SL}(n,q)]$ is odd, the corollary holds. Hence, we assume for the remainder of this paragraph that $[L : \mathbf{SL}(n,q)]$ is even. If q is a square, by Theorem 4.6, $m = 0$, which implies that $\tilde{b} = \mathrm{br}(F(\eta)/F, \mu_1)$,

and the corollary follows immediately in this case. Hence, we assume that q is not a square. Since we are assuming that $\widetilde{Z}$ is the direct sum of copies of F, it follows from Theorem 4.6 that, for some element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. By Definition 2.5 and Theorem 3.3, this implies that n_2 does not divide $[G : I]$. Since q is odd, and not a square, we have $(q-1)_2 = (p-1)_2$. Using Lemma 4.3, this implies that $|\tilde{I}|_2 = (|I|_2 d_2)/(q-1)_2$ does not divide d_2/n_2. It follows that $|\tilde{I}|$ is even. Hence, $|\bar{I}|$ is even, and $\widetilde{\alpha}$ is a square in $F(\psi)$. The result follows also in this case.

Finally, suppose that $\widetilde{Z}$ is not a direct sum of copies of F. By Theorem 4.6, q is not a square and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Furthermore, we set $K = F(\sqrt{\varepsilon p})$, where $\varepsilon \in \{1, -1\}$ and $p \equiv \varepsilon \pmod 4$, and $\widetilde{Z}$ is isomorphic to the direct sum of $r/2$ copies of K, which are permuted transitively by $D/\tilde{I}$, and where the subgroup of $D/\tilde{I}$ of order 2 acts on each copy of K as the Galois group of K/F. Furthermore, we have that $K = F(\psi_\lambda)$. Since for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, by Definition 2.5 and Theorem 3.3, we have that n_2 divides $[G : I]$. Since q is odd, and not a square, we further have $(q-1)_2 = (p-1)_2$. Using Lemma 4.3, this implies that $|\bar{I}| = (|I|d)/(q-1)$ divides d/n_2. Since $d = (q-1, n)$, we have that d/n_2 is odd, and it follows that $|\tilde{I}|$ is odd. Setting $\bar{I} = (L/J) \cap \tilde{I}$, we have that $|\bar{I}|$ is odd as well.

Assume that $[L : J]$ is even. Then, r_2 does not divide $[D : L]$ and we are in the second case of Corollary 1.5. Hence, there exists some $\beta \in F(\psi)^\times$ such that β is an $|\bar{I}|$-th root of $\widetilde{\alpha}$, $F(\psi) = F(\beta)$. Furthermore, there exists some $\beta_0 \in F(\psi)^\times$ such that β_0 is an $|\tilde{I}|_2$-th root of $\widetilde{\alpha}$, and, we have

$$[\psi] = \mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big)\bar{b} \in \mathrm{Br}\big(F(\psi)\big),$$

where the product is in the Brauer group $\mathrm{Br}(F(\psi))$, and $\bar{b}$ is the image of b under the extension of scalars map $\mathrm{Br}(F) \to \mathrm{Br}(F(\psi))$. Since $\tilde{I}$ has odd order, $\beta_0 = \widetilde{\alpha}$. From Theorem 4.6, we know that $\widetilde{\alpha}_0 = \mu\chi(j_0)^{|\tilde{I}|}$, and that $\widetilde{\alpha}$ is $\widetilde{\alpha}_0$ times the $|\tilde{I}|$-th power of some element of $F^\times$. Since $|\bar{I}|$ divides $|\tilde{I}|$, $\chi(j_0)^{|\tilde{I}|}$ is the $|\bar{I}|$ power of some element of $F(\chi)$. Since the values of χ are contained in F extended by p'-th roots of unity, μ is a $(q-1)$-th root of 1, and p does not divide $|\bar{I}|$, it follows that β is in the field of p'-th roots of unity. It follows that $F(\psi)$ is contained in the field of p'-th roots of unity. Hence, $K(\psi) \subseteq F(\eta, \psi)$ and $[F(\eta, \psi) : K(\psi)] = (p-1)/2$. By, for example, Corollary b in page 277 of [6], we have

$$\mathrm{br}\big(K(\psi)/F(\psi), \beta_0^{-1}\big) = \mathrm{br}\big(F(\eta, \psi)/F(\psi), \widetilde{\alpha}^{-(p-1)/2}\big).$$

Hence, since in our case $m = 1$,

$$\begin{aligned}[\psi] &= \mathrm{br}\big(F(\eta, \psi)/F(\psi), \widetilde{\alpha}^{-(p-1)/2}\big)\,\mathrm{br}\big(F(\eta, \psi)/F(\psi), \mu_1\widetilde{\alpha}^{(p-1)/2}\big)\\ &= \mathrm{br}\big(F(\eta, \psi)/F(\psi), \mu_1\big) \in \mathrm{Br}\big(F(\psi)\big).\end{aligned}$$

Therefore, the corollary holds if $[L : J]$ is even.

Hence, we assume that $[L : J]$ is odd. By Corollary 1.5(1) applied to χ_λ, with $L = J$, we have that $[\psi_\lambda] = \bar{b}$, where $\bar{b}$ is the extension of scalars of $\tilde{b}$ to $\bar{b} \in \mathrm{Br}(F(\psi_\lambda))$. We set $\bar{I} = (L/J) \cap \tilde{I}$. Notice that r_2 divides $[\tilde{H} : L]$. It follows that $F(\psi)$ and $[\psi]$ are given by Corollary 1.5(1). We have that $K \subseteq F(\psi)$, and $[\psi]$ is the extension of scalars of the element $\tilde{b} \in \mathrm{Br}(F)$. Hence, $[\psi]$ is the extension of scalars of $[\psi_\lambda]$, as desired. The proof of the corollary is complete.

Remark 4.10. We refer the reader to [9], where $[\psi_\lambda]$ is calculated explicitly for each λ, and its local Schur indices analyzed. We note, furthermore, that using results in, for example, [5], it follows that the local Schur $\mathrm{m}_\ell(\psi)$ of any character ψ in the situation of Corollary 4.9(2) is 1 for all rational prime ℓ, except, possibly, $\ell = p$ or $\ell = \infty$.

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