Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 277 (2004) 579–614
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized Foulkes' Conjecture and tableaux construction

Rebecca Vessenes [1]

California Institute of Technology, Department of Mathematics, M.C. 253-37, Pasadena, CA 91125, USA

Received 25 June 2003

Available online 9 April 2004

Communicated by Jan Saxl

Abstract

Foulkes conjectured that for $n = ab$ and $a \leqslant b$, every irreducible module occurring as a constituent in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_a}$ occurs with greater or equal multiplicity in $1^{\mathcal{S}_n}_{\mathcal{S}_a \wr \mathcal{S}_b}$. We generalize this to say those irreducibles also occur in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$, where $cd = n$ and $c, d \geqslant a$. We prove the generalized conjecture for $a = 2$ and $a = 3$ by explicitly constructing the corresponding tableaux. To do so we develop a theory of construction conditions for tableaux giving rise to $\mathcal{S}_b \wr \mathcal{S}_a$ modules and in doing so, completely classify all such tableaux for $a = 2$ and $a = 3$.
© 2004 Elsevier Inc. All rights reserved.

Keywords: Foulkes' Conjecture; Tableaux; Wreath products

1. Introduction

Foulkes' Conjecture is an outstanding problem in the areas of plethysms, rational homotopy theory, multisymmetric functions, and representation theory of symmetric groups. In representation terms, Foulkes' Conjecture deals with induced permutation characters of wreath products of symmetric groups. The wreath product of symmetric groups $\mathcal{S}_a$ and $\mathcal{S}_b$, denoted $\mathcal{S}_a \wr \mathcal{S}_b$, is the normalizer of the Young subgroup

$$\underbrace{\mathcal{S}_a \times \cdots \times \mathcal{S}_a}_{b}$$

E-mail address: rebeccav@its.caltech.edu.

[1] The results of this paper constitute part of the author's doctoral thesis, written under the direction of Professor David Wales at the California Institute of Technology.

0021-8693/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2004.01.017

in $\mathcal{S}_{ab}$. Let 1_H be the trivial representation of a group H and 1_H^G the induced representation from H to G. From this we can state Foulkes' Conjecture:

Conjecture 1 (Foulkes' Conjecture). *If $a \leqslant b$ then every irreducible character occurring as a constituent in $1^{\mathcal{S}_{ab}}_{\mathcal{S}_b \wr \mathcal{S}_a}$ occurs in $1^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ with multiplicity greater than or equal to its multiplicity in $1^{\mathcal{S}_{ab}}_{\mathcal{S}_b \wr \mathcal{S}_a}$.*

Foulkes made this conjecture in [7] from his work on plethysms. The $a = 2$ case was contained in Thrall's 1942 work on symmetrized Kronecker powers [18], and proved again in [13] by James and Kerber using Gaussian coefficients. Coker also gives a proof for $a = 2$ using eigenvalues in [4], while Doran proves it in [6] using transition matrices. The $a = 3$ case was proven by Dent and Siemons using mappings of unordered partitions [5]. In [1], Black and List formulated Foulkes' conjecture in terms of matrix incidences and Wu has rephrase it in terms of rational homotopy theory in [20]. Howe, in [10], used a plethystic approach to interpret Foulkes' Conjecture via canonical morphisms between symmetric power modules. Using this, Brion [3] showed the conjecture holds for b sufficiently large with respect to a and Briand [2] proved Foulkes' Conjecture for $a = 4$. Also using symmetric powers and plethysms, Stanley [17] places Foulkes' Conjecture inside a larger body of open positivity conjectures in Algebraic Combinatorics. With a more combinatorial approach, Doran gave additional formulations in [6] using tableaux spaces. We will use some of his ideas in this paper. Doran also suggested generalizing Foulkes' Conjecture to:

Conjecture 2 (Generalized Foulkes' Conjecture). *Given $n = ab$, $a \leqslant b$, if c, d are such that $cd = n$, and $c, d \geqslant a$, then every irreducible character occurring as a constituent in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_a}$ occurs in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$ with multiplicity at least as large.*

For $c = a$ and $d = b$ this becomes the standard Foulkes' Conjecture. Note that $c, d \geqslant a$ is necessary. This is easily verified by using GAP [8], which shows that some irreducibles in $1^{\mathcal{S}_{12}}_{\mathcal{S}_4 \wr \mathcal{S}_3}$ do not occur in $1^{\mathcal{S}_{12}}_{\mathcal{S}_6 \wr \mathcal{S}_2}$. Conjecture 2 holds for small n (less than 28), by computer verification, also using GAP [8]. In Section 5 we will prove it holds for $a = 2$ by construction. Namely, we will show:

Theorem 1. *Given $b \geqslant 2$, let $n = 2b$. If c, d are such that $cd = n$, and $c, d \geqslant 2$, then every irreducible occurring in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_2}$ occurs in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$ with equal or larger multiplicity.*

We can also prove the following variation on Conjecture 2 for $a = 3$. The ideas of this proof are discussed in Section 6.

Theorem 2. *Let $n = 3b = cd$, with $c, d \geqslant 3$. Then every irreducible character occurring in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_3}$ occurs in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$.*

We discuss a version of Foulkes' Conjecture for the alternating character in Section 2 along with the corresponding versions of our results. In Section 3 we describe the necessary

concepts to approach the Foulkes' Conjecture combinatorially using tableaux. Much of this framework was developed by Doran in [6]. We develop the theory and techniques behind constructing appropriate tableaux in Section 4. The proof of Theorem 1 is given in Section 5. Since Theorem 2 requires constructing a large number tableaux, we omit these constructions and just outline the proof in Section 6. Further implications are listed in Section 7.

2. The alternating character

Since Foulkes' Conjecture is based on the trivial character, it is natural to ask whether the conjecture holds for the alternating character. The first question, is what we mean by the alternating character in terms of induced modules.

Consider, the 'alternating' character of the form $((-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_a \wr \mathcal{S}_b})\uparrow^{\mathcal{S}_{ab}}$, that is, the usual alternating character of $\mathcal{S}_{ab}$ restricted to the subgroup $\mathcal{S}_a \wr \mathcal{S}_b$ which is induced back up to $\mathcal{S}_{ab}$. A brief computer check of Foulkes' Conjecture using this character shows it holds for some small values of a and b. In fact, Foulkes' Conjecture is equivalent to the following conjecture using the alternating character:

Conjecture 3 (Foulkes' Conjecture for alternating characters). *If $a \leqslant b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a})\uparrow^{\mathcal{S}_{ab}}$ occurs in $((-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_a \wr \mathcal{S}_b})\uparrow^{\mathcal{S}_{ab}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a})\uparrow^{\mathcal{S}_{ab}}$.*

Naturally, this conjecture also generalizes to:

Conjecture 4 (Generalize Foulkes' Conjecture for alternating characters). *Given $n = ab$, $a \leqslant b$, if c, d are such that $cd = n$, and $c, d \geqslant a$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a})\uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_d \wr \mathcal{S}_c})\uparrow^{\mathcal{S}_n}$ with multiplicity at least as large.*

Showing the equivalences of Conjectures 1 or 2 (Foulkes' Conjecture for trivial characters), and Conjectures 3 or 4 (Foulkes' Conjecture for alternating characters) is straight forward. We will assume Conjecture 1 (or Conjecture 2) holds and prove the alternating character version. The same argument shows the reverse equivalence.

Proof. First recall that if S is a subgroup of finite index in G, F an S-module and E a G-module over a field, then there is an isomorphism $\mathrm{Ind}_S^G(\mathrm{Res}_S(E) \otimes F) \simeq E \otimes \mathrm{Ind}_S^G(F)$. (See Chapter XVIII, §7 of [14].) Note that here we have used Ind for induction and Res for restriction of modules. Also, let $G = \mathcal{S}_n$, $S = \mathcal{S}_a \wr \mathcal{S}_b$ and $T = \mathcal{S}_b \wr \mathcal{S}_a$ or $\mathcal{S}_d \wr \mathcal{S}_c$ as appropriate. Let E be the G-module corresponding to the character (-1) on G. Since we are working over $\mathbb{C}$, we will use $\mathbb{C}$ to denote the trivial module over any group.

The characters we're comparing are $\chi_S = \mathrm{Ind}_S^G(\mathrm{Res}_S(E)) = (-1)_{\mathcal{S}_{ab}}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}\uparrow^{\mathcal{S}_{ab}}$ and $\chi_T = \mathrm{Ind}_T^G(\mathrm{Res}_T(E))$. Then $\chi_S \simeq \mathrm{Ind}_S^G(\mathrm{Res}_S(E) \otimes \mathbb{C}) \simeq E \otimes \mathrm{Ind}_S^G(\mathbb{C})$ by the isomorphism mentioned above. Similarly for χ_T. Switching notation back to characters, we get $\chi_S \simeq (-1)_G 1_S^G$ and $\chi_T \simeq (-1)_G 1_T^G$.

Since we've assumed Foulkes' Conjecture on trivial characters, we have $1_S^G = 1_T^G + \psi$ for some character ψ. Then $\chi_S = (-1)_G(1_T^G + \psi) = (-1)_T^G 1_T^G + (-1)_G\psi$. Hence $\chi_S \leqslant \chi_T$ as desired. □

Applying this argument to Theorem 1, we have:

Theorem 3. *If $2 \leqslant b$ then every irreducible character occurring in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_2})\uparrow^{\mathcal{S}_{2b}}$ occurs in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_2 \wr \mathcal{S}_b})\uparrow^{\mathcal{S}_{2b}}$ with multiplicity greater than or equal to its multiplicity in $((-1)_{\mathcal{S}_{2b}}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_2})\uparrow^{\mathcal{S}_{2b}}$.*

Similarly, Theorem 2 gives:

Theorem 4. *Given $n = 3b$, $3 \leqslant b$, if c, d are such that $cd = n$, and $c, d \geqslant 3$, then every irreducible character occurring in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_b \wr \mathcal{S}_3})\uparrow^{\mathcal{S}_n}$ occurs in $((-1)_{\mathcal{S}_n}\downarrow_{\mathcal{S}_d \wr \mathcal{S}_c})\uparrow^{\mathcal{S}_n}$.*

In addition to these theorems, the previous results on Foulkes' Conjecture now hold for the alternating character as well. Namely, Foulkes' Conjecture for alternating characters is true for $a = 2$, 3, and 4 as well as $a \ll b$ by the results mentioned in Section 1.

Given the success of replacing the trivial character in Foulkes' Conjecture with this 'alternating' character, it is natural to investigate if other definitions of an alternating character yield similar results. One suggestion was to try $(-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{A}_b} - (-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ for an induced alternating character in place of $1^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ in Foulkes' Conjecture. Alas, a simple computer check via GAP [8], shows Foulkes' Conjecture for this character fails when $a = 3$ and $b = 4$. Other variations on this character, such as $(-1)^{\mathcal{S}_{ab}}_{\mathcal{A}_a \wr \mathcal{S}_b} - (-1)^{\mathcal{S}_{ab}}_{\mathcal{S}_a \wr \mathcal{S}_b}$ also fail at those values.

Arising from our construction of the alternating character is a more general character construction. If χ is a character of $\mathcal{S}_{ab}$, consider the character formed by restriction χ to $\mathcal{S}_b \wr \mathcal{S}_a$ and then inducing back up to $\mathcal{S}_{ab}$, that is $\chi\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}\uparrow^{\mathcal{S}_{ab}}$. Foulkes' Conjecture the leads to the question:

> Let $n = ab = cd$. For which characters χ of $\mathcal{S}_n$, do the irreducible characters of $\chi\downarrow_{\mathcal{S}_b \wr \mathcal{S}_a}\uparrow^{\mathcal{S}_n}$ occur in $\chi\downarrow_{\mathcal{S}_d \wr \mathcal{S}_c}\uparrow^{\mathcal{S}_n}$ with multiplicity at least as large?

The Generalized Foulkes' Conjecture is the case $\chi = 1$. We've shown its equivalence with $\chi = -1$ in the alternating character case. The question for other characters remains open.

3. Background

3.1. Tableaux

A *partition* $\lambda = [\lambda_1, \ldots, \lambda_\ell]$ of a number n is an ordered tuple of positive integers such that $\sum \lambda_i = n$ and $\lambda_i \geqslant \lambda_{i+1} > 0$; it is denoted by $\lambda \vdash n$. The *length* of λ is ℓ. A *Ferrers*

diagram is the set $[\lambda] = \{(i, j) \mid 1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \lambda_i\}$. We view $[\lambda]$ as a (left-justified) stack of boxes with row i having λ_i boxes.

A *tableau* of *shape* λ is a filling of the Ferrers diagram $[\lambda]$ with a set of elements, usually the positive integers. It is said to have *content* $\alpha = [\alpha_1, \ldots, \alpha_k]$ if the integer i occurs exactly α_i times. A tableau is *semi-standard* if the entries are weakly increasing along the rows and strictly increasing along the columns. This notation is standard and further discussion can be found in [6,16].

Example 3.1. Let

$$P = \begin{array}{lll} 1 & 2 & 5 \\ 4 & 2 & \\ 3 & & \end{array}, \quad Q = \begin{array}{lll} 1 & 1 & 5 \\ 2 & 2 & \\ 3 & 3 & \end{array}, \quad \text{and} \quad R = \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array}.$$

Then P is a $[3, 2, 1]$-tableaux of content $\alpha = [1, 2, 1, 1, 1]$, while Q has shape $\lambda = [3, 2, 2]$ and content $\alpha = [2, 2, 2, 0, 1]$. Similarly, R has shape $[3, 3]$ with content $[2^3] = [2, 2, 2]$. Both Q and R are semi-standard, but P is not.

3.2. Combinatorial structures

There are two different permutation actions on tableaux, an action permuting the entry positions in the tableau and an action permuting the numbers filling the tableau. These actions commute with each other.

Let T be a λ-tableau, filled with the numbers 1 to a, where $\lambda \vdash n$. The permutation of entry positions corresponds to an action of $\mathcal{S}_n$ on T. View the entry positions (i.e., boxes) of T as labelled 1 to n. Then $\sigma \in \mathcal{S}_n$ acts on T by permuting the entries in the positions moved by σ. To avoid confusion between entry positions and numbers in T, we will denote all entry positions with the subscript $_T$ when necessary. The permutation action of the numbers corresponds to their permutation by $\pi \in \mathcal{S}_a$.

Example 3.2. Take $n = 6$, $\lambda = [3, 2, 1]$, and $a = 3$. Consider

$$T = \begin{array}{lll} 3 & 3 & 1 \\ 1 & 2 & \\ 2 & & \end{array}.$$

In terms of entry positions, we label T as

$$\begin{array}{lll} 1_T & 2_T & 3_T \\ 4_T & 5_T & \\ 6_T & & \end{array}.$$

So for $\sigma = (23)_T \in \mathcal{S}_6$, we have

$$\sigma T = \begin{array}{lll} 3 & 1 & 3 \\ 1 & 2 & \\ 2 & & \end{array},$$

while the action of $\pi = (23) \in \mathcal{S}_3$ gives

$$\pi T = \begin{matrix} 2 & 2 & 1 \\ 1 & 3 & \\ 3 & & \end{matrix}.$$

We generally restrict the action of $\mathcal{S}_n$ to two subgroups. Let R_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the rows of T, namely the row permutations. Denote this action by σT for $\sigma \in R_T$. Let C_T be the subgroup of $\mathcal{S}_n$ which set-wise fixes the columns of T, namely the column permutations. This action is denoted τT for $\tau \in C_T$. If λ' is the conjugate partition of λ (i.e., the partition corresponding to column lengths) we have that $R_T \approx \mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}$ and $C_T \approx \mathcal{S}_{\lambda'_1} \times \cdots \times \mathcal{S}_{\lambda'_{\ell'}}$. Viewing the subgroups under these isomorphisms, we can label the entry positions by labelling each row (respectively column) with 1 to λ_i (respectively $\lambda_{i'}$). Under these labellings we write σ (respectively τ) and a direct product of the permutations for each row (respectively column).

Example 3.3. Let

$$T = \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix}.$$

Then for R_T we view T as labelled

$$\begin{matrix} 1_T & 2_T & 3_T \\ 1_T & 2_T & \end{matrix}.$$

Likewise we use the labelling

$$\begin{matrix} 1_T & 1_T & 1_T \\ 2_T & 2_T & \end{matrix}$$

for C_T. Applying these actions to T, gives the following sets

$$\{\sigma T \mid \sigma \in R_T\} = \left\{ \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix}, \begin{matrix} 1 & 2 & 1 \\ 2 & 3 & \end{matrix}, \begin{matrix} 2 & 1 & 1 \\ 2 & 3 & \end{matrix}, \begin{matrix} 1 & 1 & 2 \\ 3 & 2 & \end{matrix}, \begin{matrix} 1 & 2 & 1 \\ 3 & 2 & \end{matrix}, \begin{matrix} 2 & 1 & 1 \\ 3 & 2 & \end{matrix} \right\}$$

with each element occurring twice,

$$\{\tau T \mid \tau \in C_T\} = \left\{ \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix}, \begin{matrix} 2 & 1 & 2 \\ 1 & 3 & \end{matrix}, \begin{matrix} 1 & 3 & 2 \\ 2 & 1 & \end{matrix}, \begin{matrix} 2 & 3 & 2 \\ 1 & 1 & \end{matrix} \right\},$$

$$\{\pi T \mid \pi \in \mathcal{S}_a\} = \left\{ \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \end{matrix}, \begin{matrix} 2 & 2 & 1 \\ 1 & 3 & \end{matrix}, \begin{matrix} 3 & 3 & 2 \\ 2 & 1 & \end{matrix}, \begin{matrix} 1 & 1 & 3 \\ 3 & 2 & \end{matrix}, \begin{matrix} 2 & 2 & 3 \\ 3 & 1 & \end{matrix}, \begin{matrix} 3 & 3 & 1 \\ 1 & 2 & \end{matrix} \right\},$$

$$C_T = \left\{ ()_T \times ()_T \times ()_T,\ (12)_T \times ()_T \times ()_T,\ ()_T \times (12)_T \times ()_T,\ (12)_T \times (12)_T \times ()_T \right\}.$$

Note that the actions of σ and τ do not commute, but π commutes with both σ and τ.

Given λ a partition of $n = ab$, let $\mathcal{W}^{\lambda,a}$ be the set of all tableaux of shape λ and content $[b^a] = [b, \ldots, b]$, where the entries are 1 to a. Let $\mathcal{S}^{\lambda,a}$ be the set of all semi-standard tableaux in $\mathcal{W}^{\lambda,a}$. These sets and the following constructions are developed by Doran in [6].

Note that when $b = 1$, $\mathcal{S}^\lambda$ give rise to the Specht modules which are discussed extensively in [12,16]. The partitions of n index the Specht modules S^λ, which in turn correspond to precisely the irreducible modules of $\mathcal{S}_n$.

From $\mathcal{W}^{\lambda,a}$ we can construct the complex vector space $W^{\lambda,a}$ with the tableaux as a basis. The action of $\mathcal{S}_a$ on the tableaux give rise to a permutation representation. Inside $W^{\lambda,a}$ we construct the following objects.

Definition 3.4. Let $T \in \mathcal{W}^{\lambda,a}$. Let $\varepsilon(\tau)$ be the sign of τ as a permutation. Inside $W^{\lambda,a}$ we have:

(a) $\mathbf{e}_T = \sum_{\sigma\in R_T} \sum_{\tau\in C_T} \varepsilon(\tau)\sigma\tau T$.
(b) $\mathbf{q}_T = \sum_{\pi\in\mathcal{S}_a} \sum_{\sigma\in R_T} \sum_{\tau\in C_T} \varepsilon(\tau)\pi\sigma\tau T$.
(c) $\mathbf{m}_T = \sum_{\pi\in\mathcal{S}_a} \sum_{\sigma\in R_T} \pi\sigma T$.

The tableaux are independent basis for $W^{\lambda,a}$, so for $T_1, T_2 \in \mathcal{W}^{\lambda,a}$, we have $T_1 + T_1 = 2T_1$ but $T_1 + T_2 = T_1 + T_2$.

Example 3.5. Let

$$T = \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix},$$

then

$$\mathbf{e}_T = 4 \times \left\{ \begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} + \begin{matrix} 2 & 2 \\ 1 & 1 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} + \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} + \begin{matrix} 2 & 1 \\ 2 & 1 \end{matrix} + \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix} \right\}.$$

Here, symmetry gives $\mathbf{q}_T = 2\mathbf{e}_T$.

Definition 3.6. From $\mathbf{e}_T$ and $\mathbf{m}_T$ we can construct the following subspaces of $W^{\lambda,a}$:

(a) $S^{\lambda,a} = \mathbb{C}[\mathbf{e}_T \mid T \in \mathcal{W}^{\lambda,a}]$.
(b) $M^{\lambda,a} = \mathbb{C}[\mathbf{m}_T \mid T \in \mathcal{W}^{\lambda,a}]$.

These spaces are $\mathcal{S}_a$-modules. We have $\{\mathbf{e}_T \mid T \in \mathcal{S}^{\lambda,a}\}$ as a basis for $S^{\lambda,a}$, and $\{\mathbf{m}_T \mid T \in \mathcal{S}^{\lambda,a}/\mathcal{S}_a\}$ as a basis for $M^{\lambda,a}$. The set $\{\mathbf{q}_T \mid T \in \mathcal{W}^{\lambda,a}\}$ generates $S^{\lambda,a} \cap M^{\lambda,a}$ but does not form a basis. Background on these spaces and the proofs of the statements may be found in [6].

In [6], Doran uses Gay's result from [9]:

Lemma 3.7 (Gay's result). *The multiplicity of the irreducible module S^λ in $1^{\mathcal{S}_{ab}}_{\mathcal{S}_{b^\lambda}\mathcal{S}_a}$ equals the multiplicity of the trivial representation in $S^{\lambda,a}$.*

From this, Doran reformulated Foulkes' Conjecture to:

Lemma 3.8. *The dimension of $S^{\lambda,a} \cap M^{\lambda,a}$ equals the multiplicity of the irreducible S^λ in $1^{\mathcal{S}_{ab}}_{\mathcal{S}_b \wr \mathcal{S}_a}$.*

Hence Foulkes' Conjecture is equivalent to proving $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leqslant \dim(S^{\lambda,b} \cap M^{\lambda,b})$ for $a \leqslant b$ and all $\lambda \vdash n$. Proving Conjecture 2 reduces to showing $\dim(S^{\lambda,a} \cap M^{\lambda,a}) \leqslant \dim(S^{\lambda,c} \cap M^{\lambda,c})$ for $ab = n = cd$ with $c, d \geqslant a$ for all $\lambda \vdash n$.

Rephrasing this in terms of tableaux, proving Theorem 1 is equivalent to exhibiting m_λ non-zero linearly independent $\mathbf{q}_T$, where T has shape λ and content $[d^c]$ and m_λ is the multiplicity of S^λ in $1^{\mathcal{S}_{2b}}_{\mathcal{S}_b \wr \mathcal{S}_2}$. Similarly, Theorem 2 is provable by exhibiting a non-zero $\mathbf{q}_T$ with T having shape λ and content $[d^c]$, for all λ such that the multiplicity of S^λ in $1^{\mathcal{S}_{3b}}_{\mathcal{S}_b \wr \mathcal{S}_3}$ is non-zero.

4. Theory of tableaux construction

Throughout this paper we will use T to represent an arbitrary tableau filled with 1 to a, σ an element of R_T, τ an element of C_T, and π an element of $\mathcal{S}_a$.

Remark 4.1. Some standard properties of $\mathbf{e}_T$ and $\mathbf{q}_T$ are:

(a) $\mathbf{e}_{\tau T} = \varepsilon(\tau)\mathbf{e}_T$.
(b) $\mathbf{e}_{\pi T} = \pi \mathbf{e}_T$.
(c) $\mathbf{q}_{\pi T} = \pi \mathbf{q}_T = \mathbf{q}_T$.
(d) $\mathbf{q}_T = \sum_{\pi \in \mathcal{S}_a} \pi \mathbf{e}_T$.
(e) $\mathbf{q}_{\tau T} = \varepsilon(\tau)\mathbf{q}_T$.

These are standard computations, which are discussed in [6,16]. This shows that we can ignore the effects of permuting entries when constructing the tableaux. Also, we may order the columns however we choose at the cost of a sign.

4.1. Filling tableaux

Definition 4.2. In T, the *weight* of a number x in row i, denoted $\omega_i(x)$, is the number of times x occurs in row i of T. When T is not clear from context, we write $\omega_i(x|T)$ in place of $\omega_i(x)$. We extend this so that $\omega_i(x_1, \ldots, x_j|T) = (\omega_i(x_1|T), \ldots, \omega_i(x_j|T))$. Implicitly, we take $\omega_i(T) = (\omega_i(1), \ldots, \omega_i(a))$, which is called the *row-weight* of row i of T. Similarly,

$$\omega(x_j) = \begin{pmatrix} \omega_1(x_j) \\ \vdots \\ \omega_\ell(x_j) \end{pmatrix}$$

is the *weight vector* of x_j of T. Hence $\omega(T)$ is the matrix corresponding to $\omega_i(j|T)$. Note that row permutations do not effect weight, so $\omega(\sigma T) = \omega(T)$.

Example 4.3. Let

$$T = \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & 4 \\ 5 & 5 & \end{array}.$$

We have

$$\omega(T) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

From this we can read that $\omega_2(3) = 1$.

Weights for a tableau are only comparable with tableaux of the same shape and content. Recall that $\mathcal{W}^{\lambda,a}$ is the set of all λ-tableaux with content $[b^a]$. Let N^a be the set of all a-tuples w with non-negative integer entries and $N^{a,\ell}$ those $a \times \ell$ matrices. Then we can view the row-weight function as a linear operator $\omega_i : \mathbb{Z}[T \mid T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w \mid w \in N^a]$, where $\{w \mid w \in N^a\}$ is a $\mathbb{Z}$-basis, or correspondingly, $\omega : \mathbb{Z}[T \mid T \in \mathcal{W}^{\lambda,a}] \to \mathbb{Z}[w \mid w \in N^{a,\ell}]$, with $\ell = \ell(\lambda)$.

This means we treat weights like linearly independent basis in $\mathbb{Z}[w \mid w \in N^{a,\ell}]$. Hence $\omega(T_1 + aT_2) = \omega(T_1) + a\omega(T_2)$. If

$$T_1 = \begin{array}{cc} 1 & 1 \\ 2 & \end{array} \quad \text{and} \quad T_2 = \begin{array}{cc} 2 & 1 \\ 1 & \end{array}$$

then $\omega_1(T_1) = (2, 0)$ and $\omega_1(T_2) = (1, 1)$. So $\omega_1(T_1 + T_2) = (2, 0) + (1, 1)$ and $\omega_1(T_1 + T_1) = (2, 0) + (2, 0) = 2 \cdot (2, 0)$. For convenience, we take $\omega(0) = 0$.

Definition 4.4. Let $\text{rowsum}_T(i)$ be the sum of all the entries in row i of T.

Lemma 4.5. $\mathbf{e}_T = 0$ *iff* T *has a repeat entry in a column.*

Proof. ($\Rightarrow$) It suffices to show that if T has no repetitions within a column, then $\mathbf{e}_T \neq 0$. By Remark 4.1 column permutations only change the sign of $\mathbf{e}_T$, so without loss of generality, we may assume the columns of T are strictly increasing. If $\mathbf{e}_T = 0$ then there exists $\sigma \in R_T$, $\tau \in C_T$ such that $\varepsilon(\tau) = -1$ and $\sigma\tau T = T$; we say that T cancels in the summation. Since T cancels, we must have $\omega(\sigma\tau T) = \omega(T)$ and so $\omega(\tau T) = \omega(T)$. This implies that $\text{rowsum}_T = \text{rowsum}_{\tau T}$ for all rows.

Let r be the first row in which τ moves an entry of T. Let $\alpha_{i_1}, \ldots, \alpha_{i_k}$ be the entries of row r moved by τ. Say τ moves β_j to α_{i_j}. Since r is the first row moved by τ, $\beta_j > \alpha_{i_j}$. Then

$$\text{rowsum}_T(r) = \sum_{j=1}^{k} \alpha_{i_j} + \sum_{i \neq i_j} \alpha_i < \sum_{j=1}^{k} \beta_j + \sum_{i \neq i_j} \alpha_i = \text{rowsum}_{\tau T}(r).$$

Contradiction. Therefore $\mathbf{e}_T \neq 0$.

($\Leftarrow$) If T has a repeated entry in a column, then there exists a transposition $\tau \in C_T$ such that $\tau T = T$. Hence $\mathbf{e}_T = \mathbf{e}_{\tau T} = \varepsilon(\tau)\mathbf{e}_T = -\mathbf{e}_T$ and so $\mathbf{e}_T = 0$. □

Knowing that any tableaux with repeated numbers in a column makes $\mathbf{e}_T = 0$ is very useful for our construction of non-zero tableaux. We summarize this fact as follows:

Lemma 4.6. *Any tableaux T filled with the numbers* 1 *to a having more than a rows will have $\mathbf{e}_T = 0$ and $\mathbf{q}_T = 0$, as will any T having repetitions within a column.*

Hence every tableaux T filled with the numbers 1, 2, and 3 with $\mathbf{q}_T \neq 0$ will have at most three rows and all column entries will be distinct. Due to this, we will assume from now on that all tableaux have distinct column entries.

Notation. Many of the tableaux we construct will have multiple identical columns. We call a group of such columns a *column block*. For both clarity and space we denote a column block by one copy of the column and the number of repetitions listed above. If the number of column copies is omitted, it is assumed to be one. For example,

$$T = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$$

would be denoted by

$$T = \begin{array}{c} \overset{3}{\overline{1}} \\ 2 \end{array}.$$

While

$$T = \begin{array}{c} \overset{\text{K}}{1} \; \overset{\text{L}}{1} \; \overset{\text{M}}{1} \; \overset{\text{N}}{2} \\ 2 \; 2 \; 3 \; 3 \\ 3 \end{array}$$

has

$$\text{K copies of } \begin{array}{c} 1 \\ 2 \\ 3 \end{array}, \quad \text{L copies of } \begin{array}{c} 1 \\ 2 \end{array}, \quad \text{M copies of } \begin{array}{c} 1 \\ 3 \end{array}, \quad \text{N copies of } \begin{array}{c} 2 \\ 3 \end{array}.$$

We call the columns of T having only one entry the *tail* of T. When specifying T by column blocks, we may omit the tail, provided the content of T is known. The rest of T is called the *body*. For instance, if

$$T = \begin{array}{c} \overset{2}{\overline{1 \; 1 \; 2 \; 2 \; 3}} \\ 3 \; 2 \; 4 \\ 4 \end{array}$$

and we know that T has content [3, 3, 3, 3], we can just list the body

$$T = \begin{array}{\overset{2}{\overline{1\ 1\ 2}}} 3\ 2\ 4 \\ 4 \end{array}$$

instead. It is assumed that any entries not specified are contained in the tail. Note that by default an omitted tail will only contain those numbers which appear in the body.

We also use this abbreviated notation when describing elements of C_T. We write $\tau \in C_T$ as a direct product of permutations on the column blocks. Since the only permutation possible on the tail is the identity, we omit the blocks corresponding to the tail. Hence for T listed above, τ is of the form

$$\overset{2}{\tau_1} \times \tau_2 \times \tau_3.$$

We write $\overset{\kappa}{\tau_i}$ if the same permutation τ_i is to be applied to κ columns within a column block. When κ is less than the size of the column block, we understand $\overset{\kappa}{\tau_i}$ to mean that τ_i is applied to all κ of the columns (determined by context) and the identity permutation is applied to the remaining columns within the block. For instance, on the previous T there are two permutations of the form

$$\overset{1}{(13)}_T \times ()_T \times (12)_T,$$

which produce

$$\begin{array}{llll} 4\ 1\ 1\ 4 & & & 1\ 4\ 1\ 4 \\ 3\ 3\ 2\ 2 & \text{and} & & 3\ 3\ 2\ 2. \\ 1\ 4 & & & 4\ 1 \end{array}$$

4.2. Showing tableaux are non-zero

Definition 4.7. A tableau T is said to be *non-zero* if $\mathbf{q}_T \neq 0$. Two tableaux are said to be *distinct* if $\mathbf{q}_{T_1} \neq \pm\mathbf{q}_{T_2}$, otherwise T_1 and T_2 are said to be *equivalent*.

Since $\mathbf{q}_T$ involves many summands, showing $\mathbf{q}_T \neq 0$ by direct summation is not practical. Instead, we use a technique called *weight-set counting*. Weight-set counting involves summing only those tableaux with a given weight; if that sum is non-zero, the entire $\mathbf{q}_T$ summation must be non zero.

Definition 4.8. Given a tableau T, let $\omega(T) = (\bar{x}_1, \ldots, \bar{x}_a)$, where $\bar{x}_j$ is the weight vector $\omega(j|T) = \bar{x}_j$ of the element j in T. A *weight assignment* of T is a pairing between the set of elements of T with the set weight vectors of $\omega(T)$. We denote the pairing of the weight vector $\bar{x}_j$ with the element k_j by $\omega(k_j|T^*) = \bar{x}_j$ (the T^* represents a possible tableau which has the weight of k_j being $\bar{x}_j$). Note that if the vectors $\bar{x}_j$ and $\bar{x}_{j'}$ are equal, the weight assignment pairing k_j with $\bar{x}_j$ is the same as the weight assignment pairing k_j

with $\bar{x}_{j'}$. What matters in a weight assignment is the vector paired with each element, not how we label the vectors. We usually indicate a weight assignment by writing $(k_1, \ldots, k_a)$, by which we mean $\omega(k_1, \ldots, k_a|T^*) = (\bar{x}_1, \ldots, \bar{x}_a) = \omega(T)$.

Given a permutation π we can create a weight assignment by assigning the element $\pi(k)$ to $\bar{x}_k$, since $\omega(\pi(k)|\pi T) = \omega(k|T) = \bar{x}_k$. Similarly, given such pairing $(k_1, \ldots, k_a)$ we can construct a permutation π by taking $\pi = (k_1, \ldots, k_a)$ in one-line notation. For example, let

$$T = \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & & \end{array}.$$

Then

$$\omega(T) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The weight assignment $(3, 1, 2)$ means

$$\omega\big(3, 1, 2\big|T^*\big) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some tableau } T^*.$$

This weight assignment corresponds to the weight permutation $\pi = (132)$ in cycle notation (from left to right).

Note however that such a listing $(k_1, \ldots, k_a)$ of a weight assignment is not necessarily unique. For instance, if $\bar{x}_1 = \bar{x}_2$ then $(k_1, k_2, k_3, \ldots, k_a)$ and $(k_2, k_1, k_3, \ldots, k_a)$ represent the same weight assignment (pairing) but give rise to different permutations. The numbers of such permutations corresponding to the same weight assignment depends only on the vector symmetries of $\omega(T)$. This number is denoted $s(\omega(T))$.

For example, if $T = 1\ 1\ 2\ 2\ 3$, then $\omega_T = (2, 2, 1)$. There are three distinct weight assignments of T corresponding to which of the three elements is assigned a weight of 1. Since there are two permutations arising from such an assignment (for instance $\omega(213|T) = (2, 2, 1)$ as well) we have $s(\omega(T)) = 2$.

Definition 4.9. A weight assignment $(k_1, \ldots, k_a)$ is *valid* for T if there exists $\tau \in C_T$ such that $\omega(k_1, \ldots, k_a|\tau T) = \omega(T)$, i.e., $T^* = \tau T$. If this happens we say τ is valid for $(k_1, \ldots, k_a)$; otherwise τ is *invalid*. Given a valid τ, we say τ is *positive* if $\varepsilon(\tau) = 1$ and *negative* if $\varepsilon(\tau) = -1$.

Example 4.10. Let

$$T = \begin{array}{lllllll} 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 1 & 1 & & & & \\ 4 & 4 & & & & & \end{array}.$$

We have

$$\omega(T) = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Recall that for C_T we use the labelling

$$\begin{matrix} 1_T & 1_T & 1_T \\ 2_T & 2_T & 2_T \\ 3_T & 3_T \end{matrix}$$

(the tail is omitted since all column permutations on it are trivial). Now $(2,3,1,4)$ is a valid weight assignment since $\tau = (12)_T \times (12)_T \times (12)_T \in C_T$ has

$$\tau T = \begin{matrix} 3 & 1 & 1 & 2 & 3 & 3 & 4 \\ 1 & 2 & 2 \\ 4 & 4 \end{matrix} \quad \text{with} \quad \omega(\tau T) = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

So $\omega(2,3,1,4|\tau T) = \omega(T)$. The weight assignment, $(2,3,1,4)$ corresponds to the permutation $\pi = (123)$ in cycle notation, meaning $\omega(1,2,3,4|\tau T) = \omega(1,2,3,4|\pi T)$.

However, $(1,4,3,2)$ is not a valid weight assignment since then we must have $\omega_3(2|\tau T) = 2$ for some τ, but there is no column permutation that will put two 2's in the third row. We will make frequent use of weight assignments in order to determine when T is non-zero.

Definition 4.11. Given T, consider the following functions:

- $\mathcal{P}(\pi(T)) =$ the number of $\tau \in C_T$ such that $\varepsilon(\tau) = 1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.
- $\mathcal{N}(\pi(T)) =$ the number of $\tau \in C_T$ such that $\varepsilon(\tau) = -1$ and $\omega(\pi^{-1}(1), \ldots, \pi^{-1}(a)|\tau T) = \omega(T)$.
- $\mathcal{P}(T) = \sum_\pi \mathcal{P}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$.
- $\mathcal{N}(T) = \sum_\pi \mathcal{N}(\pi(T))$, where π correspond to distinct weight assignments of $\omega(T)$.

Theorem 5 (Weight-set counting). *If* $\mathbf{q}_T = 0$ *then* $\mathcal{P}(T) = \mathcal{N}(T)$.

Proof. Let $\mathcal{D}$ be the set permutations corresponding to distinct weight assignments of T.

$$\mathbf{q}_T = 0 \tag{4.1}$$

$$\Rightarrow \quad \sum_\pi \sum_\sigma \sum_\tau \varepsilon(\tau)\omega(\sigma\pi\tau T) = 0 \tag{4.2}$$

$$\Rightarrow \quad \sum_\pi \sum_\tau \varepsilon(\tau)\omega(\pi\tau T) = 0 \tag{4.3}$$

$$\Rightarrow \quad \sum_\pi \sum_\tau \varepsilon(\tau)\omega(\pi\tau T) = 0 \quad \text{s.t.} \quad \omega(1,\ldots,a|\pi\tau T) = \omega(T) \tag{4.4}$$

$$\Rightarrow \quad \sum_{\pi}\sum_{\tau}\varepsilon(\tau)=0 \quad \text{s.t.} \quad \omega(1,\ldots,a|\pi\tau T)=\omega(T) \tag{4.5}$$

$$\Rightarrow \quad \sum_{\pi}\sum_{\tau}\varepsilon(\tau)=0 \quad \text{s.t.} \quad \omega\big(\pi^{-1}(1),\ldots,\pi^{-1}(a)\big|\tau T\big)=\omega(T) \tag{4.6}$$

$$\Rightarrow \quad s\big(\omega(T)\big)\sum_{\pi\in\mathcal{D}}\big(\mathcal{P}(\pi T)-\mathcal{N}(\pi T)\big)=0 \tag{4.7}$$

$$\Rightarrow \quad s\big(\omega(T)\big)\big(\mathcal{P}(T)-\mathcal{N}(T)\big)=0 \tag{4.8}$$

$$\Rightarrow \quad \mathcal{P}(T)-\mathcal{N}(T)=0. \tag{4.9}$$

If $\mathbf{q}_T=0$, taking the weight of both sides gives Eq. (4.2). Since row permutations do not effect weights, we can reduce to Eq. (4.3). As distinct weights cannot cancel, we can consider only those tableaux with the same weight as $\omega(T)$. Hence we must have line (4.4). Since we are only summing over tableaux of a fixed weight we may drop the weight from the sum and simply add the sign of τ, giving line (4.5). By definition of weight assignments, $\omega(1,\ldots,a|\pi\tau T)=\omega(\pi^{-1}(1),\ldots,\pi^{-1}(a)|\tau T)$. Hence we get Eq. (4.6). Restricting the sum to distinct weight-sets gives the factor of $s(\omega(T))$ and if we split over the sign of τ' we get line (4.7). By definition this is the same as line (4.8). Factoring out $s(\omega(T))$ yields line (4.9). Thus $\mathcal{P}(T)=\mathcal{N}(T)$. □

Another way to view this theorem is: If $\mathbf{q}_T\neq 0$ then $\sum_{\mathcal{A}_T}\varepsilon(\tau)\neq 0$, where $\mathcal{A}_T=\{\tau\in C_T \mid \omega(\tau T)=\omega(\pi T),\ \text{for some } \pi\in\mathcal{S}_a\}$. An easy application of this theorem is the following useful lemma.

Lemma 4.12. *If T is a tableau consisting of a single column block* (*and an arbitrary tail*), *for instance*

$$T=\begin{array}{c}\overset{\mathrm{K}}{\overline{1}}\\ 2\end{array},$$

then $\mathbf{q}_T\neq 0$ *iff* K *even.*

Proof. We have

$$\mathsf{w}=\omega(T)=\begin{pmatrix}\mathrm{K} & 0\\ 0 & \mathrm{K}\end{pmatrix}.$$

So there are two distinct weight assignments for T, namely $\omega(1,2|T^*)=\mathsf{w}$ and $\omega(2,1|T^*)=\mathsf{w}$. We wish to determine for which τ does $T^*=\tau T$ satisfy one of these equations. For $\omega(1,2|\tau T)=\mathsf{w}$, we must have K 2's in the second row. Hence none of the columns of T may move. The only τ satisfying this is

$$\tau=\overset{\mathrm{K}}{()}_T,$$

which is positive. For $\omega(2,1|\tau T) = \mathsf{w}$, we must have κ 1's in the second row. Thus τ must exchange row 1 and row 2 for every column in T. Therefore

$$\tau = \overset{\kappa}{(12)}_T \quad \text{and} \quad \varepsilon(\tau) = (-1)^{\kappa}.$$

Hence when κ is even, we have $\mathcal{P}(T) = 2$ and $\mathcal{N}(T) = 0$ so $\mathbf{q}_T$ is non-zero. (The same ideas apply for T having more than two rows.)

When κ is odd, however, we have $\mathcal{P}(T) = \mathcal{N}(T)$ so the Theorem 5 does not apply. Instead, let

$$\tau = \overset{\kappa}{(12)}_T .$$

Let $\pi \in \mathcal{S}_a$ be the corresponding entry transposition (in our example $\pi = (12)$). Then $\pi\tau T = T$ and $\varepsilon(\tau) = -1$. Thus $\mathbf{q}_{\pi\tau t} = \pi\mathbf{q}_{\tau T} = \mathbf{q}_{\tau T} = \varepsilon(\tau)\mathbf{q}_T = -\mathbf{q}_T$. So $\mathbf{q}_T = 0$. □

It is also true that if $\mathbf{q}_T \neq 0$, it is non-zero by weight-set counting on some weight $\omega(\tau T)$. That is we cannot have $\mathcal{P}(\tau T) = \mathcal{N}(\tau T)$ for all $\tau \in C_T$ and still have $\mathbf{q}_T \neq 0$. This fact is not needed for our results and so the proof is omitted. It can be found in the forthcoming thesis [19].

Theorem 6. *The only irreducible characters occurring in* $1^{2b}_{\mathcal{S}_b \wr \mathcal{S}_2}$ *are those corresponding to all partitions* $\lambda = [\lambda_1, \lambda_2]$ *where* λ_i *is even. Moreover, these characters occur with multiplicity* 1.

Proof. By Theorem 3.7 we need only consider those shapes with $\mathbf{q}_T$ non-zero. By Lemmas 4.1 and 4.6 any distinct non-zero tableau filled with b 1's and b 2's must be equivalent to

$$T = \begin{array}{c} \overset{\kappa}{\overline{1}} \\ 2 \end{array}$$

(not including tail). Hence there is at most one distinct non-zero tableau for any shape λ. By Lemma 4.12, when $\lambda = [2b - \kappa, \kappa]$ then $\mathbf{q}_T \neq 0$ iff κ is even. Since $\langle \mathbf{q}_T \rangle = S^{\lambda,2} \cap M^{\lambda,2}$ we must have $\dim(S^{\lambda,2} \cap M^{\lambda,2}) = 1$ if κ is even and zero otherwise.

This is a well known result, appearing in [11,15]. □

The weight-set counting of Theorem 5 is useful for much more complicated tableaux as well. To illustrate the general usage of the theorem, we list here a slightly more involved example:

Example 4.13. To see directly how weight-set counting works, consider the following example. The tableau Q_1 is listed below using the column block notation, with the conditions on the block size listed on its right:

$$Q_1 = \begin{array}{ccccc|l} \text{A} & \text{A} & \text{B} & \text{B} & \text{C} & \\ 2 & 2 & 2 & 3 & 1 & \text{A}+\text{B}+\text{C}=d, \\ 1 & 4 & 4 & 1 & 4 & 2\text{A}+\text{B}<d, \\ 3 & 3 & & & & \text{C even}, \\ & & & & & \text{A, B}>0. \end{array}$$

Q_1 has weight

$$\omega_{2,3} = \begin{pmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{pmatrix}$$

and shape $\lambda = [3d - 2\text{A} - \text{B}, d + \text{A} + \text{B}, 2\text{A}]$.

We want to show that $\mathbf{q}_{Q_1} \neq 0$, which we do by showing $\mathcal{P}(Q_1) - \mathcal{N}(Q_1) \neq 0$. Now $\mathcal{P}(Q_1)$ is the number of $\tau \in C_{Q_1}$ with $\varepsilon(\tau) = 1$ such that $\omega(i_1, i_2, i_3, i_4|\tau Q_1) = \omega(Q_1)$ for some distinct weight assignment (i_1, i_2, i_3, i_4). (Equivalently, τ is such that $\{\omega(i|\tau Q_1) \mid i = 1, 2, 3, 4\} = \{\omega(i|Q_1) \mid i = 1, 2, 3, 4\}$.) Similarly, $\mathcal{N}(Q_1)$ is those with $\varepsilon(\tau) = -1$. The easiest way to count these τ is to use weight assignments. First we determine which weight assignments might be possible using some general properties of the tableau. Then we count how many τ correspond to each weight assignment (i.e., for which τ is the weight assignment valid) and determine $\varepsilon(\tau)$. Finally, we add this signed sum to determine $\mathcal{P}(Q_1) - \mathcal{N}(Q_1)$.

First we want to determine which weight assignments are possible for Q_1. That is, determine for which 4-tuples $\mathbf{x} = (i_1, i_2, i_3, i_4)$ there might exist $\tau \in C_{Q_1}$ such that

$$\omega_{2,3}(\mathbf{x}|\tau Q_1) = \begin{pmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{pmatrix}.$$

Let

$$\mathbf{w} = \begin{pmatrix} \text{A}+\text{B} & 0 & 0 & d \\ 0 & 0 & 2\text{A} & 0 \end{pmatrix}.$$

Simply looking at Q_1, there are a few restrictions on what $\mathbf{x}$ can be.

Notice the body contains d copies of the elements 1 and 4, but fewer than d copies of 2 and 3 since $2\text{A} + \text{B} < d$. Also note that the body of τQ_1 contains the same elements as the body of Q_1. This implies that not all elements can have d copies in row two of τQ_1 for some τ. If $\omega_{2,3}(\mathbf{x}|\tau Q_1) = \mathbf{w}$ then either $\omega_2(1|\tau Q_1) = d$ or $\omega_2(4|\tau Q_1) = d$; namely, only the elements 1 and 4 may have $\omega_2(i|\tau Q_1) = d$. Hence any valid weight assignment must have $i_4 = 1$ or 4. If $\omega_2(1|\tau Q_1) = d$, then there is only one other non-zero weight to assign in row two. So as $\text{B} > 0$ the remaining columns (the second A block and the first B block) must have the same element in row two, namely 2 or 4. That means we must have either $\omega_2(2|\tau Q_1) = \text{A}+\text{B}$ or $\omega_2(4|\tau Q_1) = \text{A}+\text{B}$; so the weight assignment must have $i_1 = 2$ or 4. Similarly, if $\omega_2(4|\tau Q_1) = d$, then since $\text{B} > 0$ we must have either $\omega_2(1|\tau Q_1) = \text{A} + \text{B}$ or $\omega_2(3|\tau Q_1) = \text{A} + \text{B}$, that is $i_1 = 1$ or 3.

We also consider which elements j may have $\omega_3(j|\tau Q_1) = 2\text{A}$ and find that only with $j = 2$ or 3 may this occur; these are the only elements for which both A blocks will be the same in row three. (That is if $\omega_{2,3}(\tau Q_1) = \mathbf{w}$ then either $\omega_3(2|\tau Q_1) = 2\text{A}$ or $\omega_3(3|\tau Q_1) = 2\text{A}$, so any valid weight assignment has $i_3 = 2$ or 3.)

Table 1

$\omega_{2,3}(\mathbf{x}\|\tau Q_1) = \left(\begin{smallmatrix} \mathrm{A}+\mathrm{B} & 0 & 0 & d \\ 0 & 0 & 2\mathrm{A} & 0 \end{smallmatrix}\right)$	τQ_1	τ	#	$\varepsilon(\tau)$
$\mathbf{x} = (1,2,3,4)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 2 & 2 & 2 & 3 & 1 \\ 1 & 4 & 4 & 1 & 4 \\ 3 & 3 & & & \end{array}$	$\overset{\mathrm{A}}{()} \times \overset{\mathrm{A}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{C}}{()}$	1	$(-1)^0$
$\mathbf{x} = (1,3,2,4)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 3 & 3 & 2 & 3 & 1 \\ 1 & 4 & 4 & 1 & 4 \\ 2 & 2 & & & \end{array}$	$\overset{\mathrm{A}}{(13)} \times \overset{\mathrm{A}}{(13)} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{C}}{()}$	1	$(-1)^{2\mathrm{A}}$
$\mathbf{x} = (3,1,2,4)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 1 & 3 & 2 & 1 & 1 \\ 3 & 4 & 4 & 3 & 4 \\ 2 & 2 & & & \end{array}$	$\overset{\mathrm{A}}{(132)} \times \overset{\mathrm{A}}{(13)} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{B}}{(12)} \times \overset{\mathrm{C}}{()}$	1	$(-1)^{\mathrm{A}+\mathrm{B}}$
$\mathbf{x} = (4,2,3,1)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 2 & 2 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 & 1 \\ 3 & 3 & & & \end{array}$	$\overset{\mathrm{A}}{()} \times \overset{\mathrm{A}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{C}}{(12)}$	1	$(-1)^{\mathrm{C}}$
$\mathbf{x} = (4,3,2,1)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 3 & 3 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 & 1 \\ 2 & 2 & & & \end{array}$	$\overset{\mathrm{A}}{(13)} \times \overset{\mathrm{A}}{(13)} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{C}}{(12)}$	1	$(-1)^{2\mathrm{A}+\mathrm{C}}$
$\mathbf{x} = (2,4,3,1)$	$\begin{array}{cccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{C} \\ \hline 2 & 4 & 4 & 3 & 4 \\ 1 & 2 & 2 & 1 & 1 \\ 3 & 3 & & & \end{array}$	$\overset{\mathrm{A}}{()} \times \overset{\mathrm{A}}{(12)} \times \overset{\mathrm{B}}{(12)} \times \overset{\mathrm{B}}{()} \times \overset{\mathrm{C}}{(12)}$	1	$(-1)^{\mathrm{A}+\mathrm{B}+\mathrm{C}}$

There are six distinct weight assignments $\mathbf{x} = (i_1, i_2, i_3, i_4)$ meeting these conditions: $(1,2,3,4)$, $(1,3,2,4)$, $(3,1,2,4)$, $(4,3,2,1)$, $(4,2,3,1)$, and $(2,4,3,1)$. In the table above (see Table 1), for each weight assignment we list type of tableau τQ_1 for which it is valid, the form τ used, the number of such τ, and the sign of τ. This is an easy way to summarize the counting of τ and their signs. (We will omit the subscripts $_{Q_1}$ when writing τ for easy reading. Remember that τ acts on the *entry positions* of Q_1 and not the elements.)

To see how we obtain such a table, consider the last row. We want a tableau τQ_1 such that $\omega_{2,3}(2,4,3,1|\tau Q_1) = \mathbf{w}$. This means that $\omega_3(3|\tau Q_1) = 2\mathrm{A}$ so τ cannot move any entries in row 3. We also have $\omega_2(4|\tau Q_1) = 0$. Examining Q_1, we know that τ acts non-trivially on the second column block A, the first column block B and column block C, namely

$$\tau = \overset{\mathrm{A}}{*} \times \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{B}}{(12)}_T \times \overset{\mathrm{B}}{*} \times \overset{\mathrm{C}}{(12)}_T .$$

If τ were to act non-trivially on the first column block A or the second column block B, the number of 1's in row two of τQ_1 would decrease. Since we must have $\omega_2(1|\tau Q_1) = d$, this cannot happen. Hence

$$\tau = \overset{\mathrm{A}}{()}_T \times \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{B}}{(12)}_T \times \overset{\mathrm{B}}{()}_T \times \overset{\mathrm{C}}{(12)}_T$$

and has been completely determined for us. Then we have $\omega_{2,3}(2,4,3,1|\tau Q_1) = \mathbf{w}$, so such a τ exists and is unique. For reference, τ and τQ_1 are listed. Once τ has been determined, computing $\varepsilon(\tau) = (-1)^{\mathrm{A}+\mathrm{B}+\mathrm{C}}$ is straight forward.

Finally, to compute the weight sum for $\mathbf{w}$ we sum the product of the number of τ with the sign of τ, that is $\# \cdot \varepsilon(\tau)$. Here the sum is $1+1+(-1)^{\mathrm{A}+\mathrm{B}}+1+1+(-1)^{\mathrm{A}+\mathrm{B}}$. This sum is between 2 and 6, depending on the parity of A and B. Since it is non-zero in all cases, Theorem 5 shows $\mathbf{q}_{Q_1} \neq 0$.

4.3. Joining tableaux

Definition 4.14. The *join* of two tableaux, U and V, denoted $U \vee V$ is a way of combining tableaux together. If the entries of U and V are not disjoint, renumber V so that they are. For instance if U contains the numbers 1 to n and V contains the numbers 1 to m we first renumber V with the numbers $n+1$ to $n+m$. Then concatenate the tableaux and sort the columns by length. Note that entries of every column remain fixed, only the order of the columns change.

Example 4.15. Take

$$U = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & \end{array} \quad \text{and} \quad V = \begin{array}{ccc} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & & \end{array}.$$

We renumber V to get

$$V = \begin{array}{ccc} 5 & 6 & 6 \\ 6 & 8 & 8 \\ 7 & & \end{array}.$$

Concatenation gives

$$\begin{array}{cccccc} 1 & 1 & 3 & 5 & 6 & 6 \\ 2 & 2 & 4 & 6 & 8 & 8 \\ 3 & 4 & & 7 & & \end{array}.$$

When sorted we get

$$T = U \vee V = \begin{array}{cccccc} 1 & 1 & 5 & 3 & 6 & 6 \\ 2 & 2 & 6 & 4 & 8 & 8 \\ 3 & 4 & 7 & & & \end{array}.$$

Note that since applying permutations of $\mathcal{S}_a$ to a tableaux has no effect on $\mathbf{q}_T$ and $\mathbf{m}_T$, the numbering of a tableau is irrelevant. Also, any $\sigma \in R_T$ that only interchanges columns will commute with all $\tau \in C_T$. Hence column sorting has no effect of $\mathbf{e}_T$, $\mathbf{q}_T$, and $\mathbf{m}_T$, since there is no sign change for row permutations. This join operation also joins the weight-sets, namely $\omega_3(U \vee V) = \omega_3(U)$, $\omega_3(V) = (0,0,1,1,0,0,1,0)$.

Definition 4.16. Let $T = U \vee V$ for tableaux U filled with 1 to m and V filled with $m+1$ to a. The weights, $\omega(U)$ and $\omega(V)$ are *disjoint* (equivalently, $\omega(T)$ *splits* over U and V), if every valid weight assignment of $\omega(T)$, can be obtained from concatenating valid weight assignments of $\omega(U)$ and $\omega(V)$.

Say $\omega(1,\dots,m|U) = (\bar{x}_1,\dots,\bar{x}_m)$ (the $\bar{x}_i$ are weight vectors) and $\omega(m+1,\dots,a|V) = (\bar{x}_{m+1},\dots,\bar{x}_a)$. So $\omega(1,\dots,a|T) = (\bar{x}_1,\dots,\bar{x}_a)$. Consider a valid weight assignment of T assigning to the element j the weight vector $\bar{x}_{k_j}$. So then $\omega(1,\dots,a|\tau T) = (\bar{x}_{k_1},\dots,\bar{x}_{k_a})$ for some $\tau \in C_T$. This restricts to a valid weight assignment of $\tau|_U U$ by considering only the elements 1 to m. This restriction is unique because a weight assignment is defined by the vector-element pairing, not the label assigned to the vector. If this restriction corresponds to a weight assignment of $\omega(U)$ (i.e., the weights assigned to elements 1 to m are the same as the weights of $\omega(U)$ as vectors) then the weight assignment of T arose from valid weight assignments of U. Similarly for V.

By restricting to a weight assignment of $\omega(U)$ we mean $\{\bar{x}_{k_i} \mid i = 1,\dots,m\} = \{\bar{x}_i \mid i = 1,\dots,m\}$, i.e., the weights assigned to U are equivalent to those of $\omega(U)$. If this result is a weight assignment of $\omega(U)$, it is valid for the tableau $\tau|_U U$. If this is true for all valid weight assignments of T, then $\omega(T)$ splits (the weights of U and V are disjoint).

Example 4.17. Consider

$$U = \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \quad \text{and} \quad V = \begin{array}{cccc} 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{array}.$$

So

$$T = \begin{array}{cccccc} 1 & 1 & 3 & 3 & 3 & 3 \\ 2 & 2 & 4 & 4 & 4 & 4 \end{array} \quad \text{and} \quad \omega_2(1,2,3,4|T) = (0,2,0,4).$$

There are four valid weight assignments possible for T:

$$\omega_2\big(1,2,3,4\big|T^*\big) = (0,2,0,4),$$
$$\omega_2\big(1,2,3,4\big|T^*\big) = (2,0,0,4),$$
$$\omega_2\big(1,2,3,4\big|T^*\big) = (0,2,4,0),$$
$$\omega_2\big(1,2,3,4\big|T^*\big) = (2,0,4,0).$$

(Recall that T^* represents any possible tableau τT.)

When we restrict these weight assignments to U we get two possible assignments for U, $\omega_2(1,2|U^*) = (0,2)$ and $\omega_2(1,2|U^*) = (2,0)$. Since the original weight of U is $\omega_2(1,2|U) = (0,2)$, both of these assignments are assignments of $(0,2)$ and both are valid for U (simply take $\tau_U = ()$ and $\tau_U = (12)_T \times (12)_T$). A similar argument holds for V. Thus the $\omega(T)$ splits over U and V.

However, weight-set disjointness is highly dependent on the filling of T. Consider instead,

$$U = \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \quad \text{and} \quad V = \begin{array}{cc} 4 & 3 \\ 3 & 4 \end{array}.$$

So

$$T = \begin{matrix} 1 & 1 & 4 & 3 \\ 2 & 2 & 3 & 4 \end{matrix} \quad \text{and} \quad \omega_2(1,2,3,4|T) = (0,2,1,1).$$

Then $\omega_2(1,2,3,4|T^*) = (1,1,2,0)$ is a valid weight assignment for T by $T^* = \tau T$ with $\tau = (12)_T \times ()_T \times ()_T \times (12)_T$. However, $\omega_2(1,2|U^*) = (1,1)$ it is not a valid weight assignment of $\omega_2(U) = (2,0)$, even though there exists τ such that $\omega_2(\tau U) = (1,1)$. Hence these weights are not disjoint.

Although this definition of disjointness is a bit involved, we will later (Lemma 4.26) give a sufficient (but not necessary) condition on the tableaux which is easier to check. However, we use disjointness here to obtain the full generality of the theorem below, which is one of the fundamental tools we use to construct non-zero tableaux.

Theorem 7 (Breakdown Theorem). *Let U and V be tableaux such that*

- elements$(U) = \{1,\dots,m\}$ *and* elements$(V) = \{m+1,\dots,a\}$ (*renumber if necessary*).
- *The weights $\omega(1,\dots,m|U)$, $\omega(m+1,\dots,a|V)$ are such that $\mathbf{q}_U$ and $\mathbf{q}_V$ are non-zero by weight-set counting on ω.*
- *The weight assignments corresponding to $\omega(1,\dots,m|U)$ and $\omega(m+1,\dots,a|V)$ are disjoint.*

Then for $T = U \vee V$, we have $\mathbf{q}_T \neq 0$.

Proof. By weight-set counting on U and V we have $\mathcal{P}(U) - \mathcal{N}(U) \neq 0$ and $\mathcal{P}(V) - \mathcal{N}(V) \neq 0$. By Theorem 5, showing $\mathcal{P}(T) - \mathcal{N}(T) \neq 0$ implies $\mathbf{q}_T \neq 0$. Thus it suffices to show $\mathcal{P}(T) - \mathcal{N}(T) = (\mathcal{P}(U) - \mathcal{N}(U))(\mathcal{P}(V) - \mathcal{N}(V))$ or equivalently $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ and $\mathcal{N}(T) = \mathcal{N}(U)\mathcal{P}(V) + \mathcal{P}(U)\mathcal{N}(V)$. We will show $\mathcal{P}(T) = \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$. The $\mathcal{N}(T)$ claim follows similarly.

Consider the weight assignment $(k_1,\dots,k_a)$ of T. If $\omega(k_1,\dots,k_a|\tau T) = \omega(1,\dots,a|T)$ with $\varepsilon(\tau) = 1$ (i.e., τ is positive for $(k_1,\dots,k_a)$) then it is counted in $\mathcal{P}(T)$. Since the weight splits, we have $\omega(k_1,\dots,k_a|\tau T) = \omega(k_1,\dots,k_m|\tau|_U U)\omega(k_{m+1},\dots,k_a|\tau|_V V) = \omega(1,\dots,m|U)\omega(m+1,\dots,a|V) = \omega(1,\dots,a|T)$ with $\varepsilon(\tau) = \varepsilon(\tau|_U)\dot{\varepsilon}(\tau|_V) = 1$. Hence either $\varepsilon(\tau|_U) = \varepsilon(\tau|_V) = 1$ or $\varepsilon(\tau|_U) = \varepsilon(\tau|_V) = -1$. Since $\omega(k_1,\dots,k_m|\tau|_U U) = \omega(U)$ is a valid weight assignment (because the weights are disjoint), if $\varepsilon(\tau|_U) = 1$ then it is counted in $\mathcal{P}(U)$. Similarly, for the $\mathcal{P}(V)$ cases. The $\varepsilon(\tau|_U) = -1$ cases are counted in $\mathcal{N}(U)$. Thus $\mathcal{P}(T) \leqslant \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$.

Now any weight assignments $\omega(k_1,\dots,k_a|\tau|_U U) = \omega(1,\dots,a|U)$ in $\mathcal{P}(U)$ and $\omega(k_{m+1},\dots,a|\tau|_V V) = \omega(m+1,\dots,a|V)$ in $\mathcal{P}(V)$ must also correspond to the valid weight assignment $\omega(k_1,\dots,k_a|\tau T) = \omega(1,\dots,a|T)$ with $\tau = \tau|_U \times \tau|_V$. Moreover $\varepsilon(\tau) = \varepsilon(\tau|_U)\dot{\varepsilon}(\tau|_V) = 1$. So this weight assignment is in $\mathcal{P}(T)$. Similarly for weight assignments in $\mathcal{N}(U)$ and $\mathcal{N}(V)$. Hence $\mathcal{P}(T) \geqslant \mathcal{P}(U)\mathcal{P}(V) + \mathcal{N}(U)\mathcal{N}(V)$ thus giving equality as desired. □

The Breakdown Theorem allows us to construct a non-zero tableau from smaller non-zero tableaux. The main difficulty in applying this theorem is showing that the weight-sets are disjoint. To deal with this, we develop an idea of maximality of weights which is sufficient for weight-set disjointness. Maximality of weights also provides a tool for showing linear independence of tableaux, which we do not address in this paper.

Definition 4.18. Given a tableaux T, the *generic form* of $\omega_i(T)$ is $\mathsf{w}_i(T) = \omega_i(\pi T) = (x_1, \ldots, x_a)$ for any $\pi \in \mathcal{S}_a$ such that $x_j \geqslant x_{j+1}$ for all j. In essence, $\mathsf{w}_i(T)$ is the weights of $\omega_i(T)$ listed in decreasing order. This definition works for any row of T.

We define the generic form of a weight on the entire tableau (assuming T has at most three rows) by

$$\mathsf{w}(T) = \omega_{2,3}(\pi T) = \begin{pmatrix} x_1 & \ldots & x_a \\ y_1 & \ldots & y_a \end{pmatrix}$$

for any $\pi \in \mathcal{S}_a$ such that $y_j \geqslant y_{j+1}$ for all j and if $y_j = y_{j+1}$ then $x_j \geqslant x_{j+1}$. So we consider only the weight vectors of the second and third rows and list the vectors so that the row three weights are decreasing. If two vectors have the same weight for row three, we list the vector with the larger weight in row two first.

For instance,

$$\text{if} \quad T = \begin{matrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 3 \end{matrix} \quad \text{then} \quad \omega_{2,3}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Hence

$$\mathsf{w}(T) = \begin{pmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}.$$

Definition 4.19. We put an order on generic weights by $\mathsf{w}_i(T_1) > \mathsf{w}_i(T_2)$ if when $\mathsf{w}_i(T_1) = (x_1, \ldots, x_a)$, $\mathsf{w}_i(T_2) = (v_1, \ldots, v_a)$, there exists $k \geqslant 1$ such that $x_j = v_j$ for $j < k$ and $x_k > v_k$. We say $\mathsf{w}(T_1) > \mathsf{w}(T_2)$ if

(1) $\mathsf{w}_3(T_1) > \mathsf{w}_3(T_2)$ or
(2) $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$ and $\mathsf{w}_2(T_1) > \mathsf{w}_2(T_2)$ or
(3) $\mathsf{w}_3(T_1) = \mathsf{w}_3(T_2)$, $\mathsf{w}_2(T_1) = \mathsf{w}_2(T_2)$, and if we have

$$\mathsf{w}(T_1) = \begin{pmatrix} y_1 & \ldots & y_a \\ x_1 & \ldots & x_a \end{pmatrix} \quad \text{and} \quad \mathsf{w}(T_2) = \begin{pmatrix} z_1 & \ldots & z_a \\ x_1 & \ldots & x_a \end{pmatrix}$$

then there exists $k \geqslant 1$ such that $y_j = z_j$ for $j < k$ and $y_k > z_k$.

We can also apply this ordering to sets of weight vectors by associating to each set the generic weight vector formed by concatenating the given weights in order. So to the set

$$A = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

we associate the weight $\binom{0\ 1}{2\ 1}$.

Definition 4.20. We define the *maximum generic weight* of row i of T to be the maximum with respect to $>$ of $\{\mathsf{w}_i(\tau T) \mid \tau \in C_T\}$, where w is the generic weight defined above. Similarly the *maximum generic weight* of T is the maximum with respect to $>$ of $\{\mathsf{w}(\tau T) \mid \tau \in C_T\}$. Note that the maximum generic weight of T is based only on the weights of rows two and three. As such we ignore the weight of the first row.

Example 4.21. Let

$$T = \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}.$$

Then the generic weights $\mathsf{w}_2(\tau T)$ are $(1,1)$ and $(2,0)$, with $(2,0)$ (the generic weight of $\tau T = \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}$ or $\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}$) as maximum. Here we've suppressed writing $\mathsf{w}_3(T)$, since T has only two rows.

If

$$T = \begin{array}{ccccc} 1 & 1 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & & & \end{array}$$

then the maximum generic weight of T (of the second and third rows) is $\binom{3\ 2\ 0\ 0}{2\ 0\ 0\ 0}$ which in this case is $\omega_{2,3}(T)$ in generic form, $\mathsf{w}(T)$.

Although we do not use it in this paper, maximality provides a way to determine if a set of tableaux are linearly independent. Namely, if the max weights of a set of tableaux are distinct, they are linearly independent.

Definition 4.22. Let T be a tableaux having three or fewer rows. Let w^{m} be the maximum generic weight of T. We say w^{m} is the *max weight* for T if w^{m} occurs in $\mathbf{q}_T$. That is $\mathbf{q}_T \neq 0$ by weight-set counting on w^{m}.

Unlike the maximum weight, the max weight of a tableau may not exist, since the weight may not occur in $\mathbf{q}_T$. For instance, consider

$$T = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array}.$$

We know $\mathbf{q}_T = 0$ by Lemma 4.12, yet T has $\binom{0\ 3\ 0}{3\ 0\ 0}$ as its maximum weight.

The max weight for T is always the maximum weight for row three of T, but it need not be the maximum weight for row two of T. Consider

$$T = \begin{array}{cccc} 5 & 2 & 3 & 3 \\ 1 & 4 & 4 & 1 \\ 3 & 3 & & \end{array}.$$

The maximum generic weight of T is $\left(\begin{smallmatrix}0&2&2&0\\2&0&0&0\end{smallmatrix}\right)$, but the maximum generic weight of row two of T is $(4,0,0,0)$ which is not the generic form of $(0,2,2,0)$.

Definition 4.23. If w^{m} is the max weight for T, we say T is in *maximal form* provided $\omega_{2,3}(\pi T) = w^{\mathsf{m}}$ for some $\pi \in \mathcal{S}_a$. Note that this only requires that some permutation of the weight vectors of $\omega_{2,3}(T)$ be equal to the max weight of T.

While the max weight may not exist for all tableaux, it is easy to show weight-set disjointness for those tableaux which are in maximal form. In order to show this, we use the following lemmas regarding our ordering.

Lemma 4.24. *Given two weights W_1 and W_2 of the same length, let*

$$C_k = \left\{ \binom{x}{y} \;\middle|\; \binom{x}{y} \in W_k \right\}, \quad k = 1, 2,$$

be the sets of weight vectors (*including multiplicities*) *in each of these weights. Let $A = C_1 \setminus (C_1 \cap C_2)$, the weight vectors of W_1 not in W_2. Similarly, let $B = C_2 \setminus (C_1 \cap C_2)$. If $W_1 \geqslant W_2$, then $A \geqslant B$.*

Proof. Without loss of generality, assume W_i is written in maximal form (i.e., is equal to its generic weight). If $W_1 = W_2$ then $A = B = \emptyset$, so the result holds trivially. So suppose $W_1 > W_2$. Then either W_1 differs from W_2 at some place in the third row, or the third rows are equal and they differ at some place in the second row (at least vectorwise).

Define

$$A^y = \left\{ \binom{x_i}{y_i} \;\middle|\; \binom{x_i}{y_i} \in A,\ y_i = y \right\} \quad \text{and} \quad B^v = \left\{ \binom{u_i}{v_i} \;\middle|\; \binom{u_i}{v_i} \in B,\ v_i = v \right\}.$$

Defining C_1^y and C_2^v similarly, we have $A^y = C_1^y \setminus (C_1^y \cap C_2^y)$ and $B^v = C_2^v \setminus (C_1^v \cap C_2^v)$.

Let

$$W_1 = \begin{pmatrix} x_1 & \dots & x_m \\ y_1 & \dots & y_m \end{pmatrix} \quad \text{and} \quad W_2 = \begin{pmatrix} u_1 & \dots & u_m \\ v_1 & \dots & v_m \end{pmatrix}.$$

If W_1 differs from W_2 in the third row, then there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. Hence $|C_1^{y_i}| = |C_2^{v_i}| = |C_2^{y_i}|$ for $y_i > y_j$, so $|A^{y_i}| = |B^{v_i}| = |B^{y_i}|$. Then to show $A > B$ it suffices to show $|A^{y_j}| > |B^{y_j}|$. But since $y_j > v_j$, we have $|C_1^{y_j}| > |C_2^{y_j}|$ and hence $\mathsf{w}_3(A) > \mathsf{w}_3(B)$ so the result follows. Note that when the third rows are equal, this argument shows $\mathsf{w}_3(A) = \mathsf{w}_3(B)$.

If W_1 and W_2 are equal in the third row but the generic weights of their second rows differ, we can apply the same argument as above, where A^x, B^x, C_i^x are the appropriate weight vectors with the second row weight equal to x. This shows $\mathsf{w}_2(A) > \mathsf{w}_2(B)$. Since we've already have $\mathsf{w}_3(A) = \mathsf{w}_3(B)$, the result follows.

If W_1 and W_2 have the same generic weights in the second and third rows, then from above we know the generic weights of rows two and three of A and B are the same. By

definition, $A \cap B = \emptyset$, so the second row of the first vectors in A and B are different. Now W_1 and W_2 agree in the third row, so the first vectors where they differ in the second row must be the first vector in A and B respectively. Since $W_1 > W_2$ we have the large vector occurring in W_1 and hence in A. Thus $A > B$. □

The following lemma is an obvious property of the ordering, but is included due to the non-standard ordering used.

Lemma 4.25. *Given sets A and B of weight vectors, if $A \geqslant B$ and $B \geqslant A$ then $A = B$.*

Proof. View A and B as generic weight vectors W_A and W_B. Let

$$W_A = \begin{pmatrix} x_1 & \dots & x_m \\ y_1 & \dots & y_m \end{pmatrix} \quad \text{and} \quad W_B = \begin{pmatrix} u_1 & \dots & u_m \\ v_1 & \dots & v_m \end{pmatrix}.$$

Since $A \geqslant B$ we have $(y_1, \dots, y_m) \geqslant (v_1, \dots, v_m)$. So either the rows are equal or there exists j such that $y_j > v_j$ and $y_i = v_i$ for all $i < j$. As $B \geqslant A$ we would similarly get $v_j \geqslant y_j$, which is a contradiction. Hence $\mathsf{w}_3(A) = \mathsf{w}_3(B)$. A similar argument shows $\mathsf{w}_2(A) = \mathsf{w}_2(B)$.

If $A \neq B$, let j be the first place they differ. Then $A \geqslant B$ implies

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} > \begin{pmatrix} u_j \\ v_j \end{pmatrix}.$$

Since $y_j = v_j$, this means $x_j > u_j$. But the same argument on $B \geqslant A$ implies $u_j > x_j$, which is a contradiction. Thus $A = B$. □

Now using these lemmas we can show that max weights are necessarily disjoint.

Lemma 4.26. *If U and V are tableaux in maximal form, with U containing b copies of the elements 1 to m and V containing b copies of the elements $m+1$ to a (after renumbering as necessary), then $\omega(U)$ and $\omega(V)$ are disjoint.*

Proof. Let U and V be in maximal form. Let

$$\omega_{2,3}(U \vee V) = \begin{pmatrix} x_1 & \dots & x_m & x_{m+1} & \dots & x_a \\ y_1 & \dots & y_m & y_{m+1} & \dots & y_a \end{pmatrix}.$$

To show that U and V are disjoint, we need to show that any valid weight assignment of $U \vee V$ restricts to a valid weight assignment of U and of V. Let

$$\begin{pmatrix} x_{k_1} & \dots & x_{k_m} & x_{k_{m+1}} & \dots & x_{k_a} \\ y_{k_1} & \dots & y_{k_m} & y_{k_{m+1}} & \dots & y_{k_a} \end{pmatrix}$$

be a valid weight assignment of $U \vee V$. That means there exists τ such that

$$\omega_{2,3}(\tau[U \vee V]) = \begin{pmatrix} x_{k_1} & \dots & x_{k_m} & x_{k_{m+1}} & \dots & x_{k_a} \\ y_{k_1} & \dots & y_{k_m} & y_{k_{m+1}} & \dots & y_{k_a} \end{pmatrix}.$$

We want to show it restricts to a valid weight assignment of U, that is

$$\omega_{2,3}(\tau|_U U) = \begin{pmatrix} x_{k_1} & \cdots & x_{k_m} \\ y_{k_1} & \cdots & y_{k_m} \end{pmatrix} = \omega_{2,3}(\pi U) \quad \text{for some } \pi \in \mathcal{S}_m.$$

This is equivalent to showing

$$\left\{ \begin{pmatrix} x_i \\ y_i \end{pmatrix} \,\middle|\, i = 1, \ldots, m \right\} = \left\{ \begin{pmatrix} x_{k_i} \\ y_{k_i} \end{pmatrix} \,\middle|\, i = 1, \ldots, m \right\}.$$

Let

$$C = \left\{ \begin{pmatrix} x_i \\ y_i \end{pmatrix} \,\middle|\, i = 1, \ldots, m \right\}$$

be set of the weight vectors of $\omega_{2,3}(U)$ and

$$D = \left\{ \begin{pmatrix} x_{k_i} \\ y_{k_i} \end{pmatrix} \,\middle|\, i = 1, \ldots, m \right\}$$

the set of weight vectors of $\omega_{2,3}(\tau|_U U)$. Define $A = C \setminus (C \cap D)$ and $B = D \setminus (C \cap D)$. So A consists of those weight vectors of U assigned to V which are distinct from the weight vectors of V assigned to U under this assignment. That is, the vectors in A occur in $\omega_{2,3}(U)$ but not in $\omega_{2,3}(\tau|_U U)$. The set B is the weight vectors are those vectors coming from $\omega_{2,3}(\tau|_U U)$ which are not in $\omega_{2,3}(U)$. To prove disjointness, we need to show that $C = D$, which is equivalent to showing $A = B = \emptyset$.

Now U is in maximal form, so $\omega_{2,3}(U) \geqslant \omega_{2,3}(\tau' U)$ for all τ'. Hence $\omega_{2,3}(U) \geqslant \omega_{2,3}(\tau|_U U)$. So by Lemma 4.24, we have $A \geqslant B$. But we can also view A as the weight vectors of U in $\omega_{2,3}(\tau|_V V)$ which are not in $\omega_{2,3}(V)$. Similarly B is the set of vectors from $\omega_{2,3}(V)$ which are not in $\omega_{2,3}(\tau|_V V)$. Since V is also in maximal form, $\omega_{2,3}(V) \geqslant \omega_{2,3}(\tau|_V V)$. Hence by Lemma 4.24, we have $B \geqslant A$. Thus Lemma 4.25 shows $A = B$. However, $A \cap B = \emptyset$ by definition, so $A = B = \emptyset$. Thus the weights are disjoint. □

Example 4.27. Suppose U is a tableau in maximal form such that

$$\omega_{2,3}(1, 2, 3, 4|U) = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

and V is a tableau in maximal form with

$$\omega_{2,3}(5, 6, 7, 8, 9, 10, 11|V) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose these weights were not disjoint. That means we must be able to assign some weight $\binom{x}{y}$ of U to V and some weight $\binom{x'}{y'}$ of V to U.

First consider the vector $\binom{1}{2}$ of U (i.e., the vector with the largest weight in row three). Since V is in maximal form, we know that there can be at most one copies of any element in row three of τV for any τ. Since $2 > 1$, this vector cannot be assigned to V. Similarly,

once we know that $\binom{1}{2}$ remains a weight of U, the largest row three weight of V we can assign to U is 0. Hence $\binom{1}{1}$ remains with V.

Now consider $\binom{3}{0}$ of U. Having V in maximal form means that when $\binom{1}{1}$ are assigned to V, a vector $\binom{*}{0}$ assigned to V must have $* \leqslant 2$. Thus $\binom{3}{0}$ is assigned to U. This means $\binom{3\ 1}{0\ 2}$ must be in $\omega_{2,3}(U)$. Therefore the only vectors of U and V that can be assigned to each other are the $\binom{0}{0}$ vectors. However, since a weight assignment is based only on the vector and not its label, this is the same as a weight assignment arising from U and V. Hence the weights are disjoint.

Lemma 4.26 shows that if U and V are in maximal form, $\mathbf{q}_{U\vee V} \neq 0$ by the Breakdown Theorem. In addition, $U \vee V$ is in maximal form. We will use this fact to prove the tableaux we construct are non-zero.

5. Proof of Theorem 1

Recall Theorem 1, which says that every irreducible occurring in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_2}$ occurs in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$ with equal or greater multiplicity, where $n = 2b = cd$ and $b, c, d \geqslant 2$.

In Section 4.2 we proved Theorem 6, which showed that the irreducibles occurring in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_2}$ were exactly those corresponding to partitions $\lambda = [n - s, s]$ for s even and they occur with multiplicity one. (Since $n = 2b$ is even, it suffices to consider only the even values of s.) Hence to prove Theorem 1, it suffices to construct a non-zero tableaux filled with d copies of c elements for each partition $[n - s, s]$, where $0 \leqslant s \leqslant n/2$, s even and $n = cd$.

To do this, we will construct some non-zero generic tableaux that when assembled via the Breakdown Theorem will produce all the shapes and fillings needed. Since we are constructing generic tableaux for many partitions and fillings, we will not use a fixed c. However, we assume that every element listed in the body of the tableaux occurs d times, filling out the tail as needed. We apply weight-set counting to prove a tableaux is non-zero. We list tableaux using column block notation, with the conditions on the block size listed to the right. Underneath we list the weight and shape of the tableau. Following each tableau description is a brief explanation of why $\mathbf{q}_U \neq 0$.

Tableau U_1.

$$U_1 = \begin{array}{|c|c|c|c|}\hline \mathrm{A} & d-\mathrm{A} & d-\mathrm{A} \\ \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \sim \begin{array}{|c|}\hline \mathrm{A} \\ \hline 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{l} \mathrm{A} \text{ even}, \\ \mathrm{A} \leqslant d, \end{array}$$

$$\omega_2(U_1) = (0, \mathrm{A}),$$

$$\lambda = [2d - \mathrm{A}, \mathrm{A}].$$

For this first tableau, we listed U_1 both with and without the tail. Normally we will suppress the tail when writing these tableaux. U_1 is a non-zero by Lemma 4.12 since A is even.

Tableau U_2.

$$U_2 = \begin{array}{cccc} \overset{\mathrm{A}}{1} & \overset{\mathrm{A}}{3} & \overset{\mathrm{B}}{1} & \overset{\mathrm{B}}{3} \\ 2 & 4 & 4 & 2 \end{array} \quad \begin{array}{l} \mathrm{A}+\mathrm{B} \leqslant d, \\ \mathrm{A},\mathrm{B} > 0, \end{array}$$

$$\omega_2 = (0, \mathrm{A}+\mathrm{B}, 0, \mathrm{A}+\mathrm{B}),$$

$$\lambda = \big[4d - 2(\mathrm{A}+\mathrm{B}), 2(\mathrm{A}+\mathrm{B})\big].$$

Examining the filling of U_2 and $\mathrm{A},\mathrm{B} > 0$ we find the following constraints on any valid weight assignment:

- If $\omega_2(1|U^*) = 0$ then $\omega_2(2 \text{ and } 4|U^*) > 0$.
- If $\omega_2(2|U^*) = 0$ then $\omega_2(1 \text{ and } 3|U^*) > 0$.
- We must have $\omega_2(1 \text{ or } 2|U^*) > 0$ and $\omega_2(3 \text{ or } 4|U^*) > 0$.

(Recall that U^* corresponds to a possible tableau τU_2.)

Since any valid weight assignment of $(0, \mathrm{A}+\mathrm{B}, 0, \mathrm{A}+\mathrm{B})$ has exactly two zeros, the restrictions above show that $(1,2,3,4)$ and $(2,1,4,3)$ are the only valid weight assignments. These weights-sets correspond to applying

$$\tau = \overset{\mathrm{A}}{()}_T \times \overset{\mathrm{A}}{()}_T \times \overset{\mathrm{B}}{()}_T \times \overset{\mathrm{B}}{()}_T \quad \text{and} \quad \tau = \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{B}}{(12)}_T \times \overset{\mathrm{B}}{(12)}_T,$$

respectively. As both of these τ have positive sign, $\mathbf{q}_{U_2} \neq 0$.

Tableau U_3.

$$U_3 = \begin{array}{ccc} \overset{\mathrm{A}}{1} & \overset{\mathrm{B}}{1} & \overset{\mathrm{B}}{2} \\ 2 & 3 & 3 \end{array} \quad \begin{array}{l} \mathrm{A} \text{ even}, \\ \mathrm{A}+\mathrm{B} \leqslant d, \\ d \text{ even}, \\ \mathrm{B} = d/2, \end{array}$$

$$\omega_2 = (0, \mathrm{A}, d),$$

$$\lambda = [2d - \mathrm{A}, \mathrm{A}+d].$$

To show U_3 non-zero we will use weight-set counting on $\omega_2 = (0, \mathrm{A}, d)$. There are two cases for which we need to determine weight assignments, $\mathrm{A}+\mathrm{B} < d$ and $\mathrm{A}+\mathrm{B} = d$.

When $\mathrm{A}+\mathrm{B} < d$, only the element 3 may be assigned a row two weight of d. So the distinct weight assignments are $(1,2,3)$ and $(2,1,3)$, which occur with

$$\tau = ()_T \quad \text{and} \quad \tau = \overset{\mathrm{A}}{(12)}_T \times \overset{\mathrm{B}}{()}_t \times \overset{\mathrm{B}}{()}_T,$$

respectively. Since A is even, both τ have positive sign. Hence U_3 is non-zero.

If $\mathrm{A}+\mathrm{B} = d$, then $\mathrm{A} = \mathrm{B} = d/2$ and $d \equiv 0 \pmod 4$. While every permutation corresponds to a distinct weight assignment, every weight assignment can only be obtained by having

τ move complete column blocks. Since all of these blocks are even, τ is positive for every weight assignment and hence U_3 is non-zero.

Tableau $V(d)$.

$$V = \frac{d}{1},$$
$$\omega_1(V) = (d),$$
$$\lambda = [d].$$

This is just a single row with d ones. Since there are no column permutations, this tableau is always non-zero.

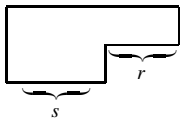
Having constructed these generic tableaux, we will use the notation $U_i(x)$ to denote the tableau U_i with the parameter $\mathrm{A} = x$ or $U_i(x, y)$ for $x = \mathrm{A}$ and $y = \mathrm{B}$ in U_i. We will use $\mathsf{f}U_i$ to denote the join of f copies of U_i. Note that these tableaux are all in maximal form.

For the proof of Theorem 1, the parity effects the construction process. To simplify notation, we define the *-function

$$x^* = \begin{cases} x, & x \text{ even}, \\ x-1, & x \text{ odd}. \end{cases}$$

We analyze T by the parameters $r = n - 2s$ and $s = \lambda_2$, the length of the second row. For reference, we consider tableaux of the following shape, with r and s even:

$$T = \quad .$$



Proof of Theorem 1. To prove Theorem 1 we need to construct non-zero tableaux of shape $\lambda = [n - s, s]$ for $s \leqslant n/2$, with s even and $n = cd$, $c, d \geqslant 2$. First we construct a general tableaux that covers most s. Suppose $s \leqslant c^* d^*/2$. We know s is even, so write $s = \mathsf{f}d^* + e$, where $0 \leqslant e < d^*$, e even. Since s, d^*, e are even, this is possible by the Euclidean algorithm.

Let $T = \mathsf{f}U_1(d^*) \vee U_1(e)$. Note that the bound on s guarantees that $2(\mathsf{f}+1) \leqslant c$ when $e > 0$, and $2\mathsf{f} \leqslant c$ when $e = 0$. This insures that there are at most c distinct elements in T. If there are fewer than c elements in T add all the remaining elements to the tail of T by joining the appropriate number of $V(d)$'s. Suppressing the tail elements from the U_1's, T looks like:

$$T = \begin{array}{cccccccccc} \overset{d^*}{1} & \overset{d^*}{3} & \cdots & \overset{d^*}{2\mathsf{f}-1} & \overset{v}{2\mathsf{f}+1} & \overset{d}{2\mathsf{f}+3} & \cdots & \overset{d}{c} \\ 2 & 4 & \cdots & 2\mathsf{f} & 2\mathsf{f}+2 & & & \end{array}.$$

The Breakdown Theorem shows T is non-zero, provided the weight-sets are disjoint. Since the tableaux are in maximal form, the weights must be disjoint by Lemma 4.26. This

covers the majority of the s. The remaining tableaux will be constructed according to the parity of c and d.

Case I (c, d even). In this case $c^*d^*/2 = cd/2$, so T constructed above covers all partitions.

Case II (d even, c odd). By the above construction, we have all tableaux with s up to $(c-1)d/2$. Thus we only need those even partitions with $s = (cd-k)/2$ with $0 \leqslant k \leqslant d-2$, $k \equiv d \pmod 4$. Take $T = \frac{c-3}{2}U_1(d) \vee U_3(\mathrm{A})$ for $0 \leqslant \mathrm{A} \leqslant d/2$ with A even. Then $s = (c-3)d/2 + \mathrm{A} + d = (cd - d + 2\mathrm{A})/2$. Thus we have $k = d - 2\mathrm{A}$, which ranges over the correct parameters. Since U_1 and U_3 are in maximal form, Lemma 4.26 implies disjointness and the Breakdown Theorem shows T is non-zero.

Case III (c even, d odd). Since $r = n - 2s = cd - 2s$, we need $\lambda = [r+s, s]$ for $r \leqslant cd$ with $r \equiv cd \pmod 4$. It suffices to construct non-zero tableaux for $r < 4d$; when $r \geqslant 4d$, let $r' = r - 4d\mathsf{z}$ with $r' < 4d$. Then, if we construct a $\lambda' = [s + r', s]$ tableaux T' filled with d copies of $c - 4\mathsf{z}$ elements, we get the needed tableau by $T = T' \vee 4\mathsf{z}V(d)$. Hence we will take $r < 4d$.

When $c \equiv 0 \pmod 4$ then $r \equiv 0 \pmod 4$, so take $T = \frac{c-4}{4}U_2(d-1,1) \vee U_2(d - r/4 - 1, 1)$. This construction gives the shape $\frac{c-4}{4}[2d, 2d] + [2d + r/2, 2d - r/2] = [cd/2 + r/2, cd/2 - r/2]$ as desired. The parameters of these tableaux are positive unless $r = 4d - 4$ since $r < 4d$, $r \equiv 0 \pmod 4$, and $d \geqslant 2$. If $r = 4d - 4$ then $d - r/4 - 1 = 0$, so use $U_1(2) \vee 2V(d)$ instead of $U_2(d - r/4 - 1, 1)$.

For $c \equiv 2 \pmod 4$ we will assume $r < 2d$, for when $2d \leqslant r < 4d$ we can join $2V(d)$ to reduce $r < 2d$ and $c \equiv 0 \pmod 4$. Take $T = \frac{c-2}{4}U_2(d-1,1) \vee U_1((2d-r)/2)$ with $V(d)$'s as needed. Note that $cd \equiv r \pmod 4$ implies that $(2d-r)/2$ is even, while $r < 2d$ insures it is positive. Hence we get $[cd/2 + r/2, cd/2 - r/2]$ as needed. The Breakdown Theorem shows these T's non-zero, provided the weight-sets are disjoint. This follows from maximality.

Note that since $cd = n$, n even, c or d is even, so we have constructed all cases. □

Although it is not directly apparent from this construction, c or d even is often a necessary requirement for all non-zero two-row tableaux with s even to exist. For instance, when $c = 3$ and $d = 7$, the shape $[11, 10]$ has s even, but all tableaux are zero by Theorem 8 in the next section.

6. Generalized Foulkes' Conjecture

In order to prove Theorem 2, we need to show every irreducible character occurring in $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_3}$ occurs in $1^{\mathcal{S}_n}_{\mathcal{S}_d \wr \mathcal{S}_c}$, where $n = 3b = cd$, with $c, d \geqslant 3$. This requires determining which irreducibles $\mathcal{S}^\lambda$, occur in $1^{\mathcal{S}_{3b}}_{\mathcal{S}_b \wr \mathcal{S}_3}$ with non-zero multiplicities. Thrall determined this, with multiplicities, in [18]. However, his results do not provide corresponding non-zero tableaux. By Lemma 3.8 of Doran, the irreducibles occur are precisely those corresponding

to partitions $[\lambda]$ such that there exists a non-zero tableau T filled with 1, 2, and 3 such that $\mathbf{q}_T \neq 0$. We can use weight-set counting of Theorem 5 to tell which non-zero partitions occur in $1^{\mathcal{S}_{3b}}_{\mathcal{S}_b \wr \mathcal{S}_3}$, but not their multiplicities. However, this technique will allow us to apply the Breakdown Theorem to the resulting tableaux.

Let T be a $[\lambda]$-tableau filled with b copies of the numbers 1, 2, and 3. By Lemma 4.6, T has at most three rows. Since entry permutations, column permutations and column exchanges do not change whether $\mathbf{q}_T$ is non-zero, we may take T to be in the following form:

$$T = \begin{array}{cccccccc} \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{O} & \mathrm{P} & \mathrm{Q} \\ \hline 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & & & \\ 3 & & & & & & \end{array} \qquad \begin{cases} \mathrm{K}+\mathrm{L}+\mathrm{M}+\mathrm{O} = b, \\ \mathrm{K}+\mathrm{L}+\mathrm{N}+\mathrm{P} = b, \\ \mathrm{K}+\mathrm{M}+\mathrm{N}+\mathrm{Q} = b, \\ \mathrm{L} \geqslant \mathrm{M} \geqslant \mathrm{N} \geqslant 0. \end{cases}$$

However, since we know there are exactly b copies of every number in T, we may omit the tail and simply write T as

$$T = \begin{array}{cccc} \mathrm{K} & \mathrm{L} & \mathrm{M} & \mathrm{N} \\ \hline 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & & & \end{array},$$

retaining the condition $\mathrm{L} \geqslant \mathrm{M} \geqslant \mathrm{N} \geqslant 0$ and assuming the tail.

Theorem 8. *With T as described above,*

$$\mathbf{q}_T = 0 \quad \Leftrightarrow \quad \begin{cases} \mathrm{K}+\mathrm{L} \textit{ odd} & \mathrm{L} > \mathrm{M} = \mathrm{N} \geqslant 0, \\ \mathrm{K}+\mathrm{N} \textit{ odd} & \mathrm{L} = \mathrm{M} \geqslant \mathrm{N} \geqslant 0, \\ \mathrm{K}+\mathrm{M} \textit{ even} & \mathrm{L} = \mathrm{M}+2,\ \mathrm{N} = \mathrm{M}-1 \geqslant 0, \\ \mathrm{K}+\mathrm{L} \textit{ even} & \mathrm{L} = \mathrm{M}+1,\ \mathrm{M} = \mathrm{N} \geqslant 0. \end{cases}$$

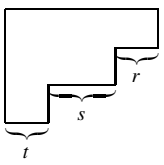
Moreover, when $\mathbf{q}_T \neq 0$, it is non-zero by weight-set counting on some τT.

The major importance of this result is the use of weight-set counting to show $\mathbf{q}_T \neq 0$. This is needed in order to use the tableaux in larger constructions under the Breakdown Theorem.

To prove $\mathbf{q}_T \neq 0$, we apply the technique of weight-set counting from Theorem 5 to various τT as in Example 4.13. For the other direction, we exploit symmetry to show the summations cancel to yield $\mathbf{q}_T = 0$. These computations are omitted for brevity.

From this we determine the shapes corresponding to non-zero tableaux. Write $[\lambda] = [r+s+t, s+t, t]$, so we can visualize T by

$$T = \quad .$$



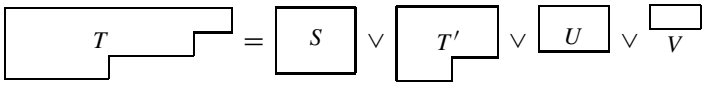
Then Theorem 8 shows:

Theorem 9. *The only partitions* $[r+s+t, s+t, t]$ *of* $n = 3b$ *which do not occur in* $1^{\mathcal{S}_n}_{\mathcal{S}_b \wr \mathcal{S}_3}$ *are those with* s *or* $r = 1$ *as well as those having* $s+t$ *odd and* s *or* $r \in \{0, 2, 4\}$. *Equivalently, a partition with* $r, s \neq 1$ *is non-zero if when* r *or* s *is in* $\{0, 2, 4\}$, *then* $s+t$ *is even. We refer to the required shapes with* r *or* s *less than* 5 *as exceptional cases.*

From Corollary 3.8, we know that in order to prove Theorem 2, we must construct non-zero tableaux filled with d copies of c elements for all required partitions of $n = cd$, $c, d \geqslant 3$ determined in Theorem 9.

Our approach is similar to the proof of Theorem 1 in Section 5. We use some generic non-zero tableaux (like U_i and V in Section 5) with c' elements. We join them together by the Breakdown Theorem to form a tableau of the appropriate shape and filling. However, unlike in Section 5, a large number of generic tableaux are needed. Since the cataloging of non-zero tableaux is quite tedious, we omit the tableaux construction. Some examples of these tableaux, such as S_1 and P_1, are listed below. For more details see [19] (forthcoming). Namely our proof here will presuppose the construction of all tableaux of the required shapes for $c' \leqslant 8$.

The general idea is to write T as follows:



for an appropriate T', with S, U, and V standard generic constructions based on the parity of d. This reduces the construction of T to a construction of T' where the shape parameters (r, s, and t) of T' are small. Hence we only need to construct tableaux for a limited number of cases corresponding to small shapes. The tableaux S, U, and V are based on the following maximal form tableaux:

$$S_0 = P_1(d) = \begin{array}{c} \overset{d}{\overline{1}} \\ 2 \\ 3 \end{array}, \quad d \text{ even}, \qquad S_1 = \begin{array}{c} \overset{\mathrm{A}\ \mathrm{A}\ \mathrm{B}\ \mathrm{B}\ \mathrm{B}\ \mathrm{B}}{\overline{5\ 6\ 6\ 6\ 5\ 5}} \\ 4\ 3\ 3\ 4\ 3\ 4 \\ 1\ 2\ 1\ 1\ 2\ 2 \end{array} \quad \begin{array}{l} \mathrm{A} = \frac{d-\mathsf{x}}{3} + \mathsf{x}, \\ \mathrm{B} = \frac{d-\mathsf{x}}{3}, \\ d \equiv \mathsf{x} \pmod 3, \\ \mathsf{x} \in \{0, 1, 2\}, \end{array}$$

$$U_1(d) = \begin{array}{c} \overset{d}{\overline{1}} \\ 2 \end{array}, \quad d \text{ even}, \qquad U_2 = \begin{array}{c} \overset{\mathrm{A}\ \mathrm{A}\ \mathrm{B}\ \mathrm{B}}{\overline{1\ 3\ 1\ 3}} \\ 2\ 4\ 4\ 2 \end{array} \quad \begin{array}{l} \mathrm{A} = d-1, \\ \mathrm{B} = 1, \end{array}$$

$$V = \begin{array}{c} \overset{d}{\overline{1}} \end{array}.$$

(Here U_1, U_2 and V occurred in Section 5.)

Let $\lambda = (r+s+t, s+t, t)$. Under the conditions of the Breakdown theorem,we can write $T = S \vee T' \vee U \vee V$, for appropriate T'. Then T' will be filled with d copies of c' elements, for some $c' < c$, which will eventually allow us to reduce to $c' \leqslant 8$. For simplicity, we will base our construction on the parity of d.

6.1. d even

To see how T may be written as $T = S \vee U \vee V \vee T'$ for an appropriate T' we first discuss the individual reductions allowing us to write $T = S \vee T'$, $T = T' \vee U$, or $T = T' \vee V$. Then successive applications on these reductions yield our desired decomposition. An analysis of these reductions also computes the resulting bounds on the shape of T'. An example application follows the reductions listed below. A reader may wish to refer to Example 6.1 while reading these reductions.

Reduction 1. Let T be any λ-tableaux with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take f to be the maximum integer such that $\mathsf{f}d \leqslant \lambda_3$ and $c - 3\mathsf{f} \geqslant 3$. Let $S = \mathsf{f}P_1(d)$ be the join of f copies of $P_1(d)$. Then by the Breakdown Theorem, we may write $T = S \vee T'$ for T' a $\lambda' = (\lambda_1 - d\mathsf{f}, \lambda_2 - d\mathsf{f}, \lambda_3 - d\mathsf{f})$-tableau filled with d copies of $c' = c - 3\mathsf{f}$ elements, provided the weight-sets of S and T' are disjoint. The choice of f means that in T', $t' = \lambda'_3 = \lambda_3 - d\mathsf{f} < d$ or $c' = c - 3\mathsf{f} < 6$. Thus we need only consider tableaux T' with $t = \lambda_3 < d$ or $c < 6$. The $c < 6$ condition occurs, since if $c < 6$ then $c - 3 < 3$, so we cannot remove any copies of $P_1(d)$. We need the requirement $c - 3\mathsf{f} \geqslant 3$ so that there are at least three elements available with which to fill the remaining tableaux, T'.

Reduction 2. Let T be any λ-tableaux with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take g to be the maximum integer such that $\mathsf{g}d \leqslant \lambda_2 - \lambda_3 = s$ and $c - 2\mathsf{g} \geqslant 3$. Let $U = \mathsf{g}U_1(d)$ be the join of g copies of $U_1(d)$. Then by the Breakdown Theorem, we may write $T = T' \vee U$ for T' a $\lambda' = (\lambda_1 - d\mathsf{g}, \lambda_2 - d\mathsf{g}, \lambda_3)$-tableau filled with d copies of $c' = c - 2\mathsf{g}$ elements, provided the weight-sets of U and T' are disjoint. The choice of g means that in T', $s' = \lambda'_2 - \lambda'_3 = \lambda_2 - d\mathsf{g} - \lambda'_3 < d$ or $c' = c - 2\mathsf{g} < 5$. However, we will need the existence of a non-zero T' in the specified shape. As was shown in Theorem 8, this is not always the case for some s. Specifically, when $s < 5$ non-zero tableaux do not exist for some shapes when $c = 3$. (Consider $\lambda = [6 + d, 2 + d, 1] = [9, 5, 1]$ with $d = 3$ and $c = 5$. Applying Reduction 2 yield $\lambda' = [5, 2, 1]$ with $c = 3$. All such tableaux are zero by Theorem 9 since $s = 1$.) To account for this, we modify the construction above to use $\mathsf{g} - 1$ copies of $U_1(d)$ when $\mathsf{g} > 0$ and $s' < 5$. In such a case, the modified T' now has $s' < d + 5$. Thus we need only consider arbitrary tableaux with $s < d + 5$ or $c < 5$.

Reduction 3. Let T be any λ-tableaux with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, filled with d copies of c elements. Take h to be the maximum integer such that $\mathsf{h}d \leqslant \lambda_1 - \lambda_2 = r$ and $c - \mathsf{h} \geqslant 3$. Let $V = \mathsf{h}V_1(d)$ be the join of h copies of $V_1(d)$. Then by the Breakdown Theorem, we may write $T = T' \vee V$ for T' a $\lambda' = (\lambda_1 - d\mathsf{h}, \lambda_2, \lambda_3)$-tableau filled with d copies of $c' = c - \mathsf{h}$ elements, provided the weight-sets of V and T' are disjoint. The choice of h means that in T', $r' = \lambda'_1 - \lambda'_2 = \lambda_1 - d\mathsf{h} - \lambda_2 < d$ or $c' = c - \mathsf{h} < 4$. However, we will need the existence of non-zero T' in the specified shape. As was shown in Theorem 8, this is not always the case for some r. Specifically, when $r < 5$ non-zero tableaux do not exist for some shapes when $c = 3$. To account for this, we modify the construction above to use $\mathsf{h} - 1$ copies of $V(d)$ when $\mathsf{h} > 0$ and $r' < 5$. In such a case, the modified T' now has $r' < d + 5$. Thus we need only consider arbitrary tableaux with $r < d + 5$ or $c < 4$.

Conclusion. When d is even, we can apply these reductions successively. Take an arbitrary λ-tableaux T filled with d copies of c elements, and assume $c \geqslant 6$. We use $T^{(i)}$ to represent the appropriate T' obtained in these reductions. By Reduction 1, $T = S \vee T^{(1)}$ where $S = \mathsf{f}P_1(d)$ and $T^{(1)}$ has $t = \lambda_3(T^{(1)}) < d$ and is filled with $c^{(1)} = c - 3\mathsf{f}$ elements.

Now, if $c^{(1)} \geqslant 6$ apply Reduction 2 to $T^{(1)}$. Since $c^{(1)} \geqslant 6$, then by Reduction 2, write $T^{(1)} = T^{(2)} \vee U$ where $U = \mathsf{g}U_1(d)$ and $T^{(2)}$ has $t < d$ (since $T^{(1)}$ does) and $s < d + 5$. Here $T^{(2)}$ is filled with $c^{(2)} = c^{(1)} - 2\mathsf{g}$ elements.

Finally if $c^{(2)} \geqslant 6$ apply Reduction 3. This gives $T^{(2)} = T^{(3)} \vee V$, where $V = \mathsf{h}V_1(d)$ and $T^{(3)}$ has $t < d$, $s < d + 5$, and $r < d + 5$. Here $T^{(3)}$ is filled with $c^{(3)} = c^{(2)} - \mathsf{h}$ elements.

Hence $T = S \vee T^{(i)} \vee U \vee V$ where either $T^{(i)}$ is filled with fewer than 6 elements, or $T^{(i)}$ has $t < d$, $s < d + 5$, and $r < d + 5$. In the second case, $T^{(i)}$ must be filled with $3t + 2s + r$ elements, which is less than or equal to $3(d-1) + 2(d+4) + (d+4) = 6d + 9 \leqslant 8d$ if $d > 4$. (If $d = 4$ we have $6d + 8 \leqslant 8d$ and its not possible to have $6d + 9 = 9d$ when $d = 4$. For $d = 3$ additional reductions apply.) So we only need those tableaux with $c \leqslant 8$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T, because the reductions do not reduce r or s to less than 5. Hence $T^{(i)} = T'$ has a required shape of Theorem 9 since all partitions of n with r and $s \geqslant 5$ are needed.

This reduction depends on the Breakdown Theorem. Our usage only requires verification that the weight-sets are disjoint. However, the tableaux S, U, and V are in maximal form. Hence for appropriately chosen tableaux (i.e., ones in maximal form), an application of Lemma 4.26 can easily prove weight-set disjointness.

Example 6.1. To see how this reduction works, let us consider a specific shape, $\lambda = [9d - 2, 5d, d + 2]$ where $d \geqslant 6$, d even and $c = 15$. This shape has $t = d + 2$, $s = 4d - 2$, and $r = 4d - 2$. First we apply Reduction 1, which joins $P_1(d)$ in order to have $t < d$. This gives

$$[9d - 2, 5d, d + 2] = P_1(d) \vee [8d - 2, 4d, 2].$$

Then we apply Reduction 2 to the shape $[8d - 2, 4d, 2]$, which has $s = 4d - 2$ to reduce to $s < d + 5$ by joining three copies of $U_1(d)$. So we get

$$[8d - 2, 4d, 2] = [5d - 2, d, 2] \vee 3U_1(d).$$

Applying Reduction 3 to shape $[5d - 2, d, 2]$ which has $r = 4d - 2$ we normally want to reduce r to be between 5 and $d + 5$. Here we won't necessarily reduce r fully, so that the resulting tableau will be familiar. Instead we will reduce to $r = 2d - 2$ (which may be reduced further depending on d) by joining two copies of $V(d)$. This yields

$$[5d - 2, d, 2] = [3d - 2, d, 2] \vee 2V(d).$$

Hence, when we combine all these reductions we get

$$[9d - 2, 5d, d + 2] = P_1(d) \vee [3d - 2, d, 2] \vee 3U_1(d) \vee 2V(d).$$

A non-zero tableau of shape $[3d-2, d, 2]$ is Q_1 of Example 4.13 with $\mathrm{A}=1$, $\mathrm{B}=1$, $\mathrm{C}=d-2$. Therefore, writing

$$T = P_1(d) \vee Q_1(1,1,d-2) \vee 3U_1(d) \vee 2V(d)$$

and omitting the extra tail of Q_1 we have

$$T = \begin{array}{cccccccccccc} \overline{d} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{d-2} & \overline{d} & \overline{d} & \overline{d} & \overline{d} & \overline{d} \\ 1 & 5 & 5 & 5 & 6 & 4 & 8 & 10 & 12 & 14 & 15 \\ 2 & 4 & 7 & 7 & 4 & 7 & 9 & 11 & 13 & & \\ 3 & 6 & 6 & & & & & & & & \end{array}.$$

As Q_1 is in maximal form, $\mathbf{q}_T \neq 0$. (Here Q_1 is one of the tableaux we have constructed and tested. It is similar to the many other tableaux constructions for different shapes and fillings we have omitted listing in this paper. See [19] for the complete collection.)

6.2. *d odd*

When d is odd, we proceed exactly as in the even case, except the tableaux we use are slightly different. Namely we use S_1 instead of P_1 and U_2 instead of U_1. These adjustments are necessary for Reductions 1 and 2 since $P_1(d)$ and $U_1(d)$ are zero for d odd. These modified procedures are denoted Reductions $1'$ and $2'$. Reduction 3 remains unchanged however.

The same argument as in the even case works for the d odd cases, though the numbers are adjusted slightly. Take an arbitrary λ-tableaux T filled with d copies of c elements, but this time assume $c \geqslant 9$. Then by applications of Reductions $1'$, $2'$, and 3, $T = S \vee U \vee V \vee T^{(i)}$; either $T^{(i)}$ is filled with fewer than 9 elements, or $T^{(i)}$ has $t < 2d$, $s < 2d+5$, and $r < d+5$. In the second case, $T^{(i)}$ must be filled with $3t+2s+r$ elements, which is less than or equal to $3(2d-1)+2(2d+4)+(d+4) = 11d+9$ as $d \geqslant 3$. Moreover, if r or $s < 5$ in $T^{(i)}$, then r or $s < 5$ in T. Hence $T^{(i)}$ has a required shape of Theorem 9. However we wish to have T fillable with $c \leqslant 8$. To do this we have additional reduction techniques. However, these technique are very sensitive to the parameters r, s in T, so we will categorize them by such. The additional maximal form tableaux we use are:

$$U_1(d-1) = \begin{array}{c} \overline{d-1} \\ 1 \\ 2 \end{array}, \quad d \text{ odd}, \quad \omega_2 = (0, d-1),$$

$$P_1(d-1) = \begin{array}{c} \overline{d-1} \\ 1 \\ 2 \\ 3 \end{array}, \quad d \text{ odd}, \quad \omega_{2,3} = \begin{pmatrix} 0 & d-1 & 0 \\ 0 & 0 & d-1 \end{pmatrix},$$

$$P_2(d-2,1) = \begin{array}{ccccc} \overline{d-2} & & & & \\ 1 & 1 & 1 & 3 & \\ 2 & 2 & 3 & 2 & \\ 3 & & & & \end{array}, \quad d \text{ odd}, \quad \omega_{2,3} = \begin{pmatrix} 0 & d & 1 \\ 0 & 0 & d-2 \end{pmatrix}.$$

We use a single copy of U_1 and P_1 whenever the parameters s and t are large enough and the reduction does not cause r to reduce to an invalid case. In the remaining situations, P_2 is used since it does not change r. This suffices to guarantee that $T^{(i)}$ will need at most 8 elements.

When d is odd, we need all tableaux with $c \leqslant 8$. The use of the Breakdown Theorem requires the tableaux to be disjoint from multiple copies of S_1, $U_2(d-1,1)$ and $V(d)$. The further reductions also require the tableaux to be disjoint from $P_1(d-1)$, $U_1(d-1)$, $P_1(d-1) \vee U_1(d-1)$, $P_2(d-2,1)$, $U_1(d-1) \vee P_2(d-2,1)$. Since these tableaux are in maximal form, if we construct the tableaux with $c \leqslant 8$ in maximal form, disjointness will follow from Lemma 4.26. The maximality condition is relatively easy to satisfy. However, many different tableaux are required in this construction due to the large number of different parameters involved. Hence such constructions are once again omitted. More details will be supplied in [19] (forthcoming). This concludes the sketch of Theorem 2.

7. Discussion

Although the construction of the tableaux is tedious and omitted here (details provided in forthcoming [19]), the techniques developed in these proofs illustrate new approaches to Young Tableaux. These concepts could be carried forth in contexts involving tableaux other than its usage here. Furthermore, these techniques discussed below, have wider applications.

The two main theoretical techniques of this paper are that of weight-set counting of Theorem 5 and application of the Breakdown Theorem.

The theory and technique of weight-set counting developed in this paper can be implemented in general, as can the concept of maximal form. While we only used tableaux with three or fewer rows, the theoretical foundations of weight-set counting have been laid for tableaux of an arbitrary number of rows. Although the computations are impractical for a random tableaux, the counting works smoothly for tableaux with suitable symmetries, particularly those tableaux in maximal form.

Moreover, the technique of weight-set counting is not dependent on a filling of content $[b^a]$. However, for non-uniform contents, one must watch carefully the action of $\mathcal{S}_a$; the weight-set counting may need to count all rows and the definition of maximality will need adjustment.

Similarly, the usage of the Breakdown Theorem in constructing larger tableaux will also work for other contents and row quantities. The use of the Lemma 4.26, to show weight-set disjointness by maximality, however, has only be defined for three row partitions. It should be possible to generalize it for other partitions. In addition, though not addressed in this paper, maximal form may be use to demonstrate the linear independence of $\{\mathbf{q}_T\}$.

Hence, the methods of proving Theorem 2 should also apply to proving Conjecture 2 with other a's, not including multiplicities. Unfortunately, the computations are likely to be somewhat cumbersome, especially the establishments of non-zero shapes as done by Theorem 8. However, should those parameters be established through other techniques, the tableaux constructed for Theorem 2 should provide nearly all the needed shapes with three

or fewer rows, thus reducing the work substantially. Moreover, the reduction procedures will also apply.

Finally, our results on Foulkes' Conjecture with the trivial character also extend to the alternating character. The correspondence established between these characters is independent of our construction. Hence previous results on Foulkes' Conjecture now hold for the alternating character as well.

References

[1] S.C. Black, R.J. List, A note on plethysms, European J. Combin. 10 (1989) 111–112.
[2] E. Briand, Polynômes multisymétriques, PhD dissertation, University Rennes I, Rennes, France, October 2002.
[3] M. Brion, Stable properties of the plethysm: on two conjectures of Foulkes, Manuscripta Math. 80 (1993) 347–371.
[4] C. Coker, A problem related to Foulkes's conjecture, Graphs Combin. 9 (1993) 117–134.
[5] S.C. Dent, J. Siemons, On a conjecture of Foulkes, J. Algebra 226 (2000) 236–249.
[6] W.F. Doran IV, On Foulkes' conjecture, J. Pure Appl. Algebra 130 (1998) 85–98.
[7] H.O. Foulkes, Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form, J. London Math. Soc. 25 (1950) 205–209.
[8] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.3, http://www.gap-system.org, 2002.
[9] D. Gay, Characters of the Weyl group of SU(n) on zero weight spaces and centralizers of permutation representations, Rocky Mountain J. Math. 6 (1976) 449–455.
[10] R. Howe, (GL_n, GL_m)-duality and symmetric plethysm, Indian Acad. Sci. 97 (1987) 85–109.
[11] N.F.J. Inglis, R.W. Richardson, J. Saxl, An explicit model for the complex representations of S_n, Arch. Math. 54 (1990) 258–259.
[12] G. James, The Representation Theory of the Symmetric Group, in: Lecture Notes in Math., vol. 682, Springer, Berlin, 1978.
[13] G. James, A. Kerber, Representation Theory of the Symmetric Group, in: Encyclopedia Math. Appl., vol. 16, Addison–Wesley, Reading, MA, 1981.
[14] S. Lang, Algebra, 3rd ed., Addison–Wesley, Reading, MA, 1999.
[15] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., in: Oxford Math. Monogr., Clarendon Press, Oxford, 1995.
[16] B.E. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions, Brooks/Cole, Pacific Grove, CA, 1991.
[17] R.P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in: V. Arnold, M. Atiyah, P. Lax, B. Mazur (Eds.), Mathematics: Frontiers and Perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.
[18] R.M. Thrall, On symmetrized Kronecker powers and the structure of the free Lie ring, Amer. J. Math. 64 (1942) 371–388.
[19] R.A. Vessenes, Generalized Foulkes conjecture and tableaux construction, PhD dissertation, California Institute of Technology, Pasadena, CA, June, 2004.
[20] J. Wu, Foulkes conjecture in representation theory and its relations in rational homotopy theory, http://www.math.nus.edu.sg/~matwujie/Foulkes.pdf.