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Random variables with completely independent subcollections

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Abstract

We investigate the algebra and geometry of the independence conditions on discrete random variables in which we consider a collection of random variables and study the condition of independence of some subcollections. We interpret independence conditions as an ideal of algebraic relations. After a change of variables, this ideal is generated by generalized 2×2 minors of multi-way tables and linear forms. In particular, let Δ be a simplicial complex on some random variables and A be the table corresponding to the product of those random variables. If A is Δ-independent table then A can be written as the entrywise sum $A^I + A^0$ where A^I is a completely independent table and A^0 is identically 0 in its Δ-margins.

We compute the isolated components of the original ideal, showing that there is only one component that could correspond to probability distributions, and relate the algebra and geometry of the main component to that of the Segre embedding. If Δ has fewer than three facets, we are able to compute generators for the main component, show that it is Cohen–Macaulay, and give a full primary decomposition of the original ideal.

Keywords: Determinental ideal; Complete independence; Segre variety; Perfect ideal; Cohen–Macaulay; Principal radical system; Primary decomposition

1. Introduction

1.1. Set-theoretic version of the main result

Let $X_1, \dots, X_n$ be discrete random variables on the same population. Then there is an n-dimensional table whose $(i_1, \dots, i_n)$ entry is the probability of $X_j = i_j$ for all j. Given the table

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of probabilities A for $X_1, \dots, X_n$, and a subset $\mathcal{J} \subset \{1, \dots, n\}$ it is easy to compute the table, $A_{\mathcal{J}}$ for $X_{j_1}, \dots, X_{j_r}$ by summing over the indices not in $\mathcal{J}$.

The random variables $X_1, \dots, X_n$ are called completely independent if the probabilities satisfy

$$\mathrm{Prob}(X_1 = i_1, \dots, X_n = i_n) = \prod_j \mathrm{Prob}(X_j = i_j)$$

for all possible $i_1, \dots, i_n$. If Δ is any collection of subsets of $\{1, \dots, n\}$ then we say that an n-dimensional table is Δ-independent if for each $\mathcal{J} \in \Delta$, $A_{\mathcal{J}}$ is completely independent.

With this notation, the main result of this paper implies

Theorem 1. *If A is a Δ-independent table associated to the product variable $X_1 \times \dots \times X_n$ then A can be written as the (entrywise) sum $A^I + A^0$ where A^I is a completely independent table and A^0 is a table whose margins $A^0_{\mathcal{J}}$ are identically* 0 *for all $\mathcal{J} \in \Delta$.*

Proof. We give a short proof here. Let A^I be defined by

$$A^I_{i_1,\dots,i_n} = \prod_{k=1}^{n} \mathrm{Prob}(X_k = i_k).$$

Then for any $\mathcal{J} \in \Delta$ the table $A^I_{\mathcal{J}}$ and the table $A_{\mathcal{J}}$ are identical, by the definition of complete independence. □

1.2. The algebraic perspective

Theorem 1 has left many algebraic questions unanswered, and in this section we give another perspective on it which will lead to stronger results. Let $A = (x_{i_1,\dots,i_n})$ be the generic $a_1 \times \dots \times a_n$ table over any field $\mathbb{K}$ and Δ be any collection of subsets of $\{1, \dots, n\}$. For each $\mathcal{J} \in \Delta$, $A_{\mathcal{J}}$ is a table whose entries are sums of the variables $x_{i_1,\dots,i_n}$. Complete independence of a table, B, with entries in a ring can be expressed by the ideal, $I(B)$, generated by generalized 2×2 minors of the table. Therefore, Δ-independence of the generic table A is expressed by the ideal

$$I_\Delta(A) = \sum_{\mathcal{J} \in \Delta} I(A_{\mathcal{J}}).$$

Theorem 1 is implied by a knowledge of the minimal primes over I_Δ. We prove that there is only one minimal prime, P_Δ over I_Δ which does not contain the sum of all the variables. Therefore, P_Δ is the only minimal prime that corresponds to probability distributions. We parameterize P_Δ and give set-theoretic generators for it in terms of the generators of a related toric ideal. In the case in which Δ has fewer than three facets, we compute the generators for P_Δ and show that it is a perfect ideal.

The other minimal primes over I_Δ are also accessible, and we give a fairly complete description of them. Moreover, when Δ has fewer than three facets we show that I_Δ is a radical ideal. I_Δ is not always radical and we also give an example in which Δ has four facets and I_Δ is not radical.

1.3. Overview

In Section 2, we define the principal objects of study and develop the elementary statistical terminology needed for the sequel. Section 3 defines the change of variables which is the foundation for the rest of the exposition.

In Section 4 we show that a related toric ideal is contained in P_Δ, and in Section 5 we parameterize P_Δ and give set-theoretic generators for it. In Section 6 we treat the other minimal primes over I_Δ and show that they can be understood in terms of P_{Δ_i} for subcomplexes $\Delta_i \subset \Delta$. In Section 7 we use principal radical systems to prove that if Δ has three or fewer facets then P_Δ is generated by the set-theoretic generators given in Section 5 and is a perfect ideal. We also prove that in the same case, I_Δ is radical. Finally, Section 8 ties up the loose ends with an example in which I_Δ is not radical, two conjectures and notes on the computational limits encountered.

The main theorems are Theorems 7 and 23.

The change of variables in Section 3 and the toric ideal Q_Δ from Section 4 are the key technical points to understand from which Theorem 7 follows. Theorem 23 is an application of principal radical systems.

2. Statistics for algebraists

2.1. Random variables

A *random variable* X is a function from a set Ω, a population, to a set S_X, the values of X. We define

$$\{X = s\} = X^{-1}(s).$$

If Ω is finite, we define a new function $P_X : S_X \to \mathbb{R}_+$ by

$$P_X(s) = \mathrm{Prob}\{X = s\} = \frac{\text{cardinality}\{X = s\}}{\text{cardinality}\,\Omega}.$$

$P_X(s)$ can be interpreted as the probability that a randomly selected $\omega \in \Omega$ will have $X(\omega) = s$. A *discrete random variable* is a random variable which takes finitely many values. From now on, all our random variables will be discrete on a finite population. That is, Ω and S_X are both finite.

If $X_1, \ldots, X_n$ are random variables on the same population, then there is a product variable

$$X_1 \times \cdots \times X_n : \Omega \to S_{X_1} \times \cdots \times S_{X_n}$$

defined in the obvious way. If X_j takes $a_j < \infty$ values, then there is an $a_1 \times \cdots \times a_n$ n-dimensional (real) table

$$A = (x_{i_1,\ldots,i_n})$$

whose $(i_1, \ldots, i_n)$ entry is the probability,

$$\mathrm{Prob}\bigl\{X_1 \times \cdots \times X_n = (i_1, \ldots, i_n)\bigr\}.$$

2.2. Marginal tables and subcollections of random variables

Suppose we have an n-dimensional array $A = (x_{i_1,\ldots,i_n})$ of probabilities associated to some random variables $X_1, \ldots, X_n$. Given any $\mathcal{J} = \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ we can define an $a_{j_1} \times \cdots \times a_{j_m}$ array which is the probability array for the random variable $X_{j_1} \times \cdots \times X_{j_m}$, disregarding the other random variables. Such an array is called an *m-margin* of A.

To recover the probability of some subcollection of events happening, disregarding the other variables, we need only to sum over the variables we wish to disregard. For example, to disregard the random variable X_n, consider

$$\mathrm{Prob}\{X_1 \times \cdots \times X_{n-1} = (i_1, \ldots, i_{n-1})\} = \sum_k x_{i_1,\ldots,i_{n-1},k}.$$

In general, suppose that $A = (x_{i_1,\ldots,i_n})$ is an n-dimensional array with entries in a ring R. Let σ be an ordered n-tuple whose jth entry, σ_j, is either an integer such that $1 \leqslant \sigma_j \leqslant a_j$ or the symbol $+$. Let

$$\mathcal{J} = \mathcal{J}(\sigma) = \{j_1, \ldots, j_m\} = \{j \mid \sigma_j \neq +\}$$

and define

$$x_\sigma := \sum_{i_j = \sigma_j \text{ if } j \in \mathcal{J}} x_{i_1,\ldots,i_n}.$$

For example, $x_{1,+,3} = \sum_j x_{1,j,3}$.

This essentially allows us to create the desired array, but we need to index the array correctly. Fix some $\mathcal{J} = \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ and numbers $i_1, \ldots, i_m$ such that $1 \leqslant i_k \leqslant a_{j_k}$. We can define a sequence $\sigma(\mathcal{J})_{\{i_1,\ldots,i_m\}}$ of length n, by $\sigma(\mathcal{J})_k = +$ if $k \notin \mathcal{J}$, and $\sigma(\mathcal{J})_{j_k} = i_k$. Again, let $A = (x_{i_1,\ldots,i_n})$ be an n-dimensional array with entries in a ring R. We may define an $a_{j_1} \times \cdots \times a_{j_m}$ array $A_{\mathcal{J}}$ whose $(i_1, \ldots, i_m)$ entry is $x_{(\sigma(\mathcal{J})_{i_1,\ldots,i_m})}$. This is an m-margin of A, as described above.

Moreover, if A is an array of probabilities that is associated to random variables $X_1, \ldots, X_n$ and $\mathcal{J} \subset \{1, \ldots, n\}$, then $A_{\mathcal{J}}$ is the array of probabilities associated to the random variables $X_{j_1}, \ldots, X_{j_m}$.

2.3. Complete independence and the Segre variety

The random variables $X_1, \ldots, X_n$ are called *completely independent* if the identity

$$\mathrm{Prob}\{X_1 \times \cdots \times X_n = (i_1, \ldots, i_n)\} = \prod_{j=1}^{n} \mathrm{Prob}\{X_j = i_j\}$$

holds for all values in $S_{X_1} \times \cdots \times S_{X_n}$. We will study the situation in which certain subcollections of the variables $X_1, \ldots, X_n$ are completely independent.

Likewise an array $A = (x_{i_1,\ldots,i_n})$ with entries in a ring R will be called *completely independent* if there are elements of R, $\{y_{1,i_1}, y_{2,i_2}, \ldots, y_{n,i_n}\}$, such that the condition

$$x_{i_1,\ldots,i_n} = \prod y_{j,i_j} \tag{1}$$

holds for all choices $(i_1, \ldots, i_n)$.

An algebraic geometer will immediately recognize that (1) implies that the table A is a point on the Segre variety

$$\mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{a_n-1} \subset \mathbb{P}^{\prod a_j - 1}. \tag{2}$$

This was observed by Sturmfels in [Stu02]. This brings us to the link between statistics and commutative algebra.

2.4. The algebraic definitions

The Segre embedding is induced by the ring map

$$\sigma : \mathbb{Z}[x_{i_1,\dots,i_n}] \longrightarrow \mathbb{Z}[y_{1,i_1}, y_{2,i_2}, \dots, y_{n,i_n}],$$
$$x_{i_1,\dots,i_n} \longmapsto \prod y_{j,i_j}.$$

The kernel of σ, which is the defining ideal of the Segre variety, can be generated by generalized 2×2 minors, which we now define.

As usual, let $A = (x_{i_1,\dots,i_n})$ be an n-dimensional array with entries in a ring R. We define a 2×2 minor about the lth coordinate of A to be any relation of the form

$$\det \begin{pmatrix} x_{i_1,\dots,i_n} & x_{j_1,\dots,j_{l-1},i_l,j_{l+1},\dots,j_n} \\ x_{i_1,\dots,i_{l-1},j_l,i_{l+1},\dots,i_n} & x_{j_1,\dots,j_n} \end{pmatrix}.$$

This is an interchange of just the lth coordinate. Obviously, the ideal in R generated by all interchanges of one coordinate will generate the ideal containing all interchanges of an arbitrary number of coordinates. From [Hà02, Corollary 1.8], we know that the 2×2 minors of an n-dimensional array generate the defining ideal of the Segre embedding. Thus we define the *Segre relations* to be these generalized 2×2 minors.

We can define an $a_1 \times \cdots \times a_n$ table with entries in R to be a map

$$B : \mathbb{Z}[x_{i_1,\dots,i_n}] \longrightarrow R,$$
$$x_{i_1,\dots,i_n} \longmapsto b_{i_1,\dots,i_n},$$

where the $(i_1, \dots, i_n)$ entry in B is defined to be $b_{i_1,\dots,i_n}$. In this language, the generic table is the identity map.

We have a diagram

$$\begin{tikzcd} R \arrow[r] & R \otimes_{\mathbb{Z}[x_{i_1,\dots,i_n}]} \mathbb{Z}[y_{1,i_1}, y_{2,i_2}, \dots, y_{n,i_n}] \\ \mathbb{Z}[x_{i_1,\dots,i_n}] \arrow[u, "B"] \arrow[r, "\sigma"] & \mathbb{Z}[y_{1,i_1}, y_{2,i_2}, \dots, y_{n,i_n}] \arrow[u] \end{tikzcd}$$

and we let $I(B) \subset R$ be the kernel of the top map. This amounts to imposing the Segre relations above on the table B.

Let A be the generic $a_1 \times \cdots \times a_n$ table and let Δ be a collection of subsets of $\{1,\ldots,n\}$. Recall the definition of the marginal tables $A_{\mathcal{J}}$ from Section 2.2. We define the ideal

$$I_\Delta(A) = \sum_{\mathcal{J}\in\Delta} I(A_{\mathcal{J}}).$$

That is, $I_\Delta(A)$ is the ideal generated by the generalized 2×2 minors of each margin $A_{\mathcal{J}}$, when $\mathcal{J}\in\Delta$. We give an example at the end of this section.

This is a special case of what are called "independence ideals" in the algebraic statistics literature. See [Stu02, §8.1] for more about independence models and their corresponding ideals. One recent paper which uses similar techniques to study statistical ideals is [GSS05], which we will discuss in the next section. $I_\Delta(A)$ should be thought of as the defining ideal of the variety of tables which are completely independent in the margins given by Δ. We call a table Δ-*independent* if it lies on the variety defined by $I_\Delta(A)$.

If $\mathcal{J}'\subset\mathcal{J}$, then because of the multilinearity of the Segre relations, the complete independence of $A_{\mathcal{J}}$ implies the complete independence of $A_{\mathcal{J}'}$. Thus we may assume that Δ has the structure of a simplicial complex; that is, $\mathcal{J}'\subset\mathcal{J}\in\Delta \Rightarrow \mathcal{J}'\in\Delta$.

The rest of the paper is concerned with the primary decomposition of the ideals $I_\Delta(A)$. For any Δ we will show there is only one minimal prime which does not contain $x_{+,\ldots,+}$. This component is the most important because when A represents a probability distribution, $x_{+,\ldots,+}=1$. Thus we study that prime and relate it algebraically and geometrically to the Segre variety. When Δ is a simplicial complex with three or fewer facets, we can compute generators for the main component and show that it is perfect. In that case we will also show that $I_\Delta(A)$ is a radical ideal and give a full primary decomposition.

Throughout the exposition, we will consider the following running example for clarity: $n=3$, $a_1=a_2=a_3=2$, and

$$\Delta = \big\{\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\emptyset\big\}.$$

In this case $R=\mathbb{K}[x_{i,j,k}]$ is a polynomial ring with 8 variables and I_Δ is generated by 3 elements:

$$I_\Delta = \left\langle \det\begin{pmatrix} x_{1,1,+} & x_{1,2,+}\\ x_{2,1,+} & x_{2,2,+}\end{pmatrix}, \det\begin{pmatrix} x_{1,+,1} & x_{1,+,2}\\ x_{2,+,1} & x_{2,+,2}\end{pmatrix}, \det\begin{pmatrix} x_{+,1,1} & x_{+,1,2}\\ x_{+,2,1} & x_{+,2,2}\end{pmatrix}\right\rangle.$$

Despite its appearance, I_Δ is not a binomial ideal because $x_{1,1,+}=x_{1,1,1}+x_{1,1,2}$.

3. A linear change of variables

3.1. Set-theoretic heuristics

Let Δ be some fixed collection of subsets of $\{1,\ldots,n\}$. Our goal is to decompose the ideal $I_\Delta(A)\subset R=\mathbb{K}[x_{i_1,\ldots,i_n}]$ which is defined by the complete independence of the collection of margins of the generic table A given by Δ. First, it will be helpful and illuminating to perform a linear change of variables on R which makes I_Δ an ideal generated by quadratic binomials and linear forms. We will show that $R/I_\Delta(A)$ is a polynomial ring over a ring of smaller dimension.

Set-theoretically, suppose that one table A is Δ-independent, and another table B has the property that for each $\mathcal{J}\in\Delta$, $B_{\mathcal{J}}=0$. Then the sum (entry by entry) $A+B$ is also Δ-independent.

This is a trivial result of the fact that the equations which define Δ-independence only involve entries of the marginal tables and B is identically 0 in its Δ-margins. In this section, we will develop this idea algebraically. This is similar to the change of variables employed in [GSS05]. In their case the change of variables made their ideal binomial. However, in our case, after the change of variables the ideal has linear forms and binomials in it. Another difference is that in [GSS05], the authors used a limited version of the change of variables employed here which was well-adapted to the questions they answered.

3.2. S_Δ, T_Δ and the change of variables

We define S_Δ to be the polynomial ring over $\mathbb{K}$ with variables that are indexed by the entries in the marginal tables given by the elements of Δ. That is, for every $\mathcal{J} \in \Delta$, $A_{\mathcal{J}} = (x_{i_1,\dots,i_n})$ with $i_k = +$ for every $k \notin \mathcal{J}$. So for every $\mathcal{J} \in \Delta$, create a formal symbol $X_{i_1,\dots,i_n}$ with $i_k = \bullet$ for every $k \notin \mathcal{J}$ and $1 \leqslant i_j \leqslant a_j$ for all $j \in \mathcal{J}$. Then let S_Δ be the polynomial ring over $\mathbb{K}$ generated by these formal symbols.

To make this section clear we will use our example, in which $n = 3$ and

$$\Delta = \bigl\{\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\emptyset\bigr\}.$$

In this case,

$$S_\Delta = \mathbb{K}[X_{i,j,\bullet}, X_{i,\bullet,k}, X_{\bullet,j,k}, X_{i,\bullet,\bullet}, X_{\bullet,j,\bullet}, X_{\bullet,\bullet,k}, X_{\bullet,\bullet,\bullet}]$$

for $1 \leqslant i, j, k \leqslant 2$. Thus S_Δ has 19 variables.

For any Δ we consider the map of rings $\tau_\Delta : S_\Delta \to R$ defined by

$$X_{i_1,\dots,i_n} \longmapsto x_{i_1,\dots,i_n}$$

in which $\bullet$ changes to $+$. The kernel of τ_Δ, $K_\Delta \subset S_\Delta$, is generated by linear forms. Let $T_\Delta = S_\Delta / K_\Delta$ be the coordinate ring of *Δ-marginal tables*. Set-theoretically, a Δ-marginal table B represents the class of tables B' such that for all $\mathcal{J} \in \Delta$, $B_{\mathcal{J}} = B'_{\mathcal{J}}$.

In the example above,

$$\tau_\Delta(X_{1,1,\bullet}X_{2,\bullet,2} - X_{1,\bullet,2}X_{2,1,\bullet}) = x_{1,1,+}x_{2,+,2} - x_{1,+,2}x_{2,1,+}.$$

We will discuss the generators of K_Δ in Section 3.3.

In general, if Δ and Δ' have the property that the maximal elements of Δ and Δ' are the same, it is clear that $T_\Delta \cong T_{\Delta'}$. Since there is no ambiguity in T_Δ, we will replace all $\bullet$'s in the indices of the variables by $+$'s as usual.

Notice that if $\Delta' = \{\{1,2\},\{1,3\},\{2,3\}\}$ and Δ is the simplicial complex in our example, $S_{\Delta'}$ has 12 variables and $T_{\Delta'} \cong T_\Delta$.

On the other hand, let $L_\Delta(A)$ be the ideal generated by the entries of $A_{\mathcal{J}}$ for all $\mathcal{J} \in \Delta$ and let

$$Z_\Delta = R / \bigl(L_\Delta(A)\bigr),$$

the coordinate ring of tables whose margins given by Δ are identically zero. Since the ideal we quotient by is generated by linear forms, Z_Δ is a polynomial ring over $\mathbb{K}$. Moreover, since the image of τ_Δ is generated by the linear forms which generate $\sum_{\mathcal{L}\in\Delta} L_{\mathcal{L}}$, we have

$$T_\Delta \otimes Z_\Delta \cong \operatorname{im}\tau_\Delta \otimes Z_\Delta \cong R. \tag{3}$$

Set-theoretically, this says that the space of $a_1 \times \cdots \times a_n$ tables is a trivial bundle over the space of Δ-marginal tables.

Proposition 2. *Suppose that $I = \langle f_n \rangle$ is any ideal in R such that the f_n are written entirely in terms of the margins given by Δ, as above. Then let F_n be the polynomial in S_Δ (or T_Δ) which has the same form as f_n except that the lower-case x's are replaced by upper-case X's and the $+$'s are replaced by $\bullet$'s. Let $I^{S_\Delta} := K_\Delta + \langle F_n \rangle$.*

Then I is prime (respectively radical, perfect) if and only if I^{S_Δ} is prime (respectively radical, perfect). Moreover, the Betti diagram of I as an R-module is the same as that of I^{S_Δ} as an S_Δ-module.

Proof. Since polynomial rings are flat over the ground field, by (3)

$$R/I \cong S_\Delta/\big(I^{S_\Delta}\big) \otimes Z_\Delta,$$

which is a polynomial ring over $S_\Delta/(I^{S_\Delta})$. Thus, R/I is a domain (respectively reduced, Cohen–Macaulay) if and only if S_Δ/I^{S_Δ} is a domain (respectively reduced, Cohen–Macaulay). $\square$

3.3. Generators for K_Δ

We can also describe the generators of K_Δ. The idea is that if we have two margins $A_{\mathcal{J}}$ and $A_{\mathcal{K}}$ then they have an "intersection" which is $A_{\mathcal{J}\cap\mathcal{K}}$. In particular, the entries of $A_{\mathcal{J}\cap\mathcal{K}}$ will have a representation as sums of elements of $A_{\mathcal{J}}$ and $A_{\mathcal{K}}$, and they must agree. For ease of notation, we will assume that $\mathcal{J} = \{1, \ldots, r\}$ and $\mathcal{K} = \{s, \ldots, n\}$, so $\mathcal{L} = \{s, \ldots, r\}$. Then we have an ideal of relations

$$R_{\mathcal{J},\mathcal{K}} = \left\langle \sum_{i_m | m<s} X_{i_1,\ldots,i_r,+,\ldots,+} - \sum_{i_m | m>r} X_{+,\ldots,+,i_s,\ldots,i_n} \right\rangle$$

for all choices of $(i_s, \ldots, i_r)$.

Proposition 3. *K_Δ is generated by $\sum R_{\mathcal{J},\mathcal{K}}$ for all pairs of $\mathcal{J}, \mathcal{K} \in \Delta$.*

In our example, K_Δ is easy to understand. We list some generators here:

$$R_{\{1,2\},\{1,3\}} = \big\langle (X_{1,1,\bullet} + X_{1,2,\bullet}) - (X_{1,\bullet,1} + X_{1,\bullet,2}), (X_{2,1,\bullet} + X_{2,2,\bullet}) - (X_{2,\bullet,1} + X_{2,\bullet,2}) \big\rangle,$$
$$R_{\{1,2\},\{1\}} = \big\langle (X_{1,1,\bullet} + X_{1,2,\bullet}) - X_{1,\bullet,\bullet}, (X_{2,1,\bullet} + X_{2,2,\bullet}) - X_{2,\bullet,\bullet} \big\rangle,$$
$$R_{\{1,2\},\{3\}} = \big\langle (X_{1,1,\bullet} + X_{1,2,\bullet} + X_{2,1,\bullet} + X_{2,2,\bullet}) - (X_{\bullet,\bullet,1} + X_{\bullet,\bullet,2}) \big\rangle.$$

K_Δ can be generated by 30 linear forms, but of course this is not minimal, as illustrated by the inclusion

$$R_{\{1,3\},\{1\}} \subset R_{\{1,2\},\{1,3\}} + R_{\{1,2\},\{1\}}.$$

K_Δ can be minimally generated by 12 linear forms so $T_\Delta = S_\Delta/K_\Delta$ is a polynomial ring of dimension 7.

4. In search of a statistically significant component $I_\Delta(A)$

4.1. A related toric ideal

In the following sections we will prove that there is only one minimal prime over $I_\Delta(A)$, for a generic $a_1 \times \cdots \times a_n$ table A, which does not contain $x_{+,\ldots,+}$. This will be the only statistically significant component of I_Δ because when A is a probability distribution, $x_{+,\ldots,+} = 1$. We will identify the main component as the kernel of ring map, and relate it to a toric ideal.

The first step is to define the toric ideal. Let Δ be a collection of subsets of $\{1,\ldots,n\}$ and let

$$\begin{aligned} \eta_\Delta : S_\Delta &\longrightarrow \mathbb{K}[y_{i,j_i} \mid 1 \leqslant j_i \leqslant a_i \text{ or } j_i = \bullet], \\ X_{j_1,\ldots,j_n} &\longmapsto \prod y_{i,j_i}. \end{aligned}$$

Finally, let $Q_\Delta = \ker \eta_\Delta$. Since Q_Δ is defined as the kernel of a monomial map, it is generated by binomials. The rest of this section will be devoted to showing that Q_Δ is contained in $(I_\Delta\colon x^\infty_{+,\ldots,+})$.

In our example, where Δ has facets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, Q_Δ is generated by such binomials as

$$\det\begin{pmatrix} X_{1,1,\bullet} & X_{2,1,\bullet} \\ X_{1,\bullet,1} & X_{2,\bullet,1} \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} X_{1,1,\bullet} & X_{2,1,\bullet} \\ X_{1,\bullet,\bullet} & X_{2,\bullet,\bullet} \end{pmatrix}.$$

S_Δ/Q_Δ is a ring of dimension 6.

4.2. Some useful elements of the ideal $I(A)$

First we will construct elements in $I(A)$ which will allow us to view $+$ like any other index.

Proposition 4. *Let A be the generic $a_1 \times \cdots \times a_n$ table. Then*

$$\det\begin{pmatrix} x_{i_1,\ldots,i_n} & x_{j_1,\ldots,j_{l-1},i_l,j_{l+1},\ldots,j_n} \\ x_{i_1,\ldots,i_{l-1},+,i_{l+1},\ldots,i_n} & x_{j_1,\ldots,j_{l-1},+,j_{l+1},\ldots,j_n} \end{pmatrix} \in I(A).$$

Proof. Consider the sum

$$\sum_k \det\begin{pmatrix} x_{i_1,\ldots,i_{l-1},i_l,i_{l+1},\ldots,i_n} & x_{j_1,\ldots,j_{l-1},i_l,j_{l+1},\ldots,j_n} \\ x_{i_1,\ldots,i_{l-1},k,i_{l+1},\ldots,i_n} & x_{j_1,\ldots,j_{l-1},k,j_{l+1},\ldots,j_n} \end{pmatrix}$$

which by the multilinearity of the minors is

$$\det\begin{pmatrix} x_{i_1,\dots,i_{l-1},i_l,i_{l+1},\dots,i_n} & x_{j_1,\dots,j_{l-1},i_l,j_{l+1},\dots,j_n} \\ \sum_k x_{i_1,\dots,i_{l-1},k,i_{l+1},\dots,i_n} & \sum_k x_{j_1,\dots,j_{l-1},k,j_{l+1},\dots,j_n} \end{pmatrix}.$$

By the definition of $+$ notation from Section 2.2, this is

$$\det\begin{pmatrix} x_{i_1,\dots,i_n} & x_{j_1,\dots,j_{l-1},i_l,j_{l+1},\dots,j_n} \\ x_{i_1,\dots,i_{l-1},+,i_{l+1},\dots,i_n} & x_{j_1,\dots,j_{l-1},+,j_{l+1},\dots,j_n} \end{pmatrix}$$

which establishes the result. □

This proposition allows us to let any number of coordinates equal "+," and interchange them freely.

As an example, consider the case in which $n = 2$. For any i_1, i_2

$$x_{i_1,i_2}x_{+,+} - x_{i_1,+}x_{+,i_2} \in I_{\{1,2\}}(A).$$

If the $x_{i,j}$ are really probabilities, then $x_{+,+} = 1$ so this relation becomes $x_{i_1,i_2} = x_{i_1,+}x_{+,i_2}$, which is the independence condition for two random variables, as in (1).

4.3. *An intermediate ideal, $J_\Delta \subset Q_\Delta$*

There are some quadratic binomials in Q_Δ which play a special role in the discussion. Let $J_\Delta \subset Q_\Delta$ be generated by binomials

$$f = X_{\bar{\imath}_1} \cdot X_{\bar{\imath}_2} - X_{\bar{\jmath}_1} \cdot X_{\bar{\jmath}_2} \in Q_\Delta$$

such that $X_{\bar{\imath}_1}, X_{\bar{\jmath}_1}$ are both entries in $A_{\mathcal{J}}$ for some $\mathcal{J} \in \Delta$. Since $f \in Q_\Delta$, this implies that $X_{\bar{\imath}_2}, X_{\bar{\jmath}_2}$ are both entries in $A_{\mathcal{K}}$ for some $\mathcal{K} \in \Delta$.

In our example, J_Δ will be generated by I_Δ and the 2×2 minors of the three matrices symmetric to

$$\begin{pmatrix} X_{1,1,\bullet} & X_{1,2,\bullet} & X_{1,\bullet,\bullet} & X_{1,\bullet,1} & X_{1,\bullet,2} \\ X_{2,1,\bullet} & X_{2,2,\bullet} & X_{2,\bullet,\bullet} & X_{2,\bullet,1} & X_{2,\bullet,2} \end{pmatrix}. \tag{4}$$

Lemma 5. *Let $f = X_{\bar{\imath}_1} \cdot X_{\bar{\imath}_2} - X_{\bar{\jmath}_1} \cdot X_{\bar{\jmath}_2}$ be a generator of J_Δ such that $X_{\bar{\imath}_1}$ is an entry in $A_{\mathcal{J}}$ and $X_{\bar{\imath}_2}$ is an entry in $A_{\mathcal{K}}$. Then*

$$L_{\mathcal{J}\cap\mathcal{K}} \cdot \langle f\rangle \subset I_\Delta.$$

Proof. The proof is very technical (but elementary). In our running example, the result follows from the following line of reasoning. The matrix (4) has the property that the first 3 columns and the last 3 columns have rank 1. Since they share the middle column, either each column of (4) is a scalar multiple of the middle column, or the middle column is identically 0. Thus, either the 2×2 minors of (4) vanish or $X_{1,\bullet,\bullet} = X_{2,\bullet,\bullet} = 0$.

Now we turn to the detailed proof. Since all the calculations will happen in the margin $A_{\mathcal{J}\cup\mathcal{K}}$, we can assume that $\{1,\dots,n\} = \mathcal{J}\cup\mathcal{K}$ for ease of notation. We re-index so that $\mathcal{J} = \{1,\dots,s\}$ and $\mathcal{K} = \{r,\dots,n\}$, so $\mathcal{L} = \{r,\dots,s\}$. After this reorganization, f is the following determinant:

$$q = \det\begin{pmatrix} X_{i_{1,1},\dots,i_{1,s},+,\dots,+} & X_{+,\dots,+,j_{2,r},\dots,j_{2,n}} \\ X_{j_{1,1},\dots,j_{1,s},+,\dots,+} & X_{+,\dots,+,i_{2,r},\dots,i_{2,n}} \end{pmatrix},$$

where $i_{1,k} = j_{1,k}$ for all $k < r$ and $i_{2,k} = j_{2,k}$ for all $k > s$. Moreover, $\{i_{1,k}, i_{2,k}\} = \{j_{1,k}, j_{2,k}\}$ for each $r \leqslant k \leqslant s$. Thus, f can be thought of as the exchange of some number of indices between $X_{\bar{\imath}_1}$ and $X_{\bar{\imath}_2}$. Clearly, these exchanges can be generated by exchanges of one coordinate. Re-index again, so that f is an exchange of the rth coordinate. Then f can be written as

$$f = \det\begin{pmatrix} X_{j_1,\dots,j_{r-1},j_r,j_{r+1},\dots,j_s,+,\dots,+} & X_{+,\dots,+,k_r,j_{r+1},\dots,j_n} \\ X_{j_1,\dots,j_{r-1},k_r,j_{r+1},\dots,j_s,+,\dots,+} & X_{+,\dots,+,j_r,k_{r+1},\dots,k_n} \end{pmatrix}.$$

If $l = x_{+,\dots,+,i_r,\dots,i_s,+,\dots,+}$ is any generator of $L_{\mathcal{L}}$, then we need to show that lf is in $I(A_{\mathcal{J}}) + I(A_{\mathcal{K}})$. We will construct this product explicitly.

Consider the sum

$$\begin{aligned} & x_{j_1,\dots,j_{r-1},i_r,j_{r+1},\dots,j_s,+,\dots,+} \det\begin{pmatrix} x_{+,\dots,+,k_r,k_{r+1},\dots,k_n} & x_{+,\dots,+,k_r,i_{r+1},\dots,i_s,+,\dots,+} \\ x_{+,\dots,+,j_r,k_{r+1},\dots,k_n} & x_{+,\dots,+,j_r,i_{r+1},\dots,i_s,+,\dots,+} \end{pmatrix} \\ &\quad + x_{+,\dots,+,k_r,k_{r+1},\dots,k_n} \det\begin{pmatrix} x_{j_1,\dots,j_{r-1},j_r,j_{r+1},\dots,j_s,+,\dots,+} & x_{+,\dots,+,j_r,i_{r+1},\dots,i_s,+,\dots,+} \\ x_{j_1,\dots,j_{r-1},i_r,j_{r+1},\dots,j_s,+,\dots,+} & x_{+,\dots,+,i_r,i_{r+1},\dots,i_s,+,\dots,+} \end{pmatrix} \\ &\quad + x_{+,\dots,+,j_r,\dots,j_s,k_{s+1},\dots,k_n} \det\begin{pmatrix} x_{j_1,\dots,j_{r-1},i_r,j_{r+1},\dots,j_s,+,\dots,+} & x_{+,\dots,+,i_r,i_{r+1},\dots,i_s,+,\dots,+} \\ x_{j_1,\dots,j_{r-1},k_r,j_{r+1},\dots,j_s,+,\dots,+} & x_{+,\dots,+,k_r,i_{r+1},\dots,i_s,+,\dots,+} \end{pmatrix} \end{aligned}$$

which is also evidently equal to

$$(X_{+,\dots,+,i_r,\dots,i_s,+,\dots,+})f = lf \in I(A_{\mathcal{J}}) + I(A_{\mathcal{K}}).$$

This completes the calculation. □

In our example, in which Δ has facets $\{\{1,2\},\{1,3\},\{2,3\}\}$, $J_\Delta = Q_\Delta$. This is a result of the fact that Δ has three facets. The smallest example in which $J_\Delta \neq Q_\Delta$ is when Δ has facets

$$\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}.$$

In this case,

$$X_{1,1,+,+}X_{+,+,1,1} - X_{1,+,1,+}X_{+,1,+,1}$$

is in Q_Δ but not J_Δ.

4.4. The relationship between Q_Δ and I_Δ

We are ready for the main result of the section.

Proposition 6. *If Δ is any simplicial complex, and $\mathcal{L}$ is the intersection of the facets of Δ, then*

$$Q_\Delta \subset \left(I_\Delta : L_{\mathcal{L}}^\infty\right).$$

In particular, $Q_\Delta \subset (I_\Delta : X_{+,\dots,+}^\infty)$.

Proof.
Since all the computations are in $L_{\mathcal{L}}$ we may assume that $\mathcal{L} = \{1, \dots, n\}$. In this case we need to show that $Q_\Delta \subset (I_\Delta : \langle X_{+,\dots,+}\rangle^\infty)$. Moreover, by Lemma 5, $J_\Delta \subset (I_\Delta : X_{+,\dots,+})$, so it suffices to prove that $Q_\Delta \subset (J_\Delta : X_{+,\dots,+}^\infty)$.

Since Q_Δ is generated by binomials, let f be a binomial in Q_Δ. Then

$$f = \prod_{\bar{\imath}_k} X_{\bar{\imath}_k} - \prod_{\bar{\jmath}_k} X_{\bar{\jmath}_k},$$

where $\bar{\imath}_k = (i_{k_1}, \dots, i_{k_n})$ is a sequence of integers and $+$'s. We induct on the total number of the i_{k_m} which are not $+$. As the base case, if $i_{k_m} = +$ for all j, k then $f = X_{+,\dots,+}^n - X_{+,\dots,+}^n = 0$.

Now suppose there is some pair k, m such that $i_{k_m} \neq +$. By reordering we may assume that $i_{1_1} = 1$. Then since $f \in Q_\Delta$ there must be some k such that $j_{k_1} = 1$. We can reorder the right product so that $j_{1_1} = 1$. Then consider $X_{+,\dots,+} f$. By Proposition 4

$$X_{+,\dots,+} f = X_{1,+,\dots,+} f'$$

modulo I_Δ where f' is the same binomial as f except that $i_{1_1} = j_{1_1} = +$. Thus, by induction we have shown that

$$Q_\Delta \in \left(I_\Delta : X_{+,\dots,+}^\infty\right). \qquad \square$$

5. Δ-independence and complete independence

5.1. The Segre embedding, σ_Δ, and P_Δ

In this section we study the relationship between tables which are Δ-independent and tables which are completely independent. It is obvious that any table which is completely independent is also Δ-independent. By Proposition 2, we know that inside the variety of Δ-independent tables is a trivial bundle over the Segre variety. We will establish a close connection between the ideal $K_\Delta + Q_\Delta$ and the defining ideal of the Segre variety. In this section, we assume that Δ is a simplicial complex.

The variety of completely independent tables, or the Segre variety, can be parameterized by

$$\sigma_{\{1,\dots,n\}} : R = \mathbb{K}[x_{i_1,\dots,i_n}] \longrightarrow \mathbb{K}[y_{1,i_1}, \dots, y_{n,i_n}],$$
$$x_{i_1,\dots,i_n} \longmapsto \prod_j y_{j,i_j},$$

as in Section 2.4. This map may be composed with τ_Δ from Section 3.2 to give a map

$$\sigma_\Delta : S_\Delta \longrightarrow \mathbb{K}[y_{1,i_1}, \dots, y_{n,i_n}].$$

Let P_Δ be the kernel of σ_Δ. Thus P_Δ is a prime ideal which defines the variety of Δ-marginal tables which come from a point on the Segre variety.

We have the following commutative diagram:

$$\begin{tikzcd}
& R = \mathbb{K}[x_{i_1,\dots,i_n}] \arrow[dr, "\sigma_{\{1,\dots,n\}}"] & \\
S_\Delta \arrow[ur, "\tau_\Delta"] \arrow[rr, "\sigma_\Delta"] \arrow[dr, "\eta_\Delta"'] & & \mathbb{K}[y_{1,i_1}, \dots, y_{n,i_n}]. \\
& \mathbb{K}[y_{1,i_1}, \dots, y_{n,i_n} \mid 1 \leqslant i_j \leqslant a_i \text{ or } i_j = \bullet] \arrow[ur, "\Sigma_\bullet"'] &
\end{tikzcd}$$

5.2. The main theorem

We are ready for the main theorem, of which Theorem 1 is a corollary. The commutative diagram above summarizes all the main definitions.

Theorem 7. *If Δ is any simplicial complex,*

$$P_\Delta = \operatorname{rad}(K_\Delta + Q_\Delta),$$

where $P_\Delta = \ker \sigma_\Delta$, $K_\Delta = \ker \tau_\Delta$ and $Q_\Delta = \ker \eta_\Delta$.

Proof. First, we need to show that

$$Q_\Delta + K_\Delta \subset P_\Delta.$$

It suffices to show that $\sigma_\Delta(Q_\Delta + K_\Delta) = 0$, which is clear by the definitions.

On the other hand, let B be any point in S_Δ on $V(K_\Delta + Q_\Delta)$. Since it is a point on $V(Q_\Delta)$, it can be represented by $b_{i_1,\dots,i_n} = \prod_j y_{j,i_j}$. Suppose that $\mathcal{J} \in \Delta$ such that $B_\mathcal{J} \neq 0$. Then re-index so that $\mathcal{J} = \{1, \dots, r\}$ and $b_{1,\dots,1,\bullet,\dots,\bullet} \neq 0$. Now take any $j \in \mathcal{J}$ (and re-index so $j = 1$). Since B is a point on $V(K_\Delta)$,

$$\begin{aligned}
y_{1,\bullet} \prod_2^r y_{j,1} \prod_{r+1}^n y_{j,\bullet} = b_{\bullet,1,\dots,1,\bullet,\dots,\bullet} \\
&= \sum_1^{a_1} b_{i,1,\dots,1,\bullet,\dots,\bullet} \\
&= y_{1,+} \prod_2^r y_{j,1} \prod_{r+1}^n y_{j,\bullet}.
\end{aligned}$$

Since $\prod_2^r y_{j,1} \prod_{r+1}^n y_{j,\bullet} \neq 0$, that means that $y_{1,\bullet} = y_{1,+}$. Therefore, if j is any index such that there is a face $j \in \mathcal{J} \in \Delta$ with $B_{\mathcal{J}} \neq 0$ then $y_{j,\bullet} = y_{j,+}$.

Therefore, if for each j there is a $\mathcal{J} \in \Delta$ such that $B_{\mathcal{J}} \neq 0$. Then we can let $z_{j,i_j} = y_{j,i_j}$ and

$$b_{i_1,\dots,i_n} = \prod y_{j,i_j}$$

since $y_{j,\bullet} = y_{j,+}$ for all j. Therefore, B in $V(P_\Delta)$.

On the other hand, suppose that $\mathcal{K}$ is the maximal set such that for any face $\mathcal{J} \in \Delta$, if $\mathcal{J} \cap \mathcal{K} \neq \emptyset$, $B_{\mathcal{J}} = 0$. Re-index so that $\mathcal{K} = \{1, \dots, r\}$. If there is any face $\mathcal{J}$ such that $B_{\mathcal{J}} \neq 0$, $\mathcal{J}$ must be disjoint from $\mathcal{K}$. Re-index again so that face is $\{r+1, \dots, s\}$, and we have

$$\prod_1^r y_{j,\bullet} \cdot \prod_{r+1}^s y_{j,i_j} \cdot \prod_{s+1}^n y_{j,\bullet} \neq 0.$$

Therefore, $y_{j,\bullet} \neq 0$ for any $j \leqslant r$.

If $\mathcal{L} \in \Delta$ such that $\mathcal{L} \cap \mathcal{K} = \emptyset$ then let $\mathcal{L}' = \mathcal{L} \setminus \mathcal{K}$. Since $B_{\mathcal{L}} = 0$, $B_{\mathcal{L}'} = 0$ also. Re-index so that $\mathcal{L}' = \{r+1, \dots, s\}$. Then

$$\prod_1^r y_{j,\bullet} \cdot \prod_{r+1}^s y_{j,i_j} \cdot \prod_{s+1}^n y_{j,\bullet} = 0$$

for all y_{j,i_j}, $r < j \leqslant s$. Therefore, either there is some $r < j \leqslant s$ such that $y_{j,i_j} = 0$ or there is some $j > s$ such that $y_{j,\bullet} = 0$. The former case is impossible since that would mean $B_{\mathcal{J}} = 0$ for any $\mathcal{J} \ni j$ which contradicts the maximality of $\mathcal{K}$. Therefore, for any $\mathcal{L}$ which intersects $\mathcal{K}$, there is some $j \notin \mathcal{K} \cup \mathcal{L}$ such that $y_{j,\bullet} = 0$. Since $j \notin \mathcal{K}$ we know that $y_{j,+} = y_{j,\bullet}$.

Therefore, let

$$\begin{aligned} z_{j,1} &= y_{j,\bullet} \quad \text{for } j \in \mathcal{K}. \\ z_{j,i_j} &= 0 \quad \text{for } j \in \mathcal{K},\ i_j > 1, \\ z_{j,i_j} &= y_{j,i_j} \quad \text{for } j \notin \mathcal{K}. \end{aligned}$$

Notice that $z_{j,+} = y_{j,\bullet}$ for all j. By the previous paragraph, if $\mathcal{J} \cap \mathcal{K} \neq \emptyset$ then $B_{\mathcal{J}} = 0$. Moreover, if $\mathcal{J} \cap \mathcal{K} = \emptyset$ and $b_{i_1,\dots,i_n}$ is a coordinate of $\mathcal{J}$ then

$$\prod z_{j,i_j} = \prod_{j \in \mathcal{J}} y_{j,i_j} \prod_{j \notin \mathcal{J}} y_{j,\bullet} = b_{i_1,\dots,i_n}$$

so $B \in V(P_\Delta)$

We have thus shown that $V(K_\Delta + Q_\Delta) = V(P_\Delta)$, which implies that $\operatorname{rad}(K_\Delta + Q_\Delta) = P_\Delta$ since P_Δ is prime. $\square$

Corollary 8. *P_Δ is the only minimal prime over I_Δ which does not contain $x_{+,\dots,+}$.*

Proof. By Theorem 7, P_Δ is the only minimal prime over $K_\Delta + Q_\Delta$. Therefore, by Proposition 6, P_Δ is the only minimal prime over I_Δ which does not contain $x_{+,\dots,+}$. $\square$

Now we state Theorem 7 in a set-theoretic form, which slightly generalizes Theorem 1.

Corollary 9. *Let $\mathbb{K}$ be any field and B be any table with entries in $\mathbb{K}$ which is Δ-independent. If the sum of the entries in B is not 0, then B can be written as the (entrywise) sum $B^I + B^0$ where B^I is the completely independent table whose $(i_1, \dots, i_n)$ entry is*

$$\prod_j B_{+,\dots,+,i_j,+,\dots,+}$$

and B^0 is a table whose Δ-margins are identically 0.

5.3. Determining which subcollections are independent

In this section we will consider a collection of random variables and show how to determine which subcollections are completely independent. Suppose B is any $a_1 \times \cdots \times a_n$ table, which is the probability distribution for a random variable $X_1 \times \cdots \times X_n$ and we want to know which sets of the random variables are completely independent.

Let B^I be the table whose $i_1, \dots, i_n$ entry is

$$\left(B^I\right)_{i_1,\dots,i_n} = \prod_j B_{+,\dots,+,i_j,+,\dots,+}$$

and let $B^0 = B - B^I$, the entrywise difference of B and B^I.

There is a simplicial complex $\Delta(B)$ such that $\mathcal{J} \in \Delta(B)$ if and only if $(B^0)_{\mathcal{J}} = 0$. Therefore, by Corollary 9, $\Delta(B)$ gives exactly the collection of subsets of $\{X_1, \dots, X_n\}$ which are completely independent.

6. The other minimal primes over $I_\Delta(A)$

6.1. Some technical results

Having established that P_Δ is the only minimal prime over $I_\Delta(A)$ not containing $x_{+,\dots,+}$, it remains to discuss the minimal primes over $I_\Delta(A)$ which do contain $x_{+,\dots,+}$. The following simple, technical result, which explains the interplay between the $L_{\mathcal{L}}$ and $I(A_{\mathcal{K}})$, will be the foundation of the discussion.

Proposition 10. *Suppose that $\mathcal{L}$, $\mathcal{K}$, $\mathcal{J}_1$, $\mathcal{J}_2$ are subsets of $\{1, \dots, n\}$ such that*

$$\mathcal{K} \supset \mathcal{J}_1 \cup \mathcal{J}_2,$$
$$\mathcal{L} \supset \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{K}.$$

Then

$$L_{\mathcal{J}_1} \cdot L_{\mathcal{J}_2} \subset L_{\mathcal{L}} + I(A_{\mathcal{K}}).$$

Proof. It is clear that if $\mathcal{L} \subset \mathcal{L}'$ then $L_{\mathcal{L}} \subset L_{\mathcal{L}'}$ so we may assume that

$$\mathcal{K} \supset \mathcal{L} = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{K}.$$

Since all the calculations will be done in $A_{\mathcal{K}}$, we will assume that $\mathcal{K} = \{1, \dots, n\}$. Then we re-index so that $\mathcal{J}_1 = \{1, \dots, s\}$ and $\mathcal{J}_2 = \{r, \dots, n\}$ so $\mathcal{L} = \{r, \dots, s\}$.

Let $x_{b_1,\dots,b_s,+,\dots,+}$ and $x_{+,\dots,+,c_r,\dots,c_n}$ be arbitrary generators of $L_{\mathcal{J}_1}$ and $L_{\mathcal{J}_2}$, respectively. Now consider the following element of $I(A)$:

$$\det\begin{pmatrix} x_{b_1,\dots,b_{r-1},c_r,\dots,c_n} & x_{b_1,\dots,b_s,+,\dots,+} \\ x_{+,\dots,+,c_r,\dots,c_n} & x_{+,\dots,+,b_r,\dots,b_s,+,\dots,+} \end{pmatrix}.$$

The result is clear since $x_{+,\dots,+,b_r,\dots,b_s,+,\dots,+} \in L_{\mathcal{L}}$. □

Corollary 11. *Suppose that $\mathcal{L} \subsetneq \mathcal{K}$ and Q is a prime ideal containing $L_{\mathcal{L}} + I(A_{\mathcal{K}})$. Then there is some*

$$\mathcal{L} \subset \mathcal{K}' \subset \mathcal{K}$$

with $|\mathcal{K}'| + 1 = |\mathcal{K}|$ such that $L_{\mathcal{K}'} \subset Q$.

Proof. We induct on $|\mathcal{K}| - |\mathcal{L}|$. If $|\mathcal{K}| - |\mathcal{L}| > 1$, we re-index so that $\mathcal{K} = \{1, \dots, s\}$ and $\mathcal{L} = \{1, \dots, r\}$ with $s > r + 1$. Thus we let $\mathcal{J}_1 = \{1, \dots, r + 1\}$ and $\mathcal{J}_2 = \{1, \dots, r, r + 2, \dots, s\}$. Then we can apply Proposition 10, so either $L_{\mathcal{J}_1}$ or $L_{\mathcal{J}_2}$ is in Q. If $L_{\mathcal{J}_1}$ is in Q we are done. If $L_{\mathcal{J}_2}$ is in Q, we are in a smaller case, and thus done by induction. □

Lemma 12. *Let $\Delta = \{\mathcal{J}_1, \dots, \mathcal{J}_m\}$ be any collection subsets of $\{1, \dots, n\}$ and let $\mathcal{K} = \bigcap \mathcal{J}_i$. Suppose that Q is a prime containing $I_\Delta(A) + L_{\mathcal{K}}$. Then for each $\mathcal{J}_i$ there is some $\mathcal{J}_j$ such that Q contains $L_{\mathcal{J}_i \cap \mathcal{J}_j}$.*

Proof. Without loss of generality, let $i = 1$. By Corollary 11, there is a $\mathcal{K}' \supset \mathcal{K}$ such that $|\mathcal{K}'| + 1 = |\mathcal{J}_1|$ and Q contains $L_{\mathcal{K}'}$. Re-index so that $\mathcal{J}_1 = \{1\} \cup \mathcal{K}'$. Since $\mathcal{K}' \supset \mathcal{K}$, there must be at least one $\mathcal{J}_j$ such that $1 \notin \mathcal{J}_j$. Therefore, $\mathcal{J}_i \cap \mathcal{J}_j \subset \mathcal{K}'$, so Q contains $L_{\mathcal{J}_i \cap \mathcal{J}_j}$. □

We now give a lemma which explains the interplay between P_Δ and $x_{+,\dots,+}$.

Lemma 13. *Let $\Delta = \{\mathcal{J}, \mathcal{K}\}$ and $i \in \mathcal{J} \cap \mathcal{K}$. Then*

$$(L_{\mathcal{K} \smallsetminus \{i\}}) \cdot L_{\mathcal{J}} \subset P_\Delta + L_{\mathcal{J} \smallsetminus \{i\}}.$$

Proof. Re-index so that $i = 1$. Let $x_{j_1,\dots,j_n}$ be any generator of $L_{\mathcal{J}}$ and $x_{+,k_2,\dots,k_n}$ be any generator of $L_{\mathcal{K} \smallsetminus \{1\}}$. Consider the following element of $J_\Delta \subset P_\Delta$:

$$\det\begin{pmatrix} x_{j_1,\dots,j_n} & x_{j_1,k_2,\dots,k_n} \\ x_{+,j_2,\dots,j_n} & x_{+,k_1,\dots,k_n} \end{pmatrix}.$$

Since $x_{+,j_2,\dots,j_n} \in L_{\mathcal{J} \smallsetminus \{1\}}$, the result is clear. □

The next proposition uses the previous results in this section to show that any minimal prime over I_Δ is made up of several P_{Δ_i}. The Δ_i have the property that each facet of Δ is in exactly one Δ_i.

Proposition 14. *Let Δ be any simplicial complex and $\mathcal{F}_1, \dots, \mathcal{F}_m$ its facets. If $\mathfrak{a}$ is any minimal prime containing $I_\Delta(A)$, then there is an equivalence relation on the facets of Δ,*

$$\mathcal{F}_i \sim \mathcal{F}_j \quad \Longleftrightarrow \quad L_{\mathcal{F}_i \cap \mathcal{F}_j} \not\subset \mathfrak{a}.$$

This equivalence relation gives a partition of the facets of Δ, $\Delta_1 \amalg \cdots \amalg \Delta_r$ such that $P_{\Delta_i} \subset \mathfrak{a}$ for all i. Moreover, for each i, there is some set $\mathcal{J} \subset \bigcap \Delta_i$ such that $\mathcal{J} \not\subset \mathcal{F}$ for any facet of Δ not in Δ_i, and $\mathfrak{a}$ contains $L_{\mathcal{F} \smallsetminus \{j\}}$ for each $\mathcal{F} \in \Delta_i$ and $j \in \mathcal{J}$.

Proof. It is clear that the relation given is symmetric. Reflexivity relies on the minimality of Q. If Q is any prime containing $I_\Delta + L_{\mathcal{F}_i}$ for any facet $\mathcal{F}_i$, then in T_Δ, Q can be expressed as $Q' + L_{\mathcal{F}_i}$ where Q' is an ideal whose generators are written entirely in terms of the facets $\mathcal{F}_j$, $j \neq i$. From this perspective, it is clear that

$$Q' + \sum_{j \neq i} L_{\mathcal{F}_i \cap \mathcal{F}_j}$$

is also prime, so Q was not minimal.

Transitivity of the relation follows easily from Lemma 12. Suppose that Q contains neither $L_{\mathcal{F}_i \cap \mathcal{F}_j}$ nor $L_{\mathcal{F}_i \cap \mathcal{F}_k}$. Then applying Lemma 12 to the collection $\{\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k\}$, we conclude that Q does not contain $L_{\mathcal{F}_i \cap \mathcal{F}_j \cap \mathcal{F}_k}$. Therefore, it cannot contain $L_{\mathcal{F}_i \cap \mathcal{F}_k}$.

Let Δ_i be any collection of facets such that for $\mathcal{F}_j, \mathcal{F}_k \in \Delta_i$, $L_{\mathcal{F}_j \cap \mathcal{F}_k} \not\subset Q$. Let $\mathcal{K} = \bigcap_{\mathcal{F} \in \Delta_i} \mathcal{F}$. By Lemma 12, $L_\mathcal{K} \not\subset Q$. Therefore, by Proposition 6, $P_{\Delta_i} \subset Q$.

Finally, the last statement is a consequence of the definition of the equivalence relation, Corollary 11 and Lemma 13. □

6.2. *Classification of the other minimal primes*

Next we will show that certain ideals of the kind mentioned in Proposition 14 are actually prime. If Δ is a simplicial complex all of whose facets contain the vertex k, let $\Delta \smallsetminus k$ be the simplicial complex whose facets are $\mathcal{J} \smallsetminus \{k\}$ for each facet $\mathcal{J} \in \Delta$.

Theorem 15. *Let Δ be any simplicial complex on $\{1, \dots, n\}$ and let $\Delta_1 \amalg \cdots \amalg \Delta_r$ be a partition of the facets of Δ.*

For each Δ_i suppose there is a set $\mathcal{K}_i \subset \bigcap \Delta_i$ such that for any facet $\mathcal{J} \in \Delta$ which is not in Δ_i, $\mathcal{K}_i \smallsetminus \mathcal{J}$ is nontrivial. Then

$$\mathfrak{a} = \sum_i P_{\Delta_i} + \sum_i \sum_{k \in \mathcal{K}_i} L_{\Delta_i \smallsetminus k}$$

is a prime ideal.

Any minimal prime over I_Δ has the form of one of these ideals.

Proof. By Proposition 2, we can show this in S_Δ. Notice that if $\mathcal{J}_i \in \Delta_i$ and $\mathcal{J}_j \in \Delta_j$, $i \neq j$, then $\mathfrak{a} \supset L_{\mathcal{J}_i \cap \mathcal{J}_j}$. Therefore, $\mathfrak{a}$ can be expressed as $\mathfrak{a}_1 + \cdots + \mathfrak{a}_r$ where

$$\mathfrak{a}_i = P_{\Delta_i} + \sum_{k \in \mathcal{K}_i} L_{\Delta_i, \hat{k}}$$

is an ideal in S_Δ which is expressed only in terms of the variables in S_{Δ_i}. Therefore,

$$S_\Delta/\mathfrak{a} = S_{\Delta_1}/\mathfrak{a}_1 \otimes \cdots \otimes S_{\Delta_r}/\mathfrak{a}_r.$$

Since $\mathcal{K}_i \subset \Delta_i$, $S_{\Delta_i}/\mathfrak{a}_i$ is an integral domain for each i. This statement is true regardless of the field of definition. Therefore, $S_{\Delta_i}/\mathfrak{a}_i$ remains an integral domain when it is tensored with the algebraic closure of $\mathbb{K}$. Thus the tensor product $S_\Delta/\mathfrak{a}$ is an integral domain, so $\mathfrak{a}$ is prime.

The fact that every minimal prime is of this form is a consequence of Proposition 14. □

6.3. *The case in which Δ is a graph*

Now we will give some special cases of Theorem 15. The first is in the case in which each facet of Δ has two elements. In this case, Δ is a graph.

We need one preliminary definition. For any j, let

$$\Delta(j) = \{\mathcal{J} \in \Delta \mid j \in \mathcal{J}\}.$$

Corollary 16. *Let Δ be any graph. Any minimal prime over I_Δ is either P_Δ or can be expressed as*

$$\sum_{j \in \Gamma} P_{\Delta(j)} + \sum_{j \notin \Gamma} L_{\{j\}}$$

for some vertex cover Γ of Δ.

Proof. This is a direct application of Theorem 15. Since for each facet $\{i, j\}$ of Δ, any prime containing $I_\Delta + \langle x_{+,\dots,+}\rangle$ must contain either $L_{\{i\}}$ or $L_{\{j\}}$, the statement about Γ being a vertex cover follows. □

6.4. *The case in which Δ has two facets and our example*

The second special case we give is when Δ has only two facets.

Corollary 17. *If Δ is a simplicial complex with two facets, $\mathcal{J}_1, \mathcal{J}_2$ then the minimal primes over I_Δ are P_Δ and*

$$I_\Delta + L_{\mathcal{J}_1 \setminus i_1} + L_{\mathcal{J}_2 \setminus i_2},$$

where $i_1 \notin \mathcal{J}_2$ and $i_2 \notin \mathcal{J}_1$.

Finally, we give our running example.

Corollary 18. *Let Δ have facets $\{\{1,2\},\{1,3\},\{2,3\}\}$. Then the minimal primes over I_Δ are*

$$\begin{aligned}
&P_\Delta,\\
&Q_0 = I_\Delta + L_{\{1\}} + L_{\{2\}} + L_{\{3\}},\\
&Q_1 = I_{\{2,3\}} + P_{\{1,2\},\{1,3\}} + L_{\{2\}} + L_{\{3\}},\\
&Q_2 = I_{\{1,3\}} + P_{\{1,2\},\{2,3\}} + L_{\{1\}} + L_{\{3\}},\\
&Q_3 = I_{\{1,2\}} + P_{\{1,3\},\{2,3\}} + L_{\{1\}} + L_{\{2\}}
\end{aligned}$$

unless one of the $a_i = 2$, in which case Q_0 is not minimal.

7. Principal radical systems and tables

7.1. Principal radical systems in general

In this section we will show that if Δ is a simplicial complex with three or fewer facets, $P_\Delta = K_\Delta + Q_\Delta$ is a prime, perfect ideal and I_Δ is radical. For each of these results we will use principal radical systems. We restrict ourselves to three or fewer facets because the arguments we use do not extend to larger simplicial complexes. This is because we implicitly use the fact that $J_\Delta = Q_\Delta$ when Δ has three or fewer facets, and this is not true for larger Δ. However, there may be another principal radical system that can be used to prove the result for all Δ.

The notion of a principal radical system has proved very useful in the study of determinantal ideals. Hochster and Eagon developed it as a method for showing that any ideal of minors of a generic matrix was radical. We follow the presentation Bruns and Vetter [BV88, §12].

The main idea is to prove that an ideal is radical by adding in, one at a time, well-selected elements of the ring until we have an ideal which is obviously radical. We will now cite the theorem as stated in [BV88, §12].

Theorem 19. *Let R be a noetherian ring, and $\mathcal{F}$ a family of ideals in R. Suppose that for every member $I \in \mathcal{F}$ which is not known to be radical, there is some $x \in R$ such that $I + \langle x\rangle \in \mathcal{F}$ and one of the following conditions holds*:

(1) *x is not a zero-divisor modulo* $\operatorname{rad} I$ *and* $\bigcap_1^\infty (I + \langle x^i\rangle)/I = 0$;
(2) *there exists an ideal $J \in \mathcal{F}$, $J \supsetneq I$, such that $xJ \subset I$ and x is not a zero-divisor modulo* $\operatorname{rad} J$.

Then all the ideals $I \in \mathcal{F}$ are radical.

Note that since all of our rings are graded, $\bigcap_1^\infty (I + \langle x^i\rangle)/I = 0$ will automatically be satisfied by the Krull Intersection theorem. We now apply principal radical systems to the ideals P_Δ, starting with the simplest case, when Δ has 1 facet.

7.2. The radicality of $K_\Delta + Q_\Delta$

Lemma 20. *Let A be the generic $a_1 \times \cdots \times a_n$ table and let Γ be any collection of subsets of $\{1,\dots,n\}$. Then $I(A) + L_\Gamma$ is radical.*

Proof. We induct on $(a_1, \dots, a_n)$. The base case is that in which $a_i = 1$ for all i. In this case, the polynomial ring is $\mathbb{K}[x_{1,\dots,1}]$, which is to say it is has only one variable. If L is nonempty, then the ideal L_Γ is generated by $x_{1,\dots,1}$ and if L is empty, the ideal $I(A) + L_\Gamma$ is 0.

For any other $(a_1, \dots, a_n)$, consider the following families of ideals:

$$F_{l_1,\dots,l_n} = I(A) + L_\Gamma + \big\langle x_{i_1,\dots,i_n} \mid (i_1, \dots, i_n) \leqslant_{\text{revlex}} (l_1, \dots, l_n)\big\rangle,$$
$$G_{l_1,\dots,l_n} = I(A) + L_\Gamma + \langle x_{i_1,\dots,i_n} \mid i_j < l_j \text{ for some } j\rangle.$$

$G_{l_1,\dots,l_n}$ is radical by induction if any $l_i > 1$. Of course, $G_{1,\dots,1} = I(A) + L_\Gamma$. On the other hand, consider any $l = (l_1, \dots, l_{r-1})$. Let $s(l)$ be the least l' such that $l' > l$. Let j be the least j such that $l_j \neq a_j$. Then

$$s(l) = (1, \dots, 1, l_j + 1, l_{j+1}, \dots, l_{r-1}).$$

By definition, $F_l + \langle x_{s(l)}\rangle = F_{s(l)}$. Moreover, $G_{s(l)} \supsetneq F_l$ unless $l = (a_1, \dots, a_{n-1}, i)$, in which case $F_l = G_{s(l)}$ and is thus radical.

To show that $x_{s(l)}G_{s(l)} \subset F_l$, let $x_{i_1,\dots,i_n}$ be an arbitrary generator of $G_{s(l)}$ which is not contained in $I(A) + L_\Gamma$. By the definition of $G_{s(l)}$ there is some j such that $i_j < s(l)_j$. By re-indexing, assume $j = 1$ for ease of notation. The following minor is in $I(A)$:

$$\det\begin{pmatrix} x_{s(l)_1,\dots,s(l)_n} & x_{s(l)_1,i_2,\dots,i_n} \\ x_{i_1,s(l)_2,\dots,s(l)_n} & x_{i_1,i_2,\dots,i_n} \end{pmatrix}.$$

Since $(i_1, s(l)_2, \dots, s(l)_n) <_{\text{revlex}} s(l)$, $(i_1, s(l)_2, \dots, s(l)_n) \leqslant_{\text{revlex}} l$. Therefore, the antidiagonal product is in F_l, and since the minor is in $I(A) \subset F_l$, the diagonal product is also in F_l.

All that remains to show, then, is that $x_{s(l)}$ is a nonzero-divisor modulo $\operatorname{rad} G_{s(l)}$. Since $R/G_{s(l)}$ is isomorphic to $R/(I(A) + L_\Gamma)$ for smaller values of the a_i, this part is reduced to showing that $x_{1,\dots,1}$ is a nonzero-divisor modulo $\operatorname{rad}(I(A) + L_\Gamma)$. The minimal primes over $I(A) + L_\Gamma$ are $I(A) + L_{\Gamma'}$ where Γ' is a collection of subsets of $\{1, \dots, n\}$, each of size $(n-1)$ and such that every set in Γ is contained in a set in Γ'. These are prime because $R/(I(A) + L_{\Gamma'})$ is isomorphic to $R/I(A)$, again for smaller values of the a_i. Since $x_{1,\dots,1}$ is not in any of the minimal primes, it is a nonzero-divisor modulo $\operatorname{rad}(I(A) + L_\Gamma)$.

Therefore, we have shown that $\{F_{l_1,\dots,l_n}, G_{l_1,\dots,l_n}\}$ is a principal radical system, so $I(A) + L_\Gamma$ is radical. □

This relatively simple case actually is very similar to the more complicated cases. We will see very similar arguments again.

Proposition 21. *Let Δ be a simplicial complex with two facets, $\mathcal{J}_1, \mathcal{J}_2$ and let $\mathcal{K}$ be a subset of $\mathcal{J}_1 \cup \mathcal{J}_2$. Then the ideal $K_\Delta + Q_\Delta + L_\mathcal{K}$ is radical.*

Proof. We re-index so that $\mathcal{J}_1 = \{1, \dots, s\}$ and $\mathcal{J}_2 = \{r, \dots, n\}$.

If $\mathcal{K}$ contains $\mathcal{J}_1$ or $\mathcal{J}_2$, this reduces to Lemma 20, so we suppose that $\mathcal{K}$ contains neither $\mathcal{J}_1$ nor $\mathcal{J}_2$. The minimal primes over $K_\Delta + Q_\Delta + L_\mathcal{K}$ are all of the form $K_\Delta + Q_\Delta + L_{\Delta,\hat{i}}$ for some $r \leqslant i \leqslant s$ or $K_\Delta + Q_\Delta + L_{\mathcal{J}_1 \setminus i} + L_{\mathcal{J}_2 \setminus j}$ for some $i < r$ and $j > s$. This implies that $x_{1,\dots,1,+,\dots,+}$ is a nonzero-divisor modulo $\operatorname{rad}(K_\Delta + Q_\Delta + L_\mathcal{K})$.

Consider the following families of ideals:

$$F_{l_1,\dots,l_s} = K_\Delta + Q_\Delta + L_{\mathcal{K}} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid (i_1,\dots,i_s) \leqslant_{\text{revlex}} (l_1,\dots,l_s)\rangle,$$
$$G_{l_1,\dots,l_s} = K_\Delta + Q_\Delta + L_{\mathcal{K}} + \langle x_{i_1,\dots,i_n} \mid i_j < l_j \text{ for some } j \leqslant s\rangle.$$

The $G_{l_1,\dots,l_s}$ are defined to allow any of the $i_j = +$, so long as one of the i_j is a number and $i_j < l_j$.

As in the proof of Lemma 20, we can induct on $(a_1,\dots,a_n)$, and thus we can assume that $G_{l_1,\dots,l_s}$ as long as one of the $l_i > 1$. In fact, the entire argument from Lemma 20 is valid. We only need to note that for any l, $x_{s(l)}G_{s(l)} \subset F_l$ and $F_{a_1,\dots,a_s}$ is radical by Lemma 20. □

Theorem 22. *Suppose that Δ is a simplicial complex with no more than 3 facets. Then $K_\Delta + Q_\Delta$ is radical, hence $K_\Delta + Q_\Delta = P_\Delta$.*

Proof. If Δ has two or fewer facets, then Proposition 21 and Lemma 20 apply. Suppose that Δ has facets $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, and re-index so that $\mathcal{J}_1 = \{1,\dots,s\}$.

As in the previous two proofs, consider the following families of ideals:

$$F_{l_1,\dots,l_s} = K_\Delta + Q_\Delta + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid (i_1,\dots,i_s) \leqslant_{\text{revlex}} (l_1,\dots,l_s)\rangle,$$
$$G_{l_1,\dots,l_s} = K_\Delta + Q_\Delta + \langle x_{i_1,\dots,i_n} \mid i_j < l_j \text{ for some } j \leqslant s\rangle.$$

They form a principal radical system for the following reasons. $G_{l_1,\dots,l_s}$ is radical by induction on $(a_1,\dots,a_n)$. $F_{1,\dots,1}$ satisfies condition (1) of Theorem 19 because the radical of $K_\Delta + Q_\Delta$ is prime. For $(1,\dots,1) \leqslant l \leqslant (a_1,\dots,a_s)$, F_l satisfies condition (2) of Theorem 19 because $F_{s(l)} = F_l + \langle x_{s(l),+,\dots,+}\rangle$ and $x_{s(l),+,\dots,+} \cdot G_{s(l)} \subset F_l$ while $G_{s(l)} \supsetneq F_l$. Finally, $F_{a_1,\dots,a_s}$ is radical by Proposition 21.

Therefore, we have shown that $K_\Delta + Q_\Delta$ is prime if Δ has three or fewer faces. □

7.3. *The perfection of P_Δ*

We now use the preceding proofs to establish more about the algebraic structure of P_Δ. In particular, if Δ has three or fewer facets, we can show that it is perfect.

Theorem 23. *If Δ is a simplicial complex with three or fewer facets, then P_Δ is a perfect ideal of grade $1 - n + \sum a_i$.*

Proof. We use Proposition 2 to reduce to showing that P_Δ is perfect in the ring T_Δ. Throughout this proof we will use the same notation as in the previous proof, and treat all ideals as ideals in T_Δ.

The main tool we will use is that if M_1, M_2, and M_3 are R-modules such that

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact and M_2 and M_3 are Cohen–Macaulay of depth d and $d-1$, respectively, then M_1 is a Cohen–Macaulay module of depth d.

As usual we prove the result by induction on $(a_1,\dots,a_n)$ since if all but two of these are 1, the ideal is just the 2×2 minors of a generic matrix, for which this theorem is well known.

We re-index as in the beginning of the proof of Theorem 22, and use its notation. By that proof we know that $F_{1,\dots,1}$ is radical. Any prime over $F_{1,\dots,1}$ contains either $P_\Delta + L(A_{\mathcal{J}_1})$ or of a prime of the form

$$H_k = P_\Delta + \langle x_{i_1,\dots,i_n} \mid i_k = 1\rangle$$

for some $k < s$. Let

$$G_l = \bigcap_{k=1}^{l} H_k.$$

We will show that R/G_s has depth $(\sum a_i) - n$ by induction. R/G_1 is isomorphic to R/P_Δ with a_1 reduced by 1. Therefore, $R/G_1 = R/H_1$ is Cohen–Macaulay of depth

$$1 - n + a_1 - 1 + \sum_2^n a_i = \left(\sum a_i\right) - n.$$

Now suppose that we have shown that R/G_k has depth $(\sum a_i) - n$ for any choice of $a_1, \dots, a_n$. Then there is an exact sequence

$$0 \longrightarrow R/G_{k+1} \longrightarrow R/G_k \oplus R/H_{k+1} \longrightarrow R/(G_k + H_{k+1}) \longrightarrow 0.$$

The last term is isomorphic to R/G_k where a_{k+1} is replaced by $a_{k+1} - 1$. Thus, it has depth $(\sum a_i) - n - 1$ by induction. Both summands of the middle term have depth $(\sum a_i) - n$ by induction. Therefore, R/G_{k+1} has $(\sum a_i) - n$. This implies that R/G_s is Cohen–Macaulay with depth $(\sum a_i) - n$ as claimed.

If there is some index $j \in \mathcal{J}_1$ but $j \notin \mathcal{J}_l$ for any $l > 1$ then the only minimal primes over $F_{1,\dots,1}$ are the H_k. Since $F_{1,\dots,1}$ is radical, we know that $F_{1,\dots,1} = G_r$. Thus the previous paragraph implies that $R/F_{1,\dots,1}$ is Cohen–Macaulay of depth $(\sum a_i) - n$, and since

$$F_{1,\dots,1} = P_\Delta + \langle x_{1,\dots,1,+,\dots,+}\rangle$$

and since P_Δ is prime, $x_{1,\dots,1,+,\dots,+}$ is a nonzero-divisor modulo it. Thus R/P_Δ is Cohen–Macaulay of depth $1 - n + \sum a_i$. Note that if Δ has two facets (or one), then since neither facet can contain the other, this paragraph implies the theorem for P_Δ.

On the other hand, suppose that there is no $j \in \mathcal{J}_1$ such that $j \notin \mathcal{J}_l$ for any $l \neq 1$. This implies that Δ has three facets, $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$. We may assume that the condition holds for $\mathcal{J}_2$, $\mathcal{J}_3$ as well, so for each $j \in \{1, \dots, n\}$, j is an element of two of $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$. Therefore, $\mathcal{J}_1$ must contain the symmetric difference of $\mathcal{J}_2$ and $\mathcal{J}_3$, $(\mathcal{J}_2 \cup \mathcal{J}_3) \setminus (\mathcal{J}_2 \cap \mathcal{J}_3)$. Thus the minimal primes over $P_\Delta + L_{\mathcal{J}_1}$ are

$$D_i = P_\Delta + L_{\mathcal{J}_1} + L_{\Delta,\hat{i}}$$

for each i in $(\mathcal{J}_2 \cap \mathcal{J}_3) \setminus \mathcal{J}_1$. The D_i are prime because if $i \in (\mathcal{J}_2 \cap \mathcal{J}_3) \setminus \mathcal{J}_1$, R/D_i is isomorphic to $R/(P_{\mathcal{J}_2,\mathcal{J}_3})$ with a_i reduced by 1. Thus, these prime ideals are also perfect of grade $(\sum a_i) - n$ by induction. Our next goal is to prove that their intersection is also perfect.

Re-index so that

$$(\mathcal{J}_2 \cap \mathcal{J}_3) \smallsetminus \mathcal{J}_1 = \{1, \ldots, m\}$$

and let

$$E_l = \bigcap_{i=1}^{l} D_i.$$

Suppose that E_k is perfect of grade $(\sum a_i) - n$. Then we have an exact sequence

$$0 \longrightarrow R/E_{k+1} \longrightarrow R/E_k \oplus R/D_{k+1} \longrightarrow R/(E_k + D_{k+1}) \longrightarrow 0.$$

We know that R/E_k and R/D_{k+1} are both Cohen–Macaulay of depth $(\sum a_i) - n$, and since $R/(E_k + D_{k+1}) \cong (R/D_{k+1})/E_k$ which is isomorphic to R/E_k with a_{k+1} decreased by 1, $R/(E_k + D_{k+1})$ is Cohen–Macaulay of depth $(\sum a_i) - n - 1$. Therefore, we know that R/E_{k+1} is Cohen–Macaulay of depth $(\sum a_i) - n$. Therefore, by induction, $(P_\Delta + L_{\mathcal{J}_1})$ is perfect of grade $(\sum a_i) - n$.

Finally, we need to show that

$$F_{1,\ldots,1} = (P_\Delta + L_{\mathcal{J}_1}) \cap G_s,$$

is also perfect of grade $(\sum a_i) - n$, where $G_s = \bigcap H_k$ is defined as above. This can be established in exactly the same way as the perfection of G_s and $P_\Delta + L_{\mathcal{J}_1}$ were. Let $C_k = (P_\Delta + L_{\mathcal{J}_1}) \cap G_k$, where $G_0 = \langle 1 \rangle$. We have already established that $P_\Delta + L_{\mathcal{J}_1}$ is perfect of grade $(\sum a_i) - n$, so suppose that C_k is perfect. We have $C_{k+1} = C_k \cap H_{k+1}$ and thus an exact sequence

$$0 \longrightarrow R/C_{k+1} \longrightarrow R/C_k \oplus R/H_{k+1} \longrightarrow R/(C_k + H_{k+1}) \longrightarrow 0.$$

Like the previous proofs, R/C_k and R/H_{k+1} we already know to be Cohen–Macaulay of depth $(\sum a_i) - n$, and $R/(C_k + H_{k+1}) \cong (R/H_{k+1})/C_k$, which is isomorphic to R/C_k for a_{k+1} decreased by 1, so it is Cohen–Macaulay of depth $(\sum a_i) - n - 1$. Therefore, R/C_{k+1} is Cohen–Macaulay of depth $(\sum a_i) - n$, so by induction, $R/C_s = R/F_{1,\ldots,1}$ is Cohen–Macaulay of depth $(\sum a_i) - n$.

Since $F_{1,\ldots,1} = P_\Delta + \langle x_{1,\ldots,1,+,\ldots,+} \rangle$ and $x_{1,\ldots,1,+,\ldots,+}$ is a nonzero-divisor modulo P_Δ, this implies that P_Δ is perfect of grade $1 - n + \sum a_i$. □

7.4. *The radicality of I_Δ*

We now move from the prime ideal P_Δ to the original ideal I_Δ.

Proposition 24. *Let Δ be a simplicial complex with two facets, $\mathcal{J}_1, \mathcal{J}_2$ and let $\mathcal{K}$ be a subset of $\mathcal{J}_1 \cup \mathcal{J}_2$. Then the ideal $I_\Delta + L_\mathcal{K}$ is radical.*

Proof. We re-index so that $\mathcal{J}_1 = \{1, \ldots, s\}$ and $\mathcal{J}_2 = \{r, \ldots, n\}$.

If $\mathcal{K}$ contains $\mathcal{J}_1$ or $\mathcal{J}_2$, this reduces to Lemma 20, so we suppose that $\mathcal{K}$ contains neither $\mathcal{J}_1$ nor $\mathcal{J}_2$. We will prove the result by principal radical systems. Define

$$F_{l_1,\dots,l_s} = I_\Delta + L_\mathcal{K} + \bigl\langle x_{i_1,\dots,i_s,+,\dots,+} \mid (i_1,\dots,i_s) \leqslant_{\text{revlex}} (l_1,\dots,l_s)\bigr\rangle,$$
$$G_{l_1,\dots,l_s} = I_\Delta + L_\mathcal{K} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid i_j < l_j \text{ for some } j \leqslant s\rangle$$

and let $\mathcal{F} = \{F_{l_1,\dots,l_s} + G_{k_1,\dots,k_s}\}$, the set of all sums of F's and G's. We claim that $\mathcal{F}$ is a principal radical system.

If $l = (l_1,\dots,l_s)$ is any sequence, let $s(l)$ be the least l' such that $l' >_{\text{revlex}} l$. If j is the least j such that $l_j \neq a_j$ then

$$s(l) = (1,\dots,1,l_j+1,l_{j+1},\dots,l_s).$$

By definition, $F_l + \langle x_{s(l)}\rangle = F_{s(l)}$. Therefore, $F_l + G_k + \langle x_{s(l)}\rangle = F_{s(l)} + G_k$.

The following lemma will be the key to showing that $\mathcal{F}$ is a principal radical system.

Lemma 25. *$x_{l_1,\dots,l_s,+,\dots,+}$ is a nonzero-divisor modulo* $\operatorname{rad} G_{l_1,\dots,l_s}$.

Proof. We will do this by computing the minimal primes over $G_{l_1,\dots,l_s}$, and showing that $x_{l_1,\dots,l_s,+,\dots,+}$ is not in any of them. Let $l' = (1,\dots,1,l_r,\dots,l_s)$. Then $R/G_l \cong R/G_{l'}$ where the latter ring has the values of a_i decreased by $l_i - 1$ for each $i < r$. Therefore, we can assume that $l_i = 1$ for all $i < r$.

Suppose that $l_i > 1$ for some $i \geqslant r$, without loss of generality, assume $i = s$. Then for each $j_r,\dots,j_{s-1}$ and any $j_s < l_s$

$$x_{+,\dots,+,j_r,\dots,j_s,+,\dots,+} \in G_l.$$

Since $I(A_{\mathcal{J}_2}) \subset G_l$, any prime containing G_l must either contain $L_{\{r,\dots,s\}}$ or $x_{+,\dots,+,j_r,\dots,j_s,j_{s+1},\dots,j_n}$ for all $j_s < l_s$.

Let $H_{l_1,\dots,l_s} = G_{l_1,\dots,l_s} + \langle x_{+,\dots,+,i_r,\dots,i_n} \mid i_j < l_j \text{ for some } r \leqslant j \leqslant s\rangle$. The previous paragraph implies that any prime containing $G_{l_1,\dots,l_s}$ either contains $L_{\{r,\dots,s\}}$ or contains $H_{l_1,\dots,l_s}$. Since R/H_l is isomorphic to $R/(I_\Delta + L_\mathcal{K})$ with a_i decreased by $l_i - 1$ for each i. Therefore, to show that $x_{l_1,\dots,l_s,+,\dots,+}$ is not in a minimal prime over $H_{l_1,\dots,l_s}$ is the same as showing that $x_{1,\dots,1,+,\dots,+}$ is not in a minimal prime over $I_\Delta + L_\mathcal{K}$.

The minimal primes over $I_\Delta + L_\mathcal{K}$ are either $P_\Delta + L_{\Delta,\hat{i}}$ for some $i \in (\mathcal{J}_1 \cap \mathcal{J}_2) \setminus \mathcal{K}$ or $I_\Delta + L_{\mathcal{J}_1 \setminus i_1} + L_{\mathcal{J}_2 \setminus i_2}$ where $i_1, i_2 \notin \mathcal{K}$. It is clear that $x_{1,\dots,1,+,\dots,+}$ is not in any of these ideals.

On the other hand, we must show that $x_{1,\dots,1,+,\dots,+}$ is not in any of the minimal primes over $G_l + L_{\{r,\dots,s\}}$. Because this ideal contains $L_{\{r,\dots,s\}}$, it can be expressed, in S_Δ as $I_1 + I_2$ where $I_1 \subset \mathbb{K}[X_{i_1,\dots,i_s,+,\dots,+}]$ and $I_2 \subset \mathbb{K}[X_{+,\dots,+,i_r,\dots,i_n}]$. Therefore, we need only consider the minimal primes over

$$I(A_{\mathcal{J}_1}) + L_{\mathcal{K}\cap\mathcal{J}_1} + L_{\{r,\dots,s\}} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid i_j < l_j \text{ for some } j \leqslant s\rangle.$$

The effect of the last summand is only to reduce each a_i by $l_i - 1$, so we may assume that this term is 0. Then we are left with $I(A_{\mathcal{J}_1}) + L_{\mathcal{K}\cap\mathcal{J}_1} + L_{\{r,\dots,s\}}$, whose minimal primes are contained in $I(A_{\mathcal{J}_1}) + L_{\mathcal{J}_1\setminus i} + L_{\mathcal{J}_1\setminus j}$ where $i \notin \mathcal{K}$ and $j < r$. Thus $x_{1,\dots,1,+,\dots,+}$ is not in any minimal prime over $G_l + L_{\{r,\dots,s\}}$.

This completes the proof of the lemma, so $x_{l,+,\dots,+}$ is a nonzero-divisor modulo G_l. $\square$

Since $G_{s(l)} \supsetneq F_l$ for any l, $F_l + G_k = G_k$ whenever $k >_{\text{revlex}} l$. Moreover,

$$G_l + F_l = G_l + \langle x_{l_1,\dots,l_s,+,\dots,+} \rangle$$

so by our lemma, if $k >_{\text{revlex}} l$ $F_l + G_k$ satisfies condition (1) of Theorem 19.

On the other hand, if $k \leqslant l < (a_1, \dots, a_n)$, recall that

$$G_k + F_{s(l)} = G_k + F_l + \langle x_{s(l),+,\dots,+} \rangle.$$

$G_{s(l)} \supsetneq G_k + F_l$, and $x_{s(l),+,\dots,+} G_{s(l)} \subset G_k + F_l$. Thus, since $x_{s(l),+,\dots,+}$ is a nonzero-divisor modulo $\operatorname{rad} G_{s(l)}$ by the lemma, $G_k + F_l$ satisfies condition (2) of Theorem 19.

Finally,

$$F_{a_1,\dots,a_n} = I_\Delta + L_{\mathcal{K}} + L_{\mathcal{J}_1} = I(A_{\mathcal{J}_2}) + L_{\mathcal{K}} + L_{\mathcal{J}_1}$$

which is radical by Lemma 20.

Therefore, $\mathcal{F}$ is a principal radical system and $I_\Delta + L_{\mathcal{K}}$ is radical. □

Theorem 26. *If Δ has three or fewer facets then I_Δ is a radical ideal.*

Proof. If Δ has one or two facets, this has been proven in Lemma 20 and Proposition 24, so we may assume that Δ has three facets.

This proof is very similar to that of Proposition 24. Re-index so that $\mathcal{J}_1 = \{1, \dots, s\}$, and let

$$F_{l_1,\dots,l_s} = I_\Delta + \big\langle x_{i_1,\dots,i_s,+,\dots,+} \mid (i_1, \dots, i_s) \leqslant_{\text{revlex}} (l_1, \dots, l_s) \big\rangle,$$
$$G_{l_1,\dots,l_s} = I_\Delta + L_{\mathcal{K}} + \langle x_{i_1,\dots,i_s,+,\dots,+} \mid i_j < l_j \text{ for some } j \leqslant s \rangle.$$

Define $\mathcal{F} = \{F_{l_1,\dots,l_s} + G_{k_1,\dots,k_s}\}$, the set of all sums of F's and G's. We claim that $\mathcal{F}$ is a principal radical system.

We will prove below that $x_{l,+,\dots,+}$ is a nonzero-divisor modulo $\operatorname{rad} G_l$ and now we show how that will imply the theorem.

As in the previous proof, $F_l + G_k + \langle x_{s(l)} \rangle = F_{s(l)} + G_k$, so if $k >_{\text{revlex}} l$ then $F_l + G_k$ satisfies condition (1) of Theorem 19. Moreover, if $k \leqslant l < (a_1, \dots, a_n)$,

$$G_k + F_{s(l)} = G_k + F_l + \langle x_{s(l),+,\dots,+} \rangle.$$

$G_{s(l)} \supsetneq G_k + F_l$, and $x_{s(l),+,\dots,+} G_{s(l)} \subset G_k + F_l$. Thus, since $x_{s(l),+,\dots,+}$ is a nonzero-divisor modulo $\operatorname{rad} G_{s(l)}$ as we will show below, $G_k + F_l$ satisfies condition (2) of Theorem 19.

Finally, $F_{a_1,\dots,a_s}$ is radical because it is $I_{\mathcal{J}_2,\mathcal{J}_3} + L_{\mathcal{J}_1}$ which is radical by Proposition 24.

Therefore, the theorem will be completed with the proof of the following lemma.

Lemma 27. *$x_{l_1,\dots,l_s,+,\dots,+}$ is a nonzero-divisor modulo* $\operatorname{rad} G_{l_1,\dots,l_s}$.

Proof. Again, we prove this by computing the minimal primes over $G_{l_1,\dots,l_s}$ and showing that $x_{l_1,\dots,l_s,+,\dots,+}$ is not in any of them.

Suppose Q is a minimal prime over $G_{l_1,\dots,l_s}$ which contains $L_{\mathcal{J}_1 \cap \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3}$. Since $G_{l_1,\dots,l_s} + L_{\mathcal{J}_1 \cap \mathcal{J}_2} + L_{\mathcal{J}_1 \cap \mathcal{J}_3}$ can be expressed as $I_1 + I_2$ where $I_1 \subset \mathbb{K}[X_{i_1,\dots,i_s,+,\dots,+}] = S_{\mathcal{J}_1}$

and the generators I_2 have none of those variables in them, we can show that $x_{l_1,\dots,l_s,+,\dots,+} \notin Q$ by showing that it is not in any minimal prime over $I(A_{\mathcal{J}_1}) + L_{\mathcal{J}_1\cap\mathcal{J}_2} + L_{\mathcal{J}_1\cap\mathcal{J}_3}$, which is clear.

The second case is when Q is a minimal prime over $G_{l_1,\dots,l_s}$ which contains $L_{\mathcal{J}_1\cap\mathcal{J}_3}$ but not $L_{\mathcal{J}_1\cap\mathcal{J}_2}$. Then it must also contain $L_{\mathcal{J}_2\cap\mathcal{J}_3}$ and $P_{\{\mathcal{J}_1,\mathcal{J}_2\}}$ by Proposition 14. As in the previous paragraph,

$$G_{l_1,\dots,l_s} + P_{\mathcal{J}_1,\mathcal{J}_2} + L_{\mathcal{J}_1\cap\mathcal{J}_3} + L_{\mathcal{J}_2\cap\mathcal{J}_3}$$

can be expressed as $I_1 + I_2$ where $I_1 \subset S_{\{\mathcal{J}_1,\mathcal{J}_2\}}$ and the generators I_2 have none of those variables in them. Thus we can show that $x_{l_1,\dots,l_s,+,\dots,+} \notin Q$ by showing that it is not in any minimal prime over $P_{\{\mathcal{J}_1,\mathcal{J}_2\}} + G_{l_1,\dots,l_s} + L_{\mathcal{J}_3}$. Since this case was covered in Lemma 25, we refer to that proof.

The final case is that in which Q is a minimal prime over $G_{l_1,\dots,l_s}$ and contains neither $L_{\mathcal{J}_1\cap\mathcal{J}_2}$ nor $L_{\mathcal{J}_1\cap\mathcal{J}_3}$. Thus, it cannot contain $L_{\mathcal{J}_1\cap\mathcal{J}_3}$ either and must contain P_Δ. Moreover, if $i \in \mathcal{J}_1 \cap \mathcal{J}_2$ and $l_i > 1$, then we can re-index so $i = s$ and $\mathcal{J}_2 = \{r, \dots, n\}$. As in the proof of Lemma 25, $G_{l_1,\dots,l_s}$ contains $x_{+,\dots,+,j_r,\dots,j_s,+,\dots,+}$ for all $j_r, \dots, j_{s-1}$ and any $j_s < l_s$. Since Q does not contain $L_{\mathcal{J}_1\cap\mathcal{J}_2}$ it must be the case that $x_{+,\dots,j_r,\dots,j_n} \in Q$ as long as $j_s < l_s$. Therefore, Q must contain

$$H_{l_1,\dots,l_s} = P_\Delta + \langle x_{i_1,\dots,i_n} \mid i_j < l_j \text{ for some } j \leqslant s\rangle.$$

(Notice that this ideal was defined as $G_{l_1,\dots,l_s}$ in the proof of Theorem 22.) Since $R/H_{l_1,\dots,l_s}$ is isomorphic to R/P_Δ where each a_i has been reduced by $l_i - 1$. Therefore, $H_{l_1,\dots,l_s}$ is prime and $x_{l_1,\dots,l_s,+,\dots,+}$ is not in it.

We have shown that $x_{l_1,\dots,l_s,+,\dots,+}$ is not in any minimal prime over $G_{l_1,\dots,l_s}$ and hence is a nonzero-divisor modulo its radical. □

This completes the proof that I_Δ is radical if Δ has fewer than three facets. □

8. Conjectures, examples, and notes on computation

8.1. An example in which I_Δ is not radical

It is not true that for any Δ, $I_\Delta(A)$ is radical. Any time $Q_\Delta \neq J_\Delta$, we know that

$$x_{+,\dots,+} \cdot Q_\Delta \subset \operatorname{rad}(I_\Delta);$$

however, this will not always be contained in I_Δ. For example, when

$$\Delta = \bigl\{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\bigr\}$$

it can be shown computationally that

$$x_{+,+,+,+}(x_{1,1,+,+}x_{+,+,1,1} - x_{1,+,1,+}x_{+,1,+,1}) \notin I_\Delta.$$

In this case, it turns out that the primary decomposition is still accessible, and we give a computation of it in the case $a_i = 2$. Let

$$Q_1 = P_{\{1,2\},\{2,4\}} + P_{\{1,3\},\{3,4\}} + \langle x_{i,+,+,+} \rangle + \langle x_{+,+,+,l} \rangle,$$
$$Q_2 = P_{\{1,2\},\{1,3\}} + P_{\{2,4\},\{3,4\}} + \langle x_{+,j,+,+} \rangle + \langle x_{+,+,k,+} \rangle,$$
$$Q_3 = I_\Delta + \langle x^2_{i,+,+,+}, x^2_{+,j,+,+}, x^2_{+,+,k,+}, x^2_{+,+,+,l}, x^2_{+,+,+,+} \rangle.$$

It can be verified using Macaulay 2 [GS] that

$$I_\Delta = P_\Delta \cap Q_1 \cap Q_2 \cap Q_3.$$

8.2. Two conjectures

Section 7 has exclusively dealt with the case in which Δ has three or fewer facets. We offer the following conjectures which have been borne out in all the examples which our computers have been able to accomplish.

Conjecture 28. *If Δ is any simplicial complex,*

$$K_\Delta + Q_\Delta = P_\Delta,$$

which is a prime and perfect ideal of grade $1 - n + \sum a_i$.

We have proven this result in the case in which Δ has three or fewer facets. Moreover, we have shown that $\mathrm{rad}(K_\Delta + Q_\Delta)$ is prime in Theorem 7, which should be seen as good evidence for the primality of the ideal.

The second conjecture deals with the radicality of I_Δ.

Conjecture 29. *Let Δ be any simplicial complex. I_Δ is a radical ideal if and only if $Q_\Delta = J_\Delta$.*

8.3. Notes on computation

Finally, we discuss the computational aspects of experimenting with these families of ideals. All computations should be done in T_Δ because it reduces the number of variables in the polynomial ring. This reduction is especially noticeable when some of the $a_i > 2$. A side benefit is that the relations are usually easier to decipher when they are expressed in the variables of S_Δ. In fact, these were the reasons that first attracted me to change variables.

I used Macaulay 2 for my calculations and all of the following pertain to it. If $a_i = 2$ for all i, then we are in a position to decompose I_Δ when Δ has fewer than four vertices ($n \leqslant 4$), and can do some cases with five or six vertices. After that point, the only Δ's for which I_Δ can be decomposed have two facets.

When $a_i = 2$ it is also possible to compute a free resolution for P_Δ for some cases until $n = 5$. After that, the problem again becomes insurmountable.

If we allow $a_i > 2$, both problems become very difficult very fast. The decomposition can be checked by using Theorem 15, and intersecting the minimal primes. Computing a free resolution also becomes computationally impossible very fast. For the simplest Δ with three facets, $\{\{1,2\},\{1,3\},\{2,3\}\}$, a free resolution cannot be computed when $a_i = 3$ for each i.

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Further reading

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