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On the finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$

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Abstract

We classify up to equivalence all finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ whose restriction to the commutator subgroup is diagonalizable.

Keywords: Projective modular group; Irreducible representations; Braid group; Commutator subgroup

1. Introduction

In this paper, we study finite-dimensional representations of the projective modular group $PSL_2(\mathbb{Z})$. This group is of fundamental importance in many fields including number theory, hyperbolic geometry, topological quantum field theory, and knot theory. Our main result completely classifies up to equivalence all finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ whose restrictions to the commutator subgroup are diagonalizable. These representations are of dimension 1, 2, 3, and 6.

Recall that $PSL_2(\mathbb{Z})$ has the presentation $\langle x, y \mid x^2 = y^3 = 1\rangle$. Its commutator subgroup has index 6 and is generated by the elements $xyxy^2, xy^2xy$; see [3]. Since the index is 6, it follows from standard Clifford Theory that the dimensions of the irreducible representations we

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are studying divide 6; see, for example, [2, 2.7] for further explanation. This fact is used in our analysis.

A complete classification of the finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ with dimension $\leqslant 5$ follows from the work of Tuba and Wenzl [4] on representations of B_3. Let $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be Artin's braid group. Let $Z = \langle (\sigma_1\sigma_2)^3 \rangle$ denote the center of B_3. For appropriate choices of $A, B \in GL_n(k)$, $\sigma_1 \mapsto A$ and $\sigma_2 \mapsto B$ define a representation of B_3. If $(AB)^3 = I$, then this also defines a representation of $B_3/Z \simeq PSL_2(\mathbb{Z})$. From the main theorem of Tuba and Wenzl [4], it follows that for $d \leqslant 5$, any d-dimensional irreducible representation of B_3 is uniquely determined by the eigenvalues of A and the scalar by which $(AB)^3$ acts. Note that for any representation of B_3 induced from a representation of $PSL_2(\mathbb{Z})$, this scalar is 1. For each of the representations of B_3 induced by our 1-dimensional irreducible representations of $PSL_2(\mathbb{Z})$, the eigenvalue of A is a sixth root of unity. If the characteristic of k is not 3, then each of our 2-dimensional irreducible representations is described by the eigenvalues $\lambda, -\lambda$ where $\lambda^3 = 1$. If the characteristic of k is not 2, then the two 3-dimensional irreducible representations we found are characterized by having eigenvalues $\lambda_1, \lambda_2, \lambda_3$ or η_1, η_2, η_3 where λ_i are the roots of $x^3 - 1 = 0$ and η_i are the roots of $x^3 + 1 = 0$. Since each of our 6-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ induce a representation of B_3 where the resulting matrix A has eigenvalues λ_i where λ_i are the roots of $x^6 - 1 = 0$, we conclude that the 6-dimensional irreducible representations of B_3 are not uniquely determined by the eigenvalues of A and the scalar by which $(AB)^3$ acts.

Adriaenssens and Le Bruyn [1] have recently analyzed specific families of representations of $PSL_2(\mathbb{Z})$ from the point of view of noncommutative algebraic geometry. In Proposition 5 of their paper, Adriaenssens and Le Bruyn describe a family of irreducible representations of B_3 which contains any sufficiently general irreducible representation of B_3. Indeed, included in this family are the representations of B_3 induced by the 1-dimensional, 2-dimensional, and 3-dimensional representations of $PSL_2(\mathbb{Z})$ that are characterized in this paper. Determining whether the representations of B_3 induced by our 6-dimensional representations are in this general family is computationally intensive and unknown. Subsequently constructed in [1] are two subfamilies of the general family of representations of B_3 described above. These are induced by representations of particular quotients of the path algebra of the quiver Q associated with the formally smooth algebra $\mathbb{C}PSL_2(\mathbb{Z})$. The representations of B_3 induced by the 1-dimensional and 3-dimensional representations (but not by the 2-dimensional representations) of $PSL_2(\mathbb{Z})$ detailed in this paper are included in the first subfamily. None of the representations of B_3 induced by the 1-dimensional, 2-dimensional, or 3-dimensional representations of $PSL_2(\mathbb{Z})$ studied in our paper are in the second subfamily. Again, determining whether the 6-dimensional representations of B_3 induced by our 6-dimensional representations are in these subfamilies requires further work.

The inspiration of this paper comes from the theory of highest weight modules. The finite-dimensional irreducible representations of a complex semisimple Lie algebra are diagonalizable over its Cartan subalgebra. In our study below, we view the commutator of $PSL_2(\mathbb{Z})$ as playing a role analogous to that of the Cartan subalgebra.

Another way to view our work is as follows: Set G' as the commutator subgroup of G and G'' as the second commutator subgroup of G. It is well known (see [3]) that G' is a free group in the two generators $xyxy^2$ and xy^2xy, so G'/G'' is a free abelian group on two generators. In particular, the irreducible representations of G'/G'' are all one-dimensional. Therefore, by Clifford Theory, the restrictions to G'/G'' of the irreducible finite-dimensional representations of G/G'' must be diagonalizable. Conversely, every finite-dimensional irreducible representation of G/G'' lifts to a representation of G whose restriction to G' naturally factors through G/G''.

We see that our results amount to a complete classification, up to equivalence, of the irreducible finite-dimensional representations of G/G''. Moreover, since G/G'' is abelian-by-finite, the irreducible representations are all finite-dimensional.

Our approach is mostly elementary, relying on basic linear algebraic computations and case-by-case analysis. Toward the end of the paper, extensive Maple calculations are used to determine the 6-dimensional representations; an appendix of the Maple code used is included.

Our main results are stated in Section 4. Sections 5 and 6 are devoted to the proofs. Preliminary results and notation are given in Sections 2 and 3.

2. Definitions and notation

We begin with some relevant definitions. Let k denote an algebraically closed field and $M_n(k)$ denote the set of all $n \times n$ matrices with entries in k. Let $GL_n(k)$ denote the set of all invertible elements of $M_n(k)$.

Definition 2.1. We will say the ordered m-tuple $(A_1, A_2, \dots, A_m)$, $A_i \in M_n(k)$ for $i = 1, 2, \dots, m$, is *irreducible* if every element of $M_n(k)$ can be written as a k-linear combination of products in the A_i's, $i = 1, 2, \dots, m$.

Also, if $X_i, X_i' \in M_n(k)$ for $i = 1, 2, \dots, m$, then $(X_1', X_2', \dots, X_m')$ is *equivalent* to $(X_1, X_2, \dots, X_m)$, denoted $(X_1', X_2', \dots, X_m') \approx (X_1, X_2, \dots, X_m)$, if there exists $Q \in GL_n(k)$ such that $X_i' = QX_iQ^{-1}$ for all $i = 1, 2, \dots, m$.

Notation 2.2. The following notation will remain in effect for the entire paper: Let $G = PSL_2(\mathbb{Z})$, which we identify with $\langle x, y \mid x^2 = y^3 = 1\rangle$. Let $\rho : G \to GL_n(k)$ be an irreducible representation of G. Set $X = \rho(x)$, $Y = \rho(y)$, $\Lambda = XYXY^2$, and $\Gamma = XY^2XY$. It then follows that $X^2 = Y^3 = I$, and (X, Y) is irreducible. Denote the entry in the ith row and jth column of a matrix X by $X_{i,j}$. We will use $\langle f_1, \dots, f_t\rangle$ to denote the ideal (in a given ring) generated by $f_1, f_2, \dots, f_t$.

3. Preliminary results

Remark 3.1. Since k is algebraically closed, all irreducible solutions to $XY = YX$ in $M_n(k)$ are one-dimensional.

Lemma 3.2. *All irreducible representations of $PSL_2(\mathbb{Z})$ with $\Lambda = \Gamma$, or Λ or Γ a scalar matrix are one-dimensional.*

Proof. If $\Lambda = \Gamma$, then $XY = Y^2X\Lambda Y^2 = Y^2X\Gamma Y^2 = YX$. For an arbitrary constant c, computation with $\Lambda = cI$ yields

$$XY = \left(Y^2X\right)^2(XY)\Lambda Y^2 = \left(Y^2X\right)^2\Lambda(XY)Y^2 = YX.$$

Similarly, computation with $\Gamma = cI$ yields

$$XY = \left(Y^2X\right)(XY)\Gamma\left(Y^2XY^2\right) = \left(Y^2X\right)\Gamma(XY)\left(Y^2XY^2\right) = YX.$$

Remark 3.1 thus concludes this proof. □

Remark 3.3. Assume Λ and Γ are $n \times n$ diagonal matrices, where $\Lambda_{i,i} =: \lambda_i$ and $\Gamma_{i,i} =: \gamma_i$ for $i = 1, 2, \ldots, n$. We observe the following properties of Λ and Γ:

- $\Lambda X \Lambda = X$, $\Gamma X \Gamma = X$, $\Lambda Y \Gamma = Y$, $\Gamma Y^2 \Lambda = Y^2$.
- $\Lambda \Gamma = \Gamma \Lambda$, so $(XY)^6 = (YX)^6 = I$.
- Since Y has at least one nonzero entry per row, Λ and Γ are conjugate, and $\Lambda Y \Gamma = Y$, then $\frac{1}{\lambda_i} = \gamma_j = \lambda_k$ for each $i = 1, 2, \ldots, n$, and some $j, k \in \{1, 2, \ldots, n\}$. Note that Λ, $\Gamma \in GL_n(k)$, so $\lambda_i, \gamma_i \neq 0$ for each $i = 1, 2, \ldots, n$.

The proof of the following lemma is routine and omitted.

Lemma 3.4. *Consider the following properties*:

(1) $X, Y \in GL_n(k)$ *have exactly one nonzero entry per row and column.*
(2) (X, Y) *is irreducible.*
(3) $X^2 = Y^3 = I$.
(4) Λ, Γ *are diagonal matrices.*

Then (X, Y) *satisfies the above properties if and only if* (PXP^{-1}, PYP^{-1}) *satisfies the above properties, where* P *is a nonsingular weighted permutation matrix* (*i.e.* P *has exactly one nonzero entry per row and column*).

4. Main results

Let ρ be an irreducible representation of $PSL_2(\mathbb{Z})$ which maps the commutator subgroup of $PSL_2(\mathbb{Z})$ to diagonal matrices in $GL_n(k)$. Since the index of the commutator subgroup is 6, it follows from standard Clifford Theory (see, e.g., [2, 2.7]) that the dimension of ρ divides 6. We thus analyze the cases when $n = 1, 2, 3$, and 6.

Theorem 4.1. *Let k be an algebraically closed field, and let ζ be a primitive cube root of unity if k is not of characteristic* 3. *Let $\rho : PSL_2(\mathbb{Z}) \to GL_n(k)$ be an irreducible representation of $PSL_2(\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1\rangle$ which maps the commutator subgroup of $PSL_2(\mathbb{Z})$ to diagonal matrices in $GL_n(k)$.*

(i) *If k is not of characteristic* 2 *or* 3, *then $(\rho(x), \rho(y))$ is equivalent to one of the following*:

(1) $(1,1),\ (-1,1),\ (1,\zeta),\ (-1,\zeta),\ (1,\zeta^2),\ (-1,\zeta^2)$,

(2) $$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta^2 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}\right),$$

(3) $$\left(\pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right),$$

(4) $$\left(\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_1c_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\right),$$

for $c_1, c_2 \in k,\ c_1, c_2 \neq 0,\ (c_1, c_2) \neq (1,1),\ (-1,1),\ (1,-1),\ (-1,-1),\ (\zeta,\zeta),\ (\zeta^2,\zeta^2)$.

(ii) *If k is of characteristic* 2, *then* $(\rho(x), \rho(y))$ *is equivalent to one of the following*:

$$(1)\quad (1,1),\ (1,\zeta),\ (1,\zeta^2),$$

$$(2)\quad \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&\zeta\end{pmatrix}\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&\zeta^2\end{pmatrix}\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}\zeta&0\\0&\zeta^2\end{pmatrix}\right),$$

$$(3)\quad \left(\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&c_1&0\\0&0&0&0&0&1\\c_2&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\frac{1}{c_1c_2}&0&0&0\\0&0&0&1&0&0\end{pmatrix}\right),$$

for $c_1, c_2 \in k,\ c_1, c_2 \neq 0,\ (c_1, c_2) \neq (1,1),\ (\zeta,\zeta),\ (\zeta^2,\zeta^2)$.

(iii) *If k is of characteristic* 3, *then* $(\rho(x), \rho(y))$ *is equivalent to one of the following*:

$$(1)\quad (1,1),\ (-1,1),$$

$$(2)\quad \left(\pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right),$$

$$(3)\quad \left(\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&0&c_1&0\\0&0&0&0&0&1\\c_2&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\frac{1}{c_1c_2}&0&0&0\\0&0&0&1&0&0\end{pmatrix}\right),$$

for $c_1, c_2 \in k,\ c_1, c_2 \neq 0,\ (c_1, c_2) \neq (1,1),\ (-1,1),\ (1,-1),\ (-1,-1)$.

Furthermore, for $n < 6$, the equivalence classes in each of the cases (i)–(iii) *are distinct. For $n = 6$ and c_1, c_2 satisfying the above criteria, the following correspond to the same equivalence class*: (c_1, c_2), $(\frac{1}{c_1}, \frac{1}{c_2})$, $(c_2, \frac{1}{c_1c_2})$, $(\frac{1}{c_2}, c_1c_2)$, $(\frac{1}{c_1c_2}, c_1)$, *and* $(c_1c_2, \frac{1}{c_1})$. *If* (c_1', c_2') *does not equal any of the preceding pairs, then* (c_1', c_2') *represents an equivalence class distinct from* (c_1, c_2).

The remainder of the paper is devoted to the proof of this theorem.

Remark 4.2. Since the commutator subgroup of G is generated by $xyxy^2$ and xy^2xy, ρ maps the commutator subgroup of G to diagonal matrices in $GL_n(k)$ if and only if Λ and Γ are diagonal matrices. Thus the problem reduces to finding distinct equivalence classes of (X, Y) where $X := \rho(x)$, $Y := \rho(y)$, $X^2 = I = Y^3$, $\Lambda = XYXY^2$ is a diagonal matrix, $\Gamma = XY^2XY$ is a diagonal matrix, and (X, Y) is irreducible.

5. Cases when $n = 1$, $n = 2$, and $n = 3$

In this section, we prove (i)–(iii) of Theorem 4.1, considering separately the cases when $n = 1, 2$, and 3.

5.1. $n = 1$

For $n = 1$, Λ and Γ are trivially diagonal. We only require that $(X, Y) = (a, b)$ where $a^2 = 1 = b^3$.

Distinct (X, Y) create distinct equivalence classes since $(X', Y') \approx (X, Y)$ requires that $(X', Y') = (QXQ^{-1}, QYQ^{-1}) = (X, Y)$. In fields of characteristic 2, the only solution to $X^2 = 1$ is $X = 1$. In fields of characteristic other than 2, $X^2 = 1$ has the 2 distinct solutions $X = \pm 1$. In fields of characteristic 3, the only solution to $Y^3 = 1$ is $Y = 1$. In fields of characteristic other than 3, $Y^3 = 1$ has the 3 distinct solutions $Y = 1, \zeta, \zeta^2$. We therefore arrive at the desired result.

5.2. $n = 2$

Let $\Lambda = \left(\begin{smallmatrix} \lambda_1 & \\ & \lambda_2 \end{smallmatrix}\right)$. First assume $\lambda_1 = \lambda_2$. Then $\Lambda = \lambda_1 I$, which is a contradiction by Lemma 3.2. Thus assume $\lambda_1 \neq \lambda_2$. By explicitly solving $\Lambda X \Lambda = X$, we find that for any $i, j \in \{1, 2\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If X has more than one nonzero entry per row or column, it follows that $\lambda_1 = \lambda_2$, which is a contradiction to our assumption. Therefore since X is nonsingular, X has exactly one nonzero entry per row and column.

Since Λ and Γ are conjugate, we find Γ has diagonal entries γ_1, γ_2 where $\gamma_1 \neq \gamma_2$. Solving $\Lambda Y \Gamma = Y$, we find that for any $i, j \in \{1, 2\}$, $\lambda_i \gamma_j = 1$ if $Y_{i,j} \neq 0$. If Y has more than one nonzero entry per row or column, then either $\lambda_1 = \lambda_2$ or $\gamma_1 = \gamma_2$, both of which are contradictions. Therefore since Y is nonsingular, Y has exactly one nonzero entry per row and column. Hence X and Y are of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. If Y is of the latter type, then $Y^3 \neq I$, so Y must be diagonal. Clearly X is of the latter form by the irreducibility of (X, Y). Using $X^2 = I$, we see that $X = \left(\begin{smallmatrix} 0 & x_1 \\ \frac{1}{x_1} & 0 \end{smallmatrix}\right)$ for some nonzero $x_1 \in k$. Conjugating X and Y by the weighted permutation matrix $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x_1 \end{smallmatrix}\right)$, we see by Lemma 3.4 that $(X, Y) \approx (X_1, Y_1) := \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right)$, where (X_1, Y_1) satisfies the hypotheses of Lemma 3.4. By Remark 3.1, $a \neq b$. Also, if $a \neq b$,

$$\frac{1}{a-b}(X_1Y_1 - bX_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \frac{1}{a-b}\left(aX_1^2 - Y_1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\frac{1}{a-b}(aX_1 - X_1Y_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{a-b}\left(Y_1 - bX_1^2\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus (X_1, Y_1) is irreducible if and only if $a \neq b$.

Finally, $a^3 = b^3 = 1$ by $Y_1^3 = I$. In fields of characteristic 3, $a = b = 1$ since $a^3 = b^3 = 1$, which is a contradiction. Thus there are no irreducible representations from $PSL_2(\mathbb{Z})$ to $GL_2(k)$ if k has characteristic 3. In fields not of characteristic 3, we have

$$(a, b) = (1, \zeta), \left(1, \zeta^2\right), \left(\zeta, \zeta^2\right), (\zeta, 1), \left(\zeta^2, 1\right), \text{ or } \left(\zeta^2, \zeta\right).$$

Conjugating X_1 and Y_1 by permutation matrix X_1 yields $(X_1, Y_1) \approx \left(X_1, \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)\right)$. Thus we may assume the following cases: $(a, b) = (1, \zeta), (1, \zeta^2)$, or (ζ, ζ^2). Since $\operatorname{tr}(Y_1)$ is distinct for each of the listed cases, each yields a separate equivalence class. This gives the desired result.

5.3. $n = 3$

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & \gamma_3 \end{pmatrix}.$$

By Lemma 3.4, we may assume that either $\lambda_1 = \lambda_2 = \lambda_3$, $\lambda_1 = \lambda_2 \neq \lambda_3$, or that λ_1, λ_2, and λ_3 are distinct.

In the first case $\Lambda = \lambda_1 I$, which is a contradiction by Lemma 3.2.

We will now solve the third case. By solving $\Lambda X \Lambda = X$, we conclude for any $i, j \in \{1, 2, 3\}$ that $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If X has more than one nonzero entry per row or column, we find that $\lambda_i = \lambda_j$ for some $i \neq j$, which is a contradiction to our assumption that $\lambda_1, \lambda_2, \lambda_3$ are distinct. Therefore since X is nonsingular, X has exactly one nonzero entry per row and column. Solving $\Lambda Y \Gamma = Y$, we find that for any $i, j \in \{1, 2, 3\}$, $\lambda_i \gamma_j = 1$ if $Y_{i,j} \neq 0$. If Y has more than one nonzero entry per row or column, then either $\lambda_i = \lambda_j$ or $\gamma_i = \gamma_j$ for some $i \neq j$, both of which are contradictions. Therefore since Y is nonsingular, Y has exactly one nonzero entry per row and column. Note, $\lambda_k = \frac{1}{\lambda_i}$ for some $i = 1, 2, 3$ and each k, by Remark 3.3. Since there are at most two distinct solutions to $x = \frac{1}{x}$, then $\lambda_i \neq \frac{1}{\lambda_i}$ for some i. Conjugating X and Y by a permutation matrix if necessary, we may assume that $\lambda_2 = \frac{1}{\lambda_1}$. Thus $\lambda_3 = \frac{1}{\lambda_3}$. Therefore, we find $\lambda_3 \lambda_1 \neq 1$ and $\lambda_3 \lambda_2 \neq 1$. Hence $X_{3,1} = X_{3,2} = 0$. This forces $X_{3,3} \neq 0$. Expanding $\Gamma X \Gamma = X$, we see from $X_{3,3} \neq 0$ that $\gamma_3 = \frac{1}{\gamma_3}$. Because Λ and Γ are conjugate, $\gamma_3 = \lambda_3 = \frac{1}{\lambda_3}$. Since $\gamma_3 = \frac{1}{\lambda_3}$ and $\gamma_1, \gamma_2, \gamma_3$ are distinct, we find $\lambda_3 \gamma_2 \neq 1$ and $\lambda_3 \gamma_1 \neq 1$. Thus $Y_{3,1} = Y_{3,2} = 0$. This forces $Y_{3,3} \neq 0$. But since X and Y have exactly one nonzero entry per row and column and $X_{3,3}$ and $Y_{3,3}$ are nonzero, we find that (X, Y) is reducible, which is a contradiction.

In case 2, we assume $\lambda_1 = \lambda_2 \neq \lambda_3$. First assuming $\lambda_1 \neq \frac{1}{\lambda_1}$ and using Remark 3.3, we see $\lambda_3 = \frac{1}{\lambda_1}$. Again solving $\Lambda X \Lambda = X$, we conclude for any $i, j \in \{1, 2, 3\}$ that $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. From $\lambda_3 = \frac{1}{\lambda_1}$ and $\lambda_1 = \lambda_2 \neq \lambda_3$, we see that $\lambda_1^2 = \lambda_1 \lambda_2 = \lambda_2 \lambda_1 = \lambda_2^2 \neq \lambda_1 \lambda_3 = 1$. Then $X_{1,1} = X_{1,2} = X_{2,1} = X_{2,2} = 0$. But this implies that X is singular, which is a contradiction. Hence $\lambda_1 = \frac{1}{\lambda_1}$ and $\lambda_3 = \frac{1}{\lambda_3}$.

In fields of characteristic 2, there is only one distinct solution to $x = \frac{1}{x}$. This is a contradiction since $\lambda_1 \neq \lambda_3$. Hence if k has characteristic 2, there are no irreducible representations from $PSL_2(\mathbb{Z})$ to $GL_3(k)$.

We now consider fields k which are not of characteristic 2. Since $\lambda_i = \frac{1}{\lambda_i}$ for $i = 1, 2, 3$, then

$$\Lambda = \pm \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Because Λ and Γ are conjugate and $\Lambda \neq \Gamma$, we have 4 possibilities for (Λ, Γ). Substituting each of the possibilities for (Λ, Γ) in $\Lambda X \Lambda = X = \Gamma X \Gamma$, we solve the resulting system of equations to find that in each possibility, X is diagonal. Again substituting each of the possibilities for (Λ, Γ) in the equation $\Lambda Y \Gamma = Y$ and using $Y^3 = I$, we solve the resulting system of equations. In each possibility, Y has exactly one nonzero entry per row and column.

Since Y has exactly one nonzero entry per row and column, $Y^3 = I$, X is diagonal, and (X, Y) is irreducible, we conclude

$$Y = \begin{pmatrix} 0 & y_1 & 0 \\ 0 & 0 & y_2 \\ y_3 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & y_1 \\ y_2 & 0 & 0 \\ 0 & y_3 & 0 \end{pmatrix},$$

where $y_1 y_2 y_3 = 1$. Conjugate (X, Y) with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_1 y_2 \end{pmatrix}$$

in case 1, or with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & y_1 y_3 & 0 \end{pmatrix}$$

in case 2, so that $(X, Y) \approx (PXP^{-1}, PYP^{-1}) =: (X_1, Y_1)$, where X_1 is diagonal and

$$Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $X_1^2 = I$, all of the diagonal entries must be ± 1. Because $X_1 \neq \pm I$, there are 6 possibilities for X_1. Conjugating by various permutation matrices, we can see that

$$(X, Y) \approx (\pm X_2, Y_1), \quad \text{where } X_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus (X, Y) satisfies the hypotheses of Lemma 3.4 if and only if $(\pm X_2, Y_1)$ satisfies the hypotheses of Lemma 3.4. Since (X_2, Y_1) satisfies $X_2^2 = I = Y_1^3$ and $X_2 Y_1 X_2 Y_1^2$ and $X_2 Y_1^2 X_2 Y_1$ are diagonal, we need only check irreducibility.

If (X_2, Y_1) is irreducible, then $(-X_2, Y_1)$ is irreducible. Hence it is sufficient to check irreducibility for (X_2, Y_1). However, we note that the standard basis matrices can be composed as:

$$\frac{Y_1 - Y_1 X_2}{2}, \quad \frac{Y_1(Y_1 - Y_1 X_2)}{2}, \quad \frac{Y_1^2(Y_1 - Y_1 X_2)}{2}, \quad \frac{(Y_1 - Y_1 X_2)Y_1}{2}, \quad \frac{Y_1(Y_1 - Y_1 X_2)Y_1}{2},$$
$$\frac{Y_1^2(Y_1 - Y_1 X_2)Y_1}{2}, \quad \frac{(Y_1 - Y_1 X_2)Y_1^2}{2}, \quad \frac{Y_1(Y_1 - Y_1 X_2)Y_1^2}{2}, \quad \text{and} \quad \frac{Y_1^2(Y_1 - Y_1 X_2)Y_1^2}{2}.$$

Note that (X_2, Y_1) and $(-X_2, Y_1)$ yield separate equivalence classes since the trace of the matrices are preserved in each equivalence class and $\operatorname{tr}(X_2) \neq \operatorname{tr}(-X_2)$. We have therefore achieved the desired result.

6. Case when $n = 6$

In this section, we prove (i)–(iii) of Theorem 4.1 in the case when $n = 6$.

Lemma 6.1. *Suppose $X, Y \in GL_n(k)$ where $X^2 = Y^3 = I$, (X, Y) is irreducible, and $\Lambda = XYXY^2$ and $\Gamma = XY^2XY$ are diagonal matrices. Then X and Y must have exactly one nonzero entry per row and column.*

Proof. Assume X and Y satisfy the hypothesis of the lemma. Let Λ and Γ be diagonal matrices with diagonal entries $\Lambda_{i,i} =: \lambda_i$, $\Gamma_{i,i} =: \gamma_i$. Note that (X, Y) satisfies the hypothesis if and only if (PXP^{-1}, PYP^{-1}) satisfies the hypothesis where P is a permutation matrix. Thus we may assume the diagonal entries of Λ are in one of the following cases, where entries in distinct ordered tuples are not equal and entries in the same ordered tuple are equal:

(1) $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$,
(2) $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5), (\lambda_6)$,
(3) $(\lambda_1, \lambda_2, \lambda_3, \lambda_4), (\lambda_5, \lambda_6)$,
(4) $(\lambda_1, \lambda_2, \lambda_3), (\lambda_4, \lambda_5, \lambda_6)$,
(5) $(\lambda_1, \lambda_2, \lambda_3, \lambda_4), (\lambda_5), (\lambda_6)$,
(6) $(\lambda_1, \lambda_2, \lambda_3), (\lambda_4, \lambda_5), (\lambda_6)$,
(7) $(\lambda_1, \lambda_2), (\lambda_3, \lambda_4), (\lambda_5, \lambda_6)$,
(8) $(\lambda_1, \lambda_2, \lambda_3), (\lambda_4), (\lambda_5), (\lambda_6)$,
(9) $(\lambda_1, \lambda_2), (\lambda_3, \lambda_4), (\lambda_5), (\lambda_6)$,
(10) $(\lambda_1, \lambda_2), (\lambda_3), (\lambda_4), (\lambda_5), (\lambda_6)$,
(11) $(\lambda_1), (\lambda_2), (\lambda_3), (\lambda_4), (\lambda_5), (\lambda_6)$.

Case 1. In this case $\Lambda = \lambda_1 I$, which by Lemma 3.2 is a contradiction to the irreducibility of (X, Y).

Cases 2 and 3. First assume $\lambda_1 \neq \frac{1}{\lambda_1}$. Expanding $\Lambda X \Lambda = X$ we get a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i\lambda_j = 1$ if $X_{i,j} \neq 0$. Since $\lambda_1 \neq \frac{1}{\lambda_1}$, we find X must be singular, which is a contradiction. Thus assume $\lambda_1 = \frac{1}{\lambda_1}$. We conclude from Remark 3.3 that $\lambda_6 = \frac{1}{\lambda_6}$ and hence $\Lambda^2 = I$. Because $\Lambda\Gamma = \Gamma\Lambda$, we see that $(XY)^6 = (YX)^6 = I$. Also, $(XY)^3 = (YX)^3$ since $\Lambda^2 = I$.

Let $A = XY$ and $B = YX$. Since $A^3 = B^3$, $ABA = BAB$, and $A^6 = B^6 = I$, we can count possible monomials in A and B to find that (A, B) span at most a 24-dimensional space and thus cannot be irreducible. Note (A, B) is irreducible if and only if (X, Y) is irreducible, since we can generate X and Y from A and B and vice versa. Therefore (X, Y) is not irreducible.

Case 4. By Remark 3.3, if $\lambda_1 = \frac{1}{\lambda_1}$, then $\lambda_6 = \frac{1}{\lambda_6}$ and $\Lambda^2 = I$. As in Cases 2 and 3, this leads to a contradiction. Hence $\lambda_6 = \frac{1}{\lambda_1}$. Observe that since Λ and Γ are conjugate,

$$\Lambda + \Lambda^{-1} = \left(\lambda_1 + \frac{1}{\lambda_1}\right)I = \Gamma + \Gamma^{-1}. \tag{6.1}$$

Then it follows that

$$Y\left(\Lambda+\Lambda^{-1}\right)=Y\left(\Gamma+\Gamma^{-1}\right)=\left(\Gamma+\Gamma^{-1}\right)Y. \tag{6.2}$$

Substituting $\Lambda = XYXY^2$ and $\Gamma = XY^2XY$ into Eq. (6.2) and using $(YX)^6 = I$ yields

$$(YX)\left((YX)^2-(XY)^2\right)(XY-YX)=0. \tag{6.3}$$

Since $XY = YX\Gamma$, we find that $(YX)((YX)^2 - (XY)^2)(YX)(\Gamma - I) = 0$. Given that Λ and Γ are conjugate and λ_1 and λ_6 are not equal to 1, $(\Gamma - I)$ is invertible. Also by invertibility of (YX), we deduce $(YX)^2 = (XY)^2$. Thus $A^2 = B^2$, $A^6 = B^6 = I$, and $ABA = BAB$, where $A = XY$ and $B = YX$. Again by counting monomials in A and B, we find A and B span at most an 18-dimensional space and thus (A, B) cannot be irreducible. Since (A, B) is irreducible if and only if (X, Y) is irreducible, (X, Y) cannot be irreducible.

Case 5. First assume $\lambda_1 \neq \frac{1}{\lambda_1}$. Then by Remark 3.3, λ_5 or λ_6 must equal $\frac{1}{\lambda_1}$. We expand $\Lambda X\Lambda = X$ to get a system of equations where for any $i, j \in \{1, \dots, 6\}$, $\lambda_i\lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_5 = \frac{1}{\lambda_1}$ or $\lambda_6 = \frac{1}{\lambda_1}$, then X must be singular, which is a contradiction. Thus $\lambda_1 = \frac{1}{\lambda_1}$. If $\lambda_5 = \frac{1}{\lambda_5}$ then by Remark 3.3, $\lambda_6 = \frac{1}{\lambda_6}$. This is a contradiction since there are at most two distinct solutions to $x = \frac{1}{x}$. Hence $\lambda_5 = \frac{1}{\lambda_6}$. Since Λ and Γ are conjugate, there are $\binom{6}{4,1,1}$ permutations of the λ_i's. Thus there are 30 possible matrices for Γ with nonzero entries determined by the λ_i's. By case-by-case checking, we see that the restrictions $\Gamma X\Gamma = X = \Lambda X\Lambda$ and X is nonsingular leave 14 possible matrices for Γ (see Appendix A). Note that $XY = YX$ if $\Lambda = \Gamma$ and $(XY)^2 = (YX)^2$ if $\Lambda\Gamma = I$. In both cases, we find (X, Y) is not irreducible as shown earlier. This leaves 12 possible matrices for Γ.

Let P be a permutation matrix. Note that (X, Y) satisfies the hypothesis of Lemma 6.1 if and only if (PXP^{-1}, PYP^{-1}) does. Also, X and Y have one nonzero entry per row and column if and only if PXP^{-1} and PYP^{-1} do. Thus for any permutation matrix P, we may replace (X, Y) with (PXP^{-1}, PYP^{-1}). Further case-by-case checking gives us that for each of the 12 possible values of Γ, we are able to replace (X, Y) with (PXP^{-1}, PYP^{-1}) for an appropriate permutation matrix P, so that Λ is preserved, $\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_3 = \lambda_5$, and $\gamma_4 = \lambda_6$ (see Appendix A). Substituting Λ and Γ into $\Lambda Y\Gamma = Y$ and $\Gamma Y^2\Lambda = Y^2$, using $Y^3 = I$, and solving for Y, we find that

$$Y=\begin{pmatrix} 0 & 0 & Y_1 \\ Y_2 & 0 & 0 \\ 0 & Y_3 & 0 \end{pmatrix},$$

where Y_1, Y_2, and Y_3 are 2×2 block matrices. Similarly, by substituting Λ and Γ into $\Lambda X\Lambda = X = \Gamma X\Gamma$ and solving for X, we find that

$$X=\begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix},$$

where X_1, X_2, and X_3 are 2×2 block matrices.

Due to the fact that $Y^3 = I$, we find $Y_3 = Y_1^{-1}Y_2^{-1}$. Computation with $XY = \Lambda YX$ yields

$$X_1Y_1 = \lambda_1 Y_1 X_3, \qquad X_2Y_2 = \lambda_1 Y_2 X_1, \qquad X_3Y_3 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{pmatrix} Y_3 X_2.$$

But then:

$$X_3Y_3 = \frac{1}{\lambda_1}\left(Y_1^{-1}X_1Y_1\right)Y_3 = \frac{1}{\lambda_1}\left(Y_1^{-1}\right)\left(X_1Y_2^{-1}\right) = \frac{1}{\lambda_1}\left(Y_1^{-1}\right)\left(\frac{1}{\lambda_1}\right)\left(Y_2^{-1}X_2\right) = \left(\frac{1}{\lambda_1^2}\right)Y_3X_2.$$

This is a contradiction to $X_3Y_3 = \left(\begin{smallmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{\lambda_5} \end{smallmatrix}\right)Y_3X_2$.

Case 6. Suppose $\lambda_6 \neq \frac{1}{\lambda_6}$. Then either $\lambda_1 = \frac{1}{\lambda_6}$ or $\lambda_4 = \frac{1}{\lambda_6}$. We expand $\Lambda X \Lambda = X$ to get a system of equations where for any $i, j \in \{1, \dots, 6\}$, $\lambda_i\lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_1 = \frac{1}{\lambda_6}$ or $\lambda_4 = \frac{1}{\lambda_6}$, then X must be singular, which is a contradiction. Thus $\lambda_6 = \frac{1}{\lambda_6}$. Suppose next that $\lambda_4 \neq \frac{1}{\lambda_4}$. Then $\lambda_1 = \frac{1}{\lambda_4}$. Since $\lambda_i\lambda_j = 1$ if $X_{i,j} \neq 0$ and since $\lambda_1 = \frac{1}{\lambda_4}$, we find that X is singular, which is a contradiction. Thus assume $\lambda_4 = \frac{1}{\lambda_4}$. Because $\lambda_4 \neq \frac{1}{\lambda_1}$ and $\lambda_6 \neq \frac{1}{\lambda_1}$, it follows that $\lambda_1 = \frac{1}{\lambda_1}$. This is a contradiction since there are at most 2 distinct solutions to $x = \frac{1}{x}$, while λ_1, λ_4, and λ_6 are distinct.

Case 7. It is easy to show using Remark 3.3 that there must be an even number of ordered tuples with entries λ_i such that $\lambda_i \neq \frac{1}{\lambda_i}$. Since there are 3 tuples in this case, there must be 1 or 3 tuples with entries λ_i where $\lambda_i = \frac{1}{\lambda_i}$. Because there are at most two distinct solutions to $x = \frac{1}{x}$, there is only one tuple with entries λ_i where $\lambda_i = \frac{1}{\lambda_i}$. Without loss of generality, assume that $\lambda_1 = \frac{1}{\lambda_1}$. Therefore $\lambda_3 = \frac{1}{\lambda_5}$. Since Λ and Γ are conjugate, there are $\binom{6}{2,2,2} = 90$ possible matrices for Γ in terms of the λ_i's. From $\Gamma X \Gamma = X = \Lambda X \Lambda$, exactly 22 matrices for Γ leave X nonsingular. For each of these values of Γ, we find the corresponding form for X (see Appendix A). Similarly, we substitute Λ and each possible Γ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each of the 22 remaining values of Γ, Y must be of a certain corresponding form (see Appendix A). Further case-by-case analysis yields exactly 16 choices of Γ which do not force (X, Y) to be reducible (see Appendix A). In each of these 16 cases, the restriction that $XYXY^2$ is diagonal forces both X and Y to have exactly one nonzero entry per row and column.

Case 8. Suppose $\lambda_1 \neq \frac{1}{\lambda_1}$. Then λ_4, λ_5, or λ_6 must equal $\frac{1}{\lambda_1}$. We expand $\Lambda X \Lambda = X$ to get a system of equations where for any $i, j \in \{1, \dots, 6\}$, $\lambda_i\lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_1 = \frac{1}{\lambda_4}$, $\lambda_1 = \frac{1}{\lambda_5}$, or $\lambda_1 = \frac{1}{\lambda_6}$, then we find X must be singular, which is a contradiction. Thus $\lambda_1 = \frac{1}{\lambda_1}$. As in Case 7, it is easy to show that there must be 0, 2, or 4 tuples with entries λ_i where $\lambda_i = \frac{1}{\lambda_i}$ for each i in that tuple. Since $\lambda_1 = \frac{1}{\lambda_1}$ and since there are at most 2 distinct solutions to $x = \frac{1}{x}$, there must be two such tuples. Without loss of generality, assume that $\lambda_4 = \frac{1}{\lambda_4}$. (Note that if instead $\lambda_5 = \frac{1}{\lambda_5}$ or $\lambda_6 = \frac{1}{\lambda_6}$, we may conjugate X and Y by an appropriate permutation matrix P.) Since $\lambda_5 \neq \frac{1}{\lambda_5}$, then $\lambda_6 = \frac{1}{\lambda_5}$. Because Λ and Γ are conjugate, there are $\binom{6}{3,1,1,1} = 120$ possible matrices for Γ in terms of the λ_i's. By case-by-case analysis, we see from $\Gamma X \Gamma = X = \Lambda X \Lambda$

that exactly 20 choices of Γ leave X nonsingular. For each of the 20 possible values of Γ, we find the corresponding form of X (see Appendix A). Then we substitute Λ and each possible Γ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each possible Γ, Y must be of a certain corresponding form (see Appendix A). Among these 20 choices of Γ, further case-by-case analysis yields exactly 6 choices of Γ which do not force (X, Y) to be reducible (see Appendix A). In each of the 6 remaining values of Γ, we find that $XYXY^2$ is not diagonal, which is a contradiction.

Case 9. Suppose that $\lambda_1 = \frac{1}{\lambda_1}$. Expand $\Lambda X \Lambda = X$ to yield a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If $\lambda_3 \neq \frac{1}{\lambda_3}$, then either $\lambda_5 = \frac{1}{\lambda_3}$ or $\lambda_6 = \frac{1}{\lambda_3}$. In either case, X must be singular, which is a contradiction. Thus $\lambda_3 = \frac{1}{\lambda_3}$. Since there are at most two distinct solutions to $x = \frac{1}{x}$, we find $\lambda_6 = \frac{1}{\lambda_5}$. There are $\binom{6}{2,2,1,1} = 180$ possible matrices for Γ defined in terms of the λ_i's. Substitute Λ and each possible Γ into $\Gamma X \Gamma = X = \Lambda X \Lambda$ and solve the resulting system of equations. By case-by-case analysis, we see that exactly 20 choices of Γ leave X nonsingular. For each of these 20 possible values of Γ, we find the corresponding form of X (see Appendix A). Similarly, substitute Λ and each possible Γ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each of the 20 remaining possible values of Γ, we find the corresponding form for Y (see Appendix A). Among these values of Γ, further case-by-case analysis yields exactly 4 choices of Γ which do not force (X, Y) to be reducible (see Appendix A). In each of these cases, the corresponding forms of X and Y are

$$X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & 0 & Y_1 \\ Y_2 & 0 & 0 \\ 0 & Y_3 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & Y_1 & 0 \\ 0 & 0 & Y_2 \\ Y_3 & 0 & 0 \end{pmatrix},$$

where X_i and Y_i are 2×2 block matrices. Using a similar argument to that in Case 5, we reach a contradiction with both forms of (X, Y).

Hence $\lambda_1 \neq \frac{1}{\lambda_1}$. We then see that λ_3, λ_5, or λ_6 equals $\frac{1}{\lambda_1}$. Expand $\Lambda X \Lambda = X$ to get a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_5 = \frac{1}{\lambda_1}$ or $\lambda_6 = \frac{1}{\lambda_1}$, it follows that X is singular, which is a contradiction. Thus $\lambda_3 = \frac{1}{\lambda_1}$. Suppose $\lambda_5 = \frac{1}{\lambda_5}$. Then $\lambda_6 = \frac{1}{\lambda_6}$. From $\Lambda X \Lambda = X$, one sees that $X_{5,5}$ and $X_{6,6}$ are nonzero elements. Note $\gamma_5^2 = \gamma_6^2 = 1$ since $\Gamma X \Gamma = X$. Because Λ and Γ are conjugate, either $\gamma_5 = \lambda_5$ and $\gamma_6 = \lambda_6$, or $\gamma_5 = \lambda_6$ and $\gamma_6 = \lambda_5$. In both cases, (X, Y) is not irreducible by $\Lambda Y \Gamma = Y$, which is a contradiction. Therefore $\lambda_3 = \frac{1}{\lambda_1}$ and $\lambda_6 = \frac{1}{\lambda_5}$. There are $\binom{6}{2,2,1,1} = 180$ possible matrices for Γ. By case-by-case analysis, we see from $\Gamma X \Gamma = X = \Lambda X \Lambda$ that exactly 44 choices of Γ leave X nonsingular. For each of the 44 possible values of Γ, we find the corresponding form of X (see Appendix A). Substitute Λ and each of the possible values of Γ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each of the 44 possible values of Γ, we find the corresponding form of Y (see Appendix A). Among these possible values of Γ, further case-by-case analysis yields exactly 32 choices of Γ which do not force (X, Y) to be reducible (see Appendix A). For each of the 32 remaining choices of Γ, the fact that $XYXY^2$ is diagonal either eliminates the choice of Γ, or forces both X and Y to have exactly one nonzero entry per row and column.

Case 10. Assume that $\lambda_1 \neq \frac{1}{\lambda_1}$. Then either $\lambda_3, \lambda_4, \lambda_5$, or λ_6 must equal $\frac{1}{\lambda_1}$. We expand $\Lambda X \Lambda = X$ to get a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_1 = \frac{1}{\lambda_3}$, $\lambda_1 = \frac{1}{\lambda_4}$, $\lambda_1 = \frac{1}{\lambda_5}$, or $\lambda_1 = \frac{1}{\lambda_6}$, then X must be singular, which is a contradiction. Thus $\lambda_1 = \frac{1}{\lambda_1}$. As in cases 7 and 8, it is easy to show that there must be 1, 3, or 5 tuples with entries λ_i in which $\lambda_i = \frac{1}{\lambda_i}$. Since there are at most 2 distinct solutions to $x = \frac{1}{x}$, there must be only one such tuple. Without loss of generality, we may assume that $\lambda_4 = \frac{1}{\lambda_3}$ and thus $\lambda_6 = \frac{1}{\lambda_5}$. Since Λ and Γ are conjugate, there are $\binom{6}{2,1,1,1,1} = 360$ possible matrices for Γ in terms of the λ_i's. By case-by-case analysis, we see that the restrictions $\Gamma X \Gamma = X = \Lambda X \Lambda$ and X is nonsingular leave exactly 24 choices for Γ. For each of the 24 possible values of Γ, we find the corresponding form of X (see Appendix A). Substitute Λ and each of the possible values of Γ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each of the 24 possible values of Γ, we find the corresponding form of Y (see Appendix A). Among these 24 choices of Γ, further case-by-case analysis yields exactly 8 choices of Γ which do not force (X, Y) to be reducible (see Appendix A). In each of these 8 cases, the restriction that $XYXY^2$ is diagonal forces both X and Y to have exactly one nonzero entry per row and column.

Case 11. By solving $\Lambda X \Lambda = X$, we find that for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If X has more than one nonzero entry per row or column, then $\lambda_i = \lambda_j$ for some $i \neq j$, which is a contradiction to our assumption that all λ_i are distinct. Therefore since X is nonsingular, X has exactly one nonzero entry per row and column. Solving $\Lambda Y \Gamma = Y$, we find that for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \gamma_j = 1$ if $Y_{i,j} \neq 0$. If Y has more than one nonzero entry per row or column, then either $\lambda_i = \lambda_j$ or $\gamma_i = \gamma_j$ for some $i \neq j$, both of which are contradictions. Therefore since Y is nonsingular, Y has exactly one nonzero entry per row and column. $\square$

Lemma 6.2. *If (X, Y) satisfies the properties in Lemma* 3.4 *where $n = 6$, then*

$$(X, Y) \approx \left(\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_1 c_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right)$$

where $c_1, c_2 \in k$, $c_1, c_2 \neq 0$, and

(1) $(c_1, c_2) \neq (1, 1), (-1, -1), (-1, 1), (1, -1), (\zeta, \zeta), (\zeta^2, \zeta^2)$, *if k does not have characteristic* 2 *or* 3, *where ζ is a primitive cube root of unity.*
(2) $(c_1, c_2) \neq (1, 1), (\zeta, \zeta), (\zeta^2, \zeta^2)$, *if k has characteristic* 2, *where ζ is a primitive cube root of unity.*
(3) $(c_1, c_2) \neq (1, 1), (-1, -1), (-1, 1), (1, -1)$, *if k has characteristic* 3.

Furthermore, for c_1, c_2 satisfying the above criteria, the following correspond to the same equivalence class: (c_1, c_2), $(\frac{1}{c_1}, \frac{1}{c_2})$, $(c_2, \frac{1}{c_1 c_2})$, $(\frac{1}{c_2}, c_1 c_2)$, $(\frac{1}{c_1 c_2}, c_1)$, *and* $(c_1 c_2, \frac{1}{c_1})$. *If (c_1', c_2') does* *not equal any of the preceding tuples, then* (c_1', c_2') *represents an equivalence class distinct from* (c_1, c_2).

Proof. Assume (X, Y) satisfies the properties in Lemma 3.4. Let H be the group of 6×6 matrices with exactly one nonzero entry per row and column and with the group operation of matrix multiplication. Let M be the group of all equivalence classes of H where $A \in H$ is equivalent to $B \in H$ when

$$A_{i,j} = 0 \quad \Leftrightarrow \quad B_{i,j} = 0.$$

Denote $[A] \in M$ as the equivalence class of $A \in H$ and denote $[1]_M$ as the identity element in M. Now consider $[X] \in M$. Since $X^2 = I$, $[X]$ has order 1 or 2. We see that M is isomorphic to S_6. The conjugacy classes of S_6 with elements that have order 1 or 2 are represented by (1), (1 2), (1 2)(3 4), and (1 2)(3 4)(5 6). Thus there must be a permutation matrix, P, such that PXP^{-1} is diagonal or one of

$$X_0 = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 \end{pmatrix}, \quad \text{or}$$

$$X_2 = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & x_6 & 0 \end{pmatrix},$$

for suitable choices of $x_1, \ldots, x_6$.

Similarly, since $Y^3 = I$, there must be a permutation matrix, Q, such that QYQ^{-1} is diagonal or is equal to

$$Y_1 = \begin{pmatrix} 0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix} \quad \text{or} \quad Y_2 = \begin{pmatrix} 0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 \\ 0 & 0 & 0 & y_6 & 0 & 0 \end{pmatrix}$$

for suitable choices of $y_1, \ldots, y_6$.

Assume PXP^{-1} is diagonal. Since $P \in H$, we find $[1]_M = [PXP^{-1}] = [P][X][P^{-1}]$. Thus $[X] = [1]_M$, i.e. X is a diagonal matrix. Now take an arbitrary monomial in X and Y. Since $X^2 = I = Y^3$, this monomial can be expressed in the form $Y^{a_1}XY^{a_2}X \ldots Y^{a_n}$ where $a_i = 0, 1$, or 2 for $i = 1, \ldots, n$. Because X is diagonal and $Y^3 = I$,

$$\begin{aligned}\left[Y^{a_1}XY^{a_2}X \ldots Y^{a_n}\right] &= \left[Y^{a_1}\right][X]\left[Y^{a_2}\right][X] \ldots \left[Y^{a_n}\right] = \left[Y^{a_1}\right][1]_M\left[Y^{a_2}\right][1]_M \ldots \left[Y^{a_n}\right] \\ &= \left[Y^{a_1}\right]\left[Y^{a_2}\right] \ldots \left[Y^{a_n}\right] = \left[Y^{a_1+a_2+\cdots+a_n}\right] = [1]_M, \quad [Y], \quad \text{or} \quad \left[Y^2\right].\end{aligned}$$

Since $Y \in H$ and $Y^2 \in H$, we find that Y and Y^2 have exactly one nonzero entry per row and column. Thus (X, Y) can span at most an 18-dimensional space, which is a contradiction to (X, Y) being irreducible.

Now assume QYQ^{-1} is diagonal for some permutation matrix Q. By a similar argument to that above, we see that Y is a diagonal matrix. Now take an arbitrary monomial in X and Y. Again, this monomial can be expressed as $Y^{a_1}XY^{a_2}X \dots Y^{a_n}$ where $a_i = 0, 1$, or 2 for $i = 1, \dots, n$. Then since Y is diagonal and $X^2 = I$,

$$\begin{aligned}\left[Y^{a_1}XY^{a_2}X \dots Y^{a_n}\right] &= \left[Y^{a_1}\right][X]\left[Y^{a_2}\right][X]\dots\left[Y^{a_n}\right] = [1]_M[X][1]_M[X]\dots[1]_M \\ &= [X][X]\dots[X] = \left[X^{n-1}\right] = [1]_M \quad \text{or} \quad [X].\end{aligned}$$

Since $X \in H$, we find X has one nonzero entry per row and column. Therefore (X, Y) can span at most a 12-dimensional space, which is a contradiction to (X, Y) being irreducible. Hence $PXP^{-1} = X_0, X_1$, or X_2 and $QYQ^{-1} = Y_1$ or Y_2, for some permutation matrix Q. We define Y_0 to be PYP^{-1}. Then for each (X, Y) which satisfies the properties in Lemma 3.4, we find that (X, Y) must be equivalent to (X_0, Y_0), (X_1, Y_0), or (X_2, Y_0) where Y_0 is conjugate to Y_1 or Y_2 by a permutation matrix. Note that if P is a weighted permutation matrix, then (X, Y) satisfies the properties in Lemma 3.4 if and only if (PXP^{-1}, PYP^{-1}) satisfies the properties in Lemma 3.4. Thus (X_i, Y_0) satisfies the properties in Lemma 3.4.

After examining all possible pairs (X_i, Y_0) by using the restrictions that (X_i, Y_0) is irreducible and $X_iY_0X_iY_0^2$ is diagonal, we find $(X, Y) \approx (X_2, Y_0)$, where X_2 is of the form above and Y_0 equals:

$$\begin{pmatrix} 0&0&y_1&0&0&0\\ 0&0&0&y_2&0&0\\ 0&0&0&0&y_3&0\\ 0&0&0&0&0&y_4\\ y_5&0&0&0&0&0\\ 0&y_6&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&y_1&0&0&0\\ 0&0&0&y_2&0&0\\ 0&0&0&0&0&y_3\\ 0&0&0&0&y_4&0\\ 0&y_5&0&0&0&0\\ y_6&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&y_1&0&0\\ 0&0&y_2&0&0&0\\ 0&0&0&0&y_3&0\\ 0&0&0&0&0&y_4\\ 0&y_5&0&0&0&0\\ y_6&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&y_1&0&0\\ 0&0&y_2&0&0&0\\ 0&0&0&0&0&y_3\\ 0&0&0&0&y_4&0\\ y_5&0&0&0&0&0\\ 0&y_6&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&y_1\\ 0&0&0&0&y_2&0\\ y_3&0&0&0&0&0\\ 0&y_4&0&0&0&0\\ 0&0&0&y_5&0&0\\ 0&0&y_6&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&y_1&0\\ 0&0&0&0&0&y_2\\ y_3&0&0&0&0&0\\ 0&y_4&0&0&0&0\\ 0&0&y_5&0&0&0\\ 0&0&0&y_6&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_2 \\ 0 & y_3 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_5 & 0 & 0 \\ 0 & 0 & y_6 & 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_2 & 0 \\ 0 & y_3 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_6 & 0 & 0 \end{pmatrix},$$

for suitable choices of $y_1, \dots, y_6$. From computation with $X^2 = Y^3 = I$ and conjugation by various weighted permutation matrices, one sees that

$$(X, Y) \approx (X', Y') := \left(\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_1 c_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right)$$

for some nonzero $c_1, c_2 \in k$.

We note $X'^2 = I = Y'^3$, $X'Y'X'Y'^2$ is diagonal, and $X'Y'^2X'Y'$ is diagonal. Thus it remains to determine which values of c_1 and c_2 lead to an irreducible (X', Y') and to find the equivalence classes for (X', Y'). We first determine the equivalence classes for solutions (X', Y').

Let

$$Y^* = \begin{pmatrix} 0 & 0 & 0 & 0 & c_1' & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2' & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_1' c_2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $(X', Y^*) \approx (X', Y')$ if and only if there exists an invertible matrix Q such that $QX' = X'Q$ and $QY' = Y^*Q$.

Let

$$Q = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}$$

where A through J are 2×2 block matrices. Then by $QX' = X'Q$, we find that $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_1 \end{smallmatrix}\right)$, etc.

From $QY' = Y^*Q$, we see that:

$$A = 0 \quad \Leftrightarrow \quad E = 0 \quad \Leftrightarrow \quad J = 0,$$
$$B = 0 \quad \Leftrightarrow \quad F = 0 \quad \Leftrightarrow \quad G = 0,$$
$$C = 0 \quad \Leftrightarrow \quad D = 0 \quad \Leftrightarrow \quad H = 0.$$

From this result and again using $QY' = Y^*Q$, we find $(c_1', c_2') = (c_1, c_2)$ or $(\frac{1}{c_1}, \frac{1}{c_2})$ if $A \neq 0$. Likewise, $(c_1', c_2') = (c_2, \frac{1}{c_1 c_2})$ or $(\frac{1}{c_2}, c_1 c_2)$ if $B \neq 0$, and $(c_1', c_2') = (\frac{1}{c_1 c_2}, c_1)$ or $(c_1 c_2, \frac{1}{c_1})$ if $C \neq 0$.

Note that using:

$$Q_1 = I, \qquad Q_2 = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & \frac{1}{c_2} & & \\ & & \frac{1}{c_2} & 0 & & \\ & & & & 0 & c_1 \\ & & & & c_1 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} & & 1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \\ 1 & 0 & & & & \\ 0 & 1 & & & & \end{pmatrix}, \qquad Q_4 = \begin{pmatrix} & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & c_1c_2 \\ & & & & c_1c_2 & 0 \\ 0 & c_2 & & & & \\ c_2 & 0 & & & & \end{pmatrix},$$

$$Q_5 = \begin{pmatrix} & & & & 1 & 0 \\ & & & & 0 & 1 \\ 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & & \\ & & 0 & 1 & & \end{pmatrix}, \qquad Q_6 = \begin{pmatrix} & & & & 0 & c_1c_2 \\ & & & & c_1c_2 & 0 \\ 0 & c_2 & & & & \\ c_2 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \end{pmatrix}$$

we find $Q_i X' Q_i^{-1} = X'$ and $Q_i Y' Q_i^{-1} = Y^*$ for $i = 1, \dots, 6$, and (c_1', c_2') as above. Since Q is invertible, A, B, C cannot all be 0. Thus Y^* must have

$$(c_1', c_2') = (c_1, c_2), \quad \left(\frac{1}{c_1}, \frac{1}{c_2}\right), \quad \left(c_2, \frac{1}{c_1c_2}\right), \quad \left(\frac{1}{c_2}, c_1c_2\right), \quad \left(\frac{1}{c_1c_2}, c_1\right), \quad \text{or} \quad \left(c_1c_2, \frac{1}{c_1}\right).$$

It remains to check irreducibility of (X', Y').

Note that conjugating by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

yields $(X', Y') \approx (X'', Y')$ where

$$X'' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus (X', Y') is irreducible if and only if (X'', Y') is irreducible.

Let

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since (L, U) is irreducible, it is sufficient to generate L and U as k-linear combinations of products in X'', Y'. Note that this is a necessary and sufficient condition for (X'', Y') to be irreducible.

We now find conditions that let us generate L and U from X'' and Y'. Let $\Lambda'' = X''Y'X''Y'^2$ and $\Gamma'' = X''Y'^2X''Y'$. Since $Y'X''$, $\Lambda''Y'X''$, $\Lambda''\Gamma''Y'X''$, $\Gamma''Y'X''$, $\Lambda''^2Y'X''$, and $\Gamma''^2Y'X''$ are matrices of the form

$$\begin{pmatrix} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we attempt to form U and L as a k-linear combination of those matrices. For $a_i \in k$,

$$a_1Y'X'' + a_2\Lambda''Y'X'' + a_3\Lambda''\Gamma''Y'X'' + a_4\Gamma''Y'X'' + a_5\Lambda''^2Y'X'' + a_6\Gamma''^2Y'X'' = U$$

if and only if

$$A_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 1 & c_2 & c_1c_2^2 & c_1c_2 & c_2^2 & c_1^2c_2^2 \\ c_1 & 1 & c_2 & c_1c_2 & \frac{1}{c_1} & c_1c_2^2 \\ 1 & \frac{1}{c_1c_2} & \frac{1}{c_1^2c_2} & \frac{1}{c_1} & \frac{1}{c_1^2c_2^2} & \frac{1}{c_1^2} \\ c_2 & 1 & \frac{1}{c_1c_2} & \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_1^2c_2} \\ 1 & c_1 & \frac{c_1}{c_2} & \frac{1}{c_2} & c_1^2 & \frac{1}{c_2^2} \\ \frac{1}{c_1c_2} & 1 & c_1 & \frac{1}{c_2} & c_1c_2 & \frac{c_1}{c_2} \end{pmatrix}.$$

Similarly, for $b_i \in k$,

$$b_1Y'X'' + b_2\Lambda''Y'X'' + b_3\Lambda''\Gamma''Y'X'' + b_4\Gamma''Y'X'' + b_5\Lambda''^2Y'X'' + b_6\Gamma''^2Y'X'' = L$$

if and only if

$$A_1\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus U and L can be generated as above if $\det(A_1) = R_1/(c_1^6c_2^6) \neq 0$, where R_1 is a polynomial in c_1 and c_2. Likewise, we attempt to generate U and L using matrices $Y'X''$, $Y'X''\Lambda''$, $\Lambda''Y'X''$, $\Gamma''Y'X''$, $\Lambda''\Gamma''Y'X''$, and $\Lambda''^2Y'X''$. Using the method above, we obtain a matrix

$$B_1 = \begin{pmatrix} 1 & \frac{1}{c_1} & c_2 & c_1c_2 & c_1c_2^2 & c_2^2 \\ c_1 & \frac{1}{c_2} & 1 & c_1c_2 & c_2 & \frac{1}{c_1} \\ 1 & \frac{1}{c_2} & \frac{1}{c_1c_2} & \frac{1}{c_1} & \frac{1}{c_1^2c_2} & \frac{1}{c_1^2c_2^2} \\ c_2 & c_1c_2 & 1 & \frac{1}{c_1} & \frac{1}{c_1c_2} & \frac{1}{c_2} \\ 1 & c_1c_2 & c_1 & \frac{1}{c_2} & \frac{c_1}{c_2} & c_1^2 \\ \frac{1}{c_1c_2} & \frac{1}{c_1} & 1 & \frac{1}{c_2} & c_1 & c_1c_2 \end{pmatrix}.$$

We are able to generate L and U from the monomials above if $\det(B_1) = R_2/(c_1^5c_2^5) \neq 0$, where R_2 is a polynomial in c_1 and c_2. Thus, if we are not able to generate L and U in either way, $R_1 = R_2 = 0$.

Earlier, we concluded that (X', Y') is irreducible if and only if (X'', Y') is irreducible. Also, (X'', Y') is irreducible if and only if L and U can be generated from X'' and Y'. From this, we are able to conclude that if (X', Y') is not irreducible, then $R_1 = R_2 = 0$.

We now solve $R_1 = R_2 = 0$. By factoring R_2 we find

$$R_2 = (1 - c_1)(c_2 - 1)(c_1c_2 - 1)F_1F_2F_3$$

where F_1, F_2, F_3 are irreducible polynomials in c_1, c_2. Our explicit choices of F_1, F_2, and F_3 can be found in Appendix B. Then $R_1 = R_2 = 0$ if and only if $R_1 = 0$ and at least one of the following conditions hold: $c_1 = 1$, $c_2 = 1$, $c_1c_2 = 1$, $F_1 = 0$, $F_2 = 0$, or $F_3 = 0$.

- If $R_1 = 0$ and $c_1 = 1$ then $(c_2 - 1)^6(c_2^2 - 1)^3 = 0$.
- If $R_1 = 0$ and $c_2 = 1$ then $(c_1 - 1)^6(c_1^2 - 1)^3 = 0$.
- If $R_1 = 0$ and $c_1c_2 = 1$ then $(c_2 - 1)^6(c_2^2 - 1)^3 = 0$.

Thus if $R_1 = 0$ and either $c_1 = 1$, $c_2 = 1$, or $c_1c_2 = 1$, then $(c_1, c_2) = (1, 1)$, $(1, -1)$, $(-1, 1)$, or $(-1, -1)$ in fields not of characteristic 2 and $(c_1, c_2) = (1, 1)$ in fields of characteristic 2.

We next repeatedly use the command `sprem` in Maple to find a univariate polynomial in c_1 or c_2 which is contained in the ideal $\langle R_1, F_i\rangle$ of $k[c_1, c_2]$ for each $i = 1, 2, 3$. See Appendix B for the detailed Maple commands used in this section of the paper. Specifically, `sprem` inputs a variable x, and multivariate polynomials in x, say a and b. It computes multivariate polynomials in x with integer coefficients, say m and q, where $ma = bq + r$, and the degree of x in r is strictly less than the degree of x in b. The output of `sprem` is the multivariate function r. m is always of the form x^n for some n. Temporarily regard c_1 and c_2 as indeterminates, and temporarily replace R_1 and F_1 with their natural preimages in $\mathbb{Z}[c_1, c_2]$. Using the function `sprem` to recursively reduce the degree of c_2 when starting with initial polynomials R_1 and F_1, we see that $c_1^{34}(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6$ is in the ideal $\langle R_1, F_1\rangle$ of $\mathbb{Z}[c_1, c_2]$. Now let c_1, c_2, R_1, and F_1 again be elements of k. It follows that $c_1^{34}(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6$ is in the ideal $\langle R_1, F_1\rangle$ of $k[c_1, c_2]$. Hence if $R_1 = F_1 = 0$, either $c_1 = 0$, $c_1 = 1$, or $c_1^2 + c_1 + 1 = 0$.

Similarly, we temporarily regard c_1 and c_2 as indeterminates, and temporarily replace R_1 and F_2 with their natural preimages in $\mathbb{Z}[c_1, c_2]$. We use the Maple command `sprem` to recursively reduce the degree of c_2 when starting with initial polynomials R_1 and F_2. Our computation yields that $(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6(c_1^2 - c_1 + 1)^{11}$ is in the ideal $\langle R_1, F_2\rangle$ of $\mathbb{Z}[c_1, c_2]$. Returning c_1, c_2, R_1, and F_2 to their original form, we conclude that $(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6(c_1^2 - c_1 + 1)^{11}$ is in the ideal $\langle R_1, F_2\rangle$ of $k[c_1, c_2]$. Hence if $R_1 = F_2 = 0$, either $c_1 = 1$, $c_1^2 + c_1 + 1 = 0$, or $c_1^2 - c_1 + 1 = 0$.

Again, temporarily regard c_1 and c_2 as indeterminates, and temporarily replace R_1 and F_3 with their natural preimages in $\mathbb{Z}[c_1, c_2]$. We now solve $R_1 = F_3 = 0$ in two different ways. First, we recursively reduce the degree of c_2 by using the Maple command `sprem` with initial polynomials R_1 and F_3. We find

$$c_1^{186}(c_1 + 1)^{36}(c_1 - 1)^{56}\left(c_1^2 + c_1 + 1\right)^{57}T_1^4T_2^2$$

is in the ideal $\langle R_1, F_3\rangle \subset \mathbb{Z}[c_1, c_2]$ where T_1 is an irreducible polynomial in c_1 of degree 28 and where T_2 is an irreducible polynomial in c_1 of degree 40. Alternatively, factor R_1 into $(c_2 - c_1)(c_1^2c_2 - 1)(c_1c_2^2 - 1)R_3$, where R_3 is a nonhomogeneous polynomial in c_1 and c_2 of total degree 14. Now we use the Maple command `sprem` to recursively reduce the degree of c_2 using initial polynomials F_3 and each of the polynomials $(c_2 - c_1)$, $(c_1^2c_2 - 1)$, $(c_1c_2^2 - 1)$, and R_3. In respective order, we find that the following polynomials are in $\langle R_1, F_3\rangle \subset \mathbb{Z}[c_1, c_2]$:

- $(c_1 + 1)^2(c_1 - 1)^2(c_1^2 + c_1 + 1)^2$,
- $c_1^2(c_1 + 1)^2(c_1 - 1)^2(c_1^2 + c_1 + 1)^2$,

- $c_1^5(c_1-1)^4(c_1^2+c_1+1)^2$,
- $c_1^{110}(c_1+1)^{16}(c_1-1)^{44}(c_1^2+c_1+1)^{31}T_3^4T_4^2$

where T_3 is an irreducible polynomial of degree 16 and T_4 is an irreducible polynomial of degree 26. Now remove the temporary replacements of c_1, c_2, R_1, and F_3. By comparing solutions obtained in the two different ways, we conclude that if $R_1 = F_3 = 0$, then $c_1 = 0$, $c_1 - 1 = 0$, $c_1 + 1 = 0$, or $c_1^2 + c_1 + 1 = 0$.

Thus if $R_1 = 0$ and either F_1, F_2, or $F_3 = 0$, then $c_1 = 0$, $c_1 = 1$, $c_1 + 1 = 0$, $c_1^2 + c_1 + 1 = 0$, or $c_1^2 - c_1 + 1 = 0$. We reject the case that $c_1 = 0$ since this forces $Y^3 \neq I$. The case where $c_1 = 1$ was solved above. If k is not of characteristic 2, solving $c_1 + 1 = 0$ gives $c_1 = -1$. If $c_1 = -1$ and $R_1 = R_2 = 0$, then $c_2 = 1$ or $c_2 = -1$. These solutions are both listed above. If k is of characteristic 2 and $c_1 + 1 = 0$ or k is of characteristic 3 and $c_1^2 + c_1 + 1 = 0$ then $c_1 = 1$, which was solved above. Also, if k is of characteristic 3 and $c_1^2 - c_1 + 1 = 0$ then $c_1 = -1$, which was solved above. It remains to consider fields not of characteristic 3 where $c_1^2 + c_1 + 1 = 0$ or $c_1^2 - c_1 + 1 = 0$.

Thus assume k is not of characteristic 3, $c_1^2 + c_1 + 1 = 0$, and $R_1 = R_2 = 0$. Temporarily replace c_1, c_2, R_1, and R_2 with their natural preimages in $\mathbb{Z}[c_1, c_2]$. We use the Maple command `sprem` to recursively reduce the degree of c_1 with initial polynomials R_1 and $c_1^2 + c_1 + 1$, and R_2 and $c_1^2 + c_1 + 1$. We find that:

$$\begin{aligned}
&\big(c_2^2-c_2+1\big)\big(c_2^6-5c_2^5+23c_2^4-8c_2^3-c_2^2-2c_2+1\big)\big(c_2^2+c_2+1\big)^5\big(c_2^6-2c_2^5-c_2^4\\
&\quad -8c_2^3+23c_2^2-5c_2+1\big)\in\big\langle R_1, c_1^2+c_1+1\big\rangle, \quad \text{and}\\
&27c_2^2\big(c_2^2-2c_2+4\big)\big(4c_2^2-2c_2+1\big)(c_2-1)^2\big(c_2^2+c_2+1\big)^5\in\big\langle R_2, c_1^2+c_1+1\big\rangle.
\end{aligned}$$

Note that if $R_1 = R_2 = 0$ and $c_1^2+c_1+1 = 0$, then $c_2^2+c_2+1 = 0$. Now remove the temporary replacements of c_1, c_2, R_1, and R_2. Thus $c_1 = \zeta$ or ζ^2 and $c_2 = \zeta$ or ζ^2, where $\zeta^3 = 1$. Since $R_1 \neq 0$ when $(c_1, c_2) = (\zeta, \zeta^2)$, (ζ^2, ζ), we find that $(c_1, c_2) = (\zeta, \zeta)$ or (ζ^2, ζ^2).

Now assume k is not of characteristic 3, $c_1^2 - c_1 + 1 = 0$, and $R_1 = R_2 = 0$. Temporarily regard c_1 and c_2 as indeterminates, and temporarily replace R_1 and R_2 with their natural preimages in $\mathbb{Z}[c_1, c_2]$. Using the Maple command `sprem`, we recursively reduce the degree of c_1 with initial polynomials R_1 and $c_1^2 - c_1 + 1$, and R_2 and $c_1^2 - c_1 + 1$. As a result:

$$\begin{aligned}
&\big(c_2^2-c_2+1\big)\big(c_2^2+c_2+1\big)\big(c_2^4-c_2^2+1\big)\big(c_2^8-3c_2^7+9c_2^5+4c_2^4-18c_2^3+15c_2^2\\
&\quad -6c_2+1\big)\big(c_2^8-6c_2^7+15c_2^6-18c_2^5+4c_2^4+9c_2^3-3c_2+1\big)\in\big\langle R_1, c_1^2-c_1+1\big\rangle, \quad \text{and}\\
&c_2^2\big(3c_2^2-3c_2+1\big)\big(c_2^2-3c_2+3\big)\big(c_2^2-c_2+1\big)(c_2-1)^2\big(4c_2^4+6c_2^3+c_2^2-3c_2+1\big)\big(c_2^4-3c_2^3\\
&\quad +c_2^2+6c_2+4\big)\in\big\langle R_2, c_1^2-c_1+1\big\rangle.
\end{aligned}$$

Note that if $R_1 = R_2 = 0$ and $c_1^2-c_1+1 = 0$ then $c_2^2-c_2+1 = 0$. Now remove the temporary replacements of c_1, c_2, R_1, and R_2. Thus if $R_1 = R_2 = 0$, then $c_1 = -\zeta$ or $-\zeta^2$ and $c_2 = -\zeta$ or $-\zeta^2$, where $\zeta^3 = 1$. However, if $(c_1, c_2) = (-\zeta, -\zeta)$ or $(-\zeta^2, -\zeta^2)$, then $R_2 \neq 0$, which is a contradiction. Also, if $(c_1, c_2) = (-\zeta^2, -\zeta)$ or $(-\zeta, -\zeta^2)$, then $R_1 \neq 0$, which is a contradiction.

We now consider all solutions to $R_1 = R_2 = 0$.

- $(c_1, c_2) = (1,1), (\zeta,\zeta), (\zeta^2,\zeta^2)$ if k is of characteristic 2,
- $(c_1, c_2) = (1,1), (1,-1), (-1,1), (-1,-1)$, if k is of characteristic 3,
- $(c_1, c_2) = (1,1), (1,-1), (-1,1), (-1,-1), (\zeta,\zeta), (\zeta^2,\zeta^2)$ if k is not of characteristic 2 or 3.

Thus if (X', Y') is not irreducible, (c_1, c_2) is among the preceding pairs. The pair of matrices (X', Y') given when $(c_1, c_2) = (-1,-1), (-1,1),$ and $(1,-1)$ are all equivalent. Also, the pair of matrices (X', Y') when $(c_1, c_2) = (\zeta,\zeta)$ and (ζ^2,ζ^2) are equivalent. Thus it is sufficient to check irreducibility of (X', Y') when $(c_1, c_2) = (1,1), (\zeta,\zeta)$, and $(-1,-1)$.

If $c_1 = c_2$ and $c_1^3 = 1$, then $(X'Y')^2 = (Y'X')^2$. Let $A = X'Y'$ and $B = Y'X'$. Then (X', Y') is irreducible if and only if (A, B) is irreducible. Note $\Lambda\Gamma = \Gamma\Lambda$ yields $A^6 = B^6 = I$ by Remark 3.3. Also, $ABA = BAB$. From these relations, we can count possible monomials in A and B to find that (A, B) span at most an 18-dimensional space. Thus (A, B) is not irreducible, and hence (X', Y') is not irreducible.

If $c_1 = c_2 = -1$ and

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

then

$$\left(QX'Q^{-1}, QY'Q^{-1}\right) = \left(\begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& -1 && \\ &&&& -1 & \\ &&&&& -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

Since $(QX'Q^{-1}, QY'Q^{-1})$ is not irreducible, neither is (X', Y') when $c_1 = c_2 = -1$. This gives the desired result. $\square$

Remark 6.3. In Lemma 6.2, we proved further that if

$$(X, Y) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c_1c_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right)$$

where

(1) $(c_1, c_2) \neq (1,1), (-1,-1), (-1,1), (1,-1), (\zeta,\zeta), (\zeta^2,\zeta^2)$, if k does not have characteristic 2 or 3, where ζ is a primitive cube root of unity;
(2) $(c_1, c_2) \neq (1,1), (\zeta,\zeta), (\zeta^2,\zeta^2)$, if k has characteristic 2, where ζ is a primitive cube root of unity;
(3) $(c_1, c_2) \neq (1,1), (-1,-1), (-1,1), (1,-1)$, if k has characteristic 3,

then (X, Y) satisfies the properties in Lemma 3.4.

6.4 If $\rho: PSL_2(\mathbb{Z}) \to GL_n(k)$ is an irreducible 6-dimensional representation of $PSL_2(\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1\rangle$ with $X := \rho(x)$ and $Y := \rho(y)$, it follows from Lemma 6.1 that X and Y have one nonzero entry per row and column. In particular, $\rho : PSL_2(\mathbb{Z}) \to GL_n(k)$ is an irreducible 6-dimensional representation of $PSL_2(\mathbb{Z})$ if and only if (X, Y) satisfies the properties in Lemma 3.4. Then by Lemma 6.2 and Remark 6.3, we have classified the irreducible 6-dimensional representations of $PSL_2(\mathbb{Z})$ up to equivalence.

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Appendix A

In this appendix, we list the matrices used in the case-by-case analysis proofs in Lemma 6.1. Again, let γ_i denote $\Gamma_{i,i}$, and λ_i denote $\Lambda_{i,i}$. Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ denote the unique 6×6 permutation matrix which maps e_i to e_{a_i}, where e_i is the standard basis vector of k^6.

Case 5. In our proof, we found that $\lambda_1 = \frac{1}{\lambda_1}$ and $\lambda_5 = \frac{1}{\lambda_6}$. From $\Gamma X \Gamma = X$, we conclude that if $\gamma_i \gamma_j \neq 1$, then $X_{i,j} = 0$. From $\Lambda X \Lambda = X$, we conclude that if $\lambda_i \lambda_j \neq 1$, then $X_{i,j} = 0$. For each of the 30 matrices for Γ in terms of λ_i, we determine which entries of X must equal 0. Of the 30 possible matrices for Γ, we found that for all but 14 matrices, X is forced to be singular. Two of these Γ lead to (X, Y) irreducible. For the 12 remaining matrices for Γ, we replace (X, Y) with (PXP^{-1}, PYP^{-1}) for an appropriate permutation matrix P, so that Λ is preserved, $\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_3 = \lambda_5$, and $\gamma_4 = \lambda_6$. We list each of the 14 possible matrices for Γ, and for the 12 possible matrices for Γ which do not lead to (X, Y) irreducible, we list the corresponding permutation matrix P.

(1) $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \lambda_1$, $\gamma_5 = \lambda_5$, $\gamma_6 = \lambda_6$; $\Lambda = \Gamma$ implies (X, Y) reducible.
(2) $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \lambda_1$, $\gamma_5 = \lambda_6$, $\gamma_6 = \lambda_5$; $\Lambda\Gamma = I$ implies (X, Y) reducible.
(3) $\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_3 = \lambda_5$, $\gamma_4 = \lambda_6$; P is the identity matrix.
(4) $\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_3 = \lambda_6$, $\gamma_4 = \lambda_5$; $P = (1, 2, 4, 3, 5, 6)$.
(5) $\gamma_1 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_2 = \lambda_5$, $\gamma_4 = \lambda_6$; $P = (1, 3, 2, 4, 5, 6)$.
(6) $\gamma_1 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_2 = \lambda_5$, $\gamma_3 = \lambda_6$; $P = (1, 4, 2, 3, 5, 6)$.
(7) $\gamma_1 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_2 = \lambda_6$, $\gamma_3 = \lambda_5$; $P = (1, 4, 3, 2, 5, 6)$.
(8) $\gamma_1 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_2 = \lambda_6$, $\gamma_4 = \lambda_5$; $P = (1, 3, 4, 2, 5, 6)$.

(9) $\gamma_2 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_5$, $\gamma_4 = \lambda_6$; $P = (2, 3, 1, 4, 5, 6)$.
(10) $\gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_5$, $\gamma_3 = \lambda_6$; $P = (2, 4, 1, 3, 5, 6)$.
(11) $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_5$, $\gamma_2 = \lambda_6$; $P = (3, 4, 1, 2, 5, 6)$.
(12) $\gamma_2 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_6$, $\gamma_4 = \lambda_5$; $P = (2, 3, 4, 1, 5, 6)$.
(13) $\gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_6$, $\gamma_3 = \lambda_5$; $P = (2, 4, 3, 1, 5, 6)$.
(14) $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1$, $\gamma_1 = \lambda_6$, $\gamma_2 = \lambda_5$; $P = (3, 4, 2, 1, 5, 6)$.

Case 7. Here we list the 22 possible values for Γ and the corresponding forms of X and Y. The forms of X and Y are described by which entries (of X or Y respectively) must equal 0. There are 5 forms for X and 22 forms for Y.

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =$

(1) $(\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_5)$, (2) $(\lambda_1, \lambda_1, \lambda_3, \lambda_5, \lambda_5, \lambda_3)$, (3) $(\lambda_1, \lambda_1, \lambda_3, \lambda_5, \lambda_3, \lambda_5)$,
(4) $(\lambda_1, \lambda_1, \lambda_5, \lambda_3, \lambda_3, \lambda_5)$, (5) $(\lambda_1, \lambda_1, \lambda_5, \lambda_3, \lambda_5, \lambda_3)$, (6) $(\lambda_1, \lambda_1, \lambda_5, \lambda_5, \lambda_3, \lambda_3)$,
(7) $(\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1)$, (8) $(\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_1, \lambda_5)$, (9) $(\lambda_3, \lambda_5, \lambda_5, \lambda_1, \lambda_3, \lambda_1)$,
(10) $(\lambda_3, \lambda_5, \lambda_5, \lambda_1, \lambda_1, \lambda_3)$, (11) $(\lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_5, \lambda_1)$, (12) $(\lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_1, \lambda_5)$,
(13) $(\lambda_3, \lambda_5, \lambda_1, \lambda_5, \lambda_3, \lambda_1)$, (14) $(\lambda_3, \lambda_5, \lambda_1, \lambda_5, \lambda_1, \lambda_3)$, (15) $(\lambda_5, \lambda_3, \lambda_1, \lambda_3, \lambda_1, \lambda_5)$,
(16) $(\lambda_5, \lambda_3, \lambda_1, \lambda_3, \lambda_5, \lambda_1)$, (17) $(\lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1, \lambda_3)$, (18) $(\lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_3, \lambda_1)$,
(19) $(\lambda_5, \lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_5)$, (20) $(\lambda_5, \lambda_3, \lambda_3, \lambda_1, \lambda_5, \lambda_1)$, (21) $(\lambda_5, \lambda_3, \lambda_5, \lambda_1, \lambda_1, \lambda_3)$,
(22) $(\lambda_5, \lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_1)$.

For Γ in cases 1 and 6, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$, $(5, 3)$, $(5, 4)$, $(6, 3)$, or $(6, 4)$.

For Γ in cases 2 and 4, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 5)$, $(4, 6)$, $(5, 3)$, or $(6, 4)$.

For Γ in cases 3 and 5, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 6)$, $(4, 5)$, $(5, 4)$, or $(6, 3)$.

For Γ in cases 7, 9, 12, 14, 15, 17, 20, and 22, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 2)$, $(2, 1)$, $(3, 5)$, $(4, 6)$, $(5, 3)$, or $(6, 4)$.

For Γ in cases 8, 10, 11, 13, 16, 18, 19, and 21, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 2)$, $(2, 1)$, $(3, 6)$, $(4, 5)$, $(5, 4)$, or $(6, 3)$.

Fix Γ in one of the above cases. Then Y has the following form: if $Y_{i,j} \neq 0$, then $\lambda_i \gamma_j = 1$. Note that $\lambda_1 = \frac{1}{\lambda_1}$ and $\lambda_3 = \frac{1}{\lambda_5}$.

The selection of Γ in the cases 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, and 22 does not force (X, Y) to be reducible.

Case 8. Below are the 20 possible values for Γ and the corresponding forms of X and Y. The forms of X and Y are described by which entries (of X or Y respectively) must equal 0. There are 7 forms for X and 20 forms for Y.

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =$

(1) $(\lambda_1, \lambda_5, \lambda_6, \lambda_4, \lambda_1, \lambda_1)$, (2) $(\lambda_1, \lambda_6, \lambda_5, \lambda_4, \lambda_1, \lambda_1)$, (3) $(\lambda_5, \lambda_1, \lambda_6, \lambda_4, \lambda_1, \lambda_1)$,
(4) $(\lambda_5, \lambda_6, \lambda_1, \lambda_4, \lambda_1, \lambda_1)$, (5) $(\lambda_6, \lambda_1, \lambda_5, \lambda_4, \lambda_1, \lambda_1)$, (6) $(\lambda_6, \lambda_5, \lambda_1, \lambda_4, \lambda_1, \lambda_1)$,
(7) $(\lambda_4, \lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_1)$, (8) $(\lambda_4, \lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_1)$, (9) $(\lambda_5, \lambda_4, \lambda_6, \lambda_1, \lambda_1, \lambda_1)$,

(10) $(\lambda_5, \lambda_6, \lambda_4, \lambda_1, \lambda_1, \lambda_1)$, (11) $(\lambda_6, \lambda_4, \lambda_5, \lambda_1, \lambda_1, \lambda_1)$, (12) $(\lambda_6, \lambda_5, \lambda_4, \lambda_1, \lambda_1, \lambda_1)$,
(13) $(\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_5, \lambda_6)$, (14) $(\lambda_1, \lambda_1, \lambda_4, \lambda_1, \lambda_5, \lambda_6)$, (15) $(\lambda_1, \lambda_4, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$,
(16) $(\lambda_4, \lambda_1, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$, (17) $(\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_6, \lambda_5)$, (18) $(\lambda_1, \lambda_1, \lambda_4, \lambda_1, \lambda_6, \lambda_5)$,
(19) $(\lambda_1, \lambda_4, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$, (20) $(\lambda_4, \lambda_1, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$.

For Γ in cases 1, 2, 7, and 8, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(2, 3)$, $(3, 2)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 3, 5, 9, and 11, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 3)$, $(2, 2)$, $(3, 1)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 4, 6, 10, and 12, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 2)$, $(2, 1)$, $(3, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 13 and 17, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 14 and 18, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 15 and 19, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, $(3, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 16 and 20, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(2, 2)$, $(2, 3)$, $(3, 2)$, $(3, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

Fix Γ in one of the above cases. Then Y has the following form: if $Y_{i,j} \neq 0$, then $\lambda_i \gamma_j = 1$. Note that $\lambda_1 = \frac{1}{\lambda_1}$, $\lambda_4 = \frac{1}{\lambda_4}$, and $\lambda_5 = \frac{1}{\lambda_6}$.

The selection of Γ in the cases 7, 8, 9, 10, 11, and 12, does not force (X, Y) to be reducible.

Case 9. Here, we list the possible values for Γ and the corresponding forms of X and Y for two values of Λ.

First we assume $\lambda_1 = \frac{1}{\lambda_1}$, $\lambda_3 = \frac{1}{\lambda_3}$, and $\lambda_5 = \frac{1}{\lambda_6}$. There are 20 possible values of Γ. The forms of X and Y are described by which entries (of X or Y respectively) must equal 0. There are 4 forms for X and 20 forms for Y.

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =$

(1) $(\lambda_3, \lambda_3, \lambda_5, \lambda_6, \lambda_1, \lambda_1)$, (2) $(\lambda_3, \lambda_3, \lambda_6, \lambda_5, \lambda_1, \lambda_1)$, (3) $(\lambda_5, \lambda_6, \lambda_3, \lambda_3, \lambda_1, \lambda_1)$,
(4) $(\lambda_6, \lambda_5, \lambda_3, \lambda_3, \lambda_1, \lambda_1)$, (5) $(\lambda_1, \lambda_1, \lambda_5, \lambda_6, \lambda_3, \lambda_3)$, (6) $(\lambda_1, \lambda_1, \lambda_6, \lambda_5, \lambda_3, \lambda_3)$,
(7) $(\lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_3, \lambda_3)$, (8) $(\lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_3, \lambda_3)$, (9) $(\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_6)$,
(10) $(\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_6, \lambda_5)$, (11) $(\lambda_1, \lambda_3, \lambda_1, \lambda_3, \lambda_5, \lambda_6)$, (12) $(\lambda_1, \lambda_3, \lambda_1, \lambda_3, \lambda_6, \lambda_5)$,
(13) $(\lambda_1, \lambda_3, \lambda_3, \lambda_1, \lambda_5, \lambda_6)$, (14) $(\lambda_1, \lambda_3, \lambda_3, \lambda_1, \lambda_6, \lambda_5)$, (15) $(\lambda_3, \lambda_1, \lambda_1, \lambda_3, \lambda_5, \lambda_6)$,
(16) $(\lambda_3, \lambda_1, \lambda_1, \lambda_3, \lambda_6, \lambda_5)$, (17) $(\lambda_3, \lambda_1, \lambda_3, \lambda_1, \lambda_5, \lambda_6)$, (18) $(\lambda_3, \lambda_1, \lambda_3, \lambda_1, \lambda_6, \lambda_5)$,
(19) $(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$, (20) $(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$.

For Γ in cases 3, 4, 7, and 8, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 2)$, $(2, 1)$, $(3, 3)$, $(3, 4)$, $(4, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 1, 2, 5, and 6, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 4)$, $(4, 3)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 9, 10, 19, and 20, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 3)$, $(3, 4)$, $(4, 3)$, $(4, 4)$, $(5, 6)$, or $(6, 5)$.

For Γ in cases 11, 12, 13, 14, 15, 16, 17, and 18, X has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,6)$, or $(6,5)$.

Fix Γ in one of the above cases. Then Y has the following form: if $Y_{i,j} \neq 0$, then $\lambda_i \gamma_j = 1$. Note that the selection of Γ in the cases 1, 2, 7, and 8, does not force (X,Y) to be reducible.

Now assume $\lambda_1 = \frac{1}{\lambda_3}$ and $\lambda_5 = \frac{1}{\lambda_6}$. There are 44 possible values of Γ. The forms of X and Y are described by which entries (of X or Y respectively) must equal 0. There are 3 forms for X and 44 forms for Y.

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =$

(1) $(\lambda_1, \lambda_5, \lambda_3, \lambda_6, \lambda_1, \lambda_3)$, (2) $(\lambda_1, \lambda_5, \lambda_6, \lambda_3, \lambda_1, \lambda_3)$, (3) $(\lambda_1, \lambda_6, \lambda_3, \lambda_5, \lambda_1, \lambda_3)$
(4) $(\lambda_1, \lambda_6, \lambda_5, \lambda_3, \lambda_1, \lambda_3)$, (5) $(\lambda_3, \lambda_5, \lambda_1, \lambda_6, \lambda_1, \lambda_3)$, (6) $(\lambda_3, \lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_3)$
(7) $(\lambda_3, \lambda_6, \lambda_1, \lambda_5, \lambda_1, \lambda_3)$, (8) $(\lambda_3, \lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_3)$, (9) $(\lambda_5, \lambda_1, \lambda_3, \lambda_6, \lambda_1, \lambda_3)$,
(10) $(\lambda_5, \lambda_1, \lambda_6, \lambda_3, \lambda_1, \lambda_3)$, (11) $(\lambda_5, \lambda_3, \lambda_1, \lambda_6, \lambda_1, \lambda_3)$, (12) $(\lambda_5, \lambda_3, \lambda_6, \lambda_1, \lambda_1, \lambda_3)$,
(13) $(\lambda_6, \lambda_1, \lambda_3, \lambda_5, \lambda_1, \lambda_3)$, (14) $(\lambda_6, \lambda_1, \lambda_5, \lambda_3, \lambda_1, \lambda_3)$, (15) $(\lambda_6, \lambda_3, \lambda_1, \lambda_5, \lambda_1, \lambda_3)$,
(16) $(\lambda_6, \lambda_3, \lambda_5, \lambda_1, \lambda_1, \lambda_3)$, (17) $(\lambda_1, \lambda_5, \lambda_3, \lambda_6, \lambda_3, \lambda_1)$, (18) $(\lambda_1, \lambda_5, \lambda_6, \lambda_3, \lambda_3, \lambda_1)$,
(19) $(\lambda_1, \lambda_6, \lambda_3, \lambda_5, \lambda_3, \lambda_1)$, (20) $(\lambda_1, \lambda_6, \lambda_5, \lambda_3, \lambda_3, \lambda_1)$, (21) $(\lambda_3, \lambda_5, \lambda_1, \lambda_6, \lambda_3, \lambda_1)$,
(22) $(\lambda_3, \lambda_5, \lambda_6, \lambda_1, \lambda_3, \lambda_1)$, (23) $(\lambda_3, \lambda_6, \lambda_1, \lambda_5, \lambda_3, \lambda_1)$, (24) $(\lambda_3, \lambda_6, \lambda_5, \lambda_1, \lambda_3, \lambda_1)$,
(25) $(\lambda_5, \lambda_1, \lambda_3, \lambda_6, \lambda_3, \lambda_1)$, (26) $(\lambda_5, \lambda_1, \lambda_6, \lambda_3, \lambda_3, \lambda_1)$, (27) $(\lambda_5, \lambda_3, \lambda_1, \lambda_6, \lambda_3, \lambda_1)$,
(28) $(\lambda_5, \lambda_3, \lambda_6, \lambda_1, \lambda_3, \lambda_1)$, (29) $(\lambda_6, \lambda_1, \lambda_3, \lambda_5, \lambda_3, \lambda_1)$, (30) $(\lambda_6, \lambda_1, \lambda_5, \lambda_3, \lambda_3, \lambda_1)$,
(31) $(\lambda_6, \lambda_3, \lambda_1, \lambda_5, \lambda_3, \lambda_1)$, (32) $(\lambda_6, \lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_1)$, (33) $(\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_5, \lambda_6)$,
(34) $(\lambda_1, \lambda_3, \lambda_1, \lambda_3, \lambda_5, \lambda_6)$, (35) $(\lambda_1, \lambda_3, \lambda_3, \lambda_1, \lambda_5, \lambda_6)$, (36) $(\lambda_3, \lambda_1, \lambda_1, \lambda_3, \lambda_5, \lambda_6)$,
(37) $(\lambda_3, \lambda_1, \lambda_3, \lambda_1, \lambda_5, \lambda_6)$, (38) $(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$, (39) $(\lambda_1, \lambda_1, \lambda_3, \lambda_3, \lambda_6, \lambda_5)$,
(40) $(\lambda_1, \lambda_3, \lambda_1, \lambda_3, \lambda_6, \lambda_5)$, (41) $(\lambda_1, \lambda_3, \lambda_3, \lambda_1, \lambda_6, \lambda_5)$, (42) $(\lambda_3, \lambda_1, \lambda_1, \lambda_3, \lambda_6, \lambda_5)$,
(43) $(\lambda_3, \lambda_1, \lambda_3, \lambda_1, \lambda_6, \lambda_5)$, (44) $(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$.

For Γ in cases 1, 3, 5, 7, 10, 12, 14, 16, 17, 19, 21, 23, 26, 28, 30, 32, 35, 36, 41, and 42, X has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,3)$, $(2,4)$, $(3,1)$, $(4,2)$, $(5,6)$, or $(6,5)$.

For Γ in cases 2, 4, 6, 8, 9, 11, 13, 15, 18, 20, 22, 24, 25, 27, 29, 31, 34, 37, 40, and 43 X has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,4)$, $(2,3)$, $(3,2)$, $(4,1)$, $(5,6)$, or $(6,5)$.

For Γ in cases 33, 38, 39, and 44, X has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(4,1)$, $(4,2)$, $(5,6)$, or $(6,5)$.

Fix Γ in one of the above cases. Then Y has the following form: if $Y_{i,j} \neq 0$, then $\lambda_i \gamma_j = 1$. We note that the selection of Γ in the cases 1 through 32 does not force (X,Y) to be reducible.

Case 10. Listed below are the 24 possible values for Γ and the corresponding forms of X and Y. The forms of X and Y are described by which entries (of X or Y respectively) must equal 0. There are 2 forms for X and 24 forms for Y.

$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =$

(1) $(\lambda_1, \lambda_1, \lambda_5, \lambda_6, \lambda_3, \lambda_4)$, (2) $(\lambda_1, \lambda_1, \lambda_6, \lambda_5, \lambda_3, \lambda_4)$, (3) $(\lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_3, \lambda_4)$,
(4) $(\lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_3, \lambda_4)$, (5) $(\lambda_1, \lambda_1, \lambda_5, \lambda_6, \lambda_4, \lambda_3)$, (6) $(\lambda_1, \lambda_1, \lambda_6, \lambda_5, \lambda_4, \lambda_3)$,
(7) $(\lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_4, \lambda_3)$, (8) $(\lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_4, \lambda_3)$, (9) $(\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$,

(10) $(\lambda_1, \lambda_1, \lambda_4, \lambda_3, \lambda_5, \lambda_6)$, (11) $(\lambda_3, \lambda_4, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$, (12) $(\lambda_4, \lambda_3, \lambda_1, \lambda_1, \lambda_5, \lambda_6)$,
(13) $(\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_5)$, (14) $(\lambda_1, \lambda_1, \lambda_4, \lambda_3, \lambda_6, \lambda_5)$, (15) $(\lambda_3, \lambda_4, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$,
(16) $(\lambda_4, \lambda_3, \lambda_1, \lambda_1, \lambda_6, \lambda_5)$, (17) $(\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_1, \lambda_1)$, (18) $(\lambda_3, \lambda_4, \lambda_6, \lambda_5, \lambda_1, \lambda_1)$,
(19) $(\lambda_4, \lambda_3, \lambda_5, \lambda_6, \lambda_1, \lambda_1)$, (20) $(\lambda_4, \lambda_3, \lambda_6, \lambda_5, \lambda_1, \lambda_1)$, (21) $(\lambda_5, \lambda_6, \lambda_3, \lambda_4, \lambda_1, \lambda_1)$,
(22) $(\lambda_5, \lambda_6, \lambda_4, \lambda_3, \lambda_1, \lambda_1)$, (23) $(\lambda_6, \lambda_5, \lambda_3, \lambda_4, \lambda_1, \lambda_1)$, (24) $(\lambda_6, \lambda_5, \lambda_4, \lambda_3, \lambda_1, \lambda_1)$.

For Γ in cases 3, 4, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1,2)$, $(2,1)$, $(3,4)$, $(4,3)$, $(5,6)$, or $(6,5)$.

For Γ in cases 1, 2, 5, 6, 9, 10, 13, and 14, X has the following form: if $X_{i,j} \neq 0$, then $(i, j) = (1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,4)$, $(4,3)$, $(5,6)$, or $(6,5)$.

Fix Γ in one of the above cases. Then Y has the following form: if $Y_{i,j} \neq 0$, then $\lambda_i \gamma_j = 1$. Note that $\lambda_1 = \frac{1}{\lambda_1}$, $\lambda_3 = \frac{1}{\lambda_4}$, and $\lambda_5 = \frac{1}{\lambda_6}$. Also, the selection of Γ in the cases 3, 4, 7, 8, 17, 18, 19, and 20, does not force (X, Y) to be reducible.

Appendix B

In this appendix, we list the specific choices of polynomials and the Maple commands used in the latter half of the proof of Lemma 6.2.

Our choices of F_1, F_2, and F_3 are:

$$
\begin{aligned}
F_1 &= c_1^2c_2^2 - c_1^2c_2 + c_1^2 - c_1c_2 - c_1 + 1, \\
F_2 &= c_1^2c_2^2 - c_1c_2^2 + c_2^2 - c_1c_2 - c_2 + 1, \\
F_3 &= c_1^4c_2^4 + c_1^4c_2^3 + c_1^4c_2^2 + c_1^3c_2^4 - c_1^3c_2^3 - c_1^3c_2^2 + c_1^3c_2 + c_1^2c_2^4 - c_1^2c_2^3 - 6c_1^2c_2^2 \\
&\quad - c_1^2c_2 + c_1^2 + c_1c_2^3 - c_1c_2^2 - c_1c_2 + c_1 + c_2^2 + c_2 + 1.
\end{aligned}
$$

For multivariate polynomials A and B in the indeterminate x, sprem(A, B, x, ‘m’, ‘q’) outputs a multivariate polynomial r, where $mA = qB + r$ and m, q, r are polynomials over $\mathbb{Z}$. The variables m and q are assigned their corresponding polynomial values. Furthermore, the degree of x in r is strictly less than the degree of x in B. We use the command ‘factor’ to determine the roots of the polynomials in question.

To solve $R_1 = F_1 = 0$:
```
> A1:= sprem(R1, F1, c2, ‘m’, ‘q’);
> sprem(F1, A1, c2, ‘m’, ‘q’);
> factor(%);
```

To solve $R_1 = F_2 = 0$:
```
> A2:= sprem(R1, F2, c2, ‘m’, ‘q’);
> sprem(F2, A2, c2, ‘m’, ‘q’);
> factor(%);
```

To solve $R_1 = F_3 = 0$:
Without factoring R_1:
```
> A3:= sprem(R1, F3, c2, ‘m’, ‘q’);
```

```
> A4:= sprem(F3, A3, c2, ‘m’, ‘q’);
> A5:= sprem(A3, A4, c2, ‘m’, ‘q’);
> sprem(A4, A5, c2, ‘m’, ‘q’);
> factor(%);
```

By factoring R_1:

```
> factor(R1);
> sprem(F3, c1^2 * c2 - 1, c2, ‘m’, ‘q’);
> factor(%);
> sprem(F3, c2 - c1, c2, ‘m’, ‘q’);
> factor(%);
> A6:= sprem(F3, c1 * c2^2 - 1, c2, ‘m’, ‘q’);
> sprem(c1 * c2^2 - 1, A6, c2, ‘m’, ‘q’);
> factor(%);
> A7:= sprem(R3, F3, c2, ‘m’, ‘q’);
> A8:= sprem(F3, A7, c2, ‘m’, ‘q’);
> A9:= sprem(A7, A8, c2, ‘m’, ‘q’);
> sprem(A8, A9, c2, ‘m’, ‘q’);
> factor(%);
```

To solve $R_1 = R_2 = c_1^2 + c_1 + 1 = 0$:
We first solve $R_1 = c_1^2 + c_1 + 1 = 0$.

```
> B1:= sprem(R1, c1^2 + c1 + 1, c1, ‘m’, ‘q’);
> sprem(c1^2 + c1 + 1, B1, c1, ‘m’, ‘q’);
> factor(%);
```

Then we solve $R_2 = c_1^2 + c_1 + 1 = 0$.

```
> B2:= sprem(R2, c1^2 + c1 + 1, c1, ‘m’, ‘q’);
> sprem(c1^2 + c1 + 1, B2, c1, ‘m’, ‘q’);
> factor(%);
```

To solve $R_1 = R_2 = c_1^2 - c_1 + 1 = 0$:
We first solve $R_1 = c_1^2 - c_1 + 1 = 0$.

```
> B3:= sprem(R1, c1^2 - c1 + 1, c1, ‘m’, ‘q’);
> sprem(c1^2 - c1 + 1, B3, c1, ‘m’, ‘q’);
> factor(%);
```

Then we solve $R_2 = c_1^2 - c_1 + 1 = 0$.

```
> B4:= sprem(R2, c1^2 - c1 + 1, c1, ‘m’, ‘q’);
> sprem(c1^2 - c1 + 1, B4, c1, ‘m’, ‘q’);
> factor(%);
```

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