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Lifting units in clean rings

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ABSTRACT

Let R be a clean ring with an ideal I such that R/I is semiperfect and $K_0(I)$ is torsion-free. We prove that, under some mild conditions, units in R/I can be lifted to units in R . This implies that the matrix ring $M_n(R)$ over Bergman's example of a non-clean exchange ring R is not clean, for every n . We also obtain some other results concerning lifting units in clean rings.

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1. Introduction

The problem of lifting units from a quotient of a ring R modulo a two-sided ideal I has been of interest in several instances. Mostly the problem was considered for various classes of exchange rings, especially certain classes of C^* -algebras with real rank zero. Breuer [5,6] and Olsen [17] considered this question for von Neumann algebras. Menal and Moncasi [14], and Ara [2] considered the question for Rickart C^* -algebras and certain types of self-injective rings. Perera [18] found a general principle for verifying the unit lifting property, for a very wide class of exchange rings, namely separative exchange rings. In all those cases the condition whether a unit can be lifted was measured by vanishing of the connecting index map in K -theory.

In this paper we consider the unit lifting property for another class of exchange rings, namely the class of clean rings. We show that clean rings possess some nice properties which in general do not hold for exchange rings. For example, we prove that if R is clean and I is an ideal such that R/I is local, then units lift modulo I . This does not hold for all exchange rings.

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We also prove the following theorem:

Theorem. *Let R be a clean ring with an ideal I such that $k = R/I$ is semiperfect and $K_0(I)$ is torsion-free, and suppose that in the decomposition $k/J(k) \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, with n_i positive integers and D_i division rings, each division ring D_i is algebraic over its center. Then the map $K_1(R) \rightarrow K_1(R/I)$, induced by the canonical projection $R \rightarrow R/I$, is surjective.*

Moreover, if we assume that I in this theorem is separative (which is a very mild assumption), then units in R/I can be lifted modulo I . This implies that the matrix ring $M_n(R)$ over Bergman's example of a non-clean exchange ring R is not clean, for every n . We find this observation potentially helpful for solving the Morita invariance problem for clean rings, which is still open.

Some other corollaries are also obtained, for example, if R is a clean ring with a separative ideal I such that $K_0(I)$ is torsion-free and R/I is a finite dimensional algebra over a field, then units in R/I can be lifted to units in R . This again does not hold for all exchange rings.

In this paper, rings are associative and, unless otherwise specified, unital. For a ring R , we denote the set of all idempotents by $\text{Id}(R)$, the set of units by $U(R)$, the Jacobson radical by $J(R)$, and the $n \times n$ matrix ring by $M_n(R)$. One exception are units in $M_n(R)$, where we write $GL_n(R)$ rather than $U(M_n(R))$.

2. Preliminaries

A ring R is called an *exchange ring* if for every $a \in R$ there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. Examples of exchange rings include semiperfect rings, π -regular rings [21, Example 2.3], right self-injective rings [4, Remark 2.9(b)] and C^* -algebras with real rank zero [3, Theorem 7.2]. The class of exchange rings is closed under taking corners and matrix extensions, meaning that if R is exchange then $M_n(R)$ and eRe are exchange rings, for every $n \in \mathbb{N}$ and $e \in \text{Id}(R)$.

The definition of an exchange ring can be generalized to non-unital rings. Namely, a (possibly non-unital) ring I is said to be an *exchange ring* if for each $a \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = ra = a + s - sa$. This definition agrees with the definition for unital rings if I has a unit. In [1], Ara proved that if I is a two-sided ideal of a ring R then R is an exchange ring if and only if I and R/I are exchange and idempotents in R/I lift modulo I .

A ring R is called *clean* if every element $a \in R$ is a sum of an idempotent and a unit in R . Every clean ring is exchange and if idempotents in the ring are central, then the converse also holds [15, Proposition 1.8]. Other examples of clean rings include semiperfect rings [9], unit-regular rings [7] and endomorphism rings of continuous modules [8]. The class of clean rings is closed under taking matrix extensions, meaning that if R is clean then $M_n(R)$ is clean for every n [11]. The converse of this proposition is still an open question (see [11]). There exists an exchange ring which is not clean [12, Example 1].

If R is a ring, then $GL(R)$ is defined to be the direct limit of the directed system

$$U(R) \rightarrow GL_2(R) \rightarrow GL_3(R) \rightarrow \cdots$$

where each $a \in GL_n(R)$ is mapped to $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$. Let $E(R)$ denote the subgroup of $GL(R)$ generated by all elementary matrices. Then $E(R) = [GL(R), GL(R)]$ and $K_1(R)$ is defined to be the quotient group $GL(R)/E(R) = GL(R)_{\text{ab}}$.

We recall that if I is a two-sided ideal of a ring R , then there is an exact sequence of abelian groups

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(I) \xrightarrow{i_*} K_0(R) \xrightarrow{q_*} K_0(R/I),$$

where the maps $q_* : K_i(R) \rightarrow K_i(R/I)$ are induced by the canonical projection $q : R \rightarrow R/I$ and i_* is induced by inclusion $I \rightarrow R$. We refer the reader to [19] for more details.

If I is any ring (possibly non-unital), embedded into a unital ring R as a two-sided ideal, then $V(I)$ is defined to be a monoid of all isomorphism classes of finitely generated projective right R -modules P such that $PI = P$, equipped with the operation $[P] + [Q] = [P \oplus Q]$. This monoid can also be viewed as the set of all equivalence classes of idempotent matrices in $M_\infty(I)$ with finitely many nonzero entries in I , where two idempotents $e, f \in M_\infty(I)$ are equivalent provided their images are isomorphic as R -modules, or equivalently, there exist finite matrices x, y over I such that $e = xy$ and $f = yx$. This shows that $V(I)$ does not really depend on the involving ring R .

A monoid A is called *separative* provided that $a + a = a + b = b + b$ implies $a = b$ for every $a, b \in A$. A (possibly non-unital) ring I is *separative* if $V(I)$ is a separative monoid. Separativity is found to be the key condition for many problems related to exchange ring (see [3,4,18]). Moreover, this weak cancellation condition holds widely (the existence of a non-separative exchange ring is still an open question), and is therefore regarded as a condition that might hold for all exchange rings. We refer the reader to [18] or [3] for more details.

3. Lifting units in clean rings

We begin with the observation that units do not necessarily lift in clean rings modulo every ideal. For example, consider the ring R of all endomorphisms of an infinite dimensional vector space, with a two-sided ideal I of all endomorphisms with finite rank. Then R is clean by [16], the left shift operator in this ring is invertible modulo I , but it cannot be lifted to a unit in R .

Therefore it is natural to put some additional assumptions on the ideal I . The following proposition gives the unit lifting property under the assumption that R/I is local.

Proposition 3.1. *Let R be a clean ring and I an ideal such that R/I is a local ring. Then units in R/I lift to units in R .*

Proof. Take any $\alpha \in U(R/I)$. For an element $x \in R$, we denote by \bar{x} its homomorphic image in R/I . We need to prove that there exists $u \in U(R)$ such that $\alpha = \bar{u}$. Write $\alpha = \bar{x}$ and $\alpha^{-1} = \bar{y}$ for some $x, y \in R$. Since R is clean, we have $x = e + u$ and $y = g + v$ for some $e, g \in \text{Id}(R)$ and $u, v \in U(R)$. Passing to the factor ring, we have $\alpha = \bar{e} + \bar{u}$ and $\alpha^{-1} = \bar{g} + \bar{v}$. Note that \bar{e}, \bar{g} are idempotents in R/I , therefore $\bar{e}, \bar{g} \in \{0, 1\}$. If $\bar{e} = 0$ then $\alpha = \bar{u}$, which finishes the proof. If $\bar{g} = 0$ then $\alpha^{-1} = \bar{v}$, hence $\alpha = \bar{v}^{-1}$, which again finishes the proof. Hence we may assume that $\bar{e} = \bar{g} = 1$. Multiplying the equation $\alpha^{-1} = 1 + \bar{v}$ by α , we have $1 = \alpha + \alpha\bar{v}$, hence $\alpha = 1 + \bar{u} = \alpha + \alpha\bar{v} + \bar{u}$. Therefore $\alpha\bar{v} = -\bar{u}$ and $\alpha = -u\bar{v}^{-1}$. \square

The above proposition provides perhaps the easiest explanation why Bergman's ring [20, Example 3.1] is not clean. Indeed, in that ring there is an ideal I such that R/I is a field, but units in R/I clearly do not lift to units in R . (See [20] for the detailed description of R and I .)

Corollary 3.2. *Let R be a clean ring and I an ideal such that R/I is local. Then there is a natural isomorphism $K_0(R) \cong K_0(I) \oplus \mathbb{Z}$.*

Proof. We begin with the exact sequence of abelian groups

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \rightarrow K_0(I) \rightarrow K_0(R) \rightarrow K_0(R/I). \quad (1)$$

First, let us prove that the mapping $q_* : K_1(R) \rightarrow K_1(R/I)$ in that sequence is surjective. Take any element $[A] \in K_1(R/I)$, with $A \in GL_n(R/I)$. By [19, Theorem 2.2.5 and Corollary 2.2.6] there exists $\alpha \in U(R/I)$ such that $[A] = [\alpha]$ in $K_1(R/I)$. By Proposition 3.1 we have $u \in U(R)$ such that $\alpha = u + I$. Therefore $q_*([u]) = [\alpha] = [A]$, as desired.

Now surjectivity of q_* implies that the mapping $K_1(R/I) \rightarrow K_0(I)$ in (1) is zero, hence $K_0(I) \rightarrow K_0(R)$ is injective. Furthermore, the mapping $K_0(R) \rightarrow K_0(R/I)$ is surjective by [1, Theorem 3.5]. Thus the sequence

$$0 \rightarrow K_0(I) \rightarrow K_0(R) \rightarrow K_0(R/I) \rightarrow 0$$

is exact. Since R/I is a local ring, we have $K_0(R/I) \cong \mathbb{Z}$. As \mathbb{Z} is a projective \mathbb{Z} -module, this finally gives the desired isomorphism. \square

To prove our main result, we will need the following.

Lemma 3.3. *Let R be a clean ring with an ideal I . Then every unit in R/I of the form $1 + \alpha$, with $\alpha \in J(R/I)$, can be lifted modulo I .*

Proof. Let $\alpha \in J(R/I)$. For every $x \in R$, we denote by \bar{x} its homomorphic image in R/I . Take $a \in R$ such that $\bar{a} = \alpha$, and, by assumption, write $-a = e + u$, with $e = 1 - f \in \text{Id}(R)$ and $u \in U(R)$. We have $-fa = fu$, therefore $\bar{f} = -\overline{fa}u^{-1} = -\bar{f}\alpha\bar{u}^{-1} \in J(R/I)$. Since $J(R/I)$ contains no nonzero idempotents, we conclude that $f \in I$. This gives $1 + a + u = f \in I$, and hence $1 + \alpha = \overline{-u}$, as desired. \square

Lemma 3.4. *Let R be a clean ring with an ideal I . Denote $k = R/I$ and let $p_* : K_1(k) \rightarrow K_1(k/J(k))$ be a homomorphism induced by the canonical projection $p : k \rightarrow k/J(k)$, and $\partial : K_1(k) \rightarrow K_0(I)$ the connecting map in K -theory. Then there exists a unique homomorphism $\bar{\partial} : K_1(k/J(k)) \rightarrow K_0(I)$ such that $\partial = \bar{\partial} \circ p_*$.*

Proof. By the well-known factorization theorem, it suffices to prove that $p_* : K_1(k) \rightarrow K_1(k/J(k))$ is surjective and $\ker(p_*) \subseteq \ker(\partial)$. The former is obvious since invertible matrices can always be lifted to invertible matrices modulo the Jacobson radical. To prove the latter, take $[A] \in \ker(p_*)$, with A an invertible matrix over k . We have $[p(A)] = 1$, hence $p(A) \in E(k/J(k))$ is a product of elementary matrices. Since elementary matrices can be lifted to elementary matrices in $GL(k)$, we may assume that $p(A) = 1$. Therefore $A = 1 + B \in GL_n(k)$, with $B \in J(M_n(k))$. Now, by [11] $M_n(R)$ is a clean ring, hence by Lemma 3.3 there exists $U \in GL_n(R)$ such that $q(U) = 1 + B = A$, where $q : R \rightarrow R/I$ denotes the canonical projection. Hence $q_*([U]) = [A]$. Now recall that the sequence

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(I)$$

is exact. Hence we have $\partial([A]) = \partial(q_*([U])) = 0$, as desired. \square

Theorem 3.5. *Let R be a clean ring with an ideal I such that $k = R/I$ is semiperfect and $K_0(I)$ is torsion-free. Suppose that in the decomposition $k/J(k) \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, with n_i positive integers and D_i division rings, each division ring D_i is algebraic over its center. Then the map $K_1(R) \rightarrow K_1(R/I)$, induced by the canonical projection $R \rightarrow R/I$, is surjective.*

Proof. Suppose that the assumptions of the theorem hold. First we fix some notation. Denote by $q : R \rightarrow k$ the canonical projection with the kernel I , and by $q_* : K_1(R) \rightarrow K_1(k)$ the induced homomorphism. Next, we write $S = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ and denote by $p : k \rightarrow S$ the epimorphism with the kernel $J(k)$. Let $p_* : K_1(k) \rightarrow K_1(S)$ be the induced homomorphism. We have

$$K_1(S) \cong K_1(D_1) \times \cdots \times K_1(D_r) \cong (D_1^*)_{\text{ab}} \times \cdots \times (D_r^*)_{\text{ab}}$$

where $(D_i^*)_{\text{ab}}$ denotes the abelianization of the multiplicative group $D_i^* = D_i \setminus \{0\}$, for each i . Denote by

$$\psi : K_1(S) \rightarrow (D_1^*)_{\text{ab}} \times \cdots \times (D_r^*)_{\text{ab}}$$

the corresponding isomorphism. Note that ψ is induced by the “determinant” maps $\det : GL(D_i) \rightarrow (D_i^*)_{\text{ab}}$ (see [19, Theorem 2.2.5 and Corollary 2.2.6]).

The connecting map $K_1(k) \rightarrow K_0(I)$ will be denoted by ∂ . By Lemma 3.4, there exists a homomorphism $\bar{\partial} : K_1(S) \rightarrow K_0(I)$ such that $\partial = \bar{\partial} \circ p_*$.

Now, the theorem claims that q_* is surjective. In order to prove that claim, we first recall that the sequence

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(I)$$

is exact. Hence we need to prove that $\partial = 0$. It suffices to see that $\bar{\partial} = 0$, or equivalently, $\bar{\partial} \circ \psi^{-1} = 0$. Furthermore, it suffices to verify that $\bar{\partial}(\psi^{-1}([\alpha_1], 1, \dots, 1)) = 0$ for each $\alpha_1 \in D_1^*$. (Indeed, since by symmetry we will then have $\bar{\partial}(\psi^{-1}(1, [\alpha_2], 1, \dots, 1)) = 0$ for each $\alpha_2 \in D_2^*$, and so forth, and the conclusion will follow from the fact that $\bar{\partial} \circ \psi^{-1}$ is a homomorphism.) To simplify the notation, we denote the homomorphism

$$\rho : (D_1^*)_{\text{ab}} \rightarrow K_0(I), \quad \rho([\alpha]) = \bar{\partial}(\psi^{-1}([\alpha], 1, \dots, 1)), \quad \alpha \in D_1^*.$$

Pick any $\alpha \in D_1^*$. We are proving that $\rho([\alpha]) = 0$. Let F_1 be the center of D_1 , with F_1^* the corresponding multiplicative group. Since D_1 is algebraic over F_1 , by [13, Lemma 2(i)] there exists $m \geq 1$ such that $\alpha^m \in F_1^* D_1^{*'}$, where $D_1^{*'}$ denotes the commutator subgroup of D_1^* . Writing $\alpha^m = \alpha' x$, with $\alpha' \in F_1$ and $x \in D_1^{*'}$, we have $[\alpha^m] = [\alpha']$ in $(D_1^*)_{\text{ab}}$. Therefore $m\rho([\alpha]) = \rho([\alpha^m]) = \rho([\alpha'])$. If we prove that $\rho([\alpha']) = 0$, then, since $K_0(I)$ is torsion-free, it will follow that $\rho([\alpha]) = 0$. Hence we may assume that $\alpha \in D_1^*$ is central. Clearly, we may also assume that $\alpha \neq 1$.

Now let $A \in M_{n_1}(D_1)$ be a scalar matrix with α 's on the diagonal, and $a \in R$ an element such that $p(q(a)) = (A, 1, \dots, 1)$. Since R is a clean ring, there exist $e \in \text{Id}(R)$ and $u \in U(R)$ such that $a = e + u$. Denote $(\epsilon_1, \dots, \epsilon_r) = p(q(e))$ and $(\mu_1, \dots, \mu_r) = p(q(u))$, with $\epsilon_i \in \text{Id}(M_{n_i}(D_i))$ and $\mu_i \in GL_{n_i}(D_i)$ for each i . Since $(A, 1, \dots, 1) = (\epsilon_1, \dots, \epsilon_r) + (\mu_1, \dots, \mu_r)$, it follows that $1 - \epsilon_i = \mu_i$ is an invertible idempotent for each $i \geq 2$, hence $\epsilon_i = 0$ and $\mu_i = 1$ for $i \geq 2$.

Now ϵ_1 is an idempotent matrix over a division ring. It is known that idempotent matrices over division rings are similar to diagonal idempotents (see, for example, [10, Exercise 7 on p. 162]). Thus there exists $v \in GL_{n_1}(D_1)$ such that $\delta = v\epsilon_1 v^{-1}$ is diagonal (with 1's and 0's on the diagonal). As A is central, we have $vAv^{-1} = A$. It follows that

$$\begin{aligned} [(A - \epsilon_1, 1, \dots, 1)] &= [(v, 1, \dots, 1)(A - \epsilon_1, 1, \dots, 1)(v, 1, \dots, 1)^{-1}] \\ &= [(A - v\epsilon_1 v^{-1}, 1, \dots, 1)] = [(A - \delta, 1, \dots, 1)] \end{aligned}$$

in $K_1(S)$. Observe that $A - \delta$ is a diagonal matrix, with α 's and $(\alpha - 1)$'s on the diagonal. Hence we have

$$\begin{aligned} \psi([(A - \epsilon_1, 1, \dots, 1)]) &= \psi([(A - \delta, 1, \dots, 1)]) \\ &= (\det(A - \delta), 1, \dots, 1) = ([\alpha^{n_1-c}(\alpha - 1)^c], 1, \dots, 1), \end{aligned}$$

where c denotes the rank of δ . Therefore

$$\begin{aligned} \bar{\partial}([(A - \epsilon_1, 1, \dots, 1)]) &= \bar{\partial}(\psi^{-1}([\alpha^{n_1-c}(\alpha - 1)^c], 1, \dots, 1)) \\ &= \rho([\alpha^{n_1-c}(\alpha - 1)^c]) = (n_1 - c)\rho([\alpha]) + c\rho([\alpha - 1]). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \bar{\partial}([(A - \epsilon_1, 1, \dots, 1)]) &= \bar{\partial}([\mu_1, \dots, \mu_r]) \\ &= \bar{\partial}([p(q(u))]) = \bar{\partial}(p_*(q_*([u]))) = \partial(q_*([u])) = 0. \end{aligned}$$

These two equations together give

$$(n_1 - c)\rho([\alpha]) + c\rho([\alpha - 1]) = 0. \quad (2)$$

Repeating the above argument for the element $\alpha^{-1} \in D_1^*$ (which is also central and $\alpha^{-1} \neq 1$), we show that there exists $0 \leq d \leq n_1$ such that

$$(n_1 - d)\rho([\alpha^{-1}]) + d\rho([\alpha^{-1} - 1]) = 0. \quad (3)$$

Since $2\rho([-1]) = \rho([1]) = 0$ and $K_0(I)$ is torsion-free, we have $\rho([-1]) = 0$. Therefore $\rho([\alpha^{-1} - 1]) = \rho([-\alpha^{-1}(\alpha - 1)]) = \rho([\alpha - 1]) + \rho([\alpha^{-1}]) = \rho([\alpha - 1]) - \rho([\alpha])$. Putting this to Eq. (3), we get

$$-n_1\rho([\alpha]) + d\rho([\alpha - 1]) = 0.$$

Multiplying this equation by c and subtracting from (2) multiplied by d , we have

$$((n_1 - c)d + n_1c)\rho([\alpha]) = 0.$$

As $K_0(I)$ has no torsion, we have either $\rho([\alpha]) = 0$ or $(n_1 - c)d + n_1c = 0$. In the first case the proof is finished. In the second case, since $n_1c \geq 0$ and $(n_1 - c)d \geq 0$, we have $c = 0$ and $d = 0$. This again gives $n_1\rho([\alpha]) = 0$, therefore $\rho([\alpha]) = 0$ as desired. This concludes the proof of the theorem. \square

Corollary 3.6. *Let R and I be such that the assumptions of Theorem 3.5 hold. Then there is an isomorphism $K_0(R) \cong K_0(I) \oplus \mathbb{Z}^r$.*

Proof. Recall the exact sequence

$$K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(I) \rightarrow K_0(R) \rightarrow K_0(R/I).$$

By Theorem 3.5 the map q_* is surjective. This implies that $\partial = 0$, or equivalently, $K_0(I) \rightarrow K_0(R)$ is injective. The mapping $K_0(R) \rightarrow K_0(R/I)$ is surjective by [1, Theorem 3.5]. Thus we have the exact sequence

$$0 \rightarrow K_0(I) \rightarrow K_0(R) \rightarrow K_0(R/I) \rightarrow 0.$$

Now, since $k = R/I$ is semiperfect, we have $K_0(k) \cong K_0(k/J(k)) \cong \mathbb{Z}^r$, where r is as in Theorem 3.5. The conclusion follows from the fact that \mathbb{Z}^r is a projective \mathbb{Z} -module. \square

It is known that if R is clean then $M_n(R)$ is clean for each n (cf. [11]), but the converse is still an open question. Thus the assumption that $M_n(R)$ is clean for some n , seems to be weaker than the assumption that R is clean.

Corollary 3.7. *Let R and I be such that the assumptions of Theorem 3.5 hold, except that we assume that $M_n(R)$ is clean for some n , rather than R . Then the induced mapping $K_1(R) \rightarrow K_1(R/I)$ is surjective and there is an isomorphism $K_0(R) \cong K_0(I) \oplus \mathbb{Z}^r$.*

Proof. Suppose that $M_n(R)$ is a clean ring for some $n \geq 1$. By assumption, $K_0(I)$ is torsion-free and $k = R/I$ is semiperfect such that $k/J(k) \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, with each D_i algebraic over its center. We apply Theorem 3.5 for the ring $S = M_n(R)$ and the ideal $J = M_n(I) \leq S$. Observe that $K_0(J) \cong K_0(I)$ is torsion-free and $k' = S/J \cong M_n(k)$ is semiperfect such that $k'/J(k') \cong M_n(k/J(k)) \cong$

$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, with each D_i algebraic over its center. Hence by Theorem 3.5, the canonical map $q_* : K_1(S) \rightarrow K_1(S/J)$ is surjective. We have a commutative diagram

$$\begin{array}{ccc} K_1(S) & \xrightarrow{q_*} & K_1(S/J) \\ \cong \downarrow & & \cong \downarrow \\ K_1(R) & \xrightarrow{q_*} & K_1(R/I) \end{array}$$

hence the map $q_* : K_1(R) \rightarrow K_1(R/I)$ is also surjective. This proves the first part of the statement; the proof of the second part is the same as the proof of Corollary 3.6. \square

Perera has proved that if I is a *separative* exchange ideal of a ring R , then a unit $\alpha \in U(R/I)$ can be lifted modulo I if and only if $\partial([\alpha]) = 0$, where $\partial : K_1(R/I) \rightarrow K_0(I)$ denotes the connecting map [18, Theorem 2.4]. The separativity condition holds very widely in the class of exchange rings, maybe it even holds for all exchange rings (see [3,18]). As a corollary of Perera's result, we have:

Corollary 3.8. *Let R and I be as in Corollary 3.7. If I is separative, then units in R/I can be lifted to units in R .*

Proof. First observe that, since R is an exchange ring, it follows that I is exchange (see [1, Example (1) on p. 412]). By Corollary 3.7 the canonical map $K_1(R) \rightarrow K_1(R/I)$ is surjective, thus the connecting map $\partial : K_1(R/I) \rightarrow K_0(I)$ is zero. Now the conclusion follows immediately from [18, Theorem 2.4]. \square

Corollary 3.9. *Let R be a clean ring with an ideal I such that $K_0(I)$ is torsion-free and R/I is a finite dimensional algebra over a field. Then the induced map $K_1(R) \rightarrow K_1(R/I)$ is surjective. If I is separative, then units in R/I can be lifted modulo I .*

Proof. Follows from Wedderburn–Artin theorem and Theorem 3.5, and the fact that every finite dimensional algebra is algebraic. The second part follows from Perera's theorem. \square

The problem whether $M_n(R)$ being clean implies that R is clean, is still open. The counter-example to this question would also have to be an example of a non-clean exchange ring. So far Bergman's example is the only known such example, but it has not been verified yet whether the $n \times n$ matrix ring over that ring is clean, for any $n \geq 2$. Using the above results, we can now carry out this verification.

Corollary 3.10. *The $n \times n$ matrix ring over Bergman's example is not clean, for every $n \geq 1$.*

Proof. We use terminology developed in [20, Example 3.1]. Note that the ring R in that example is isomorphic to the opposite ring of Bergman's example, therefore it suffices to show the statement for that ring. Let R and I be as in [20]. As $R/I \cong F((X))$ is a field and units in R/I clearly do not lift to units in R , by Corollary 3.8 we only need to prove that I is a separative ring and $K_0(I)$ is torsion-free.

First we see that $K_0(I) \cong \mathbb{Z}$. This fact is a matter of routine computation and has been observed, for example, in [14, Example on p. 306]. Thus we omit the proof.

The fact that I is separative has been observed, without proof, by Ara et al. in [3, Example (3) on p. 116]. However, this time we decide to write down the proof since verifying separativity is not as common routine as is computing K_0 groups.

By [3, Lemma 4.1], it suffices to prove that $eIe = eRe$ is a separative (unital) ring for every idempotent $e \in I$. Take any idempotent $e \in I$. Then e is a matrix of the form

$$e = \begin{pmatrix} E_0 & E_0X \\ 0 & 0 \end{pmatrix},$$

where E_0 is some $n \times n$ idempotent matrix over F and X is an infinite matrix with n rows. A straightforward computation shows that eRe is just

$$eRe = \left\{ \begin{pmatrix} E_0 T E_0 & E_0 T E_0 X \\ 0 & 0 \end{pmatrix}; T \in M_n(F) \right\},$$

and that this ring is isomorphic to $E_0 M_n(F) E_0$. Now we have $E_0 M_n(F) E_0 \cong M_{n_1}(F)$ for some n_1 ; but this ring is clearly separative since separative exchange rings are closed under taking matrix rings [3, Proposition 2.2]. This finishes the proof. \square

Remark 3.11. (1) Corollary 3.10 could also be verified directly, without the use of K -theory. Of course, the main idea behind this direct proof would still follow the proof of Theorem 3.5. The key observation would be the fact that for any invertible element U in $M_n(R)$, one has

$$\text{index}(\det(\bar{U})) = 0,$$

where \bar{U} is the homomorphic image of U in $M_n(F((X)))$ and $\text{index} : F((X))^* \rightarrow \mathbb{Z}$ is a group homomorphism defined via

$$\text{index} \left(\sum_{i=n_0}^{\infty} a_i X^i \right) = \inf \{i \mid a_i \neq 0\}.$$

In fact, this index mapping turns out to be precisely the corresponding connecting map $F((X))^* \cong K_1(R/I) \rightarrow K_0(I) \cong \mathbb{Z}$.

(2) Lemma 3.3 and Lemma 3.4 could be also proved for a wider class of *weakly clean* rings (see [20, Definition 2.3] for the definition of weakly clean rings). Note that every clean ring is weakly clean and every weakly clean ring is exchange. However, for other results in this paper, the assumption that R is clean (rather than weakly clean) is crucial. The counter-example that shows that the weakly clean property does not suffice is, of course, Bergman's ring.

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